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# Calderón–Zygmund theory with noncommuting kernels via $\mathrm{H}_1^c$

by

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**Abstract.** We study an alternative definition of the $\mathrm{H}_1$-space associated to a semicommutative von Neumann algebra $L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M}$, first studied by Mei. We identify a "new" description for atoms in $\mathrm{H}_1$. We then explain how they can be used to study $\mathrm{H}_1^c$-$L_1$ endpoint estimates for Calderón–Zygmund operators with noncommuting kernels.

**Introduction.** This paper is related to the theory of semicommutative Calderón–Zygmund operators. This research line takes advantage of the hybrid nature of certain vector-valued $L_p$-spaces. Let $(\mathcal{M},\tau)$ be a von Neumann algebra of operators on a separable Hilbert space, equipped with a normal semifinite faithful trace $\tau$. Denote by $\mathcal{A}$ the weak operator closure of the space of essentially bounded (strongly measurable) functions $f:\mathbb{R}\to\mathcal{M}$ acting on $L_2(\mathbb{R};L_2(\mathcal{M}))$. The von Neumann algebra $\mathcal{A}$ can be identified with the tensor product $L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M}$ equipped with the trace

$$\varphi(f)=\int_{\mathbb{R}}\tau(f(x))\,dx.$$

For the sake of exposition, we will restrict ourselves to dimension 1, even though our arguments extend trivially to any finite dimension by considering $L_\infty(\mathbb{R}^n)\,\overline{\otimes}\,\mathcal{M}$.

The noncommutative $L_p$-spaces associated with $\mathcal{A}$ are indeed vector-valued $L_p$-spaces: more precisely [22, Chapter 3],

$$L_p(\mathcal{A})=L_p(\mathbb{R};L_p(\mathcal{M}))$$

for $1\le p<\infty$. We are interested in endpoint estimates for operators acting on $L_p(\mathcal{A})$, and in particular in the boundedness of operators from the operator-valued version of the Hardy space $\mathrm{H}_1$ into $L_1$. This question was

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widely studied in the classical setting for scalar-valued functions [19, 20] as well as for vector-valued functions [7, 12], where the existence of the atomic decomposition plays an essential role. This technique does not seem to have been often exploited in the noncommutative setting, except perhaps in [11]. Mei [18] was the first to introduce the so-called *operator-valued Hardy space* $\mathrm{H}_1(\mathbb{R},\mathcal{M})$ in this context via noncommutative equivalents of the Poisson integral, the Luzin area integral and the Littlewood–Paley $g$ function. These techniques allowed Mei to identify the dual space of $\mathrm{H}_1(\mathbb{R},\mathcal{M})$, which is denoted by $\mathrm{BMO}(\mathbb{R},\mathcal{M})$, in the spirit of the classical argument by Fefferman and Stein [6]. Moreover, some maximal inequalities and several interpolation results via a martingale approach were established. Mei's fundamental contribution has been key to the development of noncommutative forms of Calderón–Zygmund theory, both in the semicommutative context and in fully noncommutative ones via transference techniques. For the first one, the semicommutative Calderón–Zygmund theory was initiated in [21] with the first weak-$L_1$ endpoint inequalities for singular integrals; the arguments were simplified in recent years [2, 3]. The second line has many instances, among which are [10, 16].

The initial motivation for the present work was to obtain new interpolation consequences of endpoint estimates of the type $L_\infty(\mathcal{A})$-$\mathrm{BMO}(\mathbb{R},\mathcal{M})$ which rely, by duality, on the structure of the Hardy space $\mathrm{H}_1(\mathbb{R},\mathcal{M})$. Our goal led to two main tasks: a completely explicit description of $\mathrm{BMO}(\mathbb{R},\mathcal{M})$, and the study of the boundedness of Calderón–Zygmund operators on the Hardy space via atomic decomposition. This may seem mundane but another of our goals is to provide complete proofs; for instance, in many papers the extension from atoms to the whole Hardy space is overlooked but requires more delicate arguments than the main estimates (see [1]). We also point out that, up to now, there is no notion of noncommutative distributions, thus one has to be careful when defining Calderón–Zygmund operators.

The operator-valued BMO-space introduced by Mei, $\mathrm{BMO}(\mathbb{R},\mathcal{M})$, is defined as the intersection of a column space and a row space, $\mathrm{BMO}^c(\mathbb{R},\mathcal{M})$ and $\mathrm{BMO}^r(\mathbb{R},\mathcal{M})$ respectively. Considering both a column and a row space is a ubiquitous phenomenon in noncommutative analysis (see [17] for an outstanding example), and by symmetry we shall limit our discussion to the column case. $\mathrm{BMO}^c(\mathbb{R},\mathcal{M})$ is set to be the subspace of the column Hilbert-valued space $L_\infty\big(\mathcal{M};L_2^c\big(\mathbb{R},\frac{dt}{1+t^2}\big)\big)$ (see [14]) on which the seminorm

$$\text{(MBMO)}\qquad \|g\|_{\mathrm{BMO}^c}=\sup_{\substack{I\subseteq\mathbb{R}\\ |I|\ \text{finite}}}\bigg\|\bigg(\frac{1}{|I|}\int_I|g-g_I|^2\bigg)^{1/2}\bigg\|_{\mathcal{M}}$$

makes sense, where $g_I=\frac{1}{|I|}\int_I g$. The $\mathrm{BMO}^r$ seminorm is $\|g\|_{\mathrm{BMO}^r}=\|g^*\|_{\mathrm{BMO}^c}$. The space $L_\infty\big(\mathcal{M};L_2^c\big(\mathbb{R},\frac{dt}{1+t^2}\big)\big)$ denotes the closure of $\mathcal{M}\otimes L_2\big(\mathbb{R},\frac{dt}{1+t^2}\big)$

with respect to the weak* topology of the von Neumann algebra $\mathcal{M}\,\overline{\otimes}\, B\big(L_2\big(\mathbb{R},\frac{dt}{1+t^2}\big)\big)$. There is no guarantee that $\mathrm{BMO}^c(\mathbb{R},\mathcal{M})$ is a space of $\mathcal{M}$-valued functions (in the Bochner sense), and so the integral in (MBMO) may not be well-defined. Indeed,

$$L_2\bigg(\mathbb{R},\frac{dt}{1+t^2};\mathcal{M}\bigg)\subset L_\infty\bigg(\mathcal{M};L_2^c\bigg(\mathbb{R},\frac{dt}{1+t^2}\bigg)\bigg),$$

but the reverse inclusion fails in general, *a priori* preventing us from defining $\mathrm{BMO}^c$ as a space of functions. In Section 1 and the first part of Section 2, we propose a general construction of $\mathrm{BMO}^c(\mathbb{R},\mathcal{M})$ which recovers Mei's description given by (MBMO). We will also study a predual of $\mathrm{BMO}^c(\mathbb{R},\mathcal{M})$ (resp. $\mathrm{BMO}^r(\mathbb{R},\mathcal{M})$). The novelty here is that this predual space, which we denote as $\mathrm{H}_1^r(\mathcal{A})$ (resp. $\mathrm{H}_1^c(\mathcal{A})$), will be a row (resp. column) Hardy space which is exclusively constructed in terms of "new" atomic decompositions, which extend the work of the second author's PhD thesis [23].

The key to our approach is the $\mathrm{H}_1$-BMO duality product when elements in $\mathrm{H}_1$ are described in terms of atoms. In the classical case, it is well-known that the norm of $g\in\mathrm{BMO}(\mathbb{R})$ can be characterized through the expression

$$\text{(atBMO)}\qquad \|g\|_{\mathrm{BMO}}=\sup_a\Big|\int ga\Big|,$$

where the supremum is taken over $L_2$-atoms [8, 19, 20]. An analogous formula for $\mathrm{BMO}^r(\mathbb{R},\mathcal{M})$ may shed light on the structure of atoms in $\mathrm{H}_1^c(\mathcal{A})$. This is exactly what we achieve. First, it turns out that it is enough to consider only elements $g\in\mathcal{A}\cap\mathrm{BMO}^r(\mathbb{R},\mathcal{M})$. The expression (MBMO) is meaningful for $g$, and so duality yields

$$\begin{aligned}\|g\|_{\mathrm{BMO}^r}=\|g^*\|_{\mathrm{BMO}^c}&=\sup_I\bigg\|\bigg(\frac{1}{|I|}\int_I|g^*-(g^*)_I|^2\bigg)^{1/2}\bigg\|_{\mathcal{M}}\\&=\sup_{I,\,\|h\|_{L_2(\mathcal{M})}\le1}\bigg(\int_I\bigg\|\frac{1}{\sqrt{|I|}}h(g-g_I)\bigg\|_{L_2(\mathcal{M})}^2\bigg)^{1/2},\end{aligned}$$

with the supremum taken over $h$ in the unit ball of $L_2(\mathcal{M})$. Now, recalling that $g-g_I$ has zero integral over $I$, it follows that

$$\begin{aligned}\|g\|_{\mathrm{BMO}^r}&=\sup_{I,h}\bigg\|\frac{1}{\sqrt{|I|}}h(g-g_I)\chi_I\bigg\|_{L_2^c(\mathbb{R};L_2(\mathcal{M}))}\\&=\sup_{I,h}\sup_{\|f\|_{L_2}\le1}\bigg|\Big(\tau\circ\int\Big)\bigg(h(g-g_I)\frac{f\chi_I}{\sqrt{|I|}}\bigg)\bigg|.\end{aligned}$$

Comparing the latter expression with (atBMO) suggests that an atom in $\mathrm{H}_1^c(\mathbb{R},\mathcal{M})$ should be an operator of the form $a=bh$ in $L_1(\mathcal{A})$, where $h\in L_2(\mathcal{M})$ with $\|h\|_{L_2(\mathcal{M})}\le1$ and $b\in L_2(\mathcal{A})$ is supported on some interval $I$ and has an additional cancellation over $I$ that we will make precise later. In what follows, $a$ will be called a *c-atom*. Then, we define the column Hardy space $\mathrm{H}_1^c(\mathcal{A})$ as the Banach subspace of $L_1(\mathcal{A})$ of operators $f$ such that

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \quad \text{in } L_1(\mathcal{A}) \text{ for some } c\text{-atoms } (a_i)_i \text{ and } (\lambda_i)_i \in \ell_1,$$

which becomes a Banach space with respect to the norm

$$\|f\|_{\mathrm{H}_1^c} = \inf \Big\{ \sum_{i=0}^{\infty} |\lambda_i| : f = \sum_{i=0}^{\infty} \lambda_i a_i \Big\}.$$

The row space $\mathrm{H}_1^r(\mathcal{A})$ is defined analogously and $\mathrm{H}_1(\mathcal{A}) = \mathrm{H}_1^c(\mathcal{A}) + \mathrm{H}_1^r(\mathcal{A})$. By symmetry, it suffices to show $\mathrm{H}_1^c(\mathcal{A})^* = \mathrm{BMO}^r(\mathbb{R}, \mathcal{M})$. The proof of this duality result strongly relies on the extension of a well-known argument by Meyer [19]: the space

$$L_2^\circ(\mathbb{R}, (1+t^2)dt) = \Big\{ f \in L_2(\mathbb{R}, (1+t^2)dt) : \int_{\mathbb{R}} f = 0 \Big\}$$

is a dense subspace of the classical atomic Hardy space $\mathrm{H}_1(\mathbb{R})$. A further characterization of the column Hilbert-valued spaces $L_\infty(\mathcal{M}; H^c)$ will be the key to establishing an analogous result in our context. In Section 4 we will briefly explain how to achieve that the column Hardy space $\mathrm{H}_1^c(\mathcal{A})$ coincides with the one introduced by Mei [18]. This will in particular allow us to indeed use interpolation as usual.

It is worth noting that Mei's work already contains a description of $\mathrm{H}_1^c(\mathbb{R}, \mathcal{M})$ in terms of certain atomic decompositions. However, the one included in the present paper seems more useful to deal with Calderón–Zygmund operators and to make connections with vector-valued harmonic analysis. Indeed, it allows us to consider singular integrals with noncommuting kernels, something that seems to be difficult to reach at the weak-$L_1$ level [3]. Let $\mathcal{M}$ be a von Neumann algebra over a separable Hilbert space. We denote by $\mathcal{S}$ the set of compactly supported essentially bounded functions $\mathbb{R} \to L_\infty \cap L_1(\mathcal{M})$ (measurable with values in $L_1$); it will play the role of test functions. Let $T$ be a bounded operator on $L_2(\mathcal{A})$ for which there exists a kernel $K : \mathbb{R} \times \mathbb{R} \setminus \{x = y\} \to \mathcal{M}$ such that

$$\int T(f)(x) g(x)\, dx = \iint K(x,y) f(y) g(x)\, dx\, dy$$

for any $f, g \in \mathcal{S}$ such that the distance between $\operatorname{supp} \|f\|_{L_2(\mathcal{M})}$ and $\operatorname{supp} \|g\|_{L_2(\mathcal{M})}$ is strictly positive. In that case, we say that $T$ is a Calderón–Zygmund operator with kernel $K$. Also, assume that $T$ fulfills

- a right-modularity condition: $T(fh) = T(f)h$ for any $f \in L_2(\mathcal{A})$ with compact support and $h \in \mathcal{M}$; we say that $T$ is a *left Calderón–Zygmund operator*,

- the *Hörmander condition*: for some $\lambda > 0$,

$$\int_{|x-y|\geq\lambda|y'-y|} \|K(x,y) - K(x,y')\|_{\mathcal{M}}\, dx < \infty.$$

Then $T$ extends to a bounded map $T : \mathrm{H}_1^c(\mathcal{A}) \to L_1(\mathcal{A})$. The proof is inspired by [20] and is divided into two steps. The first one consists in obtaining a constant $C_\lambda > 0$ such that

$$\|T(a)\|_{L_1(\mathcal{A})} \leq C_\lambda \quad \text{for any } c\text{-atom } a.$$

For this, we first check that the extension of $T$ to $c$-atoms satisfies

$$T(a) = T(b)h \quad \text{for any } c\text{-atom } a = bh.$$

This allows us to exploit the boundedness of $T$ on $L_2(\mathcal{A})$. We rely on an approximation of $T$ and $K$ by certain uniformly bounded kernels, which also implies that $T$ extends to the whole $\mathrm{H}_1^c(\mathcal{A})$. Once we have done that, we infer that whenever $K$ is scalar and modularity works on both sides, $T$ extends to a bounded operator from $\mathrm{H}_1(\mathcal{A})$ into $L_1(\mathcal{A})$. This is in a sense a dual statement of a result in [15] where $L_\infty$-BMO estimates are given, and close in spirit to [11].

