Powers from products of terms in progressions with gaps

by

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Dedicated to the memory of Andrzej Schinzel

1. Introduction. If \( f(x) \) is a polynomial with integer coefficients, degree \( k \geq 2 \), and, say, no repeated complex roots, then the (superelliptic) Diophantine equation

\[
f(x) = z^\ell
\]

has at most finitely many solutions in integers \( x, |z| > 1 \) and \( \ell \geq \max\{2, 5-k\} \), via work of Schinzel and Tijdeman [22]. Underlying this result are lower bounds for linear forms in complex logarithms. If we replace the polynomial \( f \) with a binary form \( f(x, y) \), now assuming that this form has degree \( k \geq 3 \), then analogous results about the equation

\[
f(x, y) = z^\ell
\]

are, in general, much harder to come by (and, in particular, techniques from transcendental number theory do not apparently lead to finiteness results). Indicative of the challenges presented by such equations, one may observe that (1.2) in the case \( f(x, y) = xy(x+y) \) is essentially equivalent to Fermat’s Last Theorem.

Even equation (1.1) is, in practice, beyond the reach of current computational techniques for most polynomials \( f(x) \) – a simple example where we cannot as of now completely determine the set of solutions to (1.1) corresponds to \( f(x) = x^2 - 2 \). For certain families of polynomials, however, equation (1.1) may be treated via combinatorial methods, avoiding the use of bounds for linear forms. A particularly beautiful example of this is a remarkable result of Erdős and Selfridge [12] that the product of at least two
consecutive positive integers is never a perfect power of an integer, corresponding to equation (1.1) with

\[(1.3) \quad f(x) = x(x + 1) \ldots (x + k - 1), \quad k \geq 2.\]

There is an extensive literature exploring the limits of these arguments, when one replaces \(f(x)\) with a product of some subset of the linear terms \(x, x + 1, \ldots, x + k - 1\), say

\[g(x) = (x + y_1) \ldots (x + y_t),\]

where \(y_1, \ldots, y_t\) are distinct integers in the interval \([0, k - 1]\); see e.g. [11, 23, 24, 27]. By way of example, in the last of these papers, it is proven that, given \(\epsilon > 0\), there exists an effectively computable upper bound \(c(\epsilon)\) upon \(k\) and \(\ell\) for which the equation

\[(1.4) \quad (x + y_1) \ldots (x + y_t) = z^\ell\]

has nontrivial solutions, provided only that

\[(1.5) \quad t \geq k\ell^{-1/3+\epsilon} + \pi(k) + 2.\]

There is an even more extensive literature on attempts to generalize the result of Erdős and Selfridge to equation (1.2) with

\[(1.6) \quad f(x, y) = x(x + y) \ldots (x + (k - 1)y), \quad k \geq 3;\]

see e.g. [3, 5, 13, 14, 15, 16, 17, 20, 21, 25, 26, 28, 29]. As an intermediate result, one might consider solutions to

\[(1.7) \quad x(x + 1) \ldots (x + k - 1) = z^\ell\]

in rational rather than integral values of \(x\) – these correspond to the special case of the equation

\[x(x + y) \ldots (x + (k - 1)y) = z^\ell\]

with \(y\) an \(\ell\)th power. The finiteness of the number of solutions to equation (1.7) with \(x\) rational was established, for fixed \(k\), by the author and Siksek [4] and extended to equation (1.4), for

\[(1.8) \quad t > k - 0.26\sqrt{\frac{k}{\log k}},\]

by Edis [10]. In the special cases \(t = k - 1\) and \(4 \leq k \leq 8\), Das, Laishram and Saradha [9, Theorem 1.2] essentially solved (1.4) completely (here, we state a slightly simplified version of their result, focussing on exponents \(\ell > 3\)).

**Proposition 1.1** (Das, Laishram, Saradha (2023)). If \(4 \leq k \leq 8\) and \(0 \leq y_1 < \cdots < y_{k-1} \leq k - 1\), then equation (1.4) has no solutions in nonzero rational numbers \(x\) and \(z\) with \(\ell > 3\).
In this paper, we will concentrate on the more general equation (1.2) with $f(x, y)$ as in (1.6), only with a single term omitted from the product, i.e. with

$$(1.9) \quad f(x, y) = f_i(x, y) = \prod_{0 \leq j \leq k-1 \atop j \neq i}(x + jy),$$

where $i \in \{0, 1, \ldots, k-1\}$. For larger values of $k$, it is possible to extend the results of [5] to deduce finiteness results with rather more terms omitted, perhaps even as many as in (1.8).

Our main result is the following, which for $5 \leq k \leq 8$ generalizes and sharpens Proposition 1.1 for odd prime values of $\ell$.

**Theorem 1.2.** If $5 \leq k \leq 8$ and $i \in \{0, 1, \ldots, k-1\}$, then the only solutions to the equation

$$(1.10) \quad \prod_{0 \leq j \leq k-1 \atop j \neq i}(x + jy) = z^\ell$$

in integers $x, y, z$ and $\ell$ with $\gcd(x, y) = 1$, $\ell \geq 3$ prime and $yz \neq 0$ are given by

$$(1.11) \quad (k, i, x, y, z^\ell) = (5, 1, \pm 8, \mp 3, 2^6) \quad \text{and} \quad (5, 3, \pm 4, \mp 3, 2^6).$$

Theorem 1.2 may be readily extended to include the cases $\ell = 2$ and $\ell = 4$, the latter unconditionally, and the former under certain constraints (equation (1.10) has, for example, infinitely many coprime solutions with $k = 5$, $i \in \{1, 3\}$ and $\ell = 2$). Such results require use of Chabauty-type techniques to find rational points on certain genus 2 curves. Larger values of $k$ may also be treated and situations where there are more terms omitted from the product are accessible to these methods (by way of example, one may omit two terms from a product of eight consecutive terms in arithmetic progression and still resolve the corresponding equations for all $k \geq 2$).

The outline of this paper is the following. In Section 2 we will discuss how to pass from solutions to (1.10) to ternary equations, and then describe a variety of techniques for resolving these, primarily based upon the modularity of Galois representations attached to corresponding Frey–Hellegouarch curves. In Section 3 we prove Theorem 1.2 for prime $\ell \geq 7$ via these techniques. Finally, in Section 4 we finish the proof by treating the cases $\ell \in \{3, 5\}$.

Previous work on problems of this sort has been based upon reducing equations like (1.10) to ternary equations of signature $(\ell, \ell, \ell), (\ell, \ell, 2)$ or $(\ell, \ell, 3)$, which have subsequently been shown to have no solutions. The main novelty in the current paper is that we are able to treat more general situations where, after reduction, we are genuinely unable to completely solve the resulting ternary equations. We are, however, able to reach our desired
conclusion by using additional local information at a number of auxiliary primes. In what follows, we will try to indicate precisely where these new arguments are employed.

