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A tiling property for actions of amenable groups along Tempelman Følner sequences

by

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Summary. We show that a certain tiling property (which directly implies the pointwise ergodic theorem) holds for pmp actions of amenable groups along increasing Tempelman Følner sequences, thus providing a short and combinatorial proof of the corresponding pointwise ergodic theorem.

1. Introduction. For a group Γ acting on a probability space X and for a sequence (F_n) of finite subsets of Γ, the *pointwise ergodic property* for Γ along (F_n) says that the action of Γ is ergodic if and only if for every L^1 function f on X, the integral (the global average) of f over X is equal to the limit of the averages of f over $F_n \cdot x$ (the pointwise average) for almost every $x \in X$. The classical *ergodic theorem*, due to G. D. Birkhoff in 1931 [Bir31], says that probability measure preserving (pmp) actions of $\mathbb{Z}$ along the sequence $([0, n))$ have the pointwise ergodic property. In 2001, E. Lindenstrauss proved that actions of amenable groups along tempered Følner sequences have the pointwise ergodic property [Lin01].

A. Tserunyan [Tse18] gives a short, combinatorial proof of the classical pointwise ergodic theorem (for $\mathbb{Z}$) by reducing it to showing that the following tiling property holds for pmp actions of $\Gamma = \mathbb{Z}$ along the intervals $F_n = [0, n)$:

Definition 1 (Tiling property). We say that a pmp action of a countable group Γ on a standard probability space (X, μ) has the *tiling property* along a sequence (F_n) of finite subsets of Γ if for any $\varepsilon > 0$, $N \in \mathbb{N}$, and

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pointwise increasing sequence $(\ell_n : X \to \mathbb{N})_{n\in\mathbb{N}}$ of measurable functions with $\lim_{n\to\infty} \ell_n(x) = \infty$, there is a finite subset $T \subseteq \Gamma$ such that $|T| > N$ and a set of points $x \in X$ of measure at least $1 - \varepsilon$ for which all of $T \cdot x$ except for possibly $\varepsilon|T \cdot x|$ many points can be covered by disjoint sets of the form $F_{\ell_i(y)} \cdot y$ with $y \in X$.

Here, $T \cdot x$ and $F_{\ell_i(y)} \cdot y$ are treated as multisets if the action of Γ is not free. In other words, points are counted with multiplicity. For instance, if there exist $t_1 \neq t_2$ with $t_1 \cdot x = t_2 \cdot x$, we will have the point $t_1 \cdot x$ appearing twice in $T \cdot x$.

It is also implicit in [Tse18] that the tiling property implies the pointwise ergodic theorem for any pmp amenable group action (see Section 3 for a proof). This implication distills out the analytic part from the proofs of pointwise ergodic theorems, reducing them to combinatorial (finitary) tiling problems. Another proof of the ergodic theorem for $\mathbb{Z}$ revolving around the same idea was given in [KP06].

In the present paper, we prove that the tiling property holds for pmp actions of amenable groups along increasing Tempelman Følner sequences (F_n) by finding Vitali covers with Følner tiles on multiple scales. As a consequence, we prove the corresponding pointwise ergodic theorem:

Theorem 2 (Pointwise ergodic). *Fix a pmp action of an amenable group Γ on a standard probability space (X, μ) and an increasing Tempelman Følner sequence (F_n). Then the action of Γ on X is ergodic if and only if for every $f \in L^1(X, \mu)$,*

$$\lim_n A_f[F_n \cdot x] = \int_X f(x)\, d\mu(x) \qquad a.e.,$$

where

$$A_f[F_n \cdot x] := \frac{1}{|F_n|} \sum_{\gamma \in F_n} f(\gamma \cdot x).$$

Although this is less general than Lindenstrauss's theorem, our proof is shorter and offers the advantage that the methods used are more elementary and finitary.

Many people have shown this result for increasing Tempelman Følner sequences. The shortest proof of Theorem 2 that the authors are aware of is given in [OW83], which uses a Vitali covering lemma along with basic functional analysis: a function $f \in L^1(X, \mu)$ is approximated by functions for which the ergodic theorem holds trivially, and the error is controlled by applying the Vitali covering lemma. Other proofs include [Eme74] and [Tem67], which also use a Vitali covering lemma along with analysis. There is a different combinatroial approach to the pointwise ergodic theorem along a tempered Følner sequence of almost invariant sets which appears in work of Weiss [Wei03]. However, none of these proofs yield the tiling property described above, and hence they do not take advantage of the abstract implications of the corresponding pointwise ergodic theorem.

