ANNALES
POLONICI MATHEMATICI
Online First version

A half-space property for hypersurfaces in the hyperbolic space

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Abstract. Through the study of geometry of the hyperspheres (also known as equidistant spheres) of the hyperbolic space $\mathbb{H}^{n+1}$, we establish a nonexistence result for complete noncompact hypersurfaces immersed into $\mathbb{H}^{n+1}$ and a characterization of complete totally geodesic hypersurfaces of $\mathbb{H}^{n+1}$; namely, we characterize those complete hyperspheres of $\mathbb{H}^{n+1}$ with the following geometric property: any geodesic contained in a complete hypersurface is also a geodesic of $\mathbb{H}^{n+1}$. Our approach is based on a suitable maximum principle at infinity for complete Riemannian manifolds.

1. Introduction. The geometry of hyperspheres of the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, $n \geq 3$, is an important theme of differential geometry. Recall that the hyperspheres define a complete foliation for $\mathbb{H}^{n+1}$ and are just the totally umbilical hypersurfaces of $\mathbb{H}^{n+1}$ isometric to the hyperbolic space $\mathbb{H}^n$. In this direction, do Carmo and Lawson [CL83] have used the well known Alexandrov's reflexion method to show that a complete hypersurface Σ^n of $\mathbb{H}^{n+1}$, properly embedded with constant mean curvature, with asymptotic boundary being a sphere and such that Σ^n separates poles, must be a hypersphere. Furthermore, they reached the same conclusion for a hypersurface Σ^n of $\mathbb{H}^{n+1}$ of constant mean curvature which admits a one-to-one orthogonal projection onto a geodesic hyperplane (for more details, see also [N08]).

The approach of [CL83] inspired several other authors to obtain some rigidity results for hypersurfaces in hyperbolic space. For instance, Nelli and Zhu [NZ20] studied the uniqueness of hyperspheres of $\mathbb{H}^{n+1}$ and generalized the Bernstein Theorem due to Carmo and Lawson [CL83] to embedded hypersurfaces with some constant higher order mean curvature. Furthermore,

2020 *Mathematics Subject Classification*: Primary 53C42; Secondary 53C20.
Key words and phrases: hyperbolic space, complete noncompact hypersurfaces, converge to zero at infinity, hyperspheres.
Received 29 June 2023; revised 4 January 2024.
Published online 3 April 2024.

DOI: 10.4064/ap230629-31-1

R. Souam [S21] proved that hyperspheres of $\mathbb{H}^{n+1}$ admit no perturbations with compact support which increase their mean curvature. This is an extension of the analogous result in Euclidean spaces, due to Gromov [G19], which states that a hyperplane in Euclidean space $\mathbb{R}^{n+1}$ admits no mean convex perturbations with compact supports.

On the other hand, Aquino and de Lima [AL12] used the quadric model of the hyperbolic space $\mathbb{H}^{n+1}$ to prove that the hyperspheres are the only complete immersed hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ with constant mean curvature contained between two hyperspheres determined by a spacelike vector $a \in \mathbb{L}^{n+2}$ and whose Gauss image lies in a totally umbilical spacelike hypersurface of de Sitter space $\mathbb{S}_1^{n+1}$ determined by a. Barros, Aquino and de Lima [BAL14a] managed to improve this result, by removing the hypothesis that the hypersurface is contained between two hyperspheres. Next, Aquino [A14] studied the rigidity of complete hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ immersed with constant scalar curvature $R = -1$ and whose Gauss image lies in a totally umbilical spacelike hypersurface of de Sitter space $\mathbb{S}_1^{n+1}$, obtaining characterization results for the hyperspheres.

Moreover, Barros, Aquino and de Lima [BAL14b] showed that complete hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ with bounded nonnegative mean curvature and constant scalar curvature $R = -1$ that are contained between two hyperspheres determined by a spacelike vector $a \in \mathbb{L}^{n+2}$ must be totally geodesic hyperspheres of $\mathbb{H}^{n+1}$, provided that the norm of the vector field $a^\top$ is Lebesgue integrable on Σ^n.

Continuing this line of research, our purpose in this article is to study some geometric aspects of complete noncompact immersed hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ assuming the same controls of Aquino, Barros and de Lima [BAL14b], but using as main analytical tool a suitable version of the maximum principle at infinity for complete Riemannian manifolds established by Alias, Caminha and do Nascimento [ACN19].

