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Sampling constants for dominating sets in generalized Fock spaces

by

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Abstract. We prove several results related to a Logvinenko–Sereda-type theorem on dominating sets for generalized doubling Fock spaces. In particular, we give a precise polynomial dependence of the sampling constant on the relative density parameter γ of the dominating set. Our method is an adaptation of that used in [J. Math. Anal. Appl. 495 (2021), no. 2, art. 124755] for the Bergman spaces and is based on a Remez-type inequality and a covering lemma related to doubling measures.

1. Introduction. Sampling problems are central in signal theory: they cover, for instance, sampling sequences and so-called dominating sets which allow recovering the norm of a signal – defined by integration over a given domain – from the integration on a subdomain (precise definitions will be given later). They have been considered in a large variety of situations (see e.g. the survey [FHR17]), including the Fock space and its generalized versions (see [Sei04, MMOC03, OC98, Lin00, JPR87, LZ19]). In this paper we will focus on the second class of problems, i.e. dominating sets. Conditions guaranteeing that a set is dominating were established rather long ago (in the 70's for the Paley–Wiener space and in the 80's for the Bergman and Fock spaces, again see e.g. the survey [FHR17] and references therein). More recently, people got interested in estimates of the sampling constants which give quantitative information on the tradeoff between the cost of the sampling and the precision of the estimates. The major paper in this area is by Kovrizhkin [Kov01] who gave a method of considering this problem in the Paley–Wiener space. Subsequently, his method was adapted to several other spaces (see [HJK20, JS21, HKKO21, GW24, BJPS21]). In the present paper, inspired by the methods in [HKKO21], we will discuss the case of generalized doubling Fock spaces. For these, the paper [MMOC03] provides a wealth of

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results that allow us to translate the main steps of [HKKO21] to this new framework.

Notations. As usual, $A \lesssim B$ (respectively $A \gtrsim B$) means that there exists a constant $c > 0$ independent of the relevant variables such that $A \leq cB$ (resp. $A \geq cB$). Similarly, $A \asymp B$ stands for $A \lesssim B$ and $A \gtrsim B$.

1.1. Doubling subharmonic functions and generalized Fock spaces. A function $\varphi : \mathbb{C} \to \mathbb{R}$ of class C^2 is said to be *subharmonic* if $\Delta\varphi \geq 0$, and *doubling subharmonic* if it is subharmonic and the (non-negative) measure $\mu := \Delta\varphi$ is doubling, i.e. there exists a constant C such that for every $z \in \mathbb{C}$ and $r > 0$,

$$\mu(D(z, 2r)) \leq C\mu(D(z, r)). \tag{1.1}$$

Here $D(z, r)$ denotes the standard open euclidean disk of center $z \in \mathbb{C}$ and radius $r > 0$. The minimal constant satisfying (1.1) is denoted by C_μ and called the *doubling constant* for μ. A basic example of a doubling subharmonic function is $\varphi(z) = |z|^2$, for which $\mu = \Delta\varphi$ is equal to 4 times the Lebesgue measure and $C_\mu = 4$. The reader may consider this simpler case for the time of the introduction.

Doubling subharmonic functions induce a new metric on the complex plane $\mathbb{C}$, which is more adapted to the complex analysis we shall work with. We postpone the advanced results on doubling measures to Section 2 but let us introduce some basic objects related to them. First, we can associate with φ as above a function $\rho : \mathbb{C} \to \mathbb{R}^+$ such that

$$\mu\big(D(z, \rho(z))\big) = 1.$$

To get an idea, assuming φ is suitably regularized, we have $\Delta\varphi \asymp \rho^{-2}$ (see [MMOC03, Theorem 14 and Remark 5]). Next, denote

$$D^r(z) := D(z, r\rho(z)),$$

the disk adapted to the new metric, and write $D(z) := D^1(z)$ for the disk with unit mass for the measure μ. Finally, with these definitions in mind, we can introduce the following natural density. A measurable set E is (γ, r)-*dense* (with respect to the above metric) if, for every $z \in \mathbb{C}$,

$$\frac{|E \cap D^r(z)|}{|D^r(z)|} \geq \gamma.$$

Here $|F|$ denotes the planar Lebesgue measure of a measurable set F. We will just say that the set is *relatively dense* if there is some $\gamma > 0$ and some $r > 0$ such that the set is (γ, r)-dense. We shall see in the next subsection that relative density is the right notion to characterize dominating sets.

Let dA be the planar Lebesgue measure on $\mathbb{C}$. For a doubling subharmonic function $\varphi : \mathbb{C} \to \mathbb{R}$ and $1 \leq p < \infty$, we will denote by $L^p_\varphi(F)$

the Lebesgue space on a measurable set $F \subset \mathbb{C}$ with respect to the measure $e^{-p\varphi}\, dA$. We will also use the notation

$$\|f\|^p_{L^p_\varphi(F)} := \int_F |f|^p e^{-p\varphi}\, dA.$$

Finally, we define the *doubling Fock space* by

$$\mathcal{F}^p_\varphi := \Big\{ f \in \mathrm{Hol}(\mathbb{C}) : \|f\|^p_{L^p_\varphi(\mathbb{C})} = \int_{\mathbb{C}} |f|^p e^{-p\varphi}\, dA < \infty \Big\}.$$

These spaces appear naturally in the study of the Cauchy–Riemann equation (see [Chr91, COC11]) and are well studied objects (see e.g. [OP16] for Toeplitz operators on these spaces). When $\varphi(z) = |z|^2$, $\mathcal{F}^p_\varphi$ is the classical Fock space (see the textbook [Zhu12] for more details).

