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Some theorems related to a problem of Ruziewicz

by

Jan MYCIELSKI and Grzegorz TOMKOWICZ

Presented by Ludomir NEWELSKI

Summary. We apply deep results of Dougherty and Foreman and of Drinfeld and Margulis to give a very simple proof of the following theorem. Let $\mathbf{B}$ be the Boolean ring of Lebesgue measurable sets with the property of Baire in the sphere $\mathbf{S}^n$ or bounded Lebesgue measurable sets with the property of Baire in the Euclidean space $\mathbb{R}^{n+1}$ $(n \geq 2)$. *Then the Lebesgue measure in* $\mathbf{B}$ *is the unique finitely additive measure suitably normalized and invariant under isometries.* Moreover, we prove that there exist everywhere dense $\mathbf{G}_\delta$ sets in the sphere $\mathbb{S}^n$ $(n \geq 2)$ and in the cube $[0,1]^n$ $(n \geq 3)$ that can be packed into arbitrarily small open sets using only subdivisions into finitely many Borel pieces.

1. Introduction. The Ruziewicz problem for Borel sets asks if the Lebesgue measure is the only *finitely additive* and isometry-invariant measure μ on Borel sets in the sphere $\mathbb{S}^n$ $(n \geq 2)$ or bounded Borel sets in $\mathbb{R}^n$ $(n \geq 3)$ such that $\mu(\mathbb{S}^n) = 1$ or $\mu([0,1]^n) = 1$. A similar question for the spaces $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{S}^1$ has a negative answer (see [9, Chapter 13]).

In the present paper we will give a partial answer to the problem, showing the uniqueness of Lebesgue measure in the case of a larger family of sets than the Borel ones, the class of Lebesgue measurable sets having the property of Baire. However, our method does not work in the case of Borel sets and so the Ruziewicz problem for Borel sets remains open.

We say that a set A in a metric space X can be *packed* into the set $B \subseteq X$ if there exist a partition of A into finitely many sets $A_1, \ldots, A_k$ and

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some isometries $f_1, \dots, f_k$ such that $f_i(A_i) \subset B$ and $f_i(A_i) \cap f_j(A_j) = \emptyset$ for all $i \neq j$.

By a theorem of Drinfeld and Margulis (see Theorem 2.3 below) a positive answer to the Ruziewicz problem follows from the following conjecture:

CONJECTURE. *Every* $\mathbf{G}_\delta$*-set in* $\mathbb{S}^n$ *or* $\mathbb{R}^n$ *of Lebesgue measure* 0 *can be packed using Borel pieces into any non-empty open set in the same space.*

If one uses only pieces from an amenable group of isometries acting on $\mathbb{S}^n$ or $\mathbb{R}^n$, the above conjecture is false (even if the pieces were not required to be Borel [7, 9]). This follows from the existence of Marczewski measures, which are isometry-invariant finitely additive measures, defined over all subsets of $\mathbb{S}^n$ or $\mathbb{R}^n$, that vanish on all meager sets and normalize some compact set. The existence of a Marczewski measure requires the Axiom of Choice for uncountable families of sets. But the lack of packing of some $\mathbf{G}_\delta$-sets is also a theorem of the theory $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$ (see [7] for more details).

As shown in [7], a positive answer to the Ruziewicz problem for Borel sets is also a theorem of the theory $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$.

Using a deep theorem of Dougherty and Foreman [1] we will prove the following theorem:

THEOREM 1.1. *There exist some* $\mathbf{G}_\delta$*-sets everywhere dense in* $\mathbb{S}^n$ $(n \geq 2)$ *that can be packed into any non-empty open set, using finitely many Borel pieces. The same is true for the cube* $[0,1]^n$ $(n \geq 3)$.

The above theorem implies an answer to a problem stated in [7] about G-Tarski null sets (studied in [6, 7]) and the property of Baire (see Corollary 3.1 below and the definitions before the corollary).