This paper is organized as follows: in Section 2 we recall some basic facts concerning hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$. In Section 3, we first describe a classification of totally umbilical hypersurfaces of $\mathbb{H}^{n+1}$ (see Remark 3.1) in terms of their mean curvature and the causal character of a nonzero vector a of Lorentz–Minkowski space $\mathbb{L}^{n+2}$; next, we use a totally geodesic hypersphere of this classification to define the notions of equator $\mathbb{H}^n$, upper half-ball $\mathbb{H}_a^+$ and lower half-ball $\mathbb{H}_a^-$ of $\mathbb{H}^{n+1}$ determined by a unit spacelike vector $a \in \mathbb{L}^{n+2}$, and lastly we study the height and angle functions associated to a hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$. In Section 4, according to the behavior at infinity of the distance from a hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ to the equator $\mathbb{H}^n \looparrowright \mathbb{H}^{n+1}$, we establish a nonexistence result for complete noncompact hypersurfaces immersed in the upper half-ball $\mathbb{H}_a^+$ or lower half-ball

$\mathbb{H}_a^-$ of $\mathbb{H}^{n+1}$ determined by the spacelike vector $a \in \mathbb{L}^{n+2}$ (see Theorem 4.1) and a characterization of complete totally geodesic hypersurfaces of $\mathbb{H}^{n+1}$ (see Theorem 4.3). In Corollaries 4.5 and 4.6 we state our results in terms of normalized scalar curvature.

2. Preliminaries. We will consider the $(n+1)$-dimensional hyperbolic space as a hyperquadric of the $(n+2)$-dimensional Lorentz–Minkowski space $\mathbb{L}^{n+2}$. So, we will write $\mathbb{L}^{n+2}$ for the Euclidean space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric

$$\langle v, w\rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2}w_{n+2}.$$

and the *hyperbolic space* will be identified with

$$\mathbb{H}^{n+1} = \{p \in \mathbb{L}^{n+2} : \langle p, p\rangle = -1,\, p_{n+2} \geq 1\}$$

equipped with the Riemannian metric induced from $\mathbb{L}^{n+2}$.

In this paper we will deal with a connected *two-sided* isometrically immersed hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$, which means that there exists a unit normal vector field N globally defined on Σ^n. In this setting, we use $C^\infty(\Sigma^n)$ and $\mathfrak{X}(\Sigma^n)$ to denote the ring of real functions of class C^∞ defined on Σ^n and the $C^\infty(\Sigma^n)$-module of vector fields of class C^∞ on Σ^n, respectively. We also denote by ∇ the Levi-Civita connection of Σ^n. We recall that the unit normal vector field N can be regarded as a map $N : \Sigma^n \to \mathbb{S}_1^{n+1}$, where $\mathbb{S}_1^{n+1}$ stands for the $(n+1)$-dimensional unitary de Sitter space, that is,

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2} : \langle p, p\rangle = 1\}$$

For that reason, N is called the *Lorentzian Gauss map* of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$.

Let us denote by $A : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ the *shape operator* of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ with respect to N. At each $p \in \Sigma^n$, the shape operator A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$. Thus, for fixed $p \in \Sigma^n$, the spectral theorem allows us to choose in $T_p\Sigma$ an orthonormal basis $\{e_1, \ldots, e_n\}$ of eigenvectors of A_p, with corresponding eigenvalues $\lambda_1(p), \ldots, \lambda_n(p)$, respectively. The functions $\lambda_1, \ldots, \lambda_n : \Sigma^n \to \mathbb{R}$ (obtained in this way) are called the *principal curvatures* of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$. As is well known, we say that $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ is *totally geodesic* when its shape operator A vanishes identically and, in turn, $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ will be *totally umbilical* if there is a function $\lambda : \Sigma^n \to \mathbb{R}$ such that $A = \lambda\,\mathrm{Id}$, where Id denotes the identity map on $\mathfrak{X}(\Sigma^n)$.