1.2. Dominating sets and sampling constants. A measurable set $E \subset \mathbb{C}$ will be called *dominating* for $\mathcal{F}^p_\varphi$ if there exists $C > 0$ such that

$$\int_E |f|^p e^{-p\varphi}\, dA \geq C^p \int_{\mathbb{C}} |f|^p e^{-p\varphi}\, dA, \qquad \forall f \in \mathcal{F}^p_\varphi. \tag{1.2}$$

The largest constant C satisfying (1.2) is called the *sampling constant.*

The notion of dominating sets can be defined for several spaces of analytic functions and their characterization has been at the center of intensive research starting with a famous result of Panejah [Pan62, Pan66], and Logvinenko and Sereda [LS74] for the Paley–Wiener space, which consists of all functions in L^2 whose Fourier transform is supported on a fixed compact interval. For corresponding results in the Bergman space we refer to [Lue81, Lue85, Lue88, GW24]. As for Fock spaces, this has been done by Janson, Peetre and Rochberg [JPR87] for the classical case $\varphi(z) = |z|^2$ and by Lou and Zhuo [LZ19, Theorem A] for generalized doubling Fock spaces (see also [OC98] for a characterization of general sampling measures). It turns out that a set E is dominating for the doubling Fock spaces if and only if it is relatively dense. This characterization also holds for the other spaces of analytic functions mentioned above with an adapted notion of relative density.

Once the dominating sets have been characterized in terms of relative density, a second question of interest is to know whether the sampling constants can be estimated in terms of the density parameters (γ, r). Kovrizhkin answered this question in [Kov01] giving a sharp estimate on the sampling constants for the Paley–Wiener spaces, with a polynomial dependence on γ (see also [Rez10] for sharp constants in some particular geometric settings). Such a dependence is used for example in control theory as a key spectral estimate to get the null-controllability of certain partial differential equations (see [EV18, ES21, BJPS21] and the references therein) and it is also useful in the study of Toeplitz operators as we will see below. Kovrizhkin's method involves Remez-type and Bernstein inequalities. It has been used in several spaces in which the Bernstein inequality holds (see for instance [HJK20] for the model space, [BJPS21] for spaces spanned by Hermite functions, or [ES21] for general spectral subspaces), and also in settings where a Bernstein inequality is not at hand (e.g. Fock and polyanalytic Fock spaces [JS21] or Bergman spaces [HKKO21, GW24]). The authors of [HKKO21] developed a machinery based on a covering argument to circumvent Bernstein's inequality, and the aim of the present paper is to adapt this method to doubling Fock spaces.

1.3. Main results. It can be deduced from Lou and Zhuo's result [LZ19, Theorem A] that if E is (γ, r)-dense then there exist some constants $0 < \varepsilon_0 < 1$ and $c > 0$ depending only on r such that inequality (1.2) holds for every

$$C^p \leq c\gamma\varepsilon_0^{2(p+2)/\gamma}.$$

This last estimate gives an exponential dependence on γ. In the spirit of the work [Kov01], we improve it providing a polynomial estimate in γ with a power depending suitably on r.

THEOREM 1.1. *Let φ be a doubling subharmonic function and $1 \leq p < \infty$. Given $r > 1$, there exists L such that for every measurable set $E \subset \mathbb{C}$ which is (γ, r)-dense, we have*

$$\|f\|_{L^p_\varphi(E)} \geq \left(\frac{\gamma}{c}\right)^L \|f\|_{L^p_\varphi(\mathbb{C})} \tag{1.3}$$

for every $f \in \mathcal{F}^p_\varphi$. Here, the constants c and L depend on r, and we can choose L with

$$L \lesssim r^{\log_2(C_\mu)} + \frac{1}{p}(1 + \log(r)),$$

where C_μ is the doubling constant (see (1.1)) and the implicit constant depends only on the space (i.e. only on φ).

REMARK 1.2. When $\varphi(z) = |z|^2$, we have $\log_2(C_\mu) = 2$ and we get the same result as that given in [JS21, Theorem 4.6] for the classical Fock space with an explicit dependence on p in addition.

Observe that we are mainly interested in the case when r is large, for instance $r \geq 1$ (in case E is (γ, r)-dense for some $r < 1$, we can also show that E is $(\widetilde{\gamma}, 1)$-dense for some $\widetilde{\gamma} \asymp \gamma$, where the underlying constants are universal).

The proof of Theorem 1.1 follows the scheme presented in [HKKO21]. We will recall the necessary results from that paper. The main new ingredients come from [MMOC03] and concern a finite overlapping property and a lemma allowing us to express the subharmonic weight locally as (the real part of) a holomorphic function.

We should mention that in [LZ19, Theorem 7], in the course of proving that the relative density is a necessary condition for domination, it is shown that $\gamma \gtrsim C^p$. Hence, we cannot expect better than a polynomial dependence on γ of the sampling constant C. In this sense, our result is optimal.

As noticed in a remark in [Lue81, p. 11] and in [HKKO21, after Theorem 2] for the Bergman space, there is no reason a priori why a holomorphic function for which the integral $\int_E |f|^p e^{-p\varphi}\,dA$ is bounded for a relative dense set E should be in $\mathcal{F}^p_\varphi$. Outside the class $\mathcal{F}^p_\varphi$ relative density is in general not necessary for domination. For this reason, we always assume that we test on functions $f \in \mathcal{F}^p_\varphi$.

As a direct consequence of Theorem 1.1, we obtain a bound for the norm of the inverse of a Toeplitz operator T_φ. We remind that for any bounded measurable function φ, the *Toeplitz operator* T_φ is defined on $\mathcal{F}^2_\varphi$ by $T_\varphi f = \mathbf{P}(\varphi f)$, where $\mathbf{P}$ denotes the orthogonal projection from $L^2_\varphi(\mathbb{C})$ onto $\mathcal{F}^2_\varphi$. These operators have a long history and we refer the interested reader to the textbook [Zhu12, Chapter 6] for more information. In particular, their study (on the Fock space) has been initiated in [BC86, BC87] in view of applications in quantum mechanics. They are known to be unitarily equivalent to pseudo-differential operators on $L^2(\mathbb{R})$ through the Bargmann transform and are closely related to complex Fourier integral operators (see the recent paper [CHS23] and the references therein). They are also particular cases of Toeplitz localization operators (see for example [HMM16]).

