Corrigendum to “Explicit estimates for the summatory
function of $\Lambda(n)/n$ from the one of $\Lambda(n)$”

by

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1. Correcting [2, Lemma 2.1]. This copy of [3, Lemma 4] contains a
typo. Here is the corrected version.

**Lemma 1.1 (Correction of [3, Lemma 4] and of [2, Lemma 2.1]).** Let $g$
be a continuously differentiable function on $[a, b]$ with $2 \leq a \leq b < +\infty$. We have

\[
\int_{a}^{b} \psi(t)g(t) \, dt = \int_{a}^{b} tg(t) \, dt - \sum_{\rho} \int_{a}^{b} \frac{t^\rho}{\rho} g(t) \, dt \\
- \int_{a}^{b} \left( \log 2\pi + \frac{1}{2} \log(1 - t^{-2}) \right) g(t) \, dt.
\]

The error is in the sign of $\log 2\pi$. By [1, Chapter 17, formulae (9) and
(10)], this constant is $-\zeta'(0)/\zeta(0)$ and we have

(1.1)\quad \zeta'(0) = -\frac{1}{2} \log 2\pi, \quad \zeta(0) = -\frac{1}{2}.

2. Correcting [2, Lemma 2.2]. As a consequence, and correcting another
sign typo, [2, Lemma 2.2] should read as follows.

**Lemma 2.1 (Correction of [2, Lemma 2.2]).** We have, for $x \geq 1$,

\[
\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} - \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho - 1)} + \frac{B(x)}{x}.
\]

where the sum is over the zeroes $\rho$ of the Riemann zeta function that lie in
the critical strip $0 < \Re s < 1$ (the so-called non-trivial zeroes) and $B(x)$ is

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the bounded function given by

\[ B(x) = \log 2\pi + \frac{1}{2} \log(1 - x^{-2}) + \frac{1}{2} \left( x \log \frac{x + 1}{x - 1} - 2 \right). \]

We have \( 0 \leq B(x) \leq \log 2\pi \).

2 Theorem 1.1] need not be modified. The sign in front of \( \log 2\pi \) comes from the typo above, the sign in front of the function of \( x \) in \( B(x) \) comes from the third line of the proof of this lemma: the integral from \( x \) to \( \infty \) should have a minus sign in front. In this manner we reach

\[ \tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} - \sum_{\rho} \frac{x^{\rho - 1}}{\rho(\rho - 1)} + \int_{x}^{\infty} \left( \log 2\pi + \frac{1}{2} \log(1 - t^{-2}) \right) \frac{dt}{t^2}. \]

The claimed inequality on \( B(x) \) is obvious from the integral expression one infers from this expression. In order to reach the exact expression, it is enough to mention that

\begin{align*}
\frac{d}{dx} \left( \frac{1}{x} \log(1 - x^{-2}) + \log \frac{x + 1}{x - 1} - \frac{2}{x} \right) & = -\frac{1}{x^2} \log(1 - x^{-2}) + \frac{2}{x^4(1 - x^{-2})} - \frac{2}{x^2 - 1} + \frac{2}{x^2} \\
& = -\frac{1}{x^2} \log(1 - x^{-2})
\end{align*}

as required.

3. Correcting [2, Corollary]. As [2, Theorem 1.1] need not be modified, the spread of this error stops here, but another and more annoying error comes from the Pari/GP code used to check the finite part of [2, Corollary].

COROLLARY. We have, for \( x > 1 \),

\[ \tilde{\psi}(x) = \log x - \gamma + O^*(1.833/\log^2 x). \]

Furthermore, for \( 1 \leq x \leq 10^{10} \), we have \( \tilde{\psi}(x) = \log x - \gamma + O^*(1.31/\sqrt{x}) \).

For \( x \geq 1.52 \cdot 10^6 \), we have \( \tilde{\psi}(x) = \log x - \gamma + O^*(0.0067/\log x) \).

For \( x \geq 468 000 \), we have \( \tilde{\psi}(x) = \log x - \gamma + O^*(0.01/\log x) \).

For \( x \geq 115 \), we have \( \tilde{\psi}(x) = \log x - \gamma + O^*(1/(4 \log x)) \).