The rest of the paper is organized as follows. In Section 1, we shall introduce the column and row Hilbert-valued noncommutative $L_p$-spaces. This enables us to define $\mathrm{BMO}(\mathbb{R}, \mathcal{M})$, as well as to identify a predual $\mathrm{H}_1^c(\mathcal{A})$ in Section 2. In Section 3 we will see how the atomic decomposition provides a boundedness result for Calderón–Zygmund operators with noncommuting kernels. We then briefly explain in Section 4 how to connect our construction to Mei's. We finish with some technical but useful lemmas in the appendices.

**1. Column/row Hilbert-valued $L_p$-spaces.** In this section we collect some basic facts on classical vector-valued $L_p$-spaces [13], as well as their interaction with row and column Hilbert-valued $L_p$-spaces [14].

Before giving definitions, some facts about vector-valued functions and von Neumann algebras should be recalled (see [5, Chapter 2] and [24, Section 1.22]). Let $\mathbb{X}$ be a Banach space, and let $(\Omega, \mu)$ be a $\sigma$-finite measure space (or more generally, a localizable measure space [25]). Then a function $f : \Omega \to \mathbb{X}$ is said to be *$\mu$-measurable* whenever there exists a sequence $(f_n)_{n\geq 1}$ of simple functions $f_n = \sum_i x_i^n \chi_{A_i^n}$ for some $x_i^n \in \mathbb{X}$ and $\mu$-measurable sets $A_i^n$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{\mathbb{X}} = 0 \quad \mu\text{-almost everywhere.}$$

For $1 \leq p < \infty$, one can then define $L_p(\Omega, \mu; \mathbb{X})$ as the space of $\mu$-measurable functions $f$ for which $\|f(\cdot)\|_{\mathbb{X}} \in L_p(\Omega, \mu)$ with the norm

$$\|f\|_p = \big\| \|f(\cdot)\|_{\mathbb{X}} \big\|_{L_p(\Omega,\mu)}.$$

In our situation, we recall the classical identities:

$$L_1(\Omega,\mu;L_1(\mathcal{M})) = L_1(L_\infty(\Omega,\mu)\,\overline{\otimes}\,\mathcal{M}) = L_1(\Omega,\mu)\,\widehat{\otimes}_\pi\,L_1(\mathcal{M}),$$
$$L_2(\Omega,\mu;L_2(\mathcal{M})) = L_2(L_\infty(\Omega,\mu)\,\overline{\otimes}\,\mathcal{M}) = L_2(\Omega,\mu)\otimes_2 L_2(\mathcal{M}),$$

where $\widehat{\otimes}_\pi$ and $\otimes_2$ are the Banach space projective tensor product and the Hilbert space tensor product. On the other hand, the identification of the von Neumann algebra $L_\infty(\Omega,\mu)\,\overline{\otimes}\,\mathcal{M}$ as a space of vector-valued functions requires a more subtle construction. In general, a function $f:\Omega\to\mathbb{X}^*$ is said to be *weak* $\mu$-measurable* if $J_x\circ f$ is measurable for each $x\in\mathbb{X}$, where $J_x$ denotes the continuous functional on $\mathbb{X}^*$ given by $J_x(x^*)=x^*(x)$ for every $x^*$ in $\mathbb{X}^*$. Consider $\mathbb{X}=L_1(\mathcal{M})$ (which is assumed to be separable to avoid measurability issues) and define $L_\infty(\Omega,\mu;\mathcal{M})$ as the Banach space of all $\mathcal{M}$-valued weak* $\mu$-measurable functions which are essentially bounded, that is,

$$\operatorname*{ess\,sup}_{t\in\Omega}\|f(t)\|_{\mathcal{M}}<\infty.$$

Then $L_\infty(\Omega,\mu;\mathcal{M})$ is a von Neumann algebra under pointwise multiplication, and the map

$$f\otimes m\mapsto f(t)m,\qquad f\in L_\infty(\Omega),\ m\in\mathcal{M},$$

can be extended to a weak* isomorphism of $L_\infty(\Omega,\mu)\overline{\otimes}\mathcal{M}$ onto $L_\infty(\Omega,\mu;\mathcal{M})$.

Let $H$ be a separable Hilbert space. We can identify $B(H)$ as the space of bounded infinite matrices acting on $H$ (once a basis is fixed). When equipped with the usual trace Tr for matrices, it gives rise to the Schatten classes $S_p(H)=L_p(B(H),\mathrm{Tr})$ for any $0<p\le\infty$. Along this work, the inner product in $H$ is assumed to be linear in the second variable and antilinear in the first one. Moreover, elements of $H$ viewed in the dual Hilbert space $H^*$ will be represented by overlined letters: $\overline{h}$ will denote the continuous functional

$$\overline{h}:H\ni k\mapsto\langle h,k\rangle\qquad\text{for any } k\in H.$$

Given $\xi,\eta\in H$, we consider the *rank-one operator* $\xi\otimes\overline{\eta}$ acting on $H$ as follows:

$$(\xi\otimes\overline{\eta})(h)=\langle\eta,h\rangle\xi\qquad\text{for any } h\in H.$$

In the following, $\mathbb{1}$ denotes a fixed element of $H$ with $\|\mathbb{1}\|_H=1$, and $p_{\mathbb{1}}=\mathbb{1}\otimes\overline{\mathbb{1}}$ denotes the rank-one projection onto $\operatorname{span}\{\mathbb{1}\}$. Assume that $\mathcal{M}$ is an arbitrary semifinite von Neumann algebra equipped with a normal semifinite faithful trace $\tau$. Then we define the *column Hilbert-valued $L_p$-space*

$$L_p(\mathcal{M};H^c)=L_p(\mathcal{M}\,\overline{\otimes}\,B(H))(\mathbf{1}_{\mathcal{M}}\otimes p_{\mathbb{1}})$$

for any $0<p\le\infty$. Identify $L_p(\mathcal{M})$ as a subspace of $L_p(\mathcal{M}\,\overline{\otimes}\,B(H))$ via the map $m\mapsto m\otimes p_{\mathbb{1}}=(\mathbf{1}_{\mathcal{M}}\otimes p_{\mathbb{1}})(m\otimes\mathbf{1}_{B(H)})(\mathbf{1}_{\mathcal{M}}\otimes p_{\mathbb{1}})$. This is equivalent to

the identity

$$L_p(\mathcal{M}) = (\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}}) L_p(\mathcal{M} \overline{\otimes} B(H))(\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}}).$$

Then, for any $f \in L_p(\mathcal{M}; H^c)$,

$$f^* f \in (\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}}) L_{p/2}(\mathcal{M} \overline{\otimes} B(H))(\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}}) = L_{p/2}(\mathcal{M}).$$

This justifies defining

$$\|f\|_{L_p(\mathcal{M}; H^c)} = \|f\|_{L_p(\mathcal{M} \overline{\otimes} B(H))} = \|(f^* f)^{1/2}\|_{L_p(\mathcal{M})}$$

on $L_p(\mathcal{M}; H^c)$. Analogously, the *row Hilbert-valued* $L_p$*-space* is

$$L_p(\mathcal{M}; H^{*r}) = (\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}}) L_p(\mathcal{M} \overline{\otimes} B(H))$$

equipped with the norm

$$\|f\|_{L_p(\mathcal{M}; H^{*r})} = \|(f f^*)^{1/2}\|_{L_p(\mathcal{M})} = \|f^*\|_{L_p(\mathcal{M}; H^c)}.$$

Given $h \in H$ and $m \in L_p(\mathcal{M})$, we will write $m \otimes h \in L_p(\mathcal{M}) \otimes H$ for the element $m \otimes (h \otimes \overline{\mathbb{1}}) \in L_p(\mathcal{M}; H^c)$, and similarly $m \otimes \overline{h}$ means $m \otimes (\mathbb{1} \otimes \overline{h}) \in L_p(\mathcal{M}; H^{*r})$.With these notations, for $f = \sum_{i=1}^n m_i \otimes h_i$ and $g = \sum_{i=1}^n m_i \otimes \overline{h}_i$,

$$\|f\|_{L_p(\mathcal{M}; H^c)} = \Big\| \Big( \sum_{i,j=1}^n \langle h_i, h_j \rangle_H m_i^* m_j \Big)^{1/2} \Big\|_{L_p(\mathcal{M})},$$

$$\|g\|_{L_p(\mathcal{M}; H^{*r})} = \Big\| \Big( \sum_{i,j=1}^n \langle h_i, h_j \rangle_H m_i m_j^* \Big)^{1/2} \Big\|_{L_p(\mathcal{M})}.$$

In this way, we do not need to refer to the function $\mathbb{1}$. Moreover, $(m \otimes h)^* = m^* \otimes \overline{h}$ as operators. We will use without reference the fact that $L_p(\mathcal{M}) \otimes H$ is dense (resp. weak* dense) in $L_p(\mathcal{M}; H^c)$ for $1 \le p < \infty$ (resp. $p = \infty$), and similarly for rows.

In fact, column and row Hilbert-valued $L_p$-spaces satisfy the expected duality relations expressed via the natural duality bracket

$$\langle f, g \rangle_{r,c} = \mathrm{Tr} \otimes \tau(fg), \qquad \langle m \otimes \overline{h}, m' \otimes h' \rangle_{r,c} = \tau(mm') \langle h, h' \rangle. \tag{1.1}$$

In particular, we have linearly isometrically

$$L_p(\mathcal{M}; H^c)^* = L_{p'}(\mathcal{M}; H^{*r}) \quad \text{and} \quad L_p(\mathcal{M}; H^{*r})^* = L_{p'}(\mathcal{M}; H^c)$$

for any $1 \le p < \infty$ whenever $1/p + 1/p' = 1$. On the other hand, we recall the following fact about homogeneity of column and row Hilbert spaces (see [14, Lemma 2.4]). We state it only for columns.

Corollary 1.1. *Let $H$ and $K$ be two Hilbert spaces, and let $T : H \to K$ be a bounded linear operator. Then $\mathrm{Id}_{L_p(\mathcal{M})} \otimes T$ admits a unique continuous (resp. weak* continuous) extension $\widetilde{T}$ from $L_p(\mathcal{M}; H^c)$ into $L_p(\mathcal{M}; K^c)$ for $1 \le p < \infty$ (resp. $p = \infty$) with the same norm.*

Some obvious properties of these extension maps will be crucial in the following sections.

LEMMA 1.2. *Let $H$ and $K$ be two Hilbert spaces, let $S$, $T$, and $(T_j)_{j=1}^{\infty}$ be some bounded operators from $H$ to $K$, and let $\widetilde{S}, \widetilde{T}, (\widetilde{T}_j)_{j=1}^{\infty}$ be the corresponding extensions from $L_p(\mathcal{M}; H^c)$ to $L_p(\mathcal{M}; K^c)$. Then the following hold:*

(1) $\widetilde{ST} = \widetilde{S}\widetilde{T}$.
(2) *If $S$ and $T$ commute, then $\widetilde{S}$ and $\widetilde{T}$ also commute.*
(3) *If $\sum_{j=1}^{\infty} T_j$ converges in the norm of $B(H)$, then*

$$\widetilde{\sum_{j=1}^{\infty} T_j} = \sum_{j=1}^{\infty} \widetilde{T}_j.$$

(4) $(\widetilde{S})^* = \widetilde{\overline{S}^*}$ *for $p < \infty$.*

In the last statement, one needs a conjugation because the $^*$ on the left is for the Banach space adjoint whereas the right one is in the Hilbert space sense. Of course similar statements hold for row spaces and bounded maps $T : H^* \to K^*$.

**1.1. Noncommutative spaces $L_p(\mathcal{M}; L_2^c(\Omega))$.** Let $(\Omega, \mu)$ be a $\sigma$-finite measure space. A remarkable setting for noncommutative Hilbert-valued column/row $L_p$-spaces is the case $H = L_2(\Omega) := L_2(\Omega, \mu)$. Notice that under these conditions, the duality bracket (1.1) is given by the expression

$$\langle m_1 \otimes \overline{f}_1, m_2 \otimes f_2 \rangle_{r,c} = \tau_{\mathcal{M}}(m_1 m_2) \int_{\Omega} \overline{f}_1 f_2 \, d\mu.$$

In particular, identifying $L_2(\Omega, \mu)^*$ and $L_2(\Omega, \mu)$ and using the bilinear pairing $(f, g) \mapsto \int_{\Omega} fg \, d\mu$, we have

$$L_{p'}(\mathcal{M}; L_2^r(\Omega, \mu)) = L_p(\mathcal{M}; L_2^c(\Omega, \mu))^* \quad \text{for } 1 \le p < \infty.$$

This duality identity will allow us to drop the conjugation or the involution $*$ from now on. Moreover, for $F = \sum_{i=1}^{n} m_i \otimes f_i \in L_p(\mathcal{M}) \otimes L_2(\Omega)$ with $p < \infty$,

$$\|F\|^p_{L_p(\mathcal{M}; L_2^c(\Omega,\mu))} = \tau\Big( \int_{\Omega} \Big| \sum_{i=1}^{n} f_i(t) m_i \Big|^2 d\mu \Big)^{p/2}.$$

Since $\sum_{i=1}^{n} f_i(t) m_i$ can be interpreted as $F(t)$, it is tempting to consider elements in $L_p(\mathcal{M}; L_2^c(\Omega))$ as functions. Indeed, for $p \le 2$ (see [14, Proposition 2.5]) we have:

PROPOSITION 1.3. *For $1 \le p \le 2$, the identity on $L_p(\mathcal{M}) \otimes L_2(\Omega)$ extends to an injective contraction $L_p(\mathcal{M}; L_2^c(\Omega)) \to L_2(\Omega; L_p(\mathcal{M}))$.*

*Proof.* First note that by definition

$$L_2(\mathcal{M}) \otimes_2 L_2(\Omega) = L_2(\mathcal{M}; L_2^c(\Omega)) = L_2(\Omega; L_2(\mathcal{M})).$$

Let $p < 2$ and let $q$ be such that $1/p = 1/2 + 1/q$. The element

$$F = \sum_{i=1}^{n} m_i \otimes f_i \in L_p(\mathcal{M}) \otimes L_2(\Omega) \subset L_p(\mathcal{M}; L_2^c(\Omega))$$

corresponds to an element of $L_p(\mathcal{M} \overline{\otimes} B(L_2(\Omega)))$. Its modulus $|F|^2 = \int_\Omega |F(t)|^2 \, d\mu$ falls into $L_{p/2}(\mathcal{M})$, and its polar decomposition can be written as $F = ab$ with $b \in L_q(\mathcal{M})$ and $a \in L_2(\mathcal{M}; L_2^c(\Omega))$ with $\|a\|_2 \|b\|_q = \|F\|_p$. Actually, $a$ is also a simple function and

$$\int_\Omega \|F(t)\|_p^2 \, d\mu \leq \|b\|_q^2 \int_\Omega \|a(t)\|_2^2 \, d\mu = \|F\|_p^2.$$

This shows that the identity is indeed a contraction and thus extends to a contraction $\iota$ by density.