As remarked in [LZ19, Theorem B], for a non-negative function φ, T_φ is invertible if and only if $E_s = \{z \in \mathbb{C} : \varphi(z) > s\}$ is a dominating set for some $s > 0$. Tracking the constants we obtain

Corollary 1.3. *Let φ be a non-negative bounded measurable function. The operator T_φ is invertible if and only if there exists $s > 0$ such that $E_s = \{z \in \mathbb{C} : \varphi(z) > s\}$ is (γ, r)-dense for some $\gamma > 0$ and $r > 0$. In this case,*

$$\|T_\varphi^{-1}\| \leq \frac{\|\varphi\|_\infty^{-1}}{1 - \sqrt{1 - (s/\|\varphi\|_\infty)^2 (\gamma/c)^{2L}}}.$$

Notice that the right hand side behaves like γ^{-2L} as $\gamma \to 0$. For the sake of completeness, we will give a proof of the reverse implication at the end of Section 4.

When φ is the characteristic function χ_E of a subset E of $\mathbb{C}$, the Toeplitz operator T_{χ_E} is also called the *concentration operator* (see [Sei91]). It has been widely studied in the context of wavelets and time-frequency analysis (see for example [Sei89, MR23]). In that case, the above corollary reads as follows.

Corollary 1.4. *Let E be a subset of $\mathbb{C}$. The operator T_{χ_E} is invertible if and only if E is (γ, r)-dense for some $\gamma > 0$ and $r > 0$. In this case,*

$$\|T_{\chi_E}^{-1}\| \leq \frac{1}{1 - \sqrt{1 - (\gamma/c)^{2L}}}.$$

The paper is organized as follows. In Section 2 we recall several results from [MMOC03] concerning doubling measures and subharmonic functions. In Section 3, we introduce planar Remez-type inequalities which will be a key ingredient in the proof of Theorem 1.1. In Section 4, we prove the covering lemma and Theorem 1.1 and deduce Corollary 1.3. Finally, in Section 5 we explain briefly how to generalize our result to several complex variables.

2. Reminders on doubling measures. A non-negative doubling measure μ (see (1.1) for the definition) determines a metric on $\mathbb{C}$ to which the usual notions have to be adapted. In this section, we recall several geometric measure-theoretical tools in connection with doubling measures that we will need later and which come essentially from the paper [MMOC03, Section 2]. Actually, the results of the present section can be translated in terms of the distance induced by the metric $\rho(z)^{-2} dz \otimes d\bar{z}$ but we will not exploit this point of view here.

We start with a standard geometric estimate whose main part is due to [Chr91, Lemma 2.1].

Lemma 2.1 ([MMOC03, Lemma 1]). *Let μ be a doubling measure on $\mathbb{C}$. For any disks $D(z, r)$ and $D(z', r')$ such that $r > r'$ and $z' \in D(z, r)$, we have*

$$\frac{1}{2C_\mu}\left(\frac{r}{r'}\right)^{\kappa} \leq \frac{\mu(D(z,r))}{\mu(D(z',r'))} \leq C_\mu^2 \left(\frac{r}{r'}\right)^{\log_2(C_\mu)},$$

where $\kappa = 1/\lceil C_\mu^2 + 2 \rceil$.

Here $\lceil x \rceil$ denotes the smallest integer larger than or equal to x and correspondingly, $\lfloor x \rfloor$ is the largest integer less than or equal to x. The paper [MMOC03] refers to [Chr91] for the proof of this result. We mention that this latter paper only contains the left hand estimate. For the convenience of the reader and in order to better understand the constants, we provide a detailed proof completing Christ's argument.

Proof of Lemma 2.1. Let us start with the right hand inequality. By the triangular inequality, we have $D(z, r) \subset D(z', 2r)$. Now, note that

$$2r = 2^{\log_2(r/r')+1} r' \leq 2^{\lceil \log_2(r/r') \rceil + 1} r'.$$

Hence iterating the doubling inequality (1.1), we get

$$\begin{aligned}\mu(D(z,r)) \le \mu(D(z',2r)) &\le C_\mu^{\lceil \log_2(r/r')\rceil+1}\mu(D(z',r'))\\ &\le C_\mu^2\Big(\frac{r}{r'}\Big)^{\log_2(C_\mu)}\mu(D(z',r')).\end{aligned}$$

As mentioned above, the left hand inequality is given in [Chr91, Lemma 2.1]. We reproduce the proof and make the constant κ more precise. Since $z' \in D(z,r)$, we have $D(z',r) \subset D(z,2r)$ and so

$$\frac{\mu(D(z,r))}{\mu(D(z',r'))} \ge \frac{C_\mu^{-1}\mu(D(z,2r))}{\mu(D(z',r'))} \ge \frac{C_\mu^{-1}\mu(D(z',r))}{\mu(D(z',r'))}. \tag{2.1}$$

Hence it suffices to prove that for any $z' \in \mathbb{C}$,

$$\frac{\mu(D(z',r))}{\mu(D(z',r'))} \ge \frac{1}{2}\Big(\frac{r}{r'}\Big)^{\kappa}$$

whenever $r > r'$. Let $k \ge 3$ be an integer to be fixed later.

First assume that $r/r' = 2^k$. Then we can construct pairwise disjoint disks $D_1, \dots D_{k-2}$ such that D_j has radius $2^j r'$, $D(z',r') \subset 3D_j$ and $D_j \subset D(z',r)$. A way to do so is to choose the disks D_j centered along a radius of $D(z',r)$ and ordered in such a way that the radius increases away from z' (see Figure 1).

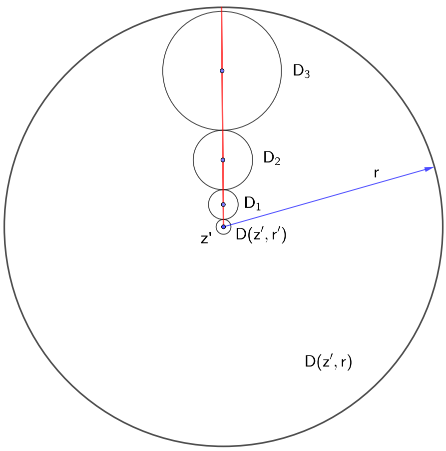

Fig. 1. The disks D_j, $D(z',r')$ and $D(z',r)$

Now, since $\mu(D(z',r')) \le \mu(3D_j) \le C_\mu^2\mu(D_j)$, we get

$$\mu(D(z',r)) \ge \sum_{j=1}^{k-2}\mu(D_j) + \mu(D(z',r')) \ge [(k-2)C_\mu^{-2}+1]\mu(D(z',r')).$$

Hence, fixing k to be the smallest integer such that $(k-2)C_\mu^{-2}+1 \geq 2$, i.e. $k = \lceil C_\mu^2 + 2 \rceil$, we obtain

$$\mu(D(z',r)) \geq 2\mu(D(z',r'))$$

whenever $r/r' = 2^k$.