A word on the proof of the tiling property. Compared to [Tse18], the tiling property is much harder to establish for general increasing Tempelman Følner sequences. For example, tiling $\mathbb{Z}^d$ with boxes of different given sizes for each center is harder than tiling $\mathbb{Z}$ with intervals. The key idea in mitigating this difficulty is to iterate the Vitali covering lemma to find covers on multiple scales. We essentially zoom very far out, cover some constant fraction of the space with large sets (this fraction comes from the Tempelman condition and is independent of how far we have zoomed out), and then zoom in on the spots we miss, and fill those in as best as we can with smaller sets, and so on and so forth. Since we cover a constant fraction on each scale, if we zoom out far enough at the beginning, once we zoom all the way back in, we will have covered nearly the whole space.

Organization. In Section 2, we provide the necessary definitions and notation that will be used throughout the paper. In Section 3, we give an explicit proof, due to Tserunyan, that Definition 1 implies the pointwise ergodic property for any pmp amenable group action. In Section 4, we establish the tiling property for pmp actions of amenable groups along increasing Tempelman Følner sequences, which then directly implies the corresponding pointwise ergodic property.

2. Definitions and notation. Let (X, μ) be a standard probability space, and consider a function $f : X \to \mathbb{R}$. For a finite set $A \subseteq X$, define the *average* of f over A to be $A_f[A] := \frac{1}{|A|} \sum_{x \in A} f(x)$. For a finite equivalence relation F on X, let $[x]_F$ be the equivalence class of x under F and define $A_f[F](x) := A_f[[x]_F]$. Given a group Γ and a finite set R, define the R-*boundary* of a set S, denoted $\partial_R S$, to be the set of points s for which $R \cdot s \cap S \neq \emptyset$ and $R \cdot s \cap S^c \neq \emptyset$.

A sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of Γ is a *Følner sequence* if $\Gamma = \bigcup_n F_n$ and $\lim_n |\partial_R F_n|/|F_n| = 0$ for all finite sets R. A group Γ is called *amenable* if it admits a Følner sequence. For this paper, we will assume that our Følner sequences are increasing.

Given a Følner sequence (F_n), we say (F_n) is *tempered* if there is some natural number C such that for all n,

$$\Big|\bigcup_{k<n} F_k^{-1} F_n\Big| \leq C|F_n|,$$

and *Tempelman* if there is C such that for all n,

$$\Big|\bigcup_{k\le n} F_k^{-1}F_n\Big| \le C|F_n|.$$

In the latter case, we will call the smallest such C the *Tempelman constant* of (F_n). Note that any Tempelman Følner sequence is tempered. Note also that $|F_k^{-1}F_n| \ge |F_n|$ for any k, and so the Tempelman constant is at least 1.

Every amenable group has a tempered Følner sequence (in fact, every Følner sequence has a tempered subsequence). In [Lin01, Example 4.2], an example is given of an amenable group without a Tempelman Følner sequence. However, [Hoc07, Theorem 3.4] gives a sufficient condition for the existence of a Tempelman Følner sequence:

Theorem 3 (Hochman 2007). *If for a countable, abelian group G, we have*

$$r(G) = \sup\{n \in \mathbb{N} : G \textit{ contains a subgroup isomorphic to } \mathbb{Z}^n\} < \infty$$

then G has at least one Tempelman Følner sequence.

3. The tiling property implies the pointwise ergodic theorem. The following result is implicitly stated in [Tse18] and was explained by Tserunyan to the second author.

Theorem 4 (Tserunyan). *Let Γ be an amenable group with a Følner sequence $(F_n)_{n\in\mathbb{N}}$. Assume Γ has the tiling property along (F_n). Then for any pmp action of Γ on a standard probability space (X,μ), the action of Γ on X is ergodic if and only if for every $f \in L^1(X,\mu)$,*

$$\lim_n A_f[F_n\cdot x] = \int_X f(x)\,d\mu(x) \qquad a.e.,$$

where $A_f[F_n\cdot x] := \frac{1}{|F_n|}\sum_{\gamma\in F_n} f(\gamma\cdot x)$.