2. The plan of attack. An important initial observation is that the existence of a solution to the equation

$$\prod_{0 \leq j \leq k-1, j \neq i} (x + jy) = z^\ell$$

with $\gcd(x, y) = 1$ implies that each term in the product on the left-hand side of the equation may be written as

$$x + jy = b_j z_j^\ell,$$

where $b_j, z_j \in \mathbb{Z}$ and the greatest prime divisor $P(b_j)$ of $b_j$ satisfies $P(b_j) < k$. By appealing to relations between the terms $x + jy$, for various values of $j$, we are able to deduce the existence of solutions to certain ternary Diophantine equations of the shape

$$AX^\ell + BY^\ell = CZ^m$$

where $m \in \{2, 3, \ell\}$.

To discuss these relations, we will have need of some notation, borrowed from [2]. Let $j$ be a positive integer. Denote by

$$\{a_1, \ldots, a_j; b_1, \ldots, b_j; c_1, \ldots, c_j\}$$

three $j$-tuples of integers with the property that there exist integers $\alpha, \beta$ and $\gamma$, not all zero, satisfying the polynomial identity

$$\alpha \prod_{i=1}^{j} (x + a_i) + \beta \prod_{i=1}^{j} (x + b_i) + \gamma \prod_{i=1}^{j} (x + c_i) = 0. \quad (2.3)$$

Further, by

$$[a_1, \ldots, a_j; b_1, \ldots, b_j]$$

we mean two distinct $j$-tuples of integers satisfying the polynomial identity

$$\prod_{i=1}^{j} (x + a_i) - \prod_{i=1}^{j} (x + b_i) = \prod_{i=1}^{j} a_i - \prod_{i=1}^{j} b_i. \quad (2.4)$$

Here and henceforth, we will write $\nu_p(m)$ for the largest integer $j$ such that $p^j$ divides a nonzero integer $m$. For our purposes, we will mostly appeal to tuples $\{r; s; t\}$, where $0 \leq r < s < t \leq k - 1$ are integers and we assume that $i \notin \{r, s, t\}$. These tuples correspond via the identity

$$(t - s)(x + ry) + (r - t)(x + sy) + (s - r)(x + ty) = 0$$
to ternary equations of the shape (2.2) with \( m = \ell \). In particular, we can associate to a solution to (1.10) a rational point on the nonsingular curve (in \( \mathbb{P}^{k-2} \)) \( C_b = C_{b,k,\ell} \), defined via the equations

\[
(2.5) \quad (t - s)b_\ell z_\ell^k + (r - t)b_s z_s^k + (s - r)b_t z_t^k = 0,
\]

where \( \{r, s, t\} \) runs through all 3-element subsets of \( \{0, \ldots, k-1\} \setminus \{i\} \). Here, \( b \) is shorthand for \( (b_0, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{k-1}) \).

An important second observation is that, if \( p \) is any prime number and \( x \) and \( y \) are integers, necessarily either \( p \mid y \), or there exists an integer \( i_p \) satisfying \( 0 \leq i_p \leq p - 1 \) such that \( p \mid x + i_p y \). Define an elliptic curve \( E_{r,s,t} \) via the equation

\[
(2.6) \quad E_{r,s,t} : Y^2 = X^3 + aX^2 + bX,
\]

where we set

\[
a = 2s - r - t \quad \text{and} \quad b = (s - r)(s - t), \quad \text{if} \ p \mid y,
\]

and

\[
a = (r + t - 2s)i_p + st + rs - 2rt, \quad b = (s - r)(s - t)(t - i_p)(r - i_p), \quad \text{otherwise}.
\]

We will also define

\[
g(r, s, t) = \gcd((t - r)b_r, (s - r)b_t),
\]

\[
(2.7) \quad M(r, s, t) = g(r, s, t)^{-3}(t - r)(t - s)(s - r)b_rb_sb_t,
\]

\[
(2.8) \quad N_{r,s,t} = 2^\kappa \prod_{p\mid M(r,s,t), p \geq 2} p,
\]

where

\[
\kappa = \begin{cases} 
0 & \text{if } \nu_2(M(r,s,t)) = 4, \\
1 & \text{if } \nu_2(M(r,s,t)) \geq 5 \text{ or } z_r z_s z_t \equiv 0 \pmod{2}, \\
3 & \text{if } \nu_2(M(r,s,t)) \in \{2, 3\}, \\
5 & \text{if } \nu_2(M(r,s,t)) = 1.
\end{cases}
\]

Our results depend crucially upon the following.

**Proposition 2.1.** Suppose that \( k \geq 4 \) is an integer, that \( x \) and \( y \) are coprime nonzero integers, that \( \ell \geq 5 \) is prime and that \( r, s \) and \( t \) are integers with \( 0 \leq r < s < t \leq k - 1 \) satisfying (2.1) for each \( j \in \{r, s, t\} \). Suppose that \( p \) is an odd prime, define \( i_p \) as before if \( p \nmid y \), \( E_{r,s,t} \) as in (2.6), \( M(r,s,t) \) as in (2.7) and \( N_{r,s,t} \) as in (2.8). Then there exists a weight 2, level \( N_{r,s,t} \) cuspidal newform \( f \) with coefficients \( c_p(f) \) such that there is a prime ideal \( \mathfrak{p} \mid \ell \) in \( K_f / \mathbb{Q} \) for which

\[
(2.9) \quad \pm a_p(E_{r,s,t}) \equiv c_p(f) \pmod{\mathfrak{p}} \text{ if } p \nmid \ell N_{r,s,t} \text{ and either } i_p \not\in \{r, s, t\} \text{ or } p \mid y,
\]
and
\[(2.10) \quad \pm (p + 1) \equiv c_p(f) \pmod{f} \quad \text{if } p \nmid \ell N_{r,s,t}, \ i_p \in \{r, s, t\}.
\]

If \(K_f = \mathbb{Q}\), we have the slightly stronger conclusion that
\[(2.11) \quad \pm a_p(E_{r,s,t}) \equiv c_p(f) \pmod{f} \quad \text{if } p \nmid N_{r,s,t} \text{ and}
\]
either \(i_p \notin \{r, s, t\}\) or \(p | y\),

and
\[(2.12) \quad \pm (p + 1) \equiv c_p(f) \pmod{f} \quad \text{if } p \nmid N_{r,s,t}, \ i_p \in \{r, s, t\}.
\]

Proof. Define
\[g_1 = g_1(r, s, t) = g(r, s, t)^{-1}(s - r)(x + ty),
\]
\[g_2 = g_2(r, s, t) = g(r, s, t)^{-1}(r - t)(x + ry),
\]
\[g_3 = g_3(r, s, t) = g(r, s, t)^{-1}(t - r)(x + sy),
\]
so that \(g_1, g_2\) and \(g_3\) are coprime nonzero integers satisfying \(g_1 = g_2 + g_3\).