Let us treat the general case $r > r'$ now. If $r/r' \geq 2^k$, we reduce the problem to the previous setting by decomposing $D(z',r)$ into a chain of disks

$$D(z',r') = D(z',r_0) \subset D(z',r_1) \subset \cdots \subset D(z',r_m) \subset D(z',r),$$

where $r_i = 2^k r_{i-1}$ for $1 \leq i \leq m$, and $m = \lfloor \log_2(r/r')/k \rfloor$. Hence we can iterate the process and get, for every $z' \in \mathbb{C}$,

$$\begin{aligned} \mu(D(z',r)) &\geq \mu(D(z',r_m)) \geq 2\mu(D(z',r_{m-1})) \geq \cdots \geq 2^m \mu(D(z',r')) \\ &= 2^{\lfloor \log_2(r/r')/k \rfloor} \mu(D(z',r')) \geq \frac{1}{2}\left(\frac{r}{r'}\right)^{1/k} \mu(D(z',r')). \end{aligned}$$

Combined with (2.1), this leads to the left hand inequality of Lemma 2.1 with

$$\kappa = \frac{1}{k} = \frac{1}{\lceil C_\mu^2 + 2 \rceil}.$$

If $r/r' < 2^k$, then it is clear that

$$\frac{1}{2}\left(\frac{r}{r'}\right)^{\kappa} < 1 \leq \frac{\mu(D(z',r))}{\mu(D(z',r'))}$$

since $r > r'$. Again, combined with (2.1), this yields the expected constant in the left hand inequality. ∎

Remark 2.2. If instead of $z' \in D(z,r)$, we have $D(z,r) \cap D(z',r') \neq \emptyset$ as in [MMOC03, Lemma 1], we get the same inequalities with C_μ replaced by C_μ^2 in the left hand inequality and C_μ^2 replaced by C_μ^3 in the right hand inequality. Also, when $z = z'$, we can remove C_μ from the left hand side and replace C_μ^2 by C_μ in the right hand side.

Remark 2.3. In the classical Fock space, μ is the Lebesgue measure (up to a multiplicative constant) and so $C_\mu = 4$. Note that in this case, we obtain equality on the right hand side and the factor C_μ^2 disappears. This is due to the invariance by translation and the homogeneity of the Lebesgue measure.

As a direct consequence of Lemma 2.1 and Remark 2.2, picking $z' = z$, and replacing r by $r\rho(z)$ and r' by $\rho(z)$, we get the following useful estimate: for all $z \in \mathbb{C}$ and $r > 1$,

$$\tfrac{1}{2} r^\kappa \leq \mu(D^r(z)) \leq C_\mu r^{\log_2(C_\mu)}. \tag{2.2}$$

We are now looking for an upper bound and a lower bound for $\rho(z)/\rho(w)$ whenever $w \in D^r(z)$.

LEMMA 2.4. *For every $z \in \mathbb{C}$, we have*

$$\forall r > 0,\ \forall w \in D^r(z), \qquad \frac{\rho(z)}{\rho(w)} \geq \frac{1}{1+r} \tag{2.3}$$

and

$$\forall r > 1,\ \forall w \in D^r(z), \qquad \frac{\rho(z)}{\rho(w)} \lesssim \max(r^{\log_2(C_\mu)/\kappa - 1}, 1) = r^{\max(\log_2(C_\mu)/\kappa-1,\, 0)}. \tag{2.4}$$

Proof. A classical fact is that ρ is a 1-Lipschitz function (see [OP16, equation (2.4)] for a simple proof):

$$|\rho(z) - \rho(w)| \leq |z - w|, \qquad \forall z, w \in \mathbb{C}.$$

Hence, for every $z \in \mathbb{C}$, we get inequality (2.3) by the triangular inequality. To obtain the upper bound, we use both inequalities of Lemma 2.1, distinguishing the cases $r\rho(z) \geq \rho(w)$ and $r\rho(z) < \rho(w)$. We get, for every $w \in D^r(z)$,

$$\frac{r\rho(z)}{\rho(w)} \lesssim \max(\mu(D^r(z))^{1/\kappa}, \mu(D^r(z))^{1/\log_2(C_\mu)}).$$

Therefore inequality (2.2) implies that for $r > 1$,

$$\frac{r\rho(z)}{\rho(w)} \lesssim \max(r^{\log_2(C_\mu)/\kappa}, r),$$

and hence (2.4) follows. ∎

Notice that for C_μ large enough, we have $\log_2(C_\mu)/\kappa - 1 \geq 0$, which means that the maximum in (2.4) is equal to $r^{\log_2(C_\mu)/\kappa-1}$. This holds exactly when $C_\mu \geq \sqrt[4]{2}$.

In the proof of our main theorem, we will need a particular covering of the complex plane. Let us explain how we can construct it. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ is *ρ-separated* if there exists $\delta > 0$ such that

$$|a_i - a_j| \geq \delta \max(\rho(a_i), \rho(a_j)), \qquad \forall i \neq j.$$

This means that the disks $D^\delta(a_n)$ are pairwise disjoint. We will cover $\mathbb{C}$ by disks satisfying a finite overlapping property and such that the sequence formed by their centers is ρ-separated. The authors of [MMOC03] construct a decomposition of $\mathbb{C}$ into rectangles R_k: $\mathbb{C} = \bigcup_k R_k$, and two such rectangles can intersect at most along sides. For these rectangles, so-called quasi-squares, there exists a constant $e > 1$ depending only on C_μ such that the ratio of sides of every R_k lies in the interval $[1/e, e]$. Denoting by a_k the center of R_k, Theorem 8(c) of [MMOC03] claims in particular that there is $r_0 \geq 1$ such that, for every k,

$$\frac{\rho(a_k)}{r_0} \leq \operatorname{diam} R_k \leq r_0 \rho(a_k).$$

In other words,

$$D^{1/(2Cr_0)}(a_k) \subset R_k \subset D^{r_0/2}(a_k) \tag{2.5}$$

with $C = \sqrt{1+e^2}$. As a consequence we get the required covering $\mathbb{C} = \bigcup D^{r_0/2}(a_k)$, and the sequence (a_k) is ρ-separated (take $\delta = 1/(2Cr_0)$).