Proof. By replacing f with $f - \int f$, we may assume without loss of generality that $\int f = 0$. We will show that $f^* := \limsup_{n\to\infty} A_f[F_n\cdot x] \le 0$ a.e., and an analogous argument shows $f_* := \liminf_{n\to\infty} A_f[F_n\cdot x] \ge 0$ a.e.

It follows immediately from the fact that (F_n) is a Følner sequence that f^* is Γ-invariant. Hence, ergodicity implies that it equals some constant C almost everywhere. Assume by way of contradiction that $C > c > 0$. Define $\ell_i : X \to \mathbb{N}$ by $x \mapsto$ the ith $n \in \mathbb{N}$ such that $A_f[F_n x] > c$ (equivalently, $A_{f-c}[F_n x] > 0$).

Fix $\delta > 0$ small enough so that for any measurable $Y \subseteq X$, $\mu(Y) < \delta$ implies $\int_Y (f-c)\,d\mu > -c/3$, and let $M \in \mathbb{N}$ be large enough so that the set $Y := f^{-1}(-M,\infty)$ has measure at least $1-\delta$.

The tiling property applied to the function ℓ_i with $\varepsilon := \frac{1}{2(M+c)} \frac{c}{3}$ gives a finite $T \subseteq \Gamma$ such that $\mu(Z) \geq 1-\varepsilon$, where Z is the set of all $x \in X$ such that at least $1-\varepsilon$ fraction of $T \cdot x$ is partitioned into sets of the form $F_{\ell_i(y)} \cdot y$.

CLAIM. For each $x \in Z$, $A_{\mathbb{1}_Y(f-c)}[T \cdot x] \geq -(M+c)\varepsilon$.

Proof of Claim. By the definition of Z, on a subset $B \subseteq T \cdot x$ that occupies at least $1-\varepsilon$ fraction of $T \cdot x$, the average of $f-c$ is positive, and hence that of $\mathbb{1}_Y(f-c)\lfloor_B$ is non-negative. On the remaining set $T \cdot x \setminus B$, the function $\mathbb{1}_Y(f-c)$ is at least $-(M+c)$, by the definition of Y. Thus, the average of $\mathbb{1}_Y(f-c)$ on the entire $T \cdot x$ is at least $-(M+c)\varepsilon$.

Now we compute using this Claim and the invariance of μ:

$$\begin{aligned}\int_Y (f-c)\,d\mu &= \int_X A_{\mathbb{1}_Y(f-c)}[T\cdot x]\,d\mu(x)\\ &= \int_Z A_{\mathbb{1}_Y(f-c)}[T\cdot x]\,d\mu(x) + \int_{X\setminus Z} A_{\mathbb{1}_Y(f-c)}[T\cdot x]\,d\mu(x)\\ &\geq -(M+c)\varepsilon - (M+c)\varepsilon = -2(M+c)\varepsilon = -\frac{c}{3}.\end{aligned}$$

This gives a contradiction:

$$\begin{aligned}0 = \int_X f\,d\mu &= c + \int_X (f-c)\,d\mu\\ &= c + \int_Y (f-c)\,d\mu + \int_{X\setminus Y} (f-c)\,d\mu\\ &> c - \frac{c}{3} - \frac{c}{3} > 0. \quad \blacksquare\end{aligned}$$

4. The tiling property for increasing Tempelman Følner sequences. In this section, we prove the following:

LEMMA 5. *The tiling property holds for pmp actions of amenable groups along increasing Tempelman Følner sequences.*

As a corollary, by Theorem 4, we obtain Theorem 2. In order to prove this lemma, we need a Vitali covering lemma. For the rest of this section, fix an amenable group Γ and a Tempelman Følner sequence F_i with Tempelman constant C, a standard probability space (X, μ) on which Γ acts in a pmp way, and $\varepsilon > 0$.