The code used previously has been lost, so we display now the code we presently use, in order to make checking easier. Let \( F \) be an increasing positive function such that \( x \mapsto xF'(x)/F(x) \) is non-increasing. This hypothesis is satisfied by the choices \( F(x) = \log x \), \( F(x) = (\log x)^2 \) and \( F(x) = \sqrt{x} \). In order to compute the minimal constants in the inequality

\[ \forall x \in [n, n + 1) \quad C_-(n) \leq F(x)(\tilde{\psi}(x) + \gamma - \log x) \leq C_+(n) \]
where \( n \) is an integer, we first notice that we may replace \( \tilde{\psi}(x) \) by \( \tilde{\psi}(n) \). The function \( x \mapsto \tilde{\psi}(n) + \gamma - \log x \) is decreasing, and three cases can occur:

- When \( \tilde{\psi}(n) + \gamma - \log n \leq 0 \), this quantity remains non-positive, and is increasing in absolute value: the minimum we seek is \( F(n+1)(\tilde{\psi}(n) + \gamma - \log(n+1)) \). The maximum is negative: \( F(n)(\tilde{\psi}(n) + \gamma - \log n) \).
- When \( \tilde{\psi}(n) + \gamma - \log(n+1) \geq 0 \), this quantity remains non-negative, and there is a competition between the growth of \( F(x) \) and the decay of \( \tilde{\psi}(x) + \gamma - \log x \). The derivative of the product is \( F'(x)(\tilde{\psi}(n) + \gamma - \log n) - F(x)/x \). It is negative when \( \tilde{\psi}(n) + \gamma \leq \frac{F(x)}{xF'(x)} + \log x \) and non-positive otherwise, meaning that our function first decreases and then increases. The maximum is thus either \( F(n)(\tilde{\psi}(x) + \gamma - \log n) \) or \( F(n+1)(\tilde{\psi}(x) + \gamma - \log(n+1)) \). The minimum on this interval is non-negative and we do not record it.
- When \( \tilde{\psi}(n) + \gamma - \log n \geq 0 \) but \( \tilde{\psi}(n) + \gamma - \log(n+1) \leq 0 \), there exists a unique point \( x_n \in [n, n+1) \) where \( \tilde{\psi}(n) + \gamma - \log x_n \) vanishes. On the negative part \( [x_n, n+1) \), the first reasoning above applies, while on the positive part \( [n, x_n) \), the second one does, by replacing the point \( n+1 \) by \( x_n \). However, since \( \tilde{\psi}(n) + \gamma - \log x_n = 0 \), the maximum is simply \( F(n)(\tilde{\psi}(n) + \gamma - \log n) \).

This leads to the following script where we may miss the best constant \( C_-(n) \) as we only compute \( \min(0, C_-(n)) \). In the computations we use, this distinction is pointless.

```plaintext
{Lambda(d) = my(dec = factor(d), P = dec[,1]);
   if(#P!=1, return(0), return(log(P[1])));
}
{check(borneinf, bornesup, myF = (t->log(t))) =
   /* We assume that tF'(t)/F(t) is non-increasing
      on the interval [borneinf, bornesup + 1].
      The values obtained are also valid for x < bornesup + 1 */
   my(mymin = 1000, mymax = -1000, psitilde = 0, val, wheremin, wheremax);
   for(d = 1, borneinf -1, psitilde += Lambda(d)/d);
   for(d = borneinf, bornesup, psitilde += Lambda(d)/d);
   val = (psitilde - log(d) + Euler)*myF(d);
   if(val < mymin, mymin = val; wheremin = d);
   if(val > mymax, mymax = val; wheremax = d);
   val = (psitilde - log(d+1) + Euler)*myF(d+1);
   if(val < mymin, mymin = val; wheremin = d + 0.9999);
   if(val > mymax, mymax = val; wheremax = d + 0.9999);
}
if(mymin > 0, print("The minimum may be smaller, see the paper."),);
print("The minimum of (psitilde - log(d) + Euler)*myF(d) ");
print("On [" , borneinf , ", " , bornesup , "] is reached at d = ", wheremin);
```
Section 6 of [2] need not be modified except that the value $1.68 \cdot 10^{-12}$ should throughout be replaced by $1.75 \cdot 10^{-12}$ with no consequence. Furthermore, in this same Section 6, the statement “When $8.950 \leq x \leq 10^{10}$, we have
\[
|\tilde{\psi}(x) - \log x + \gamma| \log x \leq \frac{1.3 \log x}{\sqrt{x}} + 1.68 \cdot 10^{-12} \log x \leq 0.0003
\]
should be replaced by “When $9.75 \cdot 10^{6} \leq x \leq 10^{10}$, we have
\[
|\tilde{\psi}(x) - \log x + \gamma| \log x \leq \frac{1.3 \log x}{\sqrt{x}} + 1.68 \cdot 10^{-12} \log x \leq 0.0067.
\]
By direct inspection, we show this bound is valid in the range $x \geq 1.52 \cdot 10^{6}$.

References