We end this section by focusing on the particular case of a doubling measure μ given by the Laplacian $\Delta\varphi$ of a subharmonic function φ, which is the case of interest in this paper. We will need to control the value of φ in a disk by the value at its center. For this, we can use [MMOC03, Lemma 13] that we state as follows: for every $\sigma > 0$, there exists $A = A(\sigma) > 0$ such that for all $k \in \mathbb{N}$,

$$\sup_{z \in D^\sigma(a_k)} |\varphi(z) - \varphi(a_k) - \mathsf{h}_{a_k}(z)| \leq A(\sigma) \tag{2.6}$$

where h_{a_k} is a harmonic function in $D^\sigma(a_k)$ with $\mathsf{h}_{a_k}(a_k) = 0$. Moreover, in view of [MMOC03, proof of Lemma 13] we have

$$A(\sigma) \lesssim \sup_{k \in \mathbb{N}} \mu(D^\sigma(a_k)) \lesssim \sigma^{\log_2(C_\mu)}, \tag{2.7}$$

where the inequalities come from [MMOC03, Lemma 5(a)] and (2.2). This last result will allow us to translate the subharmonic weight locally into a holomorphic function.

3. Remez-type inequalities. Let us introduce a central result of the paper [AR07]. Let G be a (bounded) domain in $\mathbb{C}$ and let $0 < s < |\overline{G}|$ (the Lebesgue measure of $\overline{G}$). Denoting by Pol_n the space of complex polynomials of degree at most $n \in \mathbb{N}$, we introduce the set

$$P_n(\overline{G}, s) = \{p \in \mathrm{Pol}_n : |\{z \in \overline{G} : |p(z)| \leq 1\}| \geq s\}.$$

Next, let

$$R_n(z, s) = \sup_{p \in P_n(\overline{G}, s)} |p(z)|.$$

This expression gives the largest possible value at z of a polynomial p of degree at most n and being at most 1 on a set of measure at least s. In particular, [AR07, Theorem 1] claims that for $z \in \partial G$, we have

$$R_n(z, s) \leq \left(\frac{c}{s}\right)^n \tag{3.1}$$

where the constant c depends only on the (square of the) diameter of G. This result corresponds to a generalization to the two-dimensional case of the Remez inequality which is usually given in dimension 1. In what follows we will essentially consider G to be a disk or a rectangle. By the maximum modulus principle, the above constant gives an upper estimate on G for an arbitrary polynomial of degree at most n which is bounded by 1 on a set of measure at least s. Obviously, if this set is small (s close to 0), i.e. p is controlled by 1 on a small set, then the estimate has to get worse.

REMARK 3.1. Let us make another observation. If c is the constant in (3.1) associated with the unit disk $G = \mathbb{D} = D(0,1)$, then a simple argument based on homothety shows that the corresponding constant for an arbitrary disk $D(0,r)$ is cr^2 (for $D(0,r)$ considered as the underlying domain, the constant c appearing in [AR07, Theorem 1] satisfies $c > 2m_2(D(0,r))$). So, in what follows we will use the estimate

$$R_n(z,s) \leq \left(\frac{cr^2}{s}\right)^n, \tag{3.2}$$

where c does not depend on r.

Up to a translation, the following counterpart of Kovrizhkin's result for the planar case has been given in [HKKO21]:

LEMMA 3.2. *Let $0 < r < R$ be fixed. There exists a constant $\eta > 0$ such that the following holds. Let $w \in \mathbb{C}$, let $E \subset D^r(w)$ be a planar measurable set of positive measure. For every f analytic in $D^R(w)$, if there exists $z_0 \in D^r(w)$ such that $|f(z_0)| \geq 1$ and $M = \sup_{z \in D^R(w)} |f(z)|$ then*

$$\sup_{z \in D^r(w)} |f(z)| \leq \left(\frac{cr^2 \rho(w)^2}{|E|}\right)^{\eta \log(M)} \sup_{z \in E} |f(z)|,$$

where c does not depend on r, and

$$\eta \leq c'' \frac{R^4}{(R-r)^4} \log\left(\frac{R}{R-r}\right)$$

for an absolute constant c''.

The corresponding case for p-norms is deduced exactly as in Kovrizhkin's work.

COROLLARY 3.3. *Let $0 < r < R$ be fixed. There exists a constant $\eta > 0$ such that the following holds. Let $w \in \mathbb{C}$ and let $E \subset D^r(w)$ be a planar measurable set of positive measure. For every f analytic in $D^R(w)$, if there exists $z_0 \in D^r(w)$ such that $|f(z_0)| \geq 1$ and $M = \sup_{z \in D^R(w)} |f(z)|$ then for $p \in [1, \infty)$,*

$$\|f\|_{L^p(D^r(w))} \leq \left(\frac{cr^2 \rho(w)^2}{|E|}\right)^{\eta \log(M) + 1/p} \|f\|_{L^p(E)}.$$

The estimate on η is the same as in the lemma. The constant c does not depend on r.

4. Proof of Theorem 1.1. The key ingredient in the proof of Theorem 1.1 and a main contribution of this paper is the existence and the estimate of the covering constant. Indeed, we have shown in Section 2 that the disks $(D^s(a_k))_{k\in\mathbb{N}}$ cover the complex plane $\mathbb{C}$ for $s > r_0/2$. Now, we shall prove that there exists a covering constant N depending on the covering radius $s \geq r_0/2$. This means that the number of overlapping disks cannot exceed N. We denote by χ_F the characteristic function of a measurable set F in $\mathbb{D}$. We are now in a position to prove the covering lemma.