LEMMA 6 (Vitali covering). *Given a function $l : X \to N$ and a finite subset $S \subseteq X$, there exists a set K, which is a disjoint union of sets of the form $F_{l(x)}x$, $x \in S$, such that $|K| \geq \frac{1}{C}|S \cup K|$.*

Proof. Put $S = \{x_1, \ldots, x_n\}$ and $D_0 = K_0 = \emptyset$. We will inductively define increasing sets K_i and D_i for $i \leq n$ until $S \setminus D_i = \emptyset$. Assume $S \setminus D_i \neq \emptyset$.

Let $t_i = \max_{x \in S \setminus D_i} l(x)$, and let j_i be least such that $x_{j_i} \in S \setminus D_i$ and $l(x_{j_i}) = t_i$. Put $K_{i+1} := K_i \cup F_{t_i} \cdot x_{j_i}$ and $D_{i+1} := D_i \cup F_{t_i}^{-1} F_{t_i} \cdot x_{j_i}$. Iterate i in this inductive construction, up to n times, until $S \setminus D_m = \emptyset$ for some $m \leq n$. Put $K := K_m$ and $D := D_m$.

We claim that the $F_{t_i} \cdot x_{j_i}$ selected are actually pairwise disjoint. If not, suppose that the disjointedness fails at iteration $i+1$ of the above construction. This would mean there is some $y \in F_{t_{i+1}} \cdot x_{j_{i+1}} \cap K_i$. Then $x_{j_{i+1}} \in F_{t_{i+1}}^{-1} y$. But since $y \in K_i$, there is some $t' \geq t_{i+1}$ and j' such that $y \in F_{t'} x_{j'}$. Hence $x_{j_{i+1}} \in F_{t_{i+1}}^{-1} F_{t'} x_{j'} \subseteq F_{t'}^{-1} F_{t'} x_{j'} \subseteq D_i$, contradicting our choice of $x_{j_{i+1}}$.

So at each step, we add exactly $|F_{t_i}|$ elements to K and at most $C|F_{t_i}|$ elements to D. Hence, $|K| \geq \frac{1}{C}|D| \geq \frac{1}{C}|S \cup K|$ since $|S \cup K| \leq |D|$. ■

Proof of Lemma 5. The idea is to break our space into large finite sets. We will tile each of these finite sets with Følner shapes of various sizes, using progressively smaller Følner shapes to fill in whatever holes remain after placing the larger Følner shapes.

First, fix $r \in \mathbb{N}$ large enough so that $\left(\frac{C-1}{C}\right)^r < \varepsilon/2$. Since $C \geq 1$, this is straightforward. We will ultimately pick r many "good" sizes of tiles for a large fraction of the points in X. Fix r many functions $G_i : [0,1] \to \mathbb{R}$ such that $G_1(x) \geq 2x/C$ and for $i > 1$, $G_i(x) \geq \frac{C-1}{C} G_{i-1}(x) + x(i+1)$ where each G_i is continuous and $G_i(0) = 0$. For example, $G_i(x) := \left(\frac{C-1}{C}\right)^i x + \sum_{k=1}^{i+1} kx$ is such a collection of functions. Fix α small enough so that $\beta \leq \alpha \Rightarrow G_r(\beta) < \varepsilon/2$. Put $\eta := \min(\alpha, \varepsilon)$.

For each $p \in \mathbb{N}$, let $C_n^{(p)} := \{x \in X : (\exists i \in \mathbb{N})\ l_i(x) \in [p, n]\}$. Since $l_i(x)$ increases without bound as $i \to \infty$ for each x, we have $\bigcup_n C_n^{(p)} = X$. Hence, there is some large enough p^* such that $\mu(C_{p^*}^{(p)}) > 1 - \eta^2/r$. This means that for any r values p_j $(0 \leq j < r)$,

$$\mu(\{x \in X : (\forall j < r)(\exists i \in \mathbb{N})\ l_i(x) \in [p_j, p_j^*]\}) > 1 - \eta^2.$$

We define two sequences of natural numbers of length r as follows. Let L_0 be large enough so that $|\partial_{F_r} F_n|/|F_n| < \eta$ for all $n > L_0$. For $i < r$, define $R_i := L_i^*$, and $L_{i+1} > R_i$ large enough so that

$$\frac{|\partial_{F_{R_i}} F_n|}{|F_n|} < \frac{\eta}{|F_{R_i}|}$$

for all $n \geq L_{i+1}$. Put $\delta := \eta/|F_{R_{r-1}}|$. We will think of the $[L_j, R_j]$ as ranges of allowable sizes for our tiles. Finally, let $T \subseteq \Gamma$ satisfy $|\partial_{F_{R_{r-1}}} T|/|T| < \delta$. (This property is not immediately useful for the computation that follows but will be important later on.)