To show the injectivity, consider $m \in L_q(\mathcal{M})$ and $h \in L_2(\Omega)$. The linear form associated to $m \otimes h$ satisfies $\langle F, m \otimes h\rangle_{c,r} = \int_\Omega \tau(\iota(F)(t)m)h(t)\, d\mu$. Indeed, this is clear for $F \in L_p(\mathcal{M}) \otimes L_2(\Omega)$, and for all $F$ by density. Since $L_q(\mathcal{M}) \otimes L_2(\Omega)$ is norm-dense in $L_q(\mathcal{M}; L_2^r(\Omega))$, the proof is complete. ■

As a consequence, we can identify elements in $L_p(\mathcal{M}; L_2^c(\Omega))$ with a.e. Bochner measurable functions from $\Omega$ to $L_p(\mathcal{M})$ when $1 \leq p \leq 2$. This will be convenient for some identifications. Unfortunately, when $p > 2$ we have no way to consider elements in $L_p(\mathcal{M}; L_2^c(\Omega))$ as functions. Indeed, in the following sections, the case $p = \infty$ will be specially relevant. For that reason, the extension of some useful operators on $L_2(\Omega)$ to $L_p(\mathcal{M}; L_2^c(\Omega))$ will be carefully studied.

Lemma 1.4. *Let $A, B$ be two measurable sets, and let $w$ and $w'$ be strictly positive functions belonging to $L_\infty(\Omega, \mu)$. Consider the following maps on $L_2(\Omega, \mu)$:*

$$T_w : f \mapsto w^{1/2} f, \qquad P_A : f \mapsto \chi_A f.$$

*They extend to bounded operators on $L_p(\mathcal{M}; L_2^t(\Omega, \mu))$ for $t = c, r$ and any $1 \leq p \leq \infty$ such that $\|\widetilde{T}_w\| = \|w\|_{L_\infty(\Omega,\mu)}$, $\|\widetilde{P}_A\| = \|\chi_A\|_{L_\infty(\Omega,\mu)}$. Moreover, they satisfy the following relations:*

(1) $\widetilde{T}_w \widetilde{T}_{w'} = \widetilde{T}_{w'} \widetilde{T}_w$,
(2) $\widetilde{P}_A \widetilde{T}_w = \widetilde{T}_w \widetilde{P}_A$,
(3) *whenever $w^{-1}$ is bounded,* $\widetilde{T}_w \widetilde{T}_{w^{-1}} = \mathrm{Id} = \widetilde{T}_{w^{-1}} \widetilde{T}_w$.
(4) $\widetilde{P}_A = \widetilde{P}_A \widetilde{P}_B = \widetilde{P}_B \widetilde{P}_A$ *whenever* $A \subseteq B$,
(5) $\widetilde{P}_{B\setminus A} = \widetilde{P}_B - \widetilde{P}_A$ *whenever* $A \subseteq B$.

*Proof.* By Corollary 1.1, the extension operators $\widetilde{T}_w$, $\widetilde{P}_A$ are bounded as long as the original ones are bounded on $L_2(\Omega,\mu)$. The maps $T_w$ and $P_A$ are bounded with norms $\|w^{1/2}\|_{L_\infty}$ and 1 respectively, since they are pointwise multiplication operators. Claims (1)–(4) follow from Lemma 1.2, while linearity of the map $T\mapsto\widetilde{T}$ implies (5). ∎

**2. Duality between Hardy spaces and** BMO**-spaces.** Consider the measure space $\big(\mathbb{R},\frac{dt}{1+t^2}\big)$. Set $\omega(t)=1+t^2$. Then $L_2\big(\mathbb{R},\frac{dt}{1+t^2}\big)$ is a Hilbert space with the inner product

$$\langle g,f\rangle_{1/\omega}:=\int_{\mathbb{R}}\overline{g(t)}f(t)\,\frac{dt}{1+t^2}.$$

We will consider the associated column space $L_p\big(\mathcal{M};L_2^c\big(\mathbb{R},\frac{dt}{1+t^2}\big)\big)$ for $0<p\le\infty$. We will choose $\mathbb{1}$ to be the constant function $1/\sqrt{\pi}$, which satisfies the condition $\|\mathbb{1}\|_{L_2(\mathbb{R},\frac{dt}{1+t^2})}=1$. For the sake of exposition, we define some operators on $L_\infty\big(\mathcal{M};L_2^c\big(\mathbb{R},\frac{dt}{1+t^2}\big)\big)$ which can be described in terms of the maps appearing in Lemma 1.4. Let $A$ be a measurable set with nonzero measure and define

$$R_A=\frac{1}{|A|}\widetilde{T}_\omega\widetilde{P}_A,$$

so that $R_A$ is the extension of the operator acting on $L_2\big(\mathbb{R},\frac{dt}{1+t^2}\big)$ as follows:

$$f\mapsto\frac{(1+t^2)^{1/2}}{\sqrt{|A|}}\chi_Af.$$

On the other hand, denote by $a_A$ the extension to $L_\infty\big(\mathcal{M};L_2^c\big(\mathbb{R},\frac{dt}{1+t^2}\big)\big)$ of the map

$$a_A:f\mapsto f_A\,\mathbb{1}=\bigg(\frac{1}{|A|}\int_A f\bigg)\mathbb{1}.$$

Similarly the linear form $f\mapsto f_A$ has a weak* extension from $L_\infty\big(\mathcal{M};L_2^c\big(\mathbb{R},\frac{dt}{1+t^2}\big)\big)$ to $\mathcal{M}=L_\infty(\mathcal{M};\mathbb{C}^c)$. Thus we may also use the notation $f_A$ for every $f\in L_\infty\big(\mathcal{M};L_2^c\big(\mathbb{R},\frac{dt}{1+t^2}\big)\big)$.

Some other Hilbert spaces over the real line will be considered through this work. Since the distinguished vectors are not relevant, we will always write them as $\mathbb{1}$ in all spaces.

LEMMA 2.1. *Let $A$ be a measurable set. Then the pre-adjoint maps for $R_A$ and $a_A$ on $L_1(\mathcal{M};L_2^r(\mathbb{R},dt/(1+t^2)))$ act as follows on any operator $m\otimes f\in L_1(\mathcal{M})\otimes L_2(\mathbb{R},dt/(1+t^2))$:*

$$(R_A)_* : m \otimes f \mapsto m \otimes \frac{\sqrt{1+t^2}}{\sqrt{|A|}} \chi_I f,$$

$$(a_A)_* : m \otimes f \mapsto m \otimes \langle 1, f \rangle_{1/\omega} \frac{1+t^2}{|A|} \chi_A.$$

*Moreover, the identity operator*

$$V : L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, (1+t^2)dt)) \mapsto L_\infty\Big(\mathcal{M}; L_2^r\Big(\mathbb{R}, \frac{dt}{1+t^2}\Big)\Big),$$

$$m \otimes f \mapsto m \otimes f,$$

*is contractive and admits a pre-adjoint*

$$V_* : L_1\Big(\mathcal{M}; L_2^c\Big(\mathbb{R}, \frac{dt}{1+t^2}\Big)\Big) \mapsto L_1(\mathcal{M}; L_2^c(\mathbb{R}, (1+t^2)dt)),$$

$$m \otimes f \mapsto m \otimes \frac{f}{1+t^2}.$$

*Proof.* These are consequences of the same results on Hilbert spaces using Proposition 1.2. One just has to take care of conjugations which actually play no role here. ∎

Now, we are ready to define the column and row BMO-spaces.

Definition 2.2. Given a von Neumann algebra $\mathcal{M}$ with n.s.f. trace $\tau$, define the *column* BMO-*space*, denoted as $\mathrm{BMO}^c(\mathbb{R}, \mathcal{M})$, to be the subspace of operators $f$ in $L_\infty\big(\mathcal{M}; L_2^c\big(\mathbb{R}, \frac{dt}{1+t^2}\big)\big)$ satisfying

$$\|f\|_{\mathrm{BMO}^c} := \sup_{I \subseteq \mathbb{R}} \|R_I(\mathrm{Id} - a_I) f\|_{L_\infty(\mathcal{M}; L_2^c(\mathbb{R}; \frac{dt}{1+t^2}))} < \infty, \tag{2.1}$$

where the supremum is taken over finite intervals $I$ of $\mathbb{R}$. Likewise, the *row* BMO-*space*, $\mathrm{BMO}^r(\mathbb{R}, \mathcal{M})$, is the subspace of elements in $L_\infty\big(\mathcal{M}; L_2^r\big(\mathbb{R}, \frac{dt}{1+t^2}\big)\big)$ for which the norm $\|f\|_{\mathrm{BMO}^r} := \|f^*\|_{\mathrm{BMO}^c}$ is finite.

It is clear that $\|\cdot\|_{\mathrm{BMO}^c}$ is a norm modulo $\mathcal{M}$. This expression will be convenient for abstract questions. We point out that it admits a much more tractable form. Multiplication by $\sqrt{1+t^2}$ is an isometry from $L_2(\mathbb{R})$ to $L_2\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$. For a finite interval $I$, let $\widetilde{\iota}_I$ denote the extension of the map $f \mapsto \chi_I f$ from $L_2\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$ to $L_2(\mathbb{R})$. Then clearly

$$\|R_I(\mathrm{Id} - a_I) f\|_{L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))} = \Big\| \frac{1}{\sqrt{|I|}} \widetilde{\iota}_I (f - (f_I \otimes 1)) \Big\|_{L_\infty(\mathcal{M}; L_2^c(\mathbb{R}))}. \tag{2.2}$$

In particular, for operators in $\mathcal{M} \otimes L_2\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$, we recover the expression which determines the definition for the $\mathrm{BMO}^c$-norm introduced in [18]:

Lemma 2.3. *For any operator $f$ in $\mathcal{M} \otimes L_2\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$,*

$$\|f\|_{\mathrm{BMO}^c} = \sup_{I \subseteq \mathbb{R}} \bigg\| \bigg( \frac{1}{|I|} \int_I |f - f_I|^2 \bigg)^{1/2} \bigg\|_{\mathcal{M}}.$$

Along the next section, the study of the boundedness of Calderón–Zygmund operators on operator-valued Hardy spaces will require a concrete formulation in terms of atomic decompositions. In order to justify introducing these spaces, we will check that the dual of this new description of the column (resp. row) Hardy space coincides with $\mathrm{BMO}^r(\mathbb{R}, \mathcal{M})$ (resp. $\mathrm{BMO}^c(\mathbb{R}, \mathcal{M})$).

Definition 2.4. Let $\mathcal{M}$ be a von Neumann algebra with n.s.f. trace. A *c-atom* is a function $a \in L_1(L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M})$ which admits a factorization $a = bh$ for some function $b : \mathbb{R} \to L_2(\mathcal{M})$ and a norm-one operator $h \in L_2(\mathcal{M})$, satisfying

(1) $\mathrm{supp}_{\mathbb{R}}(b) \subseteq I$ for some interval $I$,
(2) $\int_I b = 0$,
(3) $\|b\|_{L_2(\mathbb{R}; L_2(\mathcal{M}))} \leq 1/\sqrt{|I|}$.

Then the *column Hardy space* $\mathrm{H}_1^c(\mathcal{A})$ is defined to be the subspace of elements in $L_1(L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M})$ of the form

$$\sum_{i=0}^{\infty} \lambda_i a_i \quad \text{where } (\lambda_i)_i \in \ell_1 \text{ and } (a_i)_i \text{ are } c\text{-atoms} \tag{2.3}$$

with respect to the norm

$$\|f\| = \inf \Big\{ \sum_{i=0}^{\infty} |\lambda_i| : f = \sum_{i=0}^{\infty} \lambda_i a_i \text{ in } L_1(L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M}), \\ (\lambda_i)_i \in \ell_1,\ (a_i)_i \ c\text{-atoms} \Big\}.$$

With the above definition, any $c$-atom satisfies

$$\|a\|_{L_1(\mathbb{R}; L_1(\mathcal{M}))} \leq 1$$

since, by the Hölder inequality,

$$\|a\|_{L_1(L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M})} \leq \|b\|_{L_2(\mathbb{R}; L_2(\mathcal{M}))} \|h \chi_I\|_{L_2(\mathbb{R}; L_2(\mathcal{M}))} \leq |I|^{-1/2} |I|^{1/2} = 1.$$

Therefore, $\mathrm{H}_1^c(\mathcal{A})$ is contractively embedded into $L_1(L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M})$.

We choose to give an explicit decomposition of atoms as $a = bh$ rather than just saying $a \in L_1(\mathcal{M}; L_2^c(\mathbb{R}))$ with norm less than $1/\sqrt{|I|}$ and mean zero. We do so as this makes explicit the connection with vector-valued harmonic analysis and will make all the proofs transparent. We leave the following to the reader:

Proposition 2.5. *The column Hardy space $(\mathrm{H}_1^c(\mathcal{A}), \|\cdot\|_{\mathrm{H}_1^c})$ is a Banach space.*

Given a Banach space $\mathbb{X}$, let $L_2^\circ(\mathbb{R},(1+t^2)dt;\mathbb{X})$ denote the subspace of functions $f$ in $L_2(\mathbb{R},(1+t^2)dt;\mathbb{X})$ satisfying

$$\int_{\mathbb{R}} f(t)\,dt = 0.$$

Then the classical argument by Meyer [19, Chapter 5, Proposition 1] extends to the Banach-valued setting yielding the inclusion of $L_2^\circ(\mathbb{R},(1+t^2)dt;\mathbb{X})$ into the vector-valued Hardy space $\mathrm{H}_1(\mathbb{R};\mathbb{X})$ [12]. More precisely, given $f \in L_2^\circ(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))$, there is a sequence $(b_i)_i \subseteq \mathrm{H}_1(\mathbb{R};L_2(\mathcal{M}))$ of atoms and $(\lambda_i)_i \in \ell_1$ such that

$$f = \sum_{i=0}^{\infty} \lambda_i b_i \text{ in } L_1(\mathbb{R};L_2(\mathcal{M})) \text{ and } \sum_{i=0}^{\infty} |\lambda_i| \lesssim \|f\|_{L_2^\circ(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))}. \tag{2.4}$$

Since any $c$-atom $a = bh$ is the product of an $L_2$-atom $b$ in $\mathrm{H}_1(\mathbb{R};L_2(\mathcal{M}))$ and an element $h$ in $L_2(\mathcal{M})$, the argument of Meyer still works in the semi-commutative case.