Along this work we will deal with the *first two mean curvatures* of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$, namely, the *mean curvature*

$$H = \frac{1}{n}\sum_{i=1}^{n} \lambda_i$$

and the intrinsic geometric quantity defined by

$$H_2 = \frac{2}{n(n-1)} \sum_{i<j} \lambda_i \lambda_j,$$

which is related to the normalized scalar curvature R of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$. More precisely, from the Gauss equation of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ we can get

$$R = H_2 - 1. \tag{2.1}$$

Let us recall that, following the terminology de Lima and Parente [LP13], $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ is called 1-*minimal* when $H_2 \equiv 0$ on Σ^n.

Remark 2.1. The geometric motivation that gives rise to the notion of 1-minimality is described below. Let $\overline{M}^{n+1}(c)$ be a Riemannian manifold of constant sectional curvature c and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}(c)$ be an immersion of an orientable manifold Σ^n into $\overline{M}^{n+1}(c)$. It is well-known that minimal hypersurfaces of $\overline{M}^{n+1}$ arise as critical points of the *area functional* (under compactly supported variations)

$$\mathcal{A}(\Sigma^n) = \int_{\Sigma^n} d\Sigma,$$

where $d\Sigma$ is the volume element of $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}(c)$. We can consider the similar variational problem for the 1-*area functional* $\mathcal{A}_1$ given by

$$\mathcal{A}_1(\Sigma^n) = -n \int_{\Sigma^n} H \, d\Sigma,$$

where H is the mean curvature of $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}(c)$. From variational formulas (see for instance [ACC93, Lemma 2.3]) one can see that $\psi : \Sigma^n \to \overline{M}^{n+1}(c)$ is 1-minimal, namely, a critical point of the 1-area functional $\mathcal{A}_1$ if and only if H_2 vanishes identically.

One also defines the *Newton transformation* $P_1 : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ by

$$P_1 = nH \, \mathrm{Id} - A, \tag{2.2}$$

where Id denotes the identity map on $\mathfrak{X}(\Sigma^n)$. According to [CY77], associated to P_1 is the well known *Cheng–Yau operator*

$$\square : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n), \qquad f \mapsto \square(f) = \mathrm{tr}(P_1 \circ \nabla^2 f),$$

where $\nabla^2 f : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, which is given by $\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X \nabla f, Y \rangle$ for all $X, Y \in \mathfrak{X}(\Sigma^n)$.

As established in [R93], since $\mathbb{H}^{n+1}$ has constant sectional curvature, $\square$ admits the divergence form

$$\square(f) = \mathrm{div}(P_1(\nabla f)), \tag{2.3}$$

for all $f \in C^\infty(\Sigma^n)$, where div stands for divergence on Σ^n.

For a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ and $f \in C^\infty(\Sigma^n)$, it follows from the properties of the Hessian of functions that

$$\Box(\varphi \circ f) = \varphi'(f)\Box(f) + \varphi''(f)\langle P_1(\nabla f), \nabla f\rangle. \tag{2.4}$$

Hounie and Leite [HL95] studied the properties of real homogeneous hyperbolic polynomials, and in particular obtained sufficient geometric conditions to make P_1 a semi-definite positive operator. More precisely, from Lemma 1.1 and equation (1.3) of [HL95] we have the following

LEMMA 2.2. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ be a hypersurface with Lorentzian Gauss map N. If the mean curvature H of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ with respect to N is nonnegative then the Newton transformation P_1 is positive semi-definite.*

We close this section by describing the main analytical tool which is used along the proofs of our main results in the next sections. Our approach is based a suitable maximum principle at infinity for complete noncompact Riemannian manifolds due to Alías, Caminha and do Nascimento. To quote it, we need to recall the following concept established at the beginning of Section 2 of their paper: Let Σ^n be a complete noncompact Riemannian manifold and let

$$d(\,\cdot\,, o) : \Sigma^n \to [0, +\infty)$$

denote the Riemannian distance of Σ^n, measured from a fixed point $o \in \Sigma^n$. We say that a smooth function $f \in C^\infty(\Sigma^n)$ *converges to* $\varrho_0 \in \mathbb{R}$ *at infinity* when

$$\lim_{d(x,o)\to+\infty} f(x) = \varrho_0.$$

Keeping in mind this concept, the following maximum principle at infinity corresponds to [ACN19, Theorem 2.2].