LEMMA 4.1. *For $s > 1$, there exists a constant $N := N(s)$ such that*

$$\sum_k \chi_{D^s(a_k)} \leq N.$$

Moreover, there exists some universal constant $c_{\text{ov}} := c_{\text{ov}}(\varphi) > 0$ such that

$$N \leq c_{\text{ov}} s^\alpha,$$

where

$$\alpha = \min\left[2\max\left(1 + \frac{\log_2(C_\mu)}{\kappa}, 2\right), \max\left(\frac{\log_2(C_\mu)}{\kappa}, 1\right)\log_2(C_\mu)\right],$$

C_μ is the doubling constant and κ is given in Lemma 2.1.

Obviously, the constant N is at least equal to 1, and since N is non-decreasing in s, we are only interested in the behavior when s goes to ∞. We also recall that in order to cover the complex plane $\mathbb{C}$, we need to require that $s \geq r_0/2$.

Proof of Lemma 4.1. The proof is inspired by that of [AS11, Lemma 2.5]. Let $s > 1$ and $z \in \mathbb{C}$. Denote $\Gamma(z) = \{k \in \mathbb{N},\, z \in D^s(a_k)\}$. Our goal is to estimate $\#\Gamma(z)$. Write $\theta = \log_2(C_\mu)/\kappa$. Recall that for $z \in D^s(a_k)$, by (2.3) and (2.4) we have

$$\frac{\rho(z)}{1+s} \leq \rho(a_k) \lesssim \rho(z)\max(s^{\theta-1}, 1). \tag{4.1}$$

Now, pick $\delta > 0$ such that the disks $D^\delta(a_n)$ are disjoint (recall that $(a_n)_{n\in\mathbb{N}}$ is ρ-separated). By the triangular inequality, there exists a constant $C > 0$

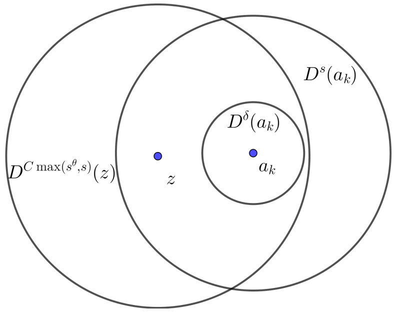

Fig. 2. The disks $D^s(a_k)$, $D^{C\max(s^\theta,s)}(z)$ and $D^\delta(a_k)$

such that $D^\delta(a_k) \subset D^{C\max(s^\theta,s)}(z)$ whenever $z \in D^s(a_k)$ (see Figure 2). Indeed, if $w \in D^\delta(a_k)$ then

$$\begin{aligned}|w-z| &\le |w-a_k| + |z-a_k| \le \delta\rho(a_k) + s\rho(a_k)\\ &\lesssim (\delta+s)\rho(z)\max(s^{\theta-1},1) \lesssim \rho(z)\max(s^\theta,s),\end{aligned}$$

where we have taken into account that $\delta > 0$ is fixed and $s > 1$.

Hence, since $D^\delta(a_k)$ are disjoint and with (4.1) in mind, it follows that

$$\begin{aligned}\#\Gamma(z) &\le \#\{k \in \mathbb{N} : D^\delta(a_k) \subset D^{C\max(s^\theta,s)}(z)\}\\ &\le \frac{|D^{C\max(s^\theta,s)}(z)|}{\inf_{k\in\Gamma(z)} |D^\delta(a_k)|} = \sup_{k\in\Gamma(z)} \frac{[C\max(s^\theta,s)\rho(z)]^2}{(\delta\rho(a_k))^2}\\ &\lesssim [\max(s^\theta,s)(1+s)]^2 \lesssim s^{2\max(1+\theta,\,2)},\end{aligned}$$

where we have used $\max(s^\theta,s) = s^{\max(\theta,1)}$ for $s > 1$.

Moreover, notice that $\mu[D^\delta(a_k)] \gtrsim 1$ since $0 < \delta < 1$ is fixed. Hence an analogous computation, with the Lebesgue measure replaced by the measure μ and with an appeal to the right hand side of inequality (2.2), leads to

$$\begin{aligned}\#\Gamma(z) &\le \#\{k \in \mathbb{N} : D^\delta(a_k) \subset D^{C\max(s^\theta,s)}(z)\}\\ &\le \frac{\mu[D^{C\max(s^\theta,s)}(z)]}{\inf_{k\in\Gamma(z)} \mu[D^\delta(a_k)]} \lesssim \max(s^\theta,s)^{\log_2(C_\mu)} \lesssim s^{\max(\theta,1)\log_2(C_\mu)}.\end{aligned}$$

Finally, taking the minimum between the two estimates, for $s > 1$ we get

$$\begin{aligned}\#\Gamma(z) &\lesssim \min(s^{2\max(1+\theta,2)},\ s^{\max(\theta,1)\log_2(C_\mu)})\\ &= s^{\min[2\max(1+\theta,2),\,\max(\theta,1)\log_2(C_\mu)]}.\end{aligned}$$

Since the estimates are uniform in z, the lemma follows. ∎

As in [HKKO21] for the Bergman space, we now introduce good disks. Fix $r_0/2 \le s < t$, where $r_0/2$ is the radius such that the disks $D^{r_0/2}(a_k)$ cover the complex plane. For $K > 1$ the set

$$I_f^{K\text{-good}} = \{k : \|f\|_{L^p_\varphi(D^t(a_k))} \le K\|f\|_{L^p_\varphi(D^s(a_k))}\}$$

will be called the set of K-*good disks* for (t,s) (in order to keep the notation light we will not include s and t as indices). This set depends on f.

The following proposition has been shown in [HKKO21] for the Bergman space, but its proof, implying essentially the finite overlapping property, is exactly the same.