Proposition 2.6. *The formal identity map*

$$L_2^\circ(\mathbb{R},(1+t^2)dt) \otimes L_1(\mathcal{M}) \to L_1(\mathbb{R}) \otimes L_1(\mathcal{M})$$

*extends to an injective and contractive map* $Q : L_1(\mathcal{M};L_2^{\circ,c}(\mathbb{R},(1+t^2)dt)) \to \mathrm{H}_1^c(\mathcal{A})$.

*Proof.* First note that simple tensors $f \otimes m \in L_2^\circ(\mathbb{R},(1+t^2)dt) \otimes L_1(\mathcal{M})$ are both in $\mathrm{H}_1^c(\mathcal{A})$ and in $L_1(\mathbb{R}) \otimes L_1(\mathcal{M})$. Indeed, by Meyer's result in the scalar case, $f = \sum_i \lambda_i a_i$ where $a_i$ are scalar-valued atoms. Choose $\alpha, \beta \in L_2(\mathcal{M})$ with $m = \alpha\beta$ and $\|\alpha\|_2 = 1$. Then $a_i \otimes \alpha$ are atoms in $\mathrm{H}_1(\mathbb{R};L_2(\mathcal{M}))$, so $a_i \otimes m$ is a multiple of a $c$-atom and $f \otimes m$ is in $\mathrm{H}_1^c(\mathcal{A})$.

The end of the argument is as in Proposition 1.3. Any simple $x = \sum_{i=1}^n f_i \otimes m_i \in L_2^\circ(\mathbb{R},(1+t^2)dt) \otimes L_1(\mathcal{M})$ can be written as $x = F\beta$ with $\beta \in L_2(\mathcal{M})$ and a simple tensor $F \in L_2(\mathcal{M};L_2^{\circ,c}(\mathbb{R},(1+t^2)dt)) = L_2^\circ(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))$ such that $\|F\|_2\|\beta\|_2 = \|x\|_{L_2(\mathcal{M};L_2^{\circ,c}(\mathbb{R},(1+t^2)dt))}$. Using Meyer's decomposition (2.4) for $F$ gives the norm estimate. One deduces the boundedness of the extension by density. Injectivity follows by the weak* density of simple tensors in the dual spaces as in Proposition 1.3. ∎

Consequently, any linear form on $\mathrm{H}_1^c(\mathcal{A})$ induces a linear form on the space $L_1(\mathcal{M};L_2^{\circ,c}(\mathbb{R},(1+t^2)dt))$. Precomposing it with the map $V_*$ of Lemma 2.1 (which obviously has dense range) allows us to represent $\mathrm{H}_1^c(\mathcal{A})^*$ as a subspace of $L_1\big(\mathcal{M};L_2^{\circ,c}(\mathbb{R},\frac{1}{1+t^2}dt)\big)^* \subset L_\infty\big(\mathcal{M};L_2^r\big(\mathbb{R},\frac{1}{1+t^2}dt\big)\big)$.

Theorem 2.7. *Given a semifinite von Neumann algebra* $\mathcal{M}$, *we have a contractive inclusion*

$$\mathrm{H}_1^c(\mathcal{A})^* \subseteq \mathrm{BMO}^r(\mathbb{R},\mathcal{M}).$$

*Proof.* Let $g \in \mathrm{H}_1^c(\mathcal{A})^*$. We already explained that

$$VQ^*(g) \in L_\infty\Big(\mathcal{M}; L_2^r\Big(\mathbb{R}, \frac{1}{1+t^2} dt\Big)\Big).$$

We estimate its $\mathrm{BMO}^r$ norm, using the fact that all operators commute with the involution:

$$\begin{aligned} \|VQ^* g\|_{\mathrm{BMO}^r} &= \sup_{I \subseteq \mathbb{R}} \|R_I(\mathrm{Id} - a_I)(VQ^* g)\|_{L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dt}{1+t^2}))} \\ &= \sup_I \sup_f |\langle f, R_I(\mathrm{Id} - a_I)(VQ^* g)\rangle|, \end{aligned}$$

where the supremum is taken over $f$ in the unit ball of $L_1\big(\mathcal{M}; L_2^c\big(\mathbb{R}, \frac{dt}{1+t^2}\big)\big)$. We can as well take it over simple tensors by density (we could consider them as functions with values in $L_1(\mathcal{M})$). Moreover, by the factorization argument of Proposition 1.3, we can write $f = \sum_{i=1}^n m_i h \otimes f_i$ with $f_i \in L_2\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$, $m_i, h \in L_2(\mathcal{M})$ such that $\|h\|_2 = \|\sum_{i=1}^n m_i \otimes f_i\|_{L_2(\mathbb{R}, \frac{dt}{1+t^2}; L_2(\mathcal{M}))} = 1$. By the formulas of Lemma 2.1,

$$QV_*(\mathrm{Id} - a_I)_* R_{I*}(f) = \sum_{i=1}^n m_i h \otimes F_i,$$

where for $i = 1, \ldots, n$,

$$F_i = \frac{\chi_I}{\sqrt{|I|}\sqrt{1+t^2}} f_i - \frac{\chi_I}{|I|}\Big\langle 1, \frac{\sqrt{1+t^2}}{\sqrt{|I|}} \chi_I f_i \Big\rangle_{1/\omega} \in L_2(\mathbb{R}, (1+t^2)\, dt).$$

Set $b = \sum_{i=1}^n m_i \otimes F_i \in L_2(\mathcal{M}; L_2(\mathbb{R}, (1+t^2) dt))$. Then clearly $\mathrm{supp}_{\mathbb{R}}(b) \subseteq I$ and $\int_I b = 0$ since $\int_I F_i = 0$. Let

$$G = \frac{\chi_I}{\sqrt{|I|}\sqrt{1+t^2}} \sum_{i=1}^n m_i \otimes f_i.$$

Since $b$ is the projection of $G$ onto the orthogonal complement of the constant functions on $I$, one gets

$$\Big(\int_I \|b(t)\|_2^2 \, dt\Big)^{1/2} \leq \|G\|_{L_2(\mathbb{R})} \leq \frac{1}{\sqrt{|I|}} \Big\|\sum_{i=1}^n m_i \otimes f_i\Big\|_{L_2(\mathbb{R}, \frac{dt}{1+t^2}; L_2(\mathcal{M}))} \leq \frac{1}{\sqrt{|I|}}.$$

In other words, $QV_*(\mathrm{Id} - a_I)_* R_{I*}(f)$ is a $c$-atom, hence

$$\|VQ^* g\|_{\mathrm{BMO}^r} \leq \|g\|_{\mathrm{H}_1^c(\mathcal{A})^*}. \quad \blacksquare$$

The reverse inclusion $\mathrm{BMO}^r(\mathbb{R}, \mathcal{M}) \subseteq \mathrm{H}_1^c(\mathcal{A})^*$ is more involved. We need to check that every operator in $\mathrm{BMO}^r(\mathbb{R}, \mathcal{M})$ induces a continuous functional on $\mathrm{H}_1^c(\mathcal{A})$. The starting point of our argument is that every operator $\varphi \in \mathrm{BMO}^r(\mathbb{R}, \mathcal{M})$ induces a functional on the algebraic vector space $\mathcal{H}$ generated by the $c$-atoms in $\mathrm{H}_1^c(\mathcal{A}) \subset L_1(L_\infty(\mathbb{R}) \,\overline{\otimes}\, \mathcal{M})$.

Recall that $L_1(\mathcal{M}; L_2^c(\mathbb{R}, dt))$ can also be interpreted as a space of functions (defined a.e.) with values in $L_1(\mathcal{M})$. By definition, as a function, any $c$-atom $a$ is the pointwise product of a compactly supported function in $L_2(\mathbb{R}; L_2(\mathcal{M}))$ and a constant element $h \in L_2(\mathcal{M})$, thus it is an element in $L_1(\mathcal{M}; L_2^c(\mathbb{R}, dt))$ and moreover $\|a\|_{L_1(\mathcal{M}; L_2^c(\mathbb{R}, dt))} \leq 1$. To emphasize, we will denote the inclusion by $\gamma : \mathcal{H} \to L_1(\mathcal{M}; L_2^c(\mathbb{R}, dt))$. The arguments we just gave also show that it is continuous for the norm on $\mathcal{H}$ given by

$$\|f\|_{\mathcal{H}} = \inf\Big\{\sum_{i=1}^N |\lambda_i| : f = \sum_{i=1}^N \lambda_i a_i,\ N \geq 1,\ \lambda_i \in \mathbb{C} \text{ and } a_i \text{ } c\text{-atoms}\Big\}.$$

We will denote by $A_g$ the operator of pointwise multiplication by $g$ on a.e. functions with values in $L_1(\mathcal{M})$ or $L_2(\mathcal{M})$. It is well-defined on all elements that have compact support in the spaces we consider. The same kind of arguments justify that $A_\omega(a) \in L_1\big(\mathcal{M}; L_2^c\big(\mathbb{R}, \frac{1}{1+t^2} dt\big)\big)$ and $A_\omega : \mathcal{H} \to L_1\big(\mathcal{M}; L_2^c\big(\mathbb{R}, \frac{1}{1+t^2} dt\big)\big)$ is of course linear. $A_\omega$ is not continuous but if $a$ is an atom supported on $I$ then $A_\omega(a) = \widetilde{M_\omega^I}(\gamma(a))$, where $M_\omega^I : L_2(\mathbb{R}) \to L_2\big(\mathbb{R}, \frac{1}{1+t^2} dt\big)$ is multiplication by $\omega\chi_I$ (which is bounded).

This allows us to define a duality pairing for $\mathcal{H}$ and $\mathrm{BMO}^r$ by setting, for $\varphi \in \mathrm{BMO}^r(\mathbb{R}, \mathcal{M})$ and $f \in \mathcal{H}$,

$$\langle \varphi, f\rangle_{\mathrm{BMO}^r, \mathcal{H}} = \langle \varphi, A_\omega(f)\rangle_{L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dt}{1+t^2})), L_1(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))}.$$

This definition is so designed that if $\varphi = \sum_{i=1}^n m_i \otimes f_i \in \mathcal{M} \otimes L_2\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$ then

$$\langle \varphi, f\rangle_{\mathrm{BMO}^r, \mathcal{H}} = \int_{\mathbb{R}} \tau(f(t) m_i) f_i(t)\, dt.$$

Thus, we recover the classical duality pairing for functions.

Lemma 2.8. *Let* $\varphi \in \mathrm{BMO}^r(\mathbb{R}, \mathcal{M})$. *If* $a$ *is a c-atom in* $\mathrm{H}_1^c(\mathcal{A})$, *then*

$$|\langle \varphi, a\rangle_{\mathrm{BMO}^r, \mathcal{H}}| \leq \|\varphi\|_{\mathrm{BMO}^r}.$$

*Proof.* Let $I$ be the interval for which $a = bh$ satisfies the definition of a $c$-atom. Then $A_\omega(a) = |I|(R_{I*})^2(a)$ and the mean zero condition exactly means that $a_{I*}(A_\omega(a)) = 0$. Thus,

$$\langle \varphi, a\rangle_{\mathrm{BMO}^r, \mathcal{H}} = \langle R_I(\mathrm{Id} - a_I)\varphi, |I| R_{I*}(a)\rangle_{L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dt}{1+t^2})), L_1(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))}.$$

It remains to show that $\big\||I| R_{I*}(a)\big\|_{L_1(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))} \leq 1$; but this comes from the factorization property in terms of operators

$$|I| R_{I*}(a) = (\sqrt{|I|} A_{\sqrt{\omega}}(b)) \cdot h,$$

where $\|\sqrt{|I|} A_{\sqrt{\omega}}(b)\|_{L_2(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))} = \sqrt{|I|}\, \|b\|_{L_2(\mathcal{M}; L_2^c(\mathbb{R}, dt))} \leq 1$ and $h$ is a norm-one operator. ■

The second step in the proof is to extend the duality pairing to $\mathrm{H}_1^c(\mathcal{A})$. This will require some care and approximations.

Given a compactly supported continuous function $\xi : \mathbb{R} \to \mathbb{R}$, we denote by $R_\xi$ the convolution with $\xi$ on $L_2(\mathbb{R})$. We could consider its extension $\widetilde{R}_\xi$ to any $L_p(\mathcal{M}; L_2^c(\mathbb{R}))$. On the other hand, using vector-valued integration, the convolution with $\xi$ in $L_p(\mathbb{R}; L_p(\mathcal{M}))$ is feasible; we denote it by $\xi * x$. When $p = 2$, we have $\widetilde{R}_\xi(x) = \xi * x$ using the identification $L_2(\mathbb{R}; L_2(\mathcal{M})) = L_2(\mathcal{M}; L_2^c(\mathbb{R}))$. Moreover, we consider here a third possible extension which will be used along this section.

Lemma 2.9. *Assume $\xi$ is a compactly supported continuous function on $\mathbb{R}$. For any $f \in L_1(\mathbb{R}; L_1(\mathcal{M}))$, we have $\xi * f \in L_1(\mathcal{M}; L_2^c(\mathbb{R}))$ (viewed as a function space). Moreover, this induces a continuous linear map $C_\xi$ : $L_1(\mathbb{R}; L_1(\mathcal{M})) \to L_1(\mathcal{M}; L_2^c(\mathbb{R}))$.*

*Proof.* Recall that $L_1(\mathbb{R}; L_1(\mathcal{M}))$ coincides with the projective tensor product $L_1(\mathbb{R}) \widehat{\otimes}_\pi L_1(\mathcal{M})$. It suffices to prove the statement for a simple tensor $f = g \otimes m$ with $g \in L_1(\mathbb{R})$ and $m \in L_1(\mathcal{M})$. Then the function $\xi * f$ corresponds to the operator $m \otimes (\xi * g) \in L_1(\mathcal{M}; L_2^c(\mathbb{R}))$. Factorizing $m = rs$ with $r, s \in L_2(\mathcal{M})$ and $\|r\|_2 = \|s\|_2 = \|m\|_1^{1/2}$, we obtain a factorization in terms of operators, $m \otimes (\xi * g) = (r \otimes (\xi * g))s$, so that $\|r \otimes (\xi * g)\|_{L_2(\mathcal{M}; L_2^c(\mathbb{R}))} = \|r\|_2 \|\xi * g\|_2$. Thus, due to the Young inequality we conclude that

$$\|m \otimes (\xi * g)\|_{L_1(\mathcal{M}; L_2^c(\mathbb{R}))} \leq \|r\|_2 \|s\|_2 \|\xi * g\|_2 \leq \|m\|_1 \|g\|_1 \|\xi\|_2.$$

We see that the norm of $C_\xi$ is controlled by $\|\xi\|_2$. ■

Fix some nonnegative even continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with support in $(-1, 1)$ such that $\int_{\mathbb{R}} \phi(t)\, dt = 1$ and denote $\phi_n(x) = n\phi(nx)$.