Define partial functions $p_i(x) := l_j(x)$ where j is smallest such that $l_j(x) \in [L_i, R_i]$, if such a j exists. Set $P := \{x \in X : (\forall i < r)\ x \in \mathrm{dom}(p_i)\}$.

Hence, $\mu(P) > 1 - \eta^2$. We now show that

$$\mu(\{x \in X : A_{\mathbb{1}_P}[T \cdot x] < 1 - \eta\}) \le \eta.$$

Assuming the contrary, set $B := \{x \in X : A_{\mathbb{1}_P}[T \cdot x] < 1 - \eta\}$ and suppose $\mu(B) > \eta$. Then

$$\begin{aligned}
1 - \eta^2 < \mu(P) &= \int_X \mathbb{1}_P(x)\, d\mu(x) \\
\text{[by the invariance of } \mu] &= \int_X \frac{1}{|T|} \sum_{\gamma \in T} \mathbb{1}_P(\gamma \cdot x)\, d\mu(x) \\
&= \int_X A_{\mathbb{1}_P}[T \cdot x]\, d\mu(x) \\
&= \int_{X \setminus B} A_{\mathbb{1}_P}[T \cdot x]\, d\mu(x) + \int_B A_{\mathbb{1}_P}[T \cdot x]\, d\mu(x) \\
&\le \mu(X \setminus B) + \mu(B)(1 - \eta) \\
&= 1 - \mu(B) + \mu(B) - \eta\mu(B) = 1 - \eta\mu(B) \\
&< 1 - \eta^2.
\end{aligned}$$

Hence, at least $1-\eta$ fraction of points $x \in X$ have $1-\eta$ fraction of points of $T \cdot x$ lying in P. Since $\eta \le \varepsilon$, it now suffices to show that for a point x such that at least $1 - \eta$ fraction of $T \cdot x$ is contained in P, we can tile $T \cdot x$ up to ε fraction with tiles of the form $F_{l_i(x)}x$.

We claim that in k steps, $1 \le k \le r$, we can tile $T \cdot x$ up to $\left(\frac{C-1}{C}\right)^k + G_k(\eta)$ fraction. As discussed earlier, we will start by tiling with our largest Følner shapes, i.e. $l_i(x) \in [L_r, R_r]$, and in each step we will move down a size.

In step k, apply Lemma 6 with $l = p_{r-k}$ and

$$\begin{aligned}
S_k := \{y \in T \cdot x \cap \operatorname{dom}(p_{r-k}) :& \\
&F_{R_{r-k}} \cdot y \text{ is contained in the set of uncovered points in } T \cdot x\},
\end{aligned}$$

so that the constructed set of tiles K_k is contained in the set of uncovered points in $T \cdot x$.

See Figure 1 for a sketch of the tiling process for $\mathbb{Z}^2$. In the first picture,

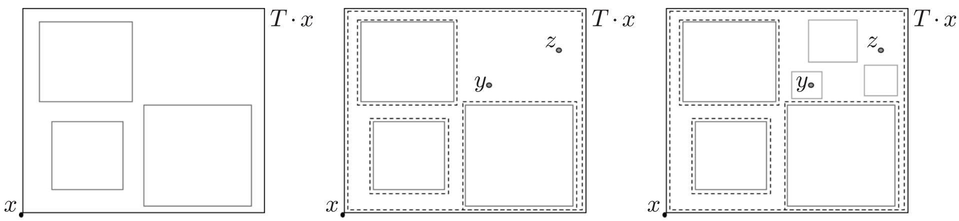

Fig. 1. Identifying the set S_2 and placing the set of tiles K_2 when $\Gamma = \mathbb{Z}^2$

we place the tiles from K_1, and in the second picture we remove strips along the boundaries of $T \cdot x$ as well as K_1. We also mark the points y and z for which p_{r-2} is not defined. Applying Lemma 6 to S_2 (the remaining points), we place smaller tiles, seen in a lighter color in the third picture.