PROPOSITION 4.2. *Let $r_0/2 \le s < t$. For every constant $c \in (0,1)$, there exists K such that for every $f \in \mathcal{F}^p_\varphi$ we have*

$$\sum_{k\in I_f^{K\text{-good}}} \|f\|^p_{L^p_\varphi(D^s(a_k))} \ge c\|f\|^p_{L^p_\varphi(\mathbb{C})}.$$

One can pick $K^p = N(t)/(1-c)$, where $N(t)$ corresponds to the overlapping constant from Lemma 4.1 for the radius t.

We are now in a position to prove the main theorem.

Proof of Theorem 1.1. Take $s = \max(r, r_0/2)$ and $t = 4s$. As noted in [HKKO21], if E is (γ, r)-dense then E is $(\widetilde{\gamma}, s)$-dense, where $\widetilde{\gamma} = c\gamma$ and c is a multiplicative constant. Hence we can assume that E is (γ, s)-dense instead of (γ, r)-dense.

Let h_{a_k} be the harmonic function introduced in (2.6) for $\sigma = t$. Since h_{a_k} is harmonic on $D^t(a_k)$, there exists a function H_{a_k} holomorphic on $D^t(a_k)$ with real part h_{a_k}.

Now, given f with $\|f\|_{L^p_\varphi(\mathbb{C})} = 1$, let $g = fe^{-(H_{a_k}+\varphi(a_k))}$ for $k \in I_f^{K\text{-good}}$, and set

$$h = c_0 g, \qquad c_0 = \left(\frac{\pi s^2 \rho(a_k)^2}{\int_{D^s(a_k)} |g|^p \, dA}\right)^{1/p}.$$

Clearly there is $z_0 \in D^s(a_k)$ with $|h(z_0)| \geq 1$.

Set $R = 2s$. We have to estimate the maximum modulus of h on $D^R(a_k)$ in terms of a local integral of h. To that purpose, we use the fact that $h \in A^p(D^t(a_k))$. Indeed, applying (2.6) we have

$$\begin{aligned}
\int_{D^t(a_k)} |h|^p \, dA &= \frac{\pi s^2 \rho(a_k)^2}{\int_{D^s(a_k)} |g|^p \, dA} \int_{D^t(a_k)} |g|^p \, dA \\
&= \frac{\pi s^2 \rho(a_k)^2}{\int_{D^s(a_k)} |f|^p e^{-p(\mathsf{h}_{a_k}+\varphi(a_k))} \, dA} \int_{D^t(a_k)} |f|^p e^{-p(\mathsf{h}_{a_k}+\varphi(a_k))} \, dA \\
&\leq \frac{\pi s^2 \rho(a_k)^2 e^{2pA(t)}}{\int_{D^s(a_k)} |f|^p e^{-p\varphi(z)} \, dA(z)} \int_{D^t(a_k)} |f|^p e^{-p\varphi(z)} \, dA(z) \\
&\leq \pi s^2 \rho(a_k)^2 e^{2pA(t)} K^p,
\end{aligned}$$

where the last inequality comes from the fact that k is K-good for (s, t). Therefore, taking into account this last estimate, we deduce from the subharmonicity of $|h|^p$ that

$$M^p := \max_{z \in D^R(a_k)} |h(z)|^p \leq \frac{1}{\pi s^2 \rho(a_k)^2} \int_{D^t(a_k)} |h|^p \, dA \leq c_s K^p,$$

where $c_s = e^{2pA(4s)}$ since $t = 4s$.

Now, setting $\widetilde{E} = E \cap D^s(a_k)$ we get, using Corollary 3.3 applied to h,

$$\int_{D^s(a_k)} |h(z)|^p \, dA(z) \leq \left(\frac{cs^2 \rho(a_k)^2}{|\widetilde{E}|}\right)^{p\eta \log(M)+1} \int_{\widetilde{E}} |h(z)|^p \, dA(z).$$

Again, by homogeneity we can replace h by g in the above inequality. Note also that $\pi s^2\rho(a_k)^2/|\widetilde{E}|$ is controlled by $1/\gamma$. This yields

$$\begin{aligned}\int\limits_{D^s(a_k)} |f|^p e^{-p\varphi(z)}\,dA(z) &\leq e^{pA(t)} \int\limits_{D^s(a_k)} |g(z)|^p\,dA(z)\\ &\leq e^{pA(t)}\left(\frac{cs^2\rho(a_k)^2}{|\widetilde{E}|}\right)^{p\eta\log(M)+1}\int\limits_{\widetilde{E}}|g(z)|^p\,dA(z)\\ &\leq e^{pA(t)}\left(\frac{c_1}{\gamma}\right)^{p\eta\log(M)+1}\int\limits_{\widetilde{E}}|g(z)|^p\,dA(z)\\ &\leq e^{2pA(t)}\left(\frac{c_1}{\gamma}\right)^{p\eta\log(M)+1}\int\limits_{\widetilde{E}}|f(z)|^p e^{-p\varphi(z)}\,dA(z),\end{aligned}$$

where c_1 is an absolute constant.

Summing over all K-good k, and using Lemma 4.1 and Proposition 4.2, we obtain the required result

$$c\|f\|_{L^p_\varphi(\mathbb{C})} \lesssim \left(\frac{c_1}{\gamma}\right)^{\eta\log(M)+1/p}\|f\|_{L^p_\varphi(E)},$$

where in view of Lemma 3.2

$$\eta \leq c'' \times 2^4\log(2)$$

and

$$\log(M) \leq \log(c_s^{1/p}K) = \log\left(\left(c_s\frac{N(4s)}{1-c}\right)^{1/p}\right) \leq \frac{1}{p}\log\left(e^{2pA(4s)}c_{\mathrm{ov}}\frac{(4s)^\alpha}{1-c}\right).$$

Here c comes from Proposition 4.2, and c_{ov} and α from Lemma 4.1. In particular, fixing $c \in (0,1)$ we obtain

$$\log(M) \leq 2A(4s) + \frac{1}{p}(C' + C''\log(s)),$$

where C', C'' depend only on the space. In view of (2.7), we get

$$\log(M) \lesssim s^{\log_2(C_\mu)} + \frac{1}{p}(1+\log(s)).$$

Finally, noticing that $r \asymp s = \max(r, r_0/2)$ for $r > 1$, we arrive at

$$\log(M) \lesssim r^{\log_2(C_\mu)} + \frac{1}{p}(1+\log(r)). \blacksquare$$

Finally, we give a short proof of the reverse implication in Corollary 1.3 which is a straightforward adaptation to the doubling Fock space of Luecking's [Lue81, proof of Corollary 3].