Lemma 2.10. *Given a c-atom $a = bh$ supported on $I$, the following hold:*

(1) *$\frac{1}{2}\phi_n * a$ is a finite convex combination of c-atoms.*
(2) *If $n \geq 2/|I|$, then $\phi_n * a - a = \lambda_n a_n$, where $a_n$ is a c-atom supported on $2I$, and $\lambda_n \in \mathbb{C}$ with $\lim_{n\to\infty} \lambda_n = 0$.*
(3) *$\gamma(\phi_n * a) = C_{\phi_n}(a)$ for all $n \geq 1$.*

*Proof.* For (1), we partition $(-1/n, 1/n)$ into $N$ disjoint intervals $K_j^n$ whose lengths are less than $|I|$ (we can have $N = 1$) and consider $b_j^n = (\phi_n \chi_{K_j^n}) * b$. For all $j$, $b_j^n$ is supported on an interval of length $2|I|$, has mean 0 and $L_2$-norm less than $2\|\phi_n \chi_{K_j^n}\|_1/\sqrt{|I|}$. Thus $b_j^n h$ is an atom up to a factor $2\|\phi_n \chi_{K_j^n}\|_1$ and $\phi_n * a = \sum_{j=1}^N b_j^n h$. But $\sum_{j=1}^N 2\|\phi_n \chi_{K_j^n}\|_1 = 2$, yielding (1).

Similarly, for (2), $\phi_n * b - b$ has support in $2I$ with mean 0. Its $L_2$-norm goes to 0, giving the result.

There are many ways of checking (3). One can identify both sides as a.e. functions from $\mathbb{R}$ to $L_1(\mathcal{M})$. Another one is to note that item (1) implies that convolution with $\phi_n$ is continuous for $\|\cdot\|_{\mathcal{H}}$. Moreover, atoms of the form $a = bh$ with $b$ an $L_2$-atom which is a simple tensor in $L_2(\mathbb{R}) \otimes L_2(\mathcal{M})$, and $h \in L_2(\mathcal{M})$, are dense in $\mathcal{H}$ for $\|\cdot\|_{\mathcal{H}}$. On such atoms, the formula is obvious and then extends by continuity. ■

PROPOSITION 2.11. *Let $\psi \in \mathrm{BMO}^r(\mathbb{R},\mathcal{M})$ satisfy $\widetilde{P}_J\psi = \psi$ for some finite interval $J$. Then for any family of atoms $(a_i)$ and $(\lambda_i) \in \ell_1$ such that $\sum_{i=1}^\infty \lambda_i a_i = 0$ in $L_1(L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M})$ we have*

$$\sum_{i=1}^{\infty} \lambda_i \langle \psi, a_i\rangle_{\mathrm{BMO}^r,\mathcal{H}} = 0.$$

*Proof.* First, by Lemma 2.10, $\phi_n * a_i \in \mathcal{H}$ for all $n, i \geq 1$. We come back to the definition

$$\langle \psi, \phi_n * a_i\rangle_{\mathrm{BMO}^r,\mathcal{H}} = \langle \widetilde{P}_J\psi, \widetilde{P}_J A_\omega(\phi_n * a_i)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},\frac{dt}{1+t^2})),L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}.$$

Using the projection $\widetilde{P}_J$, we get $\widetilde{P}_J A_\omega(\phi_n * a_i) = \widetilde{P}_J \widetilde{M}_\omega^J \gamma(\phi_n * a_i)$, and since $\gamma(\phi_n * a_i) = C_{\phi_n}(a_i)$, we find that

$$\begin{aligned}\langle \psi, \phi_n * a_i\rangle_{\mathrm{BMO}^r,\mathcal{H}} &= \langle (\widetilde{M}_\omega^J)^*\psi, C_{\phi_n}(a_i)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R})),L_1(\mathcal{M};L_2^c(\mathbb{R}))}\\ &= \langle C_{\phi_n}^*(\widetilde{M}_\omega^J)^*\psi, a_i\rangle_{\mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}),L_1(\mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}))}.\end{aligned}$$

As $\sum_{i=1}^\infty \lambda_i a_i = 0$ in $L_1(\mathbb{R}; L_1(\mathcal{M}))$, we conclude that $\sum_{i=1}^\infty \lambda_i \langle \psi, \phi_n * a\rangle = 0$.

Next, by Lemma 2.8, the duality pairing is continuous for the norm $\|\cdot\|_{\mathcal{H}}$. Lemma 2.10(1, 2) means that $\|\phi_n * a_i\|_{\mathcal{H}} \leq 2\|a_i\|_{\mathcal{H}}$ and $\|\phi_n * a_i - a_i\|_{\mathcal{H}} \to 0$ as $n \to \infty$. This yields $\sum_{i=1}^\infty \lambda_i\langle \psi, a_i\rangle = 0$ by the Lebesgue theorem. ■

In order to conclude the argument, it remains to show that any operator $\varphi \in \mathrm{BMO}^r(\mathbb{R},\mathcal{M})$ induces a well-defined functional on $\mathrm{H}_1^c(\mathcal{A})$. The estimates obtained for compactly supported $\psi \in \mathrm{BMO}^r(\mathbb{R},\mathcal{M})$ will yield a general result under another suitable approximation argument from Garnett's book [9, p. 261] as stated in [18, Lemma 1.5] in the semicommutative case. See Appendix 4 for the proof.

LEMMA 2.12. *Suppose $\varphi \in \mathrm{BMO}^c(\mathbb{R},\mathcal{M})$ and $J$ is an interval such that $\varphi_J = 0$. Let $3J$ be the interval concentric with $J$ and of length $3|J|$. Then there exist $\psi \in \mathrm{BMO}^c(\mathbb{R},\mathcal{M})$ and some universal $C > 0$ such that*

$$\widetilde{P}_{3J}\psi = \psi, \qquad \widetilde{P}_J(\psi - \varphi) = 0, \qquad \|\psi\|_{\mathrm{BMO}^c} \leq C\|\varphi\|_{\mathrm{BMO}^c}.$$

THEOREM 2.13. *For every semifinite von Neumann algebra $\mathcal{M}$, there is a contractive inclusion*

$$\mathrm{BMO}^r(\mathbb{R},\mathcal{M}) \subseteq \mathrm{H}_1^c(\mathcal{A})^*.$$

*Proof.* Let $\varphi \in \mathrm{BMO}^r(\mathbb{R},\mathcal{M})$ and let $f \in \mathrm{H}_1^c(\mathcal{A})$ admit an atomic decomposition $\sum_{i=1}^{\infty} \lambda_i a_i$. We want to define the duality pairing by

$$\langle \varphi, f\rangle = \sum_{i=1}^{\infty} \lambda_i \langle \varphi, a_i\rangle_{\mathrm{BMO}^r,\mathcal{H}}.$$

We start by showing that this is well-defined algebraically, that is, if $f = 0$ in $L_1(L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M})$ then the right hand side is also 0.

Given $\varepsilon > 0$, let $N \geq 1$ such that $\sum_{i>N} |\lambda_i| < \varepsilon$ and let $J$ be a finite interval satisfying

$$\mathrm{supp}_{\mathbb{R}}(a_i) \subseteq J \quad \text{for } i = 1, \ldots, N.$$

Without loss of generality, we can assume that $\varphi_J = 0$, so by Lemma 2.12, there exists some $\psi \in \mathrm{BMO}^r(\mathbb{R},\mathcal{M})$ satisfying $\widetilde{P}_{3J}\psi = \psi$, $\widetilde{P}_J(\varphi - \psi) = 0$ and

$$\|\psi\|_{\mathrm{BMO}^r} \leq C\|\varphi\|_{\mathrm{BMO}^r}$$

for some universal constant $C > 0$. Therefore, as a consequence of Proposition 2.11,

$$\begin{aligned}
\sum_{i=1}^{\infty} \lambda_i \langle \varphi, a_i\rangle &= \sum_{i=1}^{\infty} \lambda_i \langle \varphi, a_i\rangle - \sum_{i=1}^{\infty} \lambda_i \langle \psi, a_i\rangle \\
&= \sum_{i=1}^{N} \lambda_i \langle \varphi - \psi, a_i\rangle + \sum_{i=N+1}^{\infty} \lambda_i \langle \varphi - \psi, a_i\rangle \\
&= \sum_{i=N+1}^{\infty} \lambda_i \langle \varphi - \psi, a_i\rangle.
\end{aligned}$$

We get

$$\Big|\sum_{i=1}^{\infty} \lambda_i \langle \varphi, a_i\rangle\Big| \leq \sum_{i=N+1}^{\infty} |\lambda_i|(1+C)\|\varphi\|_{\mathrm{BMO}^r} < (1+C)\varepsilon\|\varphi\|_{\mathrm{BMO}^r}.$$

Since $\varepsilon$ is arbitrarily small, the duality pairing is well-defined. To deduce the norm estimate, one just uses Lemma 2.8. ∎

This last theorem completes the proof of the isometric identity

$$(\mathrm{H}_1^c(\mathcal{A}))^* = \mathrm{BMO}^r(\mathbb{R},\mathcal{M}).$$

Therefore, a new description for a predual of $\mathrm{BMO}^r(\mathbb{R},\mathcal{M})$ has been obtained just in terms of a new atomic decomposition, which will be crucial for the study of the boundedness of Calderón–Zygmund operators from $\mathrm{H}_1^c(\mathcal{A})$ to $L_1(L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M})$.

We end this section by briefly mentioning that there is another approach to defining the duality bracket between $\mathrm{H}_1^c(\mathcal{A})$ and $\mathrm{BMO}^r(\mathbb{R},\mathcal{M})$. One can easily show that there is a natural isometric embedding

$$\pi : L_\infty\Big(\mathcal{M}; L_2^r\Big(\mathbb{R}, \frac{1}{1+t^2}dt\Big)\Big) \subset B\Big(L_2\Big(\mathbb{R}, \frac{dt}{1+t^2} \otimes_2 L_2(\mathcal{M})\Big), L_2(\mathcal{M})\Big).$$

Then for an atom $a = bh$ and $f \in \mathrm{BMO}^r(\mathbb{R},\mathcal{M})$, one can set $\langle a, f\rangle = \tau(\pi(f)(b)h)$, which is independent of the decomposition of $a$. Then one can rely on Hilbert-valued $\mathrm{H}_1$ and BMO noting that

$$\|f\|_{\mathrm{BMO}^r(\mathbb{R},\mathcal{M})} = \sup_{\|h\|_2\le 1} \|\pi(f)^*(h)\|_{\mathrm{BMO}(\mathbb{R},L_2(\mathcal{M}))}.$$

Nevertheless, one still needs all the above approximation arguments.

**3. Calderón–Zygmund operators with operator-valued kernels.** Let $\mathcal{M}$ be a von Neumann algebra over a separable Hilbert space. In this section we establish conditions under which a kernel $K$, defined outside the diagonal, with values in $\mathcal{M}$, will induce a Calderón–Zygmund operator from $\mathrm{H}_1^c(\mathcal{A})$ into $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$. As one could expect, the resulting operator will be defined up to a pointwise multiplication operator. Recall the identification $L_\infty(I\times J)\,\overline{\otimes}\,\mathcal{M} = L_\infty(I\times J;\mathcal{M})$ where we consider weak*-measurable $\mathcal{M}$-valued functions.

Recall that we denote by $\mathcal{S}$ the set of compactly supported essentially bounded functions $\mathbb{R}\to L_\infty\cap L_1(\mathcal{M})$ that are measurable with values in $L_1$. Note that $B_{L_p(\mathcal{A})}\cap\mathcal{S}$ is norm-dense in $B_{L_p(\mathcal{A})}$ when $1\le p<\infty$ and weak*-dense for $p=\infty$. Indeed, if $f\in L_p(\mathcal{A})$ with polar decomposition $f = u|f|$, take a sequence $p_n$ of finite projections strongly converging to 1 in $\mathcal{M}$ and consider $f_n = u\max\{|f|, n\}\, p_n\chi_{[-n;n]}\in\mathcal{S}$. It is clear that $\|f_n\|_p\le\|f\|_p$ and $f_n\to f$ in norm in $L_p$ (weak*-convergence if $p=\infty$).

DEFINITION 3.1. Assume that $T$ is a bounded operator on $L_2(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$. We say that $T$ is a *Calderón–Zygmund operator* if there exists some function

$$K : \mathbb{R}\times\mathbb{R}\setminus\{x=y\}\to\mathcal{M}$$

such that for every pair of intervals $I, J$ satisfying $d(I,J)>0$, there exists $K_{I,J}\in L_\infty(I\times J)\,\overline{\otimes}\,\mathcal{M}$ such that

- $K_{I,J}(t) = K(t)$ for almost every $t\in I\times J$,
- for any $f, g\in\mathcal{S}$ supported respectively on $J$ and $I$,

$$\Big(\tau\circ\int\Big)(g\,Tf) = \langle f(y)g(x), K_{I,J}\rangle_{L_1(L_\infty(I\times J)\overline{\otimes}\mathcal{M}),L_\infty(I\times J)\overline{\otimes}\mathcal{M}}. \tag{3.1}$$

We say that $T$ is a *left Calderón–Zygmund operator* if it also satisfies, for any compactly supported $f\in L_2(L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M})$ and $h\in\mathcal{M}$,

$$T(fh) = T(f)h.$$

Let $\lambda>0$. We say that the kernel $K$ (or $T$) satisfies the *Hörmander condition* for $\lambda$ whenever

$$\int_{|x-y|\geq\lambda|y'-y|} \|K(x,y)-K(x,y')\|_{\mathcal{M}}\,dx \leq C_\lambda \tag{3.2}$$

for some constant $C_\lambda \geq 0$. Note that should the Hörmander condition hold for some $\lambda > 0$, it holds for all $\lambda' > \lambda$.

We have chosen $\mathcal{S}$ so as to have a perfectly defined duality in (3.1) as $f(y)g(x) \in L_1(\mathbb{R}^2; L_1(\mathcal{M}))$. Moreover, $\mathcal{S}$ is an algebra for the pointwise product. We could have also stated (3.1) for $L_1(\mathbb{R}; L_2(\mathcal{M}))$ instead of $\mathcal{S}$ without significant modifications.

The following argument constitutes our main estimate and it can be considered as an adaptation of the scalar case to the semicommutative setting.