Set
\[\delta = \delta(r, s, t) = \begin{cases} 1 & \text{if } g_1 \equiv g_2 \equiv 1 \pmod{4}, \text{ or } g_1 \equiv 3 \pmod{4}, \ g_2 \equiv 0 \pmod{4}, \text{ or } g_1 \equiv 0 \pmod{4}, \ g_2 \equiv 3 \pmod{4}, \\ 0 & \text{otherwise,} \end{cases}
\]

and define a corresponding Frey–Hellegouarch elliptic curve
\[(2.13) \quad F_{r,s,t} : Y^2 = X(X + (-1)^\delta g_1)(X + (-1)^\delta g_2).
\]

Then, from Kraus [19], there exists a weight 2, level \(N_{r,s,t}\) cuspidal newform \(f\) with coefficients \(c_p(f)\) generating the number field \(K_f/\mathbb{Q}\), and with the property that if \(p\) is any prime, there is a prime ideal \(\ell \mid \ell\) in \(K_f\) for which
\[(2.14) \quad a_p(F_{r,s,t}) \equiv c_p(f) \pmod{f} \quad \text{if } p \nmid \ell N_{r,s,t} z_r z_s z_t,
\]
\[(2.15) \quad \pm (p + 1) \equiv c_p(f) \pmod{f} \quad \text{if } p \nmid \ell N_{r,s,t}, \ p \mid z_r z_s z_t.
\]

If \(K_f = \mathbb{Q}\), then
\[(2.16) \quad a_p(F_{r,s,t}) \equiv c_p(f) \pmod{f} \quad \text{if } p \nmid N_{r,s,t} z_r z_s z_t,
\]
\[(2.17) \quad \pm (p + 1) \equiv c_p(f) \pmod{f} \quad \text{if } p \nmid N_{r,s,t}, \ p \mid z_r z_s z_t.
\]

Proposition 2.1 then follows from the definitions of \(i_p\) and \(E_{r,s,t}\), whence if \(p\) is a prime of good reduction for \(F_{r,s,t}\), then
\[(2.18) \quad a_p(F_{r,s,t}) = \left(\frac{(-1)^\delta x g(r, s, t)}{p}\right) a_p(E_{r,s,t}) \quad \text{if } p \nmid y,
\]
\[(2.19) \quad a_p(F_{r,s,t}) = \left(\frac{(-1)^\delta g g(r, s, t)}{p}\right) a_p(E_{r,s,t}) \quad \text{if } p \nmid y,
\]
and hence
\[(2.20) \quad a_p(F_{r,s,t}) = \pm a_p(E_{r,s,t}). \]

The real value of Proposition 2.1 is that, since \(E_{r,s,t}\) and \(F_{r,s,t}\) are defined over \(\mathbb{Q}\) and have full rational 2-torsion, if \(p \nmid N_{r,s,t}\) one can deduce the existence of a constant \(\ell_0 = \ell_0(N_{r,s,t})\) such that, for all prime \(\ell \geq \ell_0\), we necessarily have \(K_f = \mathbb{Q}\) and congruence (2.11) (i.e. \(p \nmid z_r z_s z_t\)), where the form \(f\) corresponds to an elliptic curve of conductor \(N_{r,s,t}\) which is isogenous to a curve with full rational 2-torsion. For a given \(N_{r,s,t}\), it is a relatively easy matter to quantify this statement, at least provided \(N_{r,s,t}\) is not too large. The existence of elliptic curves over \(\mathbb{Q}\) with full rational 2-torsion and conductor \(N = 2^k N_0\), where \(N_0\) is odd and squarefree, is equivalent to the existence of solutions to the \(S\)-unit equation \(x + y = 1\) with \(S = \{p : p | 2N_0\}\). Regarding this, it is worth mentioning that such equations have been completely solved for \(S\) containing all primes up to and including 53 by von Kanel and Matschke [18].

Since there are unlikely to be any elliptic curves with full rational 2-torsion for a typical conductor \(N\) (a property we can verify for a given \(N \leq 500000\) by simply checking the LMFDB), we can actually eliminate a number of \(N_{r,s,t}\) from consideration.

**Proposition 2.2.** With notation as previously, if \(\ell \geq 7\) is prime, then \(N_{r,s,t} \geq 15\) and \(N_{r,s,t} \not\in S\), where
\[S = \{22, 35, 77, 88, 110, 160, 224, 280, 352, 2464\}.\]

Further, if \(N_{r,s,t} \in T,\) where
\[T = \{15, 21, 24, 30, 33, 40, 42, 55, 56, 66, 70, 96, 105, 120, 154, 168, 210, 231, 264, 330, 440, 462, 480, 616, 672, 840, 1056, 1120, 1155, 1320, 1760, 2310, 7392\},\]

then for each \(p \in \{3, 5, 7, 11, 13\}\) with \(p \nmid N_{r,s,t}\), we have \(p \nmid z_r z_s z_t\).

**Proof.** This is a straightforward computation using Magma; code for this is available at [www.math.ubc.ca/~bennett/Prop2.2-check](http://www.math.ubc.ca/~bennett/Prop2.2-check).

Another very valuable result for us is the following.

**Proposition 2.3.** With notation as previously, if \(\ell \geq 5\) is prime and \(N_{r,s,t} = 32\), then necessarily
\[\{ |g_1|, |g_2|, |g_3| \} = \{1, 1, 2\}.\]

**Proof.** This is almost immediate from a result of Darmon and Merel [8] Main Theorem].

Given a fixed class of tuples \((b_0, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{k-1})\), we can check to see if at least one corresponding \(N_{r,s,t}\) lies in the set \(S\) defined in Proposi-
tion 2.2. Previous work on equation (1.1) for fixed values of \( k \) has proceeded along these lines, using various arguments to reduce to a small set of possible tuples, each of which can be treated via Propositions 2.2 or 2.3. The key idea in our new approach is to extract local information from various auxiliary primes \( p \) (including potentially some \( p > k \)), using in an essential way the fact that a solution to (1.1) corresponds not just to a number of solutions to various ternary equations, but that local information must be consistent for these simultaneous equations. Towards this end, we need to carefully keep track of this information, in terms of the values of \( a_p(F_{r,s,t}) \) for primes \( p \nmid N_{r,s,t} \).