Let $\mathcal{B} \cap \mathcal{L}$ be the algebra of subsets in $\mathbb{S}^n$ or $\mathbb{R}^n$, where $\mathcal{B}$ is the algebra of sets with the property of Baire and $\mathcal{L}$ the algebra of Lebesgue measurable sets. We will show, using another theorem of [1], the following theorem:

THEOREM 1.2. *The Lebesgue measure is the only finitely additive and isometry-invariant measure* μ *on the sets in* $\mathcal{B} \cap \mathcal{L}$ *of* $\mathbb{S}^n$ $(n \geq 2)$ *or bounded sets in* $\mathcal{B} \cap \mathcal{L}$ *of* $\mathbb{R}^n$ $(n \geq 3)$ *such that* $\mu(\mathbb{S}^n) = 1$ *or* $\mu([0,1]^n) = 1$*, respectively.*

The above result was also obtained by Grabowski, Máthé and Pikhurko [3, Theorem 1.14], but their proof is more complicated than ours.

2. Preliminaries. Let G be a group of isometries of $\mathbb{R}^n$ or $\mathbb{S}^n$. We will say that subsets A and B of $\mathbb{R}^n$ or $\mathbb{S}^n$ are *G-equidecomposable* (in symbols $A \equiv B$) if A can be partitioned into sets $A_1, \dots, A_k$ such that $g_1(A_1), \dots, g_k(A_k)$ is a partition of B for some elements $g_1, \dots, g_k$ of G.

Two open sets A, B in $\mathbb{R}^n$ (or $\mathbb{S}^n$) are *densely equidecomposable* if there are finitely many disjoint open subsets A_i of A and isometries σ_i such that

$\bigcup_i A_i$ is dense in A, the sets $\sigma_i(A_i)$ are pairwise disjoint, and $\bigcup \sigma_i(A_i)$ is dense in B.

We will use the following theorems proved in [1, Cor. 2.8 and Cor. 2.7]:

THEOREM 2.1 (Dougherty and Foreman without the Axiom of Choice). *Any two open subsets of $\mathbb{S}^n$ $(n \geq 2)$ or open bounded subsets of $\mathbb{R}^n$ $(n \geq 3)$ are densly equidecomposable.*

THEOREM 2.2 (Dougherty and Foreman using the Axiom of Choice). *Any two bounded subsets of $\mathbb{R}^n$ $(n \geq 3)$ or of $\mathbb{S}^n$ $(n \geq 2)$ with non-empty interiors are equidecomposable with parts having the property of Baire.*

The next theorem, which we will need in the proof of Theorem 1.2, shows that the Ruziewicz problem for Lebesgue measurable sets has an affirmative answer. It was obtained first in the case of $\mathbb{S}^n$ $(n \geq 4)$ and $\mathbb{R}^n$ $(n \geq 5)$, independently by Margulis [4] and Sullivan [8]. Then Margulis [5] settled the cases of $\mathbb{R}^3$ and $\mathbb{R}^4$, and the remaining two cases, $\mathbb{S}^2$ and $\mathbb{S}^3$, were obtained by Drinfeld [2].

THEOREM 2.3. *Let G be a group of isometries of $\mathbb{S}^n$ $(n \geq 2)$ or $\mathbb{R}^n$ $(n \geq 3)$. Then the Lebesgue measure is the only finitely additive and G-invariant measure μ on the Lebesgue measurable sets in the sphere $\mathbb{S}^n$ $(n \geq 2)$ or bounded Lebesgue measurable sets in $\mathbb{R}^n$ $(n \geq 3)$ such that $\mu(\mathbb{S}^n) = 1$ or $\mu([0,1]^n) = 1$.*

3. Proofs and a corollary

Proof of Theorem 1.1. Consider an infinite sequence $A_1, A_2, \ldots$ of disjoint spherical triangles in $\mathbb{S}^n$ $(n \geq 2)$ with diameters tending to 0. Then, by Theorem 2.1, one can pack the whole sphere $\mathbb{S}^n$ modulo a nowhere dense set into any set A_i. Denote the subset of $\mathbb{S}^n$ that can be packed into A_i by S_i. Clearly, the intersection S of all the sets S_i is a $\mathbf{G}_\delta$-set that is everywhere dense in $\mathbb{S}^n$. Moreover, S can be packed into any A_i using Borel pieces. Now, it is enough to observe that since the action of the group of all isometries of $\mathbb{S}^n$ is transitive, every open set contains a spherical triangle of the form $g(A_i)$, where g is an isometry of $\mathbb{S}$ and i is a positive integer. Moreover, isometries preserve Borel sets.