LEMMA 3.2. *Let $\mathcal{M}$ be a von Neumann algebra. Let $T$ be a left Calderón–Zygmund operator which is bounded on $L_2(\mathbb{R}; L_2(\mathcal{M}))$ and has kernel*

$$K : \mathbb{R} \times \mathbb{R} \setminus \{x = y\} \to \mathcal{M}.$$

*If $K$ satisfies the Hörmander condition for some $\lambda \geq 1$ and moreover*

$$T(a) = T(b)h \quad \text{for any } c\text{-atom } a = bh,$$

*then*

$$\|T(a)\|_{L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} \leq \max\{C_\lambda, \lambda^{1/2}\|T\|\}.$$

*Proof.* Assume that $\lambda > 1$. Let $a = bh$ be a $c$-atom in $\mathrm{H}_1^c(\mathcal{A})$ with support contained in $I = [y_0 - d, y_0 + d]$ and let $\lambda I = [y_0 - \lambda d, y_0 + \lambda d]$. Then the norm

$$\|T(a)\|_{L_1(\mathbb{R};L_1(\mathcal{M}))} = \int_{\lambda I} \|T(a)\|_{L_1(\mathcal{M})} + \int_{(\lambda I)^c} \|T(a)\|_{L_1(\mathcal{M})}$$

can be bounded in two steps. First, the continuity of $T$ on $L_2(\mathbb{R}; L_2(\mathcal{M}))$ implies that

$$\begin{aligned}\int_{\lambda I} \|T(a)\|_{L_1(\mathcal{M})} &= \int_{\lambda I} \|T(b)h\|_{L_1(\mathcal{M})} \leq \Big(\int_{\lambda I} \|T(b)\|^2_{L_2(\mathcal{M})}\Big)^{1/2}\Big(\int_{\lambda I} \|h\|^2_{L_2(\mathcal{M})}\Big)^{1/2}\\ &= \|T(b)\|_{L_2(\mathbb{R};L_2(\mathcal{M}))}|\lambda I|^{1/2}\|h\|_{L_2(\mathcal{M})}\\ &\leq \|T : L_2(\mathbb{R}; L_2(\mathcal{M})) \to L_2(\mathbb{R}; L_2(\mathcal{M}))\|\,\|b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))}|\lambda I|^{1/2}\|h\|_{L_2(\mathcal{M})}\\ &\leq \lambda^{1/2}\|T : L_2(\mathbb{R}; L_2(\mathcal{M})) \to L_2(\mathbb{R}; L_2(\mathcal{M}))\|.\end{aligned}$$

To deal with the second term, we have

$$\|T(a)\chi_{(\lambda I)^c}\|_1 = \|T(b)h\chi_{(\lambda I)^c}\|_1 = \sup_{\substack{g\in\mathcal{S},\,\|g\|_\infty\leq 1\\ \chi_{(\lambda I)^c}g=g}} \Big|\tau\int T(b)hg\Big|.$$

Assume for the moment that $b \in \mathcal{S}$ and $h \in L_\infty \cap L_1(\mathcal{M})$. Let $y_0$ be the center of $I$. Since $I$ and $(\lambda I)^c$ are disjoint measurable sets and $b$ has integral zero, it follows using (3.1) and the Hörmander condition that

$$
\begin{aligned}
\|T(a)\chi_{(\lambda I)^c}\|_1 &\le \sup_g \Big|\tau \int_{(\lambda I)^c}\int_I K_{(\lambda I)^c,I}(x,y)b(y)hg(x)\,dx\,dy\Big| \\
&\le \sup_g \Big|\tau \int_{(\lambda I)^c}\int_I (K_{(\lambda I)^c,I}(x,y) - K_{(\lambda I)^c,I}(x,y_0))b(y)hg(x)\,dx\,dy\Big| \\
&= \Big\|\int_I (K_{(\lambda I)^c,I}(x,y) - K_{(\lambda I)^c,I}(x,y_0))b(y)h\,dy\Big\|_{L_1((\lambda I)^c;L_1(\mathcal{M}))} \\
&\le \int_{(\lambda I)^c}\int_I \|(K_{(\lambda I)^c,I}(x,y) - K_{(\lambda I)^c,I}(x,y_0))b(y)h\|_{L_1(\mathcal{M})}\,dx\,dy \\
&\le \int_{(\lambda I)^c}\int_I \|(K_{(\lambda I)^c,I}(x,y) - K_{(\lambda I)^c,I}(x,y_0))\|_{\mathcal{M}}\|b(y)\|_{L_2(\mathcal{M})}\|h\|_{L_2(\mathcal{M})}\,dx\,dy \\
&\le C_\lambda \int_I \|b(y)\|_{L_2(\mathcal{M})}\|h\|_{L_2(\mathcal{M})}\,dy \le C_\lambda\|b\|_2\|h\chi_I\|_2 \le C_\lambda.
\end{aligned}
$$

The general case follows by approximation. Indeed, let $b_n \in \mathcal{S}$ go to $b$ with $\|b_n\|_2 \le \|b\|_2$ as explained above and $h_n \in L_\infty \cap L_1(\mathcal{M})$ go to $h$ in $L_2(\mathcal{M})$ with $\|h_n\|_{L_2(\mathcal{M})} \le 1$. Centering $b_n$ on $I$ decreases its $L_2$-norm, so we can as well assume it has mean 0. Let $J_k = [-k,k] \cap (\lambda I)^c$. Then for any $k \ge 1$, $T(b_n)h_n\chi_{J_k} \to T(b)h\chi_{J_k}$ in $L_1$ as $n \to \infty$ by the $L_2$-continuity of $T$. It follows that $\|T(b)h\chi_{J_k}\|_1 \le \limsup_n \|T(b_n)h_n\chi_{J_k}\|_1 \le C_\lambda$. Letting $k \to \infty$ gives $\|T(b)h\chi_{(\lambda I)^c}\|_1 \le C_\lambda$.

On the other hand, the case $\lambda = 1$ can be recovered since the argument above works for $\lambda' > 1$, and taking the limit as $\lambda' \to 1$ yields the statement of the theorem. ∎

In order to extend a left Calderón–Zygmund operator to the whole Hardy space $\mathrm{H}_1^c(\mathcal{A})$, we will proceed as in [20]. A family of bounded kernels will be constructed, yielding a bounded family of Calderón–Zygmund operators that will approximate the original operator.

LEMMA 3.3. *Let $T$ be a left Calderón–Zygmund operator whose associated kernel $K$ satisfies the Hörmander condition* (3.2) *for some $\lambda \ge 1$. Then there exist a sequence of left Calderón–Zygmund operators $(T_m)_{m\ge1}$ with kernels $(K_m)_{m\ge1} \subseteq L_\infty(\mathbb{R}\times\mathbb{R})\,\overline{\otimes}\,\mathcal{M}$ and a constant $C' > 0$ such that*

$$
\int_{|x-y|\ge 2\lambda|y'-y|} \|K_m(x,y) - K_m(x,y')\|_{\mathcal{M}}\,dx \le C'. \tag{3.3}
$$

*Moreover,*

$$
\lim_{m\to\infty} \|T_m(f) - T(f)\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} = 0
$$

*for every $f \in L_2(L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M})$.*

*Proof.* We consider an even smooth function $\phi$ supported on $(-1,1)$ with $\int\phi = 1$ as in Lemma 2.10. Then we define $R_m$ to be the operator on $L_2(\mathbb{R};L_2(\mathcal{M}))$ given by $R_m(f) = f * \phi_m$ where $\phi_m(x) = m\phi(mx)$.

We define the function $K_m : \mathbb{R}^2 \to \mathcal{M}$ by duality. Assume $z \in L_1(\mathcal{M})$ is decomposed as $z = ab$ with $a, b \in L_2(\mathcal{M})$, and define the quantity

$$\langle K_m(x,y), z\rangle_{\mathcal{M},L_1(\mathcal{M})} = \int \tau\big(T((\tau_y\phi_m)(t)a)\tau_x(\phi_m)(t)b\big)\,dt, \tag{3.4}$$

where $\tau_x\phi(t) = \phi(t-x)$. Indeed, assume $z = \alpha\beta$ with $\alpha, \beta \in L_2(\mathcal{M})$ and choose a finite projection $p$ so that $pa, p\alpha \in \mathcal{M}$; then by the right-modularity condition,

$$\int_{\mathbb{R}} \tau\big(T(\tau_y\phi_m pa)\tau_x\phi_m b\big) = \int_{\mathbb{R}} \tau\big(T(\tau_y\phi_m p)\tau_x\phi_m z\big) = \int_{\mathbb{R}} \tau\big(T(\tau_y\phi_m p\alpha)\tau_x\phi_m \beta\big).$$

Letting $p \to 1$ and using the $L_2$-continuity of $T$ yields

$$\langle T(\tau_y\phi_m a), \tau_x(\phi_m)b\rangle_{L_2(\mathbb{R};L_2(\mathcal{M}))} = \langle T(\tau_y\phi_m\alpha), \tau_x(\phi_m)\beta\rangle_{L_2(\mathbb{R};L_2(\mathcal{M}))}.$$

Linearity is proved in a similar way. We also get

$$\begin{aligned}\|K_m(x,y)\|_{\mathcal{M}} &= \sup_{\substack{a,b\in L_2(\mathcal{M})\\ \|a\|_2=\|b\|_2=1}} \Big|\Big(\int\circ\tau\Big)\big(T(\tau_y\phi_m a)\tau_x\phi_m b\big)\Big|\\ &\le \|T\|_{L_2(\mathbb{R};L_2(\mathcal{M}))}\|\phi_m\|^2_{L_2(\mathbb{R})}.\end{aligned}$$

Assuming that $f$ and $g$ are simple functions, by linearity we get

$$\begin{aligned}\langle R_mTR_mf, g\rangle &= \tau\iint T\Big(\int\phi_m(\cdot-y)f(y)\,dy\Big)(t)\phi_m(t-x)\,dt\,g(x)\,dx\\ &= \iint\tau\big(K_m(x,y)f(y)g(x)\big)\,dx\,dy.\end{aligned}$$

The formula extends to $f, g \in L_2(\mathbb{R}; L_2(\mathcal{M}))$ by continuity since $K_m$ is bounded on the whole $\mathbb{R}^2$. Therefore, $K_m$ is the kernel for $R_mTR_m$ in the sense of Definition 3.1. Moreover, $K_m$ is continuous as a function $\mathbb{R}\times\mathbb{R}\to L_\infty(\mathcal{M})$ by continuity of translations on $L_2(\mathbb{R})$ because

$$\begin{aligned}\|K_m(x,y) - K_m(x,y')\|_{\mathcal{M}} &\le \|T\|\,\|\phi_m\|_2\|(\tau_y-\tau_{y'})\phi_m\|_2\\ &\le \|T\|m^{1/2}\|\phi\|_2(2m^{3/2}|y-y'|\,\|\phi'\|_\infty)\\ &= 2m^2|y-y'|\,\|T\|\,\|\phi\|_2\|\phi'\|_\infty.\end{aligned} \tag{3.5}$$

On the other hand, the kernel $K_m$ satisfies the Hörmander condition for $2\lambda$. First, consider $x, y, y'$ such that $|x-y| \ge 2\lambda|y'-y|$ and $|x-y| > 4/m$. Then $|x-y'| \ge |x-y|/2 > 2/m$ as $\lambda \ge 1$. Thus the support of $\tau_x\phi_m$ is disjoint from that of $\tau_y\phi_m$ and $\tau_{y'}\phi_m$. Let $I_{y,y'}$ and $I_x$ denote some disjoint closed intervals containing the supports of $\tau_y\phi_m, \tau_{y'}\phi_m$ and $\tau_x\phi_m$ respectively. Then the identities (3.1) and (3.4) for the kernel $K_{I_x,I_{y,y'}}$ (extending to 0 outside $I_x\times I_{y,y'}$) give

$$K_m(x,z) = K_{I_x,I_{y,y'}} * (\phi_m\otimes\phi_m)(x,z)$$

for $z = y, y'$. Thus we can write

$$\begin{aligned}\int\limits_{|x-y|\ge 2\lambda|y'-y|\vee 4/m} &\|K_m(x,y) - K_m(x,y')\|_{\mathcal{M}}\,dx \\ &\le \int\limits_{|x-y|\ge 2\lambda|y'-y|\vee 4/m} \iint \|K(x-u,y-v) - K(x-u,y'-v)\|_{\mathcal{M}} \\ &\qquad\qquad \times \phi_m(u)\phi_m(v)\,du\,dv\,dx.\end{aligned}$$

Using the fact that for $|u|, |v| \le 1/m$ (i.e. in the support of $\phi_m$),

$$|x-u-(y-v)| \ge |x-y| - \frac{2}{m} > \frac{1}{2}|x-y| \ge \lambda|y'-v-(y-v)|,$$

the Hörmander condition for the kernel $K$ implies

$$\int\limits_{|x-y|\ge 2\lambda|y'-y|\vee 4/m} \|K_m(x,y) - K_m(x,y')\|_{\mathcal{M}}\,dx \le C.$$

On the other hand, whenever $|x-y| \le 4/m$, the estimate (3.5) yields

$$\begin{aligned}\int\limits_{4/m\ge|x-y|\ge 2\lambda|y'-y|} &\|K_m(x,y) - K_m(x,y')\|_{\mathcal{M}}\,dx \\ &\le 2m^2\|T\|\,\|\phi'\|_\infty\|\phi\|_2 \int\limits_{4/m\ge|x-y|\ge 2\lambda|y'-y|} |y'-y|\,dx \le \frac{8}{\lambda}\|T\|\,\|\phi'\|_\infty\|\phi\|_2.\end{aligned}$$

Thus $K_m$ satisfies condition (3.3). Finally, for any $f \in L_2(L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M})$,

$$\begin{aligned}\|T_m(f) - T(f)\|_{L_2} &\le \|R_mT(R_m(f) - f)\|_{L_2} + \|(R_m - \mathrm{Id})T(f)\|_{L_2} \\ &\le \|T\|\,\|R_m(f) - f\|_{L_2} + \|(R_m - \mathrm{Id})T(f)\|_{L_2} \to 0,\end{aligned}$$

as $m \to \infty$. ∎

Before proving that a left Calderón–Zygmund operator extends to a bounded map from $\mathrm{H}_1^c(\mathcal{A})$ into $L_1(\mathcal{A})$, a fundamental property is included.