Finally, we will have use of existing results in the literature on products of terms in progressions without omitted terms. The following summarizes Theorem 1.2 of [3] (in case \( k \in \{3, 4, 7, 8\}, \ell \geq 7 \)), Theorem 1.1 of [1] (in case \( k \in \{5, 6\}, \ell \geq 7 \)), Theorem 2.2 of [16] (for \( \ell = 3 \)) and Theorem 2 of [15] (for \( \ell = 5 \)), restricted to the cases \( k \leq 8 \) and prime \( \ell \geq 3 \).

**Proposition 2.4.** Suppose that \( k \) and \( \ell \) are integers with \( 3 \leq k \leq 8 \) and \( \ell \geq 3 \) prime, and that \( x \) and \( y \) are coprime integers with \( y > 0 \). If, further, \( b \) and \( z \) are nonzero integers with \( P(b) \leq P_{k,\ell} \) where \( P_{k,\ell} \) is as follows:

<table>
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<tr>
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then the only solutions to the Diophantine equation

\[
x(x + y) \cdots (x + (k - 1)y) = bz^l
\]

either have

\( y \leq 5 \) and \( |x + iy| \leq 15 \) for some \( i = 0, 1, \ldots, k - 1 \),

or have \((x, y, k)\) in the following list:

\((-40, 13, 5), (-40, 13, 6), (-39, 7, 7), (-39, 7, 8), (-36, 11, 5), (-32, 11, 5),\)
\((-32, 7, 6), (-32, 7, 7), (-27, 11, 4), (-27, 13, 5), (-27, 7, 7), (-27, 7, 8),\)
\((-25, 13, 5), (-25, 7, 6), (-25, 9, 6), (-25, 13, 6), (-22, 7, 7), (-22, 7, 8),\)
\((-20, 9, 6), (-20, 7, 7), (-16, 7, 3), (-16, 7, 4), (-16, 7, 5), (-15, 7, 4),\)
\((-15, 7, 7), (-12, 7, 4), (-12, 7, 5), (-12, 11, 5), (-12, 13, 5), (-10, 7, 6),\)
\((-10, 7, 7), (-10, 7, 8), (-9, 7, 4), (-8, 7, 3), (-8, 11, 5), (-6, 7, 3),\)
\((-6, 7, 4), (-6, 11, 4), (-5, 7, 4), (-3, 7, 6), (-3, 7, 7), (2, 7, 3).\)
3. Proof of Theorem 1.2 for prime $\ell \geq 7$. In what follows in this section, we will always suppose that $\ell \geq 7$ is prime. In all cases, from Proposition 2.4 necessarily $i \not\in \{0, k-1\}$, and hence, via symmetry, we may assume that $1 \leq i \leq (k-1)/2$.

3.1. $k = 5$. We have $i = 1$ or $i = 2$, whence we consider either

\[(3.1) \quad x(x + 2y)(x + 3y)(x + 4y) = z^\ell\]

or

\[(3.2) \quad x(x + y)(x + 3y)(x + 4y) = z^\ell.\]

3.1.1. The case $i = 1$. Suppose that we have a solution to equation (3.1) in integers $x, y, z$ and $\ell$ with $\gcd(x, y) = 1$, $yz \neq 0$ and prime $\ell \geq 7$. If $3 \nmid x$, then there necessarily exist integers $\alpha$ and $z_0 \neq 0$ such that

\[(3.3) \quad (x + 2y)(x + 3y)(x + 4y) = 2^\alpha z_0^\ell,\]

whereby, from Proposition 2.4 (or 3 Theorem 1.2), we have

\[(x, y) \in \{(-10, 3), (-8, 3), (10, -3), (8, -3)\},\]

in each case contradicting (3.1) and $\ell \geq 7$.

We may thus suppose that $3 \mid x$. Since $\ell \geq 7$ is prime, if $8 \mid x$ then necessarily, since $\nu_2(x) + 3 \equiv 0 \pmod{\ell}$, it follows that $\nu_2(x) \geq \ell - 3$, and so if $\ell \geq 11$ then $N_{0, 2, 3} = 6$, contradicting Proposition 2.2, since there are no weight 2 cuspidal newforms of level 6. Similarly, if $4 \mid x$ (so that $\nu_2(x + 4y) \geq \ell - 3$), again we have $N_{0, 3, 4} = 6$ for $\ell \geq 11$. If $2 \mid x$, then $\nu_2(x + 2y) \geq \ell - 2$ and hence $N_{0, 2, 4} = 6$ for $\ell \geq 7$. If $2 \mid x + 3y$, then $N_{0, 3, 4} = 6$ for $\ell \geq 7$. We may thus suppose that either

(i) $2 \mid y$, $3 \mid x$, $\ell \geq 7$, or
(ii) $\nu_2(x) = 2$, $\nu_2(x + 4y) = 4$, $3 \mid x$, $\ell = 7$, or
(iii) $\nu_2(x) = 4$, $\nu_2(x + 4y) = 2$, $3 \mid x$, $\ell = 7$.

In case (ii), the identity $\{0; 2; 3\}$ leads to one of the ternary equations

\[2z_2^7 + 3^5z_3^7 = z_2^7 \quad \text{or} \quad 2 \cdot 3^5z_0^7 + z_3^7 = z_2^7,\]

where $7 \nmid z_0z_2z_3$, a contradiction modulo 49 in either case. Similarly, in case (iii), from $\{0; 3; 4\}$, we have either

\[4z_0^7 + z_4^7 = 3^5z_3^7 \quad \text{or} \quad 4 \cdot 3^5z_0^7 + z_4^7 = z_3^7,\]

with $7 \nmid z_0z_3z_4$, again a contradiction modulo 49. We are thus in case (i), whereby we have

\[
\begin{array}{|c|c|c|}
\hline
r, s, t & N_{r, s, t} & r, s, t & N_{r, s, t} \\
\hline
0, 2, 3 & 96 & 0, 2, 4 & 96 \\
0, 3, 4 & 24 & \\
\hline
\end{array}
\]
Notice that, in this situation, we are not actually able to resolve the corresponding ternary equations without additional input. We provide this through consideration of the value \( i_7 \) (which is able to distinguish, up to twist, elliptic curves of conductor 24 from those of conductor 96; observe that all weight 2 cuspidal newforms of levels 24 and 96 have \( K_f = \mathbb{Q} \)).