Proof. First, assume that $\varphi \le 1$. Therefore $s \le 1$. Since E_s is (γ, r)-dense, it is a dominating set. So

$$\int_{\mathbb{C}} \varphi^2 |f|^2 e^{-2\varphi}\, dA \ge s^2 \int_{E_s} |f|^2 e^{-2\varphi}\, dA \ge s^2 C^2 \|f\|^2_{L^2_\varphi(\mathbb{C})},$$

where C is the sampling constant. Then

$$\begin{aligned}\|(I - T_\varphi)f\|^2_{L^2_\varphi(\mathbb{C})} &= \|T_{1-\varphi} f\|^2_{L^2_\varphi(\mathbb{C})} = \|\mathbf{P}[(1-\varphi)f]\|^2_{L^2_\varphi(\mathbb{C})} \le \|(1-\varphi)f\|^2_{L^2_\varphi(\mathbb{C})} \\ &\le \int_{\mathbb{C}} (1-\varphi^2)|f|^2 e^{-2\varphi}\, dA \le (1 - s^2C^2)\|f\|^2_{L^2_\varphi(\mathbb{C})},\end{aligned}$$

hence $\|I - T_\varphi\| < 1$. So T_φ is invertible and

$$\|T_\varphi^{-1}\| \le \frac{1}{1 - \|I - T_\varphi\|} \le \frac{1}{1 - \sqrt{1 - s^2C^2}}.$$

Using Theorem 1.1, we obtain

$$\|T_\varphi^{-1}\| \le \frac{1}{1 - \sqrt{1 - s^2(\gamma/c)^{2L}}}.$$

Finally, for a general φ, let $\psi = \varphi/\|\varphi\|_\infty$. Then $\psi \le 1$, $E_s = \{\psi \ge s/\|\varphi\|_\infty = s'\}$ and $T_\varphi = \|\varphi\|_\infty T_\psi$. So, applying the previous discussion to ψ, we deduce

$$\begin{aligned}\|T_\varphi^{-1}\| = \|T_\psi^{-1}\|\, \|\varphi\|_\infty^{-1} &\le \frac{\|\varphi\|_\infty^{-1}}{1 - \sqrt{1 - (s')^2(\gamma/c)^{2L}}} \\ &= \frac{\|\varphi\|_\infty^{-1}}{1 - \sqrt{1 - (s/\|\varphi\|_\infty)^2(\gamma/c)^{2L}}}. \ \blacksquare\end{aligned}$$

5. Extension to several complex variables. It is worth to make precise how the results above extend to the case of several complex variables. In this context, $\varphi : \mathbb{C}^n \to \mathbb{R}$ denotes a C^2-plurisubharmonic function, which means that the $(1,1)$-form $i\partial\bar{\partial}\varphi$ is positive. Moreover we assume that

$$(i\partial\bar{\partial}\varphi)^n = \underbrace{(i\partial\bar{\partial}\varphi) \wedge \cdots \wedge (i\partial\bar{\partial}\varphi)}_{n \text{ times}}$$

is a doubling measure on $\mathbb{C}^n$. For suitable adaptations (e.g. replacing quasi-squares by quasi-hypercubes), all the results in Section 2 hold. However, Kovrizhkin's method used to establish Lemma 3.2 cannot be generalized. Indeed, it is based on the Blaschke factorization of a bounded holomorphic function which is a phenomenon specific to the one complex variable case. A way to avoid this problem is to substitute the classical Remez inequality by the following (additive) Remez-type inequality for plurisubharmonic functions proved by Brudnyi [Bru99, Theorem 1.2].

THEOREM 5.1 (Brudnyi). *Let $a > 1$ be some fixed constant chosen so that the ball $B_{\mathbb{R}}(x,t)$ satisfies $B_{\mathbb{R}}(x,t) \subset B_{\mathbb{C}}(x, at) \subseteq B_{\mathbb{C}}(0,1)$ and let $r > 1$.*

There exist constants $c = c(a, r) > 0$ and $d = d(n) > 0$ such that

$$\sup_{B_{\mathbb{R}}(x,t)} F \le c \log\left(\frac{d|B_{\mathbb{R}}(x,t)|}{|E|}\right) + \sup_E F \tag{5.1}$$

for every measurable subset $E \subset B_{\mathbb{R}}(x,t)$ of positive measure and every plurisubharmonic function $F : \mathbb{C}^n \to \mathbb{R}$ satisfying

$$\sup_{B_{\mathbb{C}}(0,r)} F = 0 \quad \text{and} \quad \sup_{B_{\mathbb{C}}(0,1)} F \ge -1.$$

Hence, if f a holomorphic function on $\mathbb{C}^n$, we apply the above theorem to the plurisubharmonic function $(z, z') \mapsto \log(|f(z + iz')|)$ in $\mathbb{C}^{2n}$, suitably normalized. This is the method used in [JS21, Theorem 3.6], and leads to the following version of Lemma 3.2.

Lemma 5.2. *Let $0 < r < R$ be fixed. Let $w \in \mathbb{C}^n$ and let $E \subset B^r_{\mathbb{C}}(w) := B_{\mathbb{C}}(w, r\rho(w))$ be a measurable set of positive measure. For every f analytic in $B^R_{\mathbb{C}}(w)$, if there exists $z_0 \in B^r_{\mathbb{C}}(w)$ such that $|f(z_0)| \ge 1$ and $M = \sup_{B^R_{\mathbb{C}}(w)} |f|$ then*

$$\sup_{B^r_{\mathbb{C}}(w)} |f| \le \left(\frac{d|B^r_{\mathbb{C}}(w)|}{|E|}\right)^{c\log(M)} \sup_E |f|,$$

where $d = d(n)$ is the constant in Brudnyi's theorem, and c depends only on the ratio r/R.

Hence our Theorem 1.1 is unchanged except that now the constant c depends also on the dimension.

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