Proposition 3.4. *Let $\mathcal{M}$ be a von Neumann algebra and let $T$ be a left Calderón–Zygmund operator which is bounded on $L_2(\mathbb{R}; L_2(\mathcal{M}))$ and whose kernel is zero. Then $T$ corresponds to left multiplication by some operator $F \in L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M}$.*

*Proof.* Let $f \in L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M}$ be such that $\operatorname{supp}\|f\|_2$ is compact and contained in some interval $J$. Then we claim that for any $g, g' \in \mathcal{S}$,

$$\langle g', T(gf)\rangle_{L_2(\mathbb{R};L_2(\mathcal{M}))} = \langle g', T(g)f\rangle_{L_2(\mathbb{R};L_2(\mathcal{M}))}. \tag{3.6}$$

As a consequence of Definition 3.1, for any $m \in \mathcal{M} \cap L_1(\mathcal{M})$, an arbitrary interval $I$ of positive length and $\lambda < 1$,

$$\begin{aligned}&\langle g'(\chi_{(\bar I)^c}\otimes 1), T(g(\chi_{\lambda I}\otimes m))\rangle\\ &\qquad\qquad = \langle g(y)(\chi_{\lambda I}\otimes m)(y) g'(x)(\chi_{(\bar I)^c}\otimes 1)(x), K_{(\bar I)^c,\lambda I}\rangle = 0,\\ &\langle g'(\chi_{\lambda I}\otimes 1), T(g(\chi_{(\bar I)^c}\otimes m))\rangle\\ &\qquad\qquad = \langle g(y)(\chi_{(\bar I)^c}\otimes m)(y) g'(x)(\chi_{\lambda I}\otimes 1)(x), K_{\lambda I,(\bar I)^c}\rangle = 0.\end{aligned}$$

It follows that

$$\langle g'(\chi_{\lambda I}\otimes 1), T(g(\chi_{I^c}\otimes m))\rangle = \langle g'(\chi_{\lambda I}\otimes 1), (\chi_{(\bar I)^c}\otimes 1) T(g(1\otimes m))\rangle.$$

Using modularity, we get

$$\langle g'(\chi_{\lambda I}\otimes 1), T(g)(\chi_I\otimes m)\rangle = \langle g'(\chi_{\lambda I}\otimes 1), T(g(\chi_I\otimes m))\rangle.$$

By continuity, this also holds for $\lambda = 1$ and similarly

$$\langle g'(\chi_{(\bar I)^c}\otimes 1), T(g)(\chi_I\otimes m)\rangle = 0 = \langle g'(\chi_{(\bar I)^c}\otimes 1), T(g(\chi_I\otimes m))\rangle,$$

yielding $\langle g', T(g)(\chi_I\otimes m)\rangle = \langle g', T(g(\chi_I\otimes m))\rangle$. By continuity of $T$, this can be extended to any $g, g'\in L_2$. And finally, this identity also remains valid upon replacing $\chi_I\otimes m$ by any $f$ by weak*-density of elementary tensors (and Kaplansky's theorem) and by weak* continuity of all the maps involved. Therefore, claim (3.6) follows.

We have shown that $T$ commutes with right multiplication by $f\in L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}$, thus it has to be left multiplication by an element in $L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}$ as the algebra $L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}$ acts in standard form on $L_2(\mathbb{R}; L_2(\mathcal{M}))$. ■

THEOREM 3.5. *Let $\mathcal{M}$ be a von Neumann algebra. Let $T$ be a left Calderón–Zygmund operator with kernel $K:\mathbb{R}\times\mathbb{R}\setminus\{x=y\}\to\mathcal{M}$. If $K$ satisfies Hörmander's condition* (3.2)*, then $T$ extends to a bounded operator from $\mathrm{H}_1^c(\mathcal{A})$ into $L_1(\mathbb{R}; L_1(\mathcal{M}))$.*

*Proof.* We can assume that the Hörmander condition is satisfied for some $\lambda\ge 1$. First, $T$ can be defined on $c$-atoms. This is where we use modularity. Indeed, given a $c$-atom with decomposition $a = bh$ supported on $I$, $T(b)h$ is well-defined in $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ but it might depend on the decomposition. Let us justify that it does not. First, for any $m\ge 1$, since $K_m(x,\cdot)$ belongs to $L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}$ (actually, it is continuous), we have

$$T_m(a)(x) = \int K_m(x,y) b(y) h\,dy = \int K_m(x,y) b(y)\,dt\cdot h = T_m(b)(x)\cdot h.$$

Consider another decomposition $a = b'h'$ supported on $J$. Then we have $T_m(b)h = T_m(b')h'$ in $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$. Thus, $T(a)$ can indeed be defined as $T(a) = T(b)h = T(b')h'$ since the norm of the difference

$$\begin{aligned}\|T(b)h - T(b')h'\|_{L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} &\le \|T(b)h - T_m(b)h\|_{L_1} + \|T_m(b')h' - T(b')h'\|_{L_1}\\ &\le \sqrt{|I|}\,\|h\|_{L_2(\mathcal{M})}\|T(b) - T_m(b)\|_{L_2}\\ &\quad + \sqrt{|J|}\,\|h'\|_{L_2(\mathcal{M})}\|T(b') - T_m(b')\|_{L_2}\end{aligned}$$

goes to zero as $m \to \infty$ by Lemma 3.3.

In particular, $T$ and $T_m$ satisfy the assumptions of Lemma 3.2 and we can conclude that for some $C' \ge 0$ independent of $m$ (due to (3.3)),

$$\|T(a) - T_m(a)\|_{L_1(\mathbb{R};L_1(\mathcal{M}))} \le C'. \tag{3.7}$$

Moreover, $R_m a = \phi_m * a$ and $\phi_m * a - a$ are multiples of $c$-atoms by Lemma 2.10 and $\|\phi_m * a - a\|_{\mathrm{H}_1^c(\mathcal{A})} \to 0$ as $m \to \infty$. Thus

$$\begin{aligned}\|T(a)-T_m(a)\|_{L_1} &\le \|R_mTR_m(a)-R_mT(a)\|_{L_1}+\|R_mT(a)-T(a)\|_{L_1}\\ &\le \|T(R_ma-a)\|_{L_1}+\|(R_m-\mathrm{Id})Ta\|_{L_1},\end{aligned} \tag{3.8}$$

which goes to 0 as $m \to \infty$.

In order to define $T$ on $\mathrm{H}_1^c(\mathcal{A})$, assume that $f = \sum_{i=1}^\infty \lambda_i a_i$ where $a_i$'s are $c$-atoms and $(\lambda_i) \in \ell_1$. Then we check that

$$T(f) = \sum_{i=1}^\infty \lambda_i T(a_i) \in L_1(L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M})$$

does not depend on the decomposition. Indeed, assume that $f = 0$. Then the extension above is well-defined and

$$\sum_{i=1}^\infty \lambda_i T_m(a_i)(x) = \int K_m(x,y)\sum_{i=1}^\infty \lambda_i a_i(y)\,dy = 0$$

since $K_m(x,\cdot)$ belongs to $L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M}$. Therefore,

$$\begin{aligned}\Big\|\sum_{i=1}^\infty \lambda_i T(a_i)\Big\|_{L_1(\mathbb{R};L_1(\mathcal{M}))} &\le \Big\|\sum_{i=1}^\infty \lambda_i (T(a_i) - T_m(a_i))\Big\|_{L_1(\mathbb{R};L_1(\mathcal{M}))}\\ &\le \sum_{i=1}^\infty |\lambda_i|\,\|T(a_i) - T_m(a_i)\|_{L_1(\mathbb{R};L_1(\mathcal{M}))}\\ &\le \sum_{i=1}^\infty |\lambda_i|\,\|T(a_i) - T_m(a_i)\|_{L_1(\mathbb{R};L_1(\mathcal{M}))}.\end{aligned}$$

The series tends to 0 as $m \to \infty$ as a consequence of (3.7) and (3.8). The boundedness of $T$ is then clear by Lemma 3.2. ■

Remark 3.6. We have stated the results for Calderón–Zygmund operators acting from $L_2(L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M})$ into itself. Actually, they extend without any modification to maps from $L_2(L_\infty(\mathbb{R})\,\overline{\otimes}\,\mathcal{M})$ to $L_2(L_\infty(\mathbb{R})\,\overline{\otimes}\,\widehat{\mathcal{M}})$ with

$\widehat{\mathcal{M}}$-valued kernels as long as $(\mathcal{M},\tau) \subset (\widehat{\mathcal{M}},\widehat{\tau})$ is an inclusion of von Neumann algebras with $\widehat{\tau}_{|\mathcal{M}} = \tau$ and $T$ is right modular with respect to $\mathcal{M}$.

REMARK 3.7. As already mentioned in the Introduction, all of our results hold for functions in $\mathbb{R}^n$. They also extend to more general measure metric spaces so long as the underlying measure is doubling, that is,

$$\mu(B(x,2r)) \le C_\mu \mu(B(x,r)),$$

for all $x \in \operatorname{supp}(\mu)$ and all $r > 0$, and one has suitable approximations (such as in Lemma 2.12 or 3.3). When $\mu$ is a measure on $\mathbb{R}^n$ which fails the doubling condition, the definition of the appropriate BMO-type space and a predual is more involved and due to Tolsa in the classical case [26]. A semicommutative definition in that context can be found in [4]. We let the interested reader try to carry over our arguments to that setting.

**4. An example.** We sketch how to use Theorem 3.5 to recover some results of [18]. A similar approach was used in [27] to obtain characterizations of some Hardy spaces relying on column-Hilbert spaces Calderón–Zygmund kernels, and indeed constitutes a particular case of our work.

Let $(\mathcal{N},\tau)$ be a semifinite von Neumann algebra and consider

$$\mathcal{M} = \mathcal{N} \mathbin{\overline{\otimes}} B(L_2(\mathbb{R}^+, tdt)^2)$$

with its natural tensor product trace. We fix a function $\mathbb{1}$ of norm 1 in $L_2(\mathbb{R}^+, tdt)^2$.

Let $P(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ be the Poisson kernel for the upper half-plane. We can define a continuous convolution kernel $K : \mathbb{R}\times\mathbb{R}\setminus\{x=y\} \to \mathcal{M} = \mathcal{N} \mathbin{\overline{\otimes}} B(L_2(\mathbb{R}^+, tdt)^2)$ by $K(x,v) = \mathcal{K}(x-v)$ where $\mathcal{K}$ is defined as

$$\mathcal{K}(x) = 1_{\mathcal{N}} \otimes \left(\left(\frac{\partial}{\partial x}P(x,\cdot), \frac{\partial}{\partial y}P(x,\cdot)\right)\otimes \mathbb{1}\right)$$

for $x \neq 0$. The map $\mathcal{K}$ is continuous and bounded on each closed interval not containing 0. Since $\mathcal{N}$ does not play any role, $\mathcal{K}$ actually takes values in a column Hilbert space. It is a standard fact that it satisfies the Mikhlin condition. All this implies that $K$ satisfies the Hörmander condition.

The Calderón–Zygmund operator $T$ associated to $K$ can be defined. At the $L_2$-level, it acts only on a column subspace of the $S_2(L_2(\mathbb{R}^+, tdt)^2)$ component. Thus, by homogeneity of Hilbertian operator spaces, we deduce that it is bounded from the classical result with values in Hilbert spaces (i.e. going from $L_2(\mathbb{R})$ to $L_2(\mathbb{R}; L_2(\mathbb{R}^+, tdt))$). Since $T$ is right-modular with respect to $\mathcal{N}$ inside the tensor algebra $\mathcal{N}\overline{\otimes}B(L_2(\mathbb{R}^+, tdt)^2)$ (with $n \mapsto n\otimes p_{\mathbb{1}}$ which preserves the traces), we indeed have a left Calderón–Zygmund operator from $L_\infty(\mathbb{R}) \mathbin{\overline{\otimes}} \mathcal{N}$ to $L_\infty(\mathbb{R}) \mathbin{\overline{\otimes}} \mathcal{M}$ by Remark 3.6.

Let $f$ be a compactly supported simple function on $\mathbb{R}$ with values in $\mathcal{N}\cap L_1(\mathcal{N})$. If $F$ denotes the harmonic extension of $f$ to the upper half-plane, the column Littlewood–Paley function associated to $f$ is

$$G_c(f)(x) = \Big( \int_{\mathbb{R}^+} |\nabla F(x,y)|^2 y\, dy \Big)^{1/2}.$$

We can see $f$ as a function with values in $\mathcal{M}$ by considering $f \otimes (\mathbb{1} \otimes \mathbb{1})$ as we already did. Some easy computations yield

$$\big|T(f \otimes p_{\mathbb{1}})\big|^2 = G_c(f)^2 \otimes p_{\mathbb{1}}.$$

After giving suitable definitions for all the objects, Theorem 3.5 implies that there is a constant $C$ such that

$$\|G_c(f)\|_{L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} \le C\|f\|_{\mathrm{H}_1^c(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{N})} \tag{4.1}$$

for every $f \in \mathrm{H}_1^c(\mathcal{A})$. In the same way, one can also get (4.1) with the Luzin square function $S_c(f)$ instead of the Littlewood–Paley one.

Consequently, our definition for $\mathrm{H}_1^c(\mathcal{A})$ matches that of [18]: we have just proved an inclusion and the dual spaces are the same (see Theorem 2.4(i) and Corollary 2.7 there).

As another direct application, we recover a dual result of [15] and close to [11]: for scalar kernels satisfying the Hörmander condition, the associated Calderón–Zygmund operators are bounded on $L_p(\mathbb{R}; L_p(\mathcal{M}))$ when $1 < p < 2$.

**Appendix A. Vector-valued Hardy spaces.** This appendix contains an explicit argument for the construction of the map

$$Q : L_2^\circ(\mathbb{R}, (1+t^2)dt; L_2(\mathcal{M})) \to \mathrm{H}_1(\mathbb{R}; L_2(\mathcal{M}))$$

as stated along the discussion preceding Proposition 2.6. Actually, the map $Q$ can be considered whenever $L_2(\mathcal{M})$ is replaced by any Banach space. Moreover, a brief study of *molecules*, which give an additional description of the vector-valued Hardy space, is included.

Given a Banach space $\mathbb{X}$ we say that a function $a$ belonging to $L_1(\mathbb{R}; \mathbb{X})$ is an $L_2(\mathbb{R}; \mathbb{X})$-*atom* in $\mathrm{H}_1(\mathbb{R}; \mathbb{X})$ whenever it satisfies the following conditions:

- $\operatorname{supp}(a) \subseteq I$ for some interval $I$,
- $\int_I a = 0$,
- $\|a\|_{L_2(\mathbb{R};\mathbb{X})} \le 1/\sqrt{|I|}$.

Then $\mathrm{H}_1(\mathbb{R}; \mathbb{X})$ is defined as the subspace of those functions $f$ in $L_1$ admitting a decomposition

$$f = \sum_{i=1}^\infty \lambda_i a_i \quad \text{in } L_1(\mathbb{R}; \mathbb{X}) \tag{A.1}$$

for some absolutely summable sequence $(\lambda_i)_{i=1}^\infty$ and some family of $L_2(\mathbb{R};\mathbb{X})$-atoms $(a_i)_{i=1}^\infty$. According to [12], the definition of $\mathrm{H}_1(\mathbb{R};\mathbb{X})$ can be given via maximal functions in the same way as in the scalar-valued case. This justifies the sort of convergence considered at (A.1). It can be checked that the norm

$$\|f\|_{\mathrm{H}_1(\mathbb{R};\mathbb{X})} = \inf\Big\{\sum_{i=1}^\infty |\lambda_i| : f = \sum_{i=1}^\infty \lambda_i a_i \text{ for some } (\lambda_i)_i \in \ell_1 \text{ and } L_2\text{-atoms } (a_i)_i\Big\}$$

is equivalent to the one defined via maximal functions with values in arbitrary Banach spaces. On the other hand, we include below a characterization of $\mathrm{H}_1(\mathbb{R};\mathbb{X})$ via *molecules* which streamlines the proof for the classical case [19, Ch. 5, Sec. 5].