Appealing to Proposition 2.2, we may suppose that \( p | y \), or that \( i_p = i \), or that \( i_p \geq k \), for each \( p \in \{5, 7, 11, 13\} \). In particular, it follows that \( 7 | y(x + y)(x + 5y)(x + 6y) \). We compute that
\[
a_7(E_{0,2,4}) = 0 \quad \text{if} \quad i_7 \in \{5, 6\} \text{ or } 7 | y,
\]
while
\[
a_7(E_{2,3,4}) = 0 \quad \text{if} \quad i_7 = 1,
\]
contradicting (2.11) and \( N_{0,2,4} = N_{2,3,4} = 96 \) (since each elliptic curve \( E \) of conductor 96 has \( c_7(E) = \pm 4 \)).

3.1.2. The case \( i = 2 \). Next, suppose that we have a solution to equation (3.2) in integers \( x, y, z \) and \( \ell \) with \( \gcd(x, y) = 1, yz \neq 0 \) and \( \ell \geq 7 \) prime. For this equation, we will begin by appealing to an identity of the shape (2.4), specifically, in our shorthand, \( [1, 3; 0, 4] \), corresponding via
\[
(x + y)(x + 3y) - x(x + 4y) = 3y^2
\]
to a ternary equation of the shape \( AX^\ell + BY^\ell = CZ^2 \). If \( 2 | x(x + y) \), then \( C \in \{1, 3\} \) and \( AB = 2^\alpha 3^\beta \), with \( \alpha \geq \ell \geq 7 \). For example, if \( 3 | d(x + 2y) \), we obtain an equation of the shape \( X^\ell + Y^\ell = 3Z^2 \) with \( 2 | XY \), contradicting [6] Theorem 1.1]. If \( 3 | x(x + y) \), our equation is of the shape (after dividing by 3) \( X^\ell + 3^{\ell - 2}Y^\ell = Z^2 \) with, again, \( 2 | XY \). In this case, we directly contradict [6] Theorem 1.5] if \( \ell \geq 11 \), and the proof of the same theorem (in this restricted case), if \( \ell = 7 \).

We may thus suppose that \( 2 | y \). If, further \( 3 \nmid x(x + y) \), the same identity yields a solution to the equation \( X^\ell + Y^\ell = 3Z^2 \) in nonzero integers, again contradicting [6] Theorem 1.1]. It follows that \( 3 | x(x + y) \), whereby we may employ the identity \( [1, 1, 4; 0, 3, 3] \), leading to a nonzero solution to a ternary equation of the shape \( X^\ell + 3^\beta Y^\ell = 4Z^3 \) with \( \beta \geq \ell \geq 7 \), contradicting [7] Theorem 1.5].

3.2. \( k = 6 \)

3.2.1. \( i = 1 \). Suppose we have
\[
x(x + 2y)(x + 3y)(x + 4y)(x + 5y) = z^\ell
\]
for integers \( x, y, z \) and \( \ell \) with \( \gcd(x, y) = 1, yz \neq 0 \) and prime \( \ell \geq 7 \). If \( 5 | x \), then the identity \( [2, 3; 0, 5] \) leads to a nonzero solution to \( AX^\ell + BY^\ell = CZ^2 \) with \( P(ABC) \leq 3 \) and \( 5 | XY \), contrary to [3] Proposition 3.1]. We thus have \( P(b_0b_2b_3b_4b_5) \leq 3 \). If \( 3 \nmid x \), we may apply Proposition 2.4 to
(x + 2y)(x + 3y)(x + 4y)(x + 5y) to deduce a contradiction from \( \ell \geq 7 \). We may thus suppose that \( 3 \mid x \). Since \( N_{3,4,5} \mid 6 \) if \( 8 \mid x + 4y \), \( N_{2,3,4} \mid 6 \) if \( 8 \mid x + 3y \), \( N_{2,3,5} \mid 6 \) if \( 8 \mid x + 2y \), and \( N_{2,4,5} \mid 6 \) if \( 8 \mid x + 5y \), we may conclude that either \( 2 \mid y \) or \( 2^{\ell-3} \mid x \). In the latter case, from considering \( N_{0,2,3} \), necessarily \( \ell = 7 \) and \( \nu_2(x) = 4 \) (or else we find that \( N_{0,2,3} \mid 6 \)). As previously, from \( \{0; 3; 4\} \), we have either

\[
4z_0^7 + z_4^7 = 3^5z_3^7 \quad \text{or} \quad 4 \cdot 3^5z_0^7 + z_4^7 = z_3^7,
\]

with \( 7 \nmid z_0z_4z_5 \), once more a contradiction modulo 49.

We may thus suppose that \( 2 \nmid y \) and, from Proposition 2.2 that \( 7 \mid y(x + y)(x + 6y) \). But then \( N_{0,2,4} = N_{2,3,4} = 96 \), so that

\[
a_7(E_{0,2,4}) \equiv a_7(E_{2,3,4}) \equiv \pm 4 \pmod {\ell},
\]

contradicting \( a_7(E_{0,2,4}) = 0 \) (if \( 7 \mid y(x + 6y) \)) and \( a_7(E_{2,3,4}) = 0 \) (if \( 7 \mid x + y \)).

\subsection*{3.2.2. \( i = 2 \).} Suppose we have

\[
x(x + y)(x + 3y)(x + 4y)(x + 5y) = z^\ell,
\]

again for integers \( x, y, z \) and \( \ell \) with \( \gcd(x, y) = 1 \), \( yz \neq 0 \) and prime \( \ell \geq 7 \). As previously, if \( 5 \mid x \), then the identity [1, 4; 0, 5] leads to a nonzero solution to \( AX^\ell + BY^\ell = CZ^2 \) with \( P(ABC) \leq 3 \) and \( 5 \mid XY \), once more contrary to 3 Proposition 3.1. We may thus suppose that \( P(b_0b_1b_3b_4b_5) \leq 3 \). The identities \( \{0; 1; 3\}, \{0; 1; 4\}, \{0; 3; 4\}, \{3; 4; 5\} \) show that \( \nu_2(x + jy) \leq 2 \) for \( j = 0, 1, 3 \) and 4, respectively. It follows that either \( 2 \nmid y \) or \( 2^{\ell-3} \mid x + 5y \), whence, as before, we may conclude that \( \ell = 7 \) and \( \nu_2(x + 5y) = 4 \). Appealing to the identity \( \{3; 4; 5\} \) and Proposition 2.2 we may suppose that \( 3 \mid x(x + y) \).