LEMMA 2.3. *Let Σ^n be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(\Sigma^n)$. Assume that there exists a nonnegative, non-identically-vanishing function $f \in C^\infty(\Sigma^n)$ which converges to zero at infinity and $\langle \nabla f, X\rangle \geq 0$. If* $\operatorname{div} X \geq 0$ *on Σ^n, then $\langle \nabla f, X\rangle \equiv 0$ on Σ^n and* $\operatorname{div} X \equiv 0$ *on $\Sigma^n \setminus f^{-1}(0)$.*

3. The height and angle functions. We start this section by recalling the description of totally umbilical hypersurfaces of $\mathbb{H}^{n+1}$ (see, for instance, [M99, Section 4, Example 3]). For this, we fix a nonzero vector $a \in \mathbb{L}^{n+2}$ with $\langle a, a\rangle \in \{-1, 0, 1\}$ and consider the smooth function $h_a : \mathbb{H}^{n+1} \to \mathbb{R}$ defined by $h_a(p) = \langle p, a\rangle$. A straightforward computation shows that for every $\varrho \in \mathbb{R}$ with $\varrho^2 + \langle a, a\rangle > 0$, the level set

$$L^n_\varrho = h_a^{-1}(\varrho) = \{p \in \mathbb{H}^{n+1} : \langle p, a\rangle = \varrho\}$$

is a totally umbilical two-sided hypersurface in $\mathbb{H}^{n+1}$, with Lorentzian Gauss map $N_\varrho : L^n_\varrho \to \mathbb{S}^{n+1}_1$ defined by

$$N_\varrho(p) = \frac{1}{\sqrt{\varrho^2 + \langle a, a\rangle}}\,(a + \varrho p)$$

for any $p \in L^n_\varrho$. Hence, the shape operator $A_\varrho : \mathfrak{X}(L^n_\varrho) \to \mathfrak{X}(L^n_\varrho)$ of $L^n_\varrho \looparrowright \mathbb{H}^{n+1}$ is given by

$$A_\varrho(X) = -\frac{\varrho}{\sqrt{\varrho^2 + \langle a, a\rangle}} X$$

for all $X \in \mathfrak{X}(L^n_\varrho)$, and consequently $L^n_\varrho \looparrowright \mathbb{H}^{n+1}$ has constant mean curvature

$$H_\varrho = \frac{1}{n}\,\mathrm{tr}(A_\varrho) = -\frac{\varrho}{\sqrt{\varrho^2 + \langle a, a\rangle}}.$$

Remark 3.1. From the discussion above, we have the following description of totally umbilical hypersurfaces $L^n_\varrho \looparrowright \mathbb{H}^{n+1}$ in terms of their mean curvature and the causal character of $a \in \mathbb{L}^{n+2}$:

(i) if $a \in \mathbb{L}^{n+2}$ is a unit spacelike vector, then $L^n_\varrho \looparrowright \mathbb{H}^{n+1}$ is a *hypersphere* (also known as *equidistant sphere*) of $\mathbb{H}^{n+1}$, that is, L^n_ϱ is isometric to hyperbolic space $\mathbb{H}^n(\sqrt{\varrho^2+1})$, with constant mean curvature H_ϱ satisfying

$$H^2_\varrho = \frac{\varrho^2}{\varrho^2+1} \in [0, 1);$$

(ii) if $a \in \mathbb{L}^{n+2}$ is a nonzero null vector, then $\varrho \neq 0$ and $L^n_\varrho \looparrowright \mathbb{H}^{n+1}$ is a *horosphere* of $\mathbb{H}^{n+1}$, that is, L^n_ϱ is isometric to Euclidean space $\mathbb{R}^n$, with constant mean curvature H_ϱ satisfying $H^2_\varrho = 1$;