Let $\mathbb{X}$ be a Banach space and consider as before the function $\omega(x) = 1 + x^2$. Then the space of Bochner measurable functions

$$L_2(\mathbb{R}, \omega dx; \mathbb{X}) = \Big\{f \in L_0(\mathbb{R};\mathbb{X}) : \int_{\mathbb{R}} \|f(x)\|_{\mathbb{X}}^2 \omega(x)\, dx < \infty\Big\}$$

is contained in $L_1(\mathbb{R};\mathbb{X})$. We will consider the subspace

$$M(\mathbb{X}) = \Big\{f \in L_2(\mathbb{R}, \omega dx; \mathbb{X}) : \int_{\mathbb{R}} f = 0\Big\}.$$

LEMMA A.1. *$M(\mathbb{X})$ is a dense linear subspace of* $\mathrm{H}_1(\mathbb{R};\mathbb{X})$.

*Proof.* Let $f \in M(\mathbb{X})$ and define

$$f_0 = f\chi_{\{|x|\le 1\}} \quad \text{and} \quad f_j = f\chi_{\{2^{j-1}<|x|\le 2^j\}} \quad \text{for any } j > 0.$$

These functions satisfy

$$\begin{aligned}\|f_j\|_{L_2(\mathbb{R};\mathbb{X})} &= \Big(\int_{2^{j-1}<|x|\le 2^j} \|f(x)\|_{\mathbb{X}}^2\, dx\Big)^{1/2} \\ &\le \Big(\int_{2^{j-1}<|x|\le 2^j} \|f(x)\|_{\mathbb{X}}^2 \omega(x)\, dx\Big)^{1/2} 2^{-(j-1)} =: R_j 2^{-j}\end{aligned}$$

so that the sequence $(R_j)_{j\ge 0}$ belongs to $\ell_2$. Let $I_j$ be the integral $\int_{\mathbb{R}} f_j$. Then, by the Cauchy–Schwarz inequality and a similar computation, it follows that, for all $j \ge 0$,

$$\begin{aligned}\|I_j\|_{\mathbb{X}} &\le \Big(\int_{2^{j-1}<|x|\le 2^j} \|f(x)\|_{\mathbb{X}}^2 \omega(x)\, dx\Big)^{1/2} \Big(\int_{2^{j-1}<|x|\le 2^j} \omega(x)^{-1}\, dx\Big)^{1/2} \\ &\le R_j 2^{-j/2}.\end{aligned}$$

Therefore, this yields some estimates for $S_j = \sum_{k\geq j} I_j$. Indeed,

$$\|S_j\|_{\mathbb{X}} \leq c \sum_{k\geq j} R_k 2^{-k/2}.$$

Now, let us replace the functions $f_j$ by some *perturbed* atoms $a_j$ given by

$$a_j(x) = f_j(x) + S_{j+1}2^{-(j+2)}\chi_{|x|\leq 2^{j+1}}(x) - S_j 2^{-(j+1)}\chi_{|x|\leq 2^j}(x).$$

Then the sequence $(a_j)_{j\geq 0}$ satisfies

$$\begin{aligned}\int_{\mathbb{R}} a_j(x)\,dx &= \int_{2^{j-1}<|x|\leq 2^j} f(x)\,dx + S_{j+1} - S_j \\ &= \int_{2^{j-1}<|x|\leq 2^j} f(x)\,dx - I_j = 0.\end{aligned}$$

Moreover, it is easy to check that, by hypothesis, the support of $a_j$ is contained in $[-2^{j+1}, 2^{j+1}]$, and by the triangle inequality,

$$\|a_j\|_{L_2(\mathbb{R};\mathbb{X})} \leq R_j 2^{-j} + 2\sum_{k\geq j} R_k 2^{-k/2}2^{-(j+1)/2} = \lambda_j.$$

Then $(\lambda_i)_{i\in\mathbb{N}} \in \ell_1$ by previous computations and since

$$\sum_{j=0}^{\infty}\sum_{k\geq j} 2^{-(j+1)/2}R_j 2^{-k/2} \lesssim \sum_{j=0}^{\infty} R_j 2^{-j} \lesssim \|f\|_{L_2(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))}.$$

Moreover, redefining $a_i$ as $a_i/\lambda_i$, we obtain the expression

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i$$

where each $a_i$ is an atom with support $[-2^{j+1}, 2^{j+1}]$ and convergence is in the sense of $L_1(\mathbb{R};\mathbb{X})$. In conclusion, $f$ belongs to $\mathrm{H}_1(\mathbb{R};\mathbb{X})$. ∎

Definition A.2. A *molecule* $f$ with values in $\mathbb{X}$, centered at $x_0$ and of width $d > 0$, is defined to be a function belonging to $M(\mathbb{X})$ which is normalized by

$$\Big(\int_{\mathbb{R}} \|f(x)\|_{\mathbb{X}}^2 \Big(1 + \frac{|x-x_0|^2}{d^2}\Big)\,dx\Big)^{1/2} \leq d^{-1/2}.$$

The $\mathrm{H}_1(\mathbb{R};\mathbb{X})$-norm is invariant by translations and homogeneous for composition with homotheties. It follows that a molecule $f$ sits in $\mathrm{H}_1(\mathbb{R};\mathbb{X})$, with a norm controlled by an absolute constant. One can check that an atom is a molecule. Thus, one can use molecules instead of atoms in the definition of $\mathrm{H}_1(\mathbb{R};\mathbb{X})$.

**Appendix B. Proof of Lemma 2.12.** This result was stated in the commutative setting by Garnett [9], and also by Mei without proof [18]. For that reason, a general version of the argument is developed here. Before giving the explicit construction, some preliminary classical results are showed. We will use the easier form (2.2) and to lighten the notation we drop $\otimes \mathbb{1}$.

Lemma B.1. *Suppose* $f \in L_\infty\big(\mathcal{M}; L_2^c\big(\mathbb{R}, \frac{dt}{1+t^2}\big)\big)$ *and let* $I \subset J$ *be finite intervals. Then*

$$\Big\| \frac{1}{\sqrt{|I|}} \widetilde{\iota}_I(f - f_I) \Big\|_{L_\infty(\mathcal{M}; L_2^c(\mathbb{R}))} \le \sqrt{\frac{|J|}{|I|}} \Big\| \frac{1}{\sqrt{|J|}} \widetilde{\iota}_J(f - f_J) \Big\|_{L_\infty(\mathcal{M}; L_2^c(\mathbb{R}))}.$$

*Proof.* This is clear because one goes from $\iota_J(f - f_J)$ just by restricting and centering with respect to $I$ and all these operations are contractions at the $L_2$-level. ∎

Let $J = (a, b]$ be a finite interval with center $c_J = \frac{a+b}{2}$. Write

$$J = J_0 \cup \bigcup_{n=1}^{\infty} J_n \cup \bigcup_{n=1}^{\infty} J_n',$$

where $d(J_n, \partial J) = |J_n|$ for $n \ge 0$ and $d(J_n', \partial J) = |J_n'|$ for $n \ge 1$. Then $J_0$ coincides with the middle third of $J$, that is,

$$J_0 = \frac{1}{3} J = \Big(c_J - \frac{1}{2} \cdot \frac{1}{3}|J|, c_J + \frac{1}{2} \cdot \frac{1}{3}|J|\Big],$$

while for any $n \ge 1$,

$$J_n = \Big(c_J + \frac{|J|}{3} \sum_{k=0}^{n-1} \frac{1}{2^k}, c_J + \frac{|J|}{3} \sum_{k=0}^{n} \frac{1}{2^k}\Big],$$

$$J_n' = \Big(c_J - \frac{|J|}{3} \sum_{k=0}^{n} \frac{1}{2^k}, c_J - \frac{|J|}{3} \sum_{k=0}^{n-1} \frac{1}{2^k}\Big].$$

Thus, $|J_n| = |J_n'| = \frac{|J|}{3} \frac{1}{2^n}$. Let $K_n$ (respectively $K_n'$) be the reflection of $J_n$ (respectively $J_n'$) across $b$ (respectively $a$). Then

$$K_n = \Big[b + \frac{|J|}{3} - \frac{|J|}{3} \sum_{k=1}^{n} \frac{1}{2^k}, b + \frac{|J|}{3} - \frac{|J|}{3} \sum_{k=1}^{n-1} \frac{1}{2^k}\Big),$$

$$K_n' = \Big[a - \frac{|J|}{3} + \frac{|J|}{3} \sum_{k=1}^{n-1} \frac{1}{2^k}, a - \frac{|J|}{3} + \frac{|J|}{3} \sum_{k=1}^{n} \frac{1}{2^k}\Big),$$

so $|K_n| = |K_n'| = |J_n| = |J_n'|$. Finally, define $L = [b + |J|/3, \infty)$ and $L' = (-\infty, a - |J|/3)$. This construction yields the desired operator $\psi$, which is given by the following expression assuming the normalization $\varphi_J = 0$ (recall that $\varphi$ is defined up to a constant factor):

$$\psi = \widetilde{P}_J \varphi + \sum_{n \ge 1} \varphi_{J_n} \otimes \chi_{K_n} + \sum_{n \ge 1} \varphi_{J_n'} \otimes \chi_{K_n'}.$$

Let $J^+ = \bigcap_{n\geq 1} J_n$ and $J^- = \bigcap_{n\leq 1} J_n$. It is worth mentioning that if $\mathbb{E}^+$ denotes the conditional expectation on $L_2(J^+)$ corresponding to the partition $\{J_n : n \geq 1\}$, then $\sum_{n\geq 1} \varphi_{J_n} \otimes \chi_{K_n}$ is nothing but $\widetilde{R}_+(\widetilde{\mathbb{E}^+})(\widetilde{P}_{J^+}\varphi)$ where $R_+$ is the reflection with respect to $b$. A similar formula holds for $\sum_{n\geq 1} \varphi_{J'_n} \otimes \chi_{K'_n}$.

We turn to the proof of Lemma 2.12. We need to bound, for all finite intervals $I$, the quantity

$$\Big\| \frac{1}{\sqrt{|I|}} \iota_I(\psi - \psi_I) \Big\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}))}.$$

We distinguish several cases according to $I$:

Case 1: $I \subset J$. There is nothing to do as $\iota_I(\psi - \psi_I) = \iota_I(\varphi - \varphi_I)$.

Case 2: $b \in I$. Using Lemma B.1, we can assume that $I$ is symmetric around $b$ losing a factor $\sqrt{2}$.

If $a \notin I$, then $I \subset \bigcup_{n=n_0}^{\infty}(J_n \cup K_n)$ with $J_{n_0-1} \cap I = \emptyset$. Similarly, by enlarging it and losing another factor $\sqrt{2}$, we can assume $I$ has the form $\bigcup_{n=n_0}^{\infty}(J_n \cup K_n)$ because $|J_{n_0}| \leq |I|$. Then $\psi_I = \psi_{I^+} = \psi_{I^-} = \varphi_{I^-}$ where $I^- = \bigcup_{n=n_0}^{\infty} J_n$ and $I^+ = \bigcup_{n=n_0}^{\infty} K_n$, $|I| = 2|I^-|$. Note that $\widetilde{R}_+$ yields an isometry and

$$\Big\| \frac{1}{\sqrt{|I^+|}} \iota_{I^+}(\psi - \psi_{I^+}) \Big\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}))} = \Big\| \frac{1}{\sqrt{|I^-|}} \widetilde{\mathbb{E}^+} \iota_{I^-}(\varphi - \varphi_{I^-}) \Big\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}))}.$$

Since $\widetilde{\mathbb{E}^+}$ is a contraction commuting with $\iota_{I^-}$, we get

$$\begin{aligned}\Big\| \frac{1}{\sqrt{|I|}} \iota_I(\psi - \psi_I) \Big\|^2_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}))} &\leq \Big\| \frac{1}{\sqrt{|I^+|}} \iota_{I^+}(\psi - \psi_{I^+}) \Big\|^2_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}))} \\ &\leq \|\varphi\|_{\mathrm{BMO}^c}.\end{aligned}$$

If $a \in I$, we can replace it with $3J$ which has a comparable size. Then a similar reasoning also gives control by $\big\| \frac{1}{\sqrt{|J|}} \iota_J(\varphi - \varphi_J) \big\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}))}$.

Case 3: $a \in I$. One can proceed as in Case 1.

Case 4: $I \subset (b, \infty)$. If $I \supset K_n$ for some $n$, then $|I| \geq |K_n|$ and losing a factor $\sqrt{2}$ we can replace $I$ by the bigger interval $\bigcup_{k\geq n} K_k \cup I$ which has size less than $2|I|$. We are then back in Case 2.

If $I \subset K_n$ for some $n$, there is nothing to do as $\iota_I(\psi - \psi_I) = 0$.

If $I \subset K_n \cup K_{n+1}$ for some $n \geq 1$, by losing a constant factor we can assume that $I$ is symmetric with respect to the left border of $K_n$, say $b_n$. If the size of $I$ is smaller than $2|K_{n+1}|$, then $\iota_I(\psi - \psi_I)$ is a simple function taking two values $\pm\frac{1}{2}(\psi_{K_n} - \psi_{K_{n+1}})$. Then the norm of $\frac{1}{\sqrt{|I|}} \iota_I(\psi - \psi_I)$ is $\big\| \frac{1}{2}(\psi_{K_n} - \psi_{K_{n+1}}) \big\|$. It is the same as if we consider $(b_n - |K_{n+1}|, b_n + |K_{n+1}|)$

and we are back to the situation where $K_{n+1} \subset I$ (or we can use the argument of Case 2).

The only remaining situation is when $I \subset K_1 \cup L$. Then an easy calculation shows that the norm involved is controlled by $\|\sqrt{2}\,\psi_{K_1}\| = \|\sqrt{2}\,\varphi_{J_1}\|$. This is where we use the fact that $\varphi_J = 0$ to find that (recall $6|J_1| = |J|$)

$$\|\varphi_{J_1}\| \leq \sqrt{6}\left\|\frac{1}{\sqrt{|J|}}\widetilde{\mathbb{E}^+} i_J(\varphi - 0)\right\| \leq \sqrt{6}\,\|\varphi\|_{\mathrm{BMO}^c}.$$

Case 5: $I \subset (-\infty, a)$. This can be handled similarly.

We have dealt with all possible situations for $I$. This concludes the proof of Lemma 2.12 with $C$ at most $2\sqrt{6}$.

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