If \( 3 \mid x \), \( \ell = 7 \) and \( \nu_2(x + 5y) = 4 \), then \( \{1; 4; 5\} \) leads to

\[
z_1^7 + 12z_5^7 = z_4^7,
\]

a contradiction modulo 29, unless \( 29 \mid z_1z_4z_5 \), which would in turn contradict \( 2.12 \) and \( N_{1,4,5} = 24 \). If, on the other hand, \( 3 \mid x + y, \ell = 7 \) and \( \nu_2(x + 5y) = 4, \{0; 1; 3\} \) leads, if \( 3 \parallel x + 4y \), to a solution to \( X^\ell + Y^\ell = 2Z^\ell \) with \( 3 \mid Z \), contrary to 8. We may thus suppose that \( 3 \parallel x + y \), whence \( \{0; 1; 4\} \) leads to

\[
z_0^7 + 3^5z_4^7 = 16z_1^7,
\]

which is again insoluble modulo 29, via \( 2.12 \) and the fact that \( N_{0,1,4} = 24 \).

It follows that \( 2 \nmid y \) and \( 3 \mid x(x + y) \). But then the identity [1, 1, 4; 0, 3, 3] leads to a nonzero solution to a ternary equation of the shape

\[
X^\ell + 3^\beta Y^\ell = 4Z^3 \quad \text{with} \quad \beta \geq \ell \geq 7, \quad \gcd(X, Y) = 1,
\]

once again contradicting 7 Theorem 1.5.

\subsection*{3.3. \( k = 7 \).} We have \( i \in \{1, 2, 3\} \), which we consider in turn.
3.3.1. \( i = 1 \). Suppose we have a solution of
\[
x(x + 2y)(x + 3y)(x + 4y)(x + 5y)(x + 6y) = z^\ell
\]
in integers \( x, y, z \) and \( \ell \) with \( \gcd(x, y) = 1, yz \neq 0 \) and \( \ell \geq 7 \) prime. If \( 5 \mid x \), we may apply Proposition 2.4 to
\[
(x + 2y)(x + 3y)(x + 4y)(x + 5y)(x + 6y)
\]
to deduce a contradiction. Thus \( 5 \mid x \) and hence we may appeal to the identity \([2, 3; 0, 5]\) to find a nonzero solution to the ternary equation
\[
(3.5) \quad AX^\ell + BY^\ell = CZ^2, \quad P(ABC) \leq 3, \quad 5 \mid XY, \gcd(X, Y) = 1,
\]
again contrary to \([3]\) Proposition 3.1.

3.3.2. \( i = 2 \). Suppose next that we have a solution of
\[
x(x + y)(x + 3y)(x + 4y)(x + 5y)(x + 6y) = z^\ell
\]
in integers \( x, y, z \) and \( \ell \), with \( \gcd(x, y) = 1, yz \neq 0 \) and \( \ell \geq 7 \) prime.

Necessarily \( \gcd(x(x + y), 15) > 1 \) or else we reduce to
\[
(x + 3y)(x + 4y)(x + 5y)(x + 6y) = 2^\alpha z_1^\ell,
\]
contradicting Proposition 2.4. If \( 5 \mid x \), then \([1, 4; 0, 5]\) leads to \([3.5]\), while the same is true of identity \([3, 4; 1, 6]\) if \( 5 \mid x + y \). We may thus suppose that \( 3 \mid x(x + y) \) and, from Proposition 2.2, that either \( 5 \mid y \) or \( i_5 = 2 \). In each case, \([1, 1, 4; 0, 3, 3]\) leads to \([3.4]\) and a contradiction.

3.3.3. \( i = 3 \). Finally, suppose that we have a solution of
\[
x(x + y)(x + 2y)(x + 4y)(x + 5y)(x + 6y) = z^\ell
\]
in integers \( x, y, z \) and \( \ell \) with \( \gcd(x, y) = 1, yz \neq 0 \) and \( \ell \geq 7 \) prime. Combining the identities \([1, 4; 0, 5]\) and \([2, 5; 1, 6]\) with Proposition 2.2, it follows that necessarily \( 5 \mid y \) or \( i_5 = 3 \). If \( 3 \mid y \), applying Proposition 2.4 to \( x(x + y)(x + 2y) \) leads to a contradiction. If \( 3 \mid x \), then \([1, 5; 0, 6]\) leads to a solution to the equation
\[
(3.6) \quad X^\ell + 2^\alpha Y^\ell = 5Z^2, \quad \alpha = 0 \text{ or } \alpha \geq 2, 3 \mid XY, \gcd(X, Y) = 1,
\]
contrary to \([6]\) Theorem 1.1 if \( \alpha = 0 \) and to \([6]\) Proposition 4.3 (applied with \( p = 3 \)), in case \( \alpha \geq 2 \). We thus have \( 3 \mid (x + y)(x + 2y) \), whereby \([2, 2, 5; 1, 4, 4]\) leads to \([3.4]\) and hence a contradiction.

3.4. \( k = 8 \). To finish the proof of Theorem 1.2 in case \( \ell \geq 7 \) is prime, it remains to treat the case \( k = 8 \) (whence we may assume that \( i \in \{1, 2, 3\} \)).

3.4.1. \( i = 1 \). Suppose first that \( 7 \mid x \). If \( i = 1 \), Proposition 2.2 implies that either \( 5 \mid y \), or \( i_5 \in \{0, 2\} \). The first of these with \([3, 4; 0, 7]\) contradicts \([3]\) Proposition 3.1. If \( i_5 = 0 \), then \([3; 4; 6]\) and the fact that \( a_7(E_{3,4,6}) = 0 \) (whereby \( N_{3,4,6} \neq 96 \)) lead via Proposition 2.1 to the conclusion that either \( 2 \mid x + y \) or \( \max \{\nu_2(x + 4y), \nu_2(x + 6y)\} \geq 3 \). From \( a_7(E_{2,3,4}) = -4 \) (whence
$N_{2,3,4} \neq 24$), Proposition 2.1 thus implies that necessarily $\nu_2(x + 6y) \geq 3$ (and, from $\{2; 3; 4\}$, $3 \mid x(x + y)(x + 2y)$). Identity $[2, 5; 0, 7]$ thus yields a solution to an equation of the shape

\[(3.7) \quad AX^\ell + BY^\ell = Z^2, \quad AB = 2 \cdot 3^\beta \cdot 5^\gamma, \quad 7 \mid X, Y, \gcd(X, Y) = 1,\]

and hence arguing as in [3] proof of Proposition 3.1, treating levels up to $N = 1920$, yields a contradiction. If, on the other hand, $i_5 = 2$, from $a_7(E_{2,3,4}) = a_7(E_{3,4,5}) = -4$ we necessarily have $2 \mid y$, contradicting $a_7(E_{3,4,6}) = 0$.

We may thus suppose, via Proposition 2.2, that either $7 \mid y$ or $i_7 = 1$. If $5 \nmid x$, then applying Proposition 2.4 to

\[(x + 2y)(x + 3y)(x + 4y)(x + 5y)(x + 6y)(x + 7y)\]

leads to a contradiction. It follows that $5 \mid x$, so that $\{2, 3; 0, 5\}$ leads to equation (3.5) and the desired contradiction.