(iii) if $a \in \mathbb{L}^{n+2}$ is a unit timelike vector, then $\varrho^2 > 1$ and $L^n_\varrho \looparrowright \mathbb{H}^{n+1}$ is a *geodesic sphere* of $\mathbb{H}^{n+1}$, that is, L^n_ϱ is isometric to Euclidean sphere $\mathbb{S}^n(\sqrt{\varrho^2-1})$, with constant mean curvature H_ϱ satisfying

$$H^2_\varrho = \frac{\varrho^2}{\varrho^2-1} \in (1, +\infty).$$

Now, we will describe some particular regions in the hyperbolic space $\mathbb{H}^{n+1}$. From Remark 3.1(i), when $a \in \mathbb{L}^{n+2}$ is a unit spacelike vector, the level set $L^n_0 = \mathbb{H}^n \looparrowright \mathbb{H}^{n+1}$ defines a complete totally geodesic hypersurface in $\mathbb{H}^{n+1}$. In analogy with a similar situation in the $(n+1)$-dimensional unit Euclidean sphere $\mathbb{S}^{n+1}$, we will refer to L^n_0 as the *equator* of $\mathbb{H}^{n+1}$ determined by $a \in \mathbb{L}^{n+2}$. This equator divides $\mathbb{H}^{n+1}$ into two connected components, the *upper half-ball*, which is given by

$$\mathbb{H}^+_a = \{p \in \mathbb{H}^{n+1} : \langle p, a\rangle > 0\},$$

and the *lower half-ball*, given by

$$\mathbb{H}_a^- = \{p \in \mathbb{H}^{n+1} : \langle p, a\rangle < 0\}.$$

Let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ be a hypersurface with Lorentzian Gauss map N as described in Section 2. Given a nonzero vector $a \in \mathbb{L}^{n+2}$, inspired by the behavior of the hypersurfaces $L_\varrho^n \looparrowright \mathbb{H}^{n+1}$ studied above, we will consider two particular functions naturally attached to $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$, namely, the *height* and *angle* functions with respect to a, which are defined by

$$\ell_a : \Sigma^n \to \mathbb{R}, \qquad p \mapsto \ell_a(p) = \langle \psi(p), a\rangle,$$

and

$$f_a : \Sigma^n \to \mathbb{R}, \qquad p \mapsto f_a(p) = \langle N(p), a\rangle,$$

respectively. In the case of hyperspheres $L_\varrho^n \looparrowright \mathbb{H}^{n+1}$ described in Remark 3.1(i), the height function ℓ_a is exactly the constant $\varrho \in \mathbb{R}$.

REMARK 3.2. Let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ be as described above. When $a \in \mathbb{L}^{n+2}$ is a unit spacelike vector, for each $p \in \Sigma^n$, $|\ell_a(p)|$ is exactly the distance $d(p)$ from $\psi(p)$ to the equator $\mathbb{H}^n$ of $\mathbb{H}^{n+1}$ determined by $a \in \mathbb{L}^{n+2}$. Thus, in this case, we can geometrically interpret the absolute value of ℓ_a as being the *distance* $d : \Sigma^n \to [0, +\infty)$ from Σ^n to the equator $\mathbb{H}^n$. Thus, $d = 0$ if and only if $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ is the equator $\mathbb{H}^n \looparrowright \mathbb{H}^{n+1}$.

A direct computation shows that

$$\nabla \ell_a = a^\top \quad \text{and} \quad \nabla f_a = -A(a^\top), \tag{3.1}$$

where A is the shape operator of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ and $a^\top$ is the orthogonal projection of a onto $\mathfrak{X}(\Sigma^n)$, that is,

$$a^\top = a - f_a N + \ell_a \psi.$$

Furthermore, the formulas for the Laplacian operator Δ and the Cheng–Yau operator $\square$ acting on the height function of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ are (cf. [R93])

$$\Delta(\ell_a) = nHf_a + n\ell_a, \tag{3.2}$$

$$\square(\ell_a) = n(n-1)H_2 f_a + n(n-1)H\ell_a, \tag{3.3}$$

where H and H_2 are the two mean curvatures of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$.

4. Main results. In our first result we provide sufficient conditions to guarantee the nonexistence of 1-minimal hypersurfaces contained either in the upper half-ball $\mathbb{H}_a^+$ or in the lower half-ball $\mathbb{H}_a^-$ of hyperbolic space $\mathbb{H}^{n+1}$ determined by a unit spacelike vector $a \in \mathbb{L}^{n+1}$.