The same proof works in the case of $\mathbb{R}^n$ for $n \geq 3$. ∎

We say that a set $A \subset \mathbb{R}^n$ or $A \subset \mathbb{S}^n$ is *G-Tarski null* if it can be packed into arbitrarily small balls using transformations in G. Recall that the theory $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$ has a model called $L(\mathbb{R})$ (see [7] for more details). In the model every subset has the property of Baire. Our Theorem 1.1 solves the Problem in [7] about existence of non-meager G-Tarski null sets in $\mathbb{R}^n$ or $\mathbb{S}^n$.

Corollary 3.1. *Let G be the group of rotations of $\mathbb{S}^n$ or the group of isometries of $\mathbb{R}^{n+1}$ $(n \geq 2)$. Then, in the theory ZF + DC + AD, there exists a G-Tarski null set that has the property of Baire but is not meager.*

Proof. Consider the $\mathbf{G}_\delta$-set constructed in the proof of Theorem 1.1, which is clearly comeager. Since Theorem 1.1 is a theorem of ZF + DC, the result follows. ∎

In what follows we will denote by Bor_b the ring of bounded Borel sets in $\mathbb{R}^n$ or $\mathbb{S}^n$ and by $\mathcal{L}_b$ the ring of bounded sets in $\mathcal{L}$. The following lemma, suggested by the referee, will be helpful in the proof of Theorem 1.2:

Lemma 3.2. *Let Σ be some ring such that $\mathrm{Bor}_b \subseteq \Sigma \subseteq \mathcal{L}_b$ and let μ be an isometry-invariant finitely additive normalized measure that is absolutely continuous on Σ with respect to the Lebesgue measure. Then μ is the Lebesgue measure.*

Proof. Observe that for any set $A \in \mathcal{L}_b$ there exist Borel sets $B_1 \subseteq A \subseteq B_2$ such that $B_2 \setminus B_1$ has Lebesgue measure zero. Thus μ can be extended to an invariant measure on $\mathcal{L}_b$ and so, by Theorem 2.3, μ is the Lebesgue measure. ∎

Proof of Theorem 1.2. Consider an isometry-invariant normalized measure μ defined on $\mathcal{B} \cap \mathcal{L}_b$. By Lemma 3.2 it is enough to show that μ is absolutely continuous with respect to λ_n. To get the absolute continuity of μ it is enough to show that every subset of $\mathcal{B} \cap \mathcal{L}_b$ of Lebesgue measure zero can be packed into an arbitrarily small spherical triangle (resp. small cube) using pieces that are in $\mathcal{B} \cap \mathcal{L}$. Indeed, every finitely additive isometry-invariant measure agrees on spherical triangles (resp. cubes) with Lebesgue measure, so by isometry-invariance, packing of a given set into any such triangle (resp. cube) implies that the measure of such a set equals 0. So take a bounded subset A of $\mathcal{B} \cap \mathcal{L}$ such that $\lambda_n(A) = 0$. This set is contained in a bounded subset $B \in \mathcal{B}$ with non-empty interior. Now, by Theorem 2.2, B is equidecomposable using Baire pieces with an arbitrarily small spherical triangle C (resp. small cube). Next, we observe that any subset of a Lebesgue null set has Lebesgue measure zero, thus the pieces witnessing the equidecomposability of B and C belong to the algebra $\mathcal{B} \cap \mathcal{L}$. ∎

Remark 3.3. We observe that since a subset of a Borel set of Lebesgue measure zero might not be Borel, the method applied in the proof of Theorem 1.2 does not work in the case of the ring of bounded Borel sets.

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Jan Mycielski
Department of Mathematics
University of Colorado
Boulder, Colorado 80309-0395, USA
E-mail: jmyciel@euclid.colorado.edu

Grzegorz Tomkowicz
Centrum Edukacji G^2
41-902 Bytom, Poland
E-mail: gtomko@vp.pl