3.4.2. $i = 2$. Again, begin by assuming that $7 \mid x$. If $5 \mid x + y$, then $\{3, 4; 1, 6\}$ leads to (3.5) and hence a contradiction. We may thus suppose that $5 \nmid x + y$. Proposition 2.2 implies again that either $5 \mid y$ or $i_5 = 0$. The first of these with $\{1, 6; 0, 7\}$ contradicts [3] Proposition 3.1. If $i_5 = 0$, then if $5 \mid x$, we find that $N_{1,5,6} = 2^\kappa \cdot 3^\delta$ (where $\kappa \in \{0, 1, 3, 5\}$ and $\delta \in \{0, 1\}$) and $5 \mid z_1 z_5 z_6$, contrary to Proposition 2.2. Similarly, if $5 \mid x + 5y$, then $N_{0,1,6} = 2^\kappa \cdot 3^\delta \cdot 7$ with $5 \mid z_0 z_1 z_6$, again contrary to Proposition 2.2.

We thus find, via Proposition 2.2, that either $7 \mid y$ or $i_7 = 2$, and that $i_5 \notin \{2, 3, 4\}$. If $i_5 \in \{0, 1\}$, then $\{1, 4; 0, 5\}$ or $\{3, 4, 1, 6\}$, respectively, leads to (3.5) and a contradiction. It follows that $5 \mid y$ and so necessarily

\[N_{0,1,3}, N_{0,3,4}, N_{1,3,4}, N_{4,5,6} \in \{24, 96\}.\]

Assume first that $7 \mid y$. From $a_7(E_{0,1,3}) = 4$, we conclude that $x$ is odd, while $a_7(E_{4,5,6}) = 0$ implies that $x + y$ is even (to be precise, we have $\nu_2(x + 5y) \in \{1, 2\}$). Now $\{0; 3; 4\}$ leads to the conclusion that $\nu_2(x + 3y) = 1$, so that $\nu_2(x + 5y) = 2$, whence $\nu_2(x + y) \geq \ell - 4$, contradicting $a_7(E_{1,3,4}) = -4$ and $\ell \geq 7$.

Next, assume that $i_7 = 2$. From $a_7(E_{0,1,3}) = 0$ and $a_7(E_{4,5,6}) = -4$, we see that $\nu_2(x) \in \{1, 2\}$, whence, from $\{1; 3; 4\}$, we have $\nu_2(x) = 1$. From $a_7(E_{4,5,6}) = -4$, necessarily $\nu_2(x + 6y) = 2$, contradicting $\nu_2(x + 6y) \geq \ell - 3$.

3.4.3. $i = 3$. Finally, suppose that $i = 3$. Again, suppose first that $7 \mid x$. If $5 \mid x + y$, then $\{2; 5; 1, 6\}$ leads to (3.5) and hence a contradiction. We may thus suppose that $5 \nmid x + y$. From Proposition 2.2, either $5 \mid y$, or $i_5 \in \{0, 2, 3\}$. If $5 \mid y$ or $i_5 = 3$, then $\{1, 6; 0, 7\}$ contradicts [3] Proposition 3.1. If $i_5 = 0$ or $i_5 = 2$, arguing as in the preceding subcase, using the identities $\{1; 5; 6\}$ and $\{0; 1; 6\}$, or $\{1; 2; 6\}$ and $\{0; 5; 7\}$, respectively, leads to a contradiction.

We may thus suppose that $7 \mid x$, whence, via Proposition 2.2, either $7 \mid y$ or $i_7 = 3$, and $i_5 \notin \{3, 4\}$. If $i_5 \in \{0, 1, 2\}$, then $\{1, 4; 0, 5\}$, $\{2, 5, 1, 6\}$ or
[4, 5; 2, 7], respectively, leads to (3.5) and a contradiction. We thus have $5 \mid y$, whence

$$N_{0,1,2}, N_{0,1,4}, N_{0,2,4}, N_{0,2,6}, N_{4,5,6} \in \{24, 96\}.$$  

Suppose first that $7 \mid y$. From $a_7(E_{0,2,4}) = 0$, it follows that $2 \mid x$, whereby, from $a_7(E_{0,1,4}) = 0$, $4 \mid x$. Now the fact that $a_7(E_{0,1,2}) = 0$ implies that $8 \mid x$, contradicting $a_7(E_{4,5,6}) = 0$.

Finally, assume that $i_7 = 3$. If $2 \mid x$, then, from $a_7(E_{0,1,4}) = 0$, necessarily $\max \{\nu_2(x), \nu_2(x + 4y)\} \geq 4$, contradicting $a_7(E_{0,2,4}) = -4$. On the other hand, if $2 \nmid x$, the fact that $a_7(E_{0,2,6}) = 0$ leads to a contradiction.

This completes the proof of Theorem 1.2 in case $\ell \geq 7$ is prime. It is worth noting that we have really only used here the values of $i_p$ for $p \in \{5, 7\}$. To extend Theorem 1.2 to larger values of $k$ (and, potentially, more omitted terms) requires much more extensive analysis.

4. Proof of Theorem 1.2 for $\ell \in \{3, 5\}$. It remains then to treat the cases $\ell = 3$ and $\ell = 5$.

4.1. The case $\ell = 3$. In this case, we will carefully consider the curves $C_{\ell}$ defined via equation(s) (2.5). Notice that, for $\ell = 3$, equation (2.5) takes the shape

$$(4.1) \quad AX^3 + BY^3 + CZ^3 = 0$$

for suitable integers $A, B, C, X, Y$ and $Z$. Writing

$$(4.2) \quad u = -A^3BXZ, \quad v = A^3BY^3, \quad w = A^2Z^3 \quad \text{and} \quad d = ABC,$$

we thus, at least provided $AZ \neq 0$, deduce the existence of a rational point on the elliptic curve $E_d$ define via

$$(4.3) \quad E_d : v^2w + dvw^2 = u^3.$$  

Since $E_d$ and $E_{d'}$ are isomorphic precisely when $d/d'$ is a cube, we will focus on cubefree values of $d$ with $P(d) \leq 7$. We note that the corresponding result in \[3\] Lemma 5.1 is incorrect as stated, since $E_{30}$ has rank 2 over $\mathbb{Q}$; this mistake does not actually affect the statements or proofs in \[3\].