Now, S_k is almost all of the uncovered points in $T \cdot x$ except possibly:

(1) A small strip along the boundary of $T\cdot x$, of size $|\partial_{F_{R_{r-k}}} T|\,|F_{R_{r-k}}| < \eta|T|$ since
$$\frac{|\partial_{F_{R_{r-k}}} T|}{|T|} < \frac{\eta}{|F_{R_{r-k}}|}.$$

(2) The set of points on which p_{r-k} is not defined, which has fewer than $\eta|T|$ points.

(3) A small strip along the boundary of the covered points from each of the previous $k-1$ steps. Fix $j < k$, and consider the set K_j of covered points from the jth step. We might miss a strip of size $|\partial_{F_{R_{r-k}}} K_j|\,|F_{R_{r-k}}|$. Note that since K_j consists of a disjoint union of tiles which are Følner shapes of sizes in $[L_{r-j}, R_{r-j}]$, we have
$$\frac{|\partial_{F_{R_{r-k}}} K_j|}{|K_j|} \leq \frac{|\partial_{F_{R_{r-j}}} F_{R_{r-j}}|}{|F_{R_{r-j}}|},$$
so
$$\begin{aligned} |\partial_{F_{R_{r-k}}} K_j|\,|F_{R_{r-k}}| &\leq \frac{|\partial_{F_{R_{r-j}}} F_{R_{r-j}}|}{|F_{R_{r-j}}|}|K_j|\,|F_{R_{r-k}}| \\ &\leq \frac{\eta}{|F_{R_{r-j-1}}|}|K_j|\,|F_{R_{r-k}}| \leq \eta|T|, \end{aligned}$$
where the penultimate inequality (in the displayed formula) comes from our choice of L_{r-j} to be large enough that $n \geq L_{r-j}$ implies
$$\frac{|\partial_{F_{R_{r-j}}} F_n|}{|F_n|} < \frac{\eta}{|F_{R_{r-j-1}}|},$$
and the final inequality comes from $K_j \subseteq T\cdot x$ and the fact that $r-k \leq r-j-1$ for any $j < k$.

In total, S_k is missing at most $(k+1)\eta|T|$ uncovered points from $T \cdot x$. If $k = 1$, we notice that K_1 covers at least $1/C$ fraction of $S_1 \cup K_1$, and $|S_1 \cup K_1| \geq (1-2\eta)|T|$. So K_1 covers at least $\frac{1}{C}(1-2\eta)|T|$ points, and we are left with $\frac{C-1+2\eta}{C}|T|$ points, so we cover all but $\frac{C-1}{C} + G_1(\eta)$ fraction of $T \cdot x$.

If $k \geq 2$, assume $\bigcup_{i<k} K_i$ covers all but $\left(\frac{C-1}{C}\right)^{k-1} + G_{k-1}(\eta)$ fraction of $T \cdot x$. Notice that
$$|S_k \cup K_k| \leq \left(\left(\frac{C-1}{C}\right)^{k-1} + G_{k-1}(\eta)\right)|T|,$$

since both S_k and K_k are contained in the set of uncovered points of $T \cdot x$. Since K_k covers at least $1/C$ fraction of $|S_k \cup K_k|$, at most $\frac{C-1}{C}$ fraction of $|S_k \cup K_k|$ is left uncovered. So $\bigcup_{i \le k} K_k$ covers all of $T \cdot x$ but at most

$$(k+1)\eta|T| + \frac{C-1}{C}|S_k \cup K_k| \le \left((k+1)\eta + \left(\frac{C-1}{C}\right)^k + \frac{C-1}{C}G_{k-1}(\eta)\right)|T|$$
$$\le \left(\left(\frac{C-1}{C}\right)^k + G_k(\eta)\right)|T|$$

points. This concludes the proof of our claim.

Iterate this algorithm r times so that all but $\left(\frac{C-1}{C}\right)^r + G_r(\eta)$ fraction of $T \cdot x$ has been covered. Since, by hypothesis, $\left(\frac{C-1}{C}\right)^r$, $G_r(\eta) < \varepsilon/2$, this concludes the proof of Lemma 5. ■

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