**Lemma 4.1.** Let $d = 2^{e_2}3^{e_3}5^{e_5}7^{e_7}$ for $e_2, e_3, e_5, e_7 \in \{0, 1, 2\}$. For $d \in S_0$, where

$$(4.4) \quad S_0 = \{1, 2, 3, 4, 5, 10, 14, 18, 21, 25, 36, 45, 60, 100, 147, 150, 175, 196, 225, 245, 252, 300, 315, 350, 882, 980, 1050, 1470, 1575, 1764, 2940, 7350, 11025, 14700\},$$

we have $\text{rk} E_d(\mathbb{Q}) = 0$. In all other cases, $\text{rk} E_d(\mathbb{Q}) \in \{1, 2\}$. If $d \in S_0$, we have

$$E_d(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z},$$
with the single exception of \(d = 2\) where
\[
E_2(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/6\mathbb{Z}.
\]

Proof. This is mostly a standard exercise in 2-descent; for our purposes, we can simply appeal to Magma for the 81 curves in question; see www.math.ubc.ca/~bennett/L.2-check.

Tracing this result back to ternary equations of the shape (4.1), we obtain the following.

**Proposition 4.2.** There are no solutions in coprime integers \(X, Y, Z\) with \(|XYZ| > 1\) to the equations
\[
AX^3 + BY^3 = Z^3
\]
for all positive integers \(A, B\) with \(AB \in S_0\), where \(S_0\) is as defined in (4.4).

Here, the solutions to (4.5) with \(AB = 2\) and \(|XYZ| = 1\) correspond to the extra torsion points on \(E_2\).

We will proceed, for each \(k \in \{5, 6, 7, 8\}\) and \(1 \leq i \leq (k - 1)/2\), by considering the \(\binom{k-1}{3}\) tuples \(\{r; s; t\}\) with \(0 \leq r < s < t \leq 5\) and \(i \notin \{r, s, t\}\), for each possible tuple \((b_0, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{k-1})\); note that we may suppose that the \(b_i\) here are positive, cubefree and satisfy \(P(b_i) < k\).

In case \(k = 5\) and \(i = 1\), the tuple \((b_0, b_2, b_3, b_4) = (1, 2, 1, 4)\) leads to the solutions (1.11), and all other tuples correspond, via Proposition 4.2, to insoluble cubic equations. If \(k = 5\) and \(i = 2\), we may check that again all tuples lead to insoluble cubic equations, except for
\[
(b_0, b_1, b_3, b_4) = (1, 1, 1, 1), (1, 2, 4, 1) \text{ and } (1, 4, 2, 1).
\]
For these tuples, each of
\[
\{0; 1; 3\}, \{0; 1; 4\}, \{0; 3; 4\}, \{1; 3; 4\}
\]
corresponds to values of \(d\) for which \(E_d\) has positive rank. For these cases, however, the identity [1, 1, 4; 0, 3, 3] leads to (4.5) with \(AB \in \{1, 4\}\).

For all larger values of \(6 \leq k \leq 8\) and \(1 \leq i \leq (k - 1)/2\), we find, in every case, tuples \(\{r; s; t\}\) which, together with Proposition 4.2, lead to a contradiction. Code to verify this is available from the author on request.

**4.2. The case \(\ell = 5\).** To complete the proof of Theorem 1.1, it remains to treat the case \(\ell = 5\). Here, we have little work to do due to the strong results of Hajdu and Kovacs [15], as noted in Proposition 2.4. In fact, they proved an even sharper result [15, Theorem 4] that is especially well-suited to our purposes.

**Proposition 4.3 (Hajdu and Kovacs, 2011).** Let \(4 \leq k \leq 8\) and suppose that \(X_0 < X_1 < \cdots < X_{k-1}\) is a nontrivial primitive arithmetic progression
with
\[ X_0 = b_0 z_0^5, \quad X_{i_1} = b_{i_1} z_{i_1}^5, \quad X_{i_2} = b_{i_2} z_{i_2}^5, \quad X_{k-1} = b_{k-1} z_{k-1}^5, \]
where the \( b_i \) and \( z_i \) are integers, the indices satisfy \( 0 < i_1 < i_2 < k - 1 \), and \( P(b_0 b_{i_1} b_{i_2} b_{k-1}) \leq 5 \). Then the initial term \( X_0 \) and the common difference \( X_1 - X_0 \) of the progression, up to symmetry, are as follows:

\[
\begin{align*}
&k = 4: (-9,7), (-6,7), (-6,11), (-5,7); \\
&k = 5: (-32,17), (-25,13), (-20,11), (-16,13), (-12,7), \\
&\quad (-12,11), (-12,13), (-10,7), (-8,7), (-8,11), \\
&\quad (-4,7), (-3,7), (-1,7), (2,7), (4,7), (4,23); \\
&k = 6: (-125,61), (-81,17), (-30,31), (-25,8), (-25,11), (-25,13), \\
&\quad (-25,17), (-20,9), (-20,13), (-20,19), (-20,29), (-15,7), (-15,11), \\
&\quad (-15,13), (-15,23), (-10,7), (-10,11), (-8,7), (-5,7), (-3,7), \\
&\quad (-1,11), (-1,13), (1,7), (5,11); \\
&k = 7: (-54,19), (-54,29), (-48,23), (-30,11), (-30,13), (-27,17), \\
&\quad (-24,13), (-18,7), (-18,11), (-18,13), (-18,19), (-16,11), (-15,7), \\
&\quad (-12,7), (-12,11), (-10,7), (-6,7), (-6,11), (-4,9), (-3,13), \\
&\quad (-2,7), (-2,17), (2,13), (3,7), (6,7), (8,7), (9,11), (18,7); \\
&k = 8: (-405,131), (-125,41), (-100,49), (-32,11), (-27,11), \\
&\quad (-27,13), (-25,19), (-24,7), (-16,13), (-10,13), (-9,7), (-5,11), \\
&\quad (-4,7), (-2,11), (-1,13), (-1,7), (1,7), (3,11), (4,11), (5,7), (6,17).
\end{align*}
\]

This result almost immediately allows us to conclude that equation \( (1.10) \) has no nontrivial solutions with \( \ell = 5 \) and \( 5 \leq k \leq 7 \). If \( k = 8 \), the same conclusion follows unless \( 7 \mid x \) (so that also \( 7 \mid x + 7y \)). In this case, we apply Proposition 4.3 to the progression
\[
x + y, x + 2y, x + 3y, x + 4y, x + 5y, x + 6y
\]
and again, after a little work, conclude as desired. This completes the proof of Theorem 1.2.

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**References**


Powers from products of terms in progressions with gaps


[21] N. Saradha and T. N. Shorey, On the equation \(n(n + d) \cdots (n + (i_0 - 1)d)(n + (i_0 + 1)d) \cdots (n + (k - 1)d) = y^p\) with \(0 < i_0 < k - 1\), Acta Arith. 129 (2007), 1–21.


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