THEOREM 4.1. *Let $\mathbb{H}^n$ be the equator of hyperbolic space $\mathbb{H}^{n+1}$ determined by a unit spacelike vector $a \in \mathbb{L}^{n+2}$. There does not exist a complete* *noncompact* 1*-minimal hypersurface* $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1} \setminus \mathbb{H}^n$ *having nonnegative mean curvature* H *and such that the distance* d *from* Σ^n *to the equator* $\mathbb{H}^n \looparrowright \mathbb{H}^{n+1}$ *converges to* $\varrho_0 \in (0, +\infty)$ *at infinity, with* $d \geq \varrho_0$ *on* Σ^n*.*

Proof. Assume that such a hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1} \setminus \mathbb{H}^n$ exists. Whether the hypersurface is immersed in $\mathbb{H}^-_a$ or in $\mathbb{H}^+_a$, we can always choose the Lorentzian Gauss map in such a way that the mean curvature of the hypersurface is nonnegative. Without loss of generality, we can assume $\psi : \Sigma^n \looparrowright \mathbb{H}^+_a$. Moreover, if $\ell_a = \langle \psi, a \rangle$ is the height function of $\psi : \Sigma^n \looparrowright \mathbb{H}^+_a$, from Remark 3.2 and the hypothesis about the distance d we see that ℓ_a converges to $\varrho_0 \in (0, +\infty)$ at infinity, with $\ell_a \geq \varrho_0$ on Σ^n.

Let us consider the nonnegative function $f = (\ell_a - \varrho_0)^2 \in C^\infty(\Sigma^n)$ and the vector field $X = P_1(\nabla f) \in \mathfrak{X}(\Sigma^n)$. From (3.1),

$$\langle \nabla f, X \rangle = 2(\ell_a - \varrho_0)\langle \nabla \ell_a, P_1(\nabla f) \rangle = 2(\ell_a - \varrho_0)^2 \langle a^\top, P_1(a^\top) \rangle \tag{4.1}$$

on Σ^n. Since $H \geq 0$ on Σ^n, from Lemma 2.2 we know that the Newton transformation P_1 given in (2.2) is positive semi-definite, and consequently from (4.1) we obtain $\langle \nabla f, X \rangle \geq 0$ on M^n. Furthermore, since $\square$ admits the divergence form (2.3) and $H_2 = 0$ on Σ^n, from (2.4) and (3.3) we have

$$\begin{aligned} \operatorname{div} X = \square((\ell_a - \varrho_0)^2) &= 2(\ell_a - \varrho_0)\square(\ell_a) + 2\langle P_1(\nabla \ell_a), \nabla \ell_a \rangle \\ &\geq 2n(n-1)(\ell_a - \varrho_0)H\ell_a \end{aligned} \tag{4.2}$$

on Σ^n. By observing that $\psi(\Sigma^n) \subset \mathbb{H}^+_a$, we see that ℓ_a is positive on Σ^n, and since $\ell_a \geq \varrho_0$ and $H \geq 0$ on Σ^n, we conclude from (4.2) that

$$\operatorname{div} X \geq 2n(n-1)(\ell_a - \varrho_0)H\ell_a \geq 0 \tag{4.3}$$

on Σ^n. Thus, since f converges to zero at infinity, we can apply Lemma 2.3 to see that $\operatorname{div} X \equiv 0$ on $\Sigma^n \setminus f^{-1}(0)$, and consequently, from (4.3), $(\ell_a - \varrho_0)H \equiv 0$ on $\Sigma^n \setminus f^{-1}(0)$. So, either $\ell_a \equiv \varrho_0$ or $H \equiv 0$ on $\Sigma^n \setminus f^{-1}(0)$.

In the first case, since we already have $\ell_a \equiv \varrho_0$ on the level set

$$f^{-1}(0) = \{p \in \Sigma^n : (\ell_a - \varrho_0)^2 = 0\},$$

we have $\ell_a \equiv \varrho_0$ everywhere of Σ^n, and therefore, taking into account the causal character of $a \in \mathbb{L}^{n+2}$, from Remark 3.1 we conclude that $\psi(\Sigma^n)$ is the hypersphere $\mathbb{H}^n(\sqrt{\varrho_0^2 + 1})$ of $\mathbb{H}^{n+1}$. Since $H_2 \equiv 0$ on Σ^n, $\psi(\Sigma^n)$ isometric to the totally umbilical spacelike hypersurface $\mathbb{H}^n(\sqrt{\varrho_0^2 + 1})$ of $\mathbb{H}^{n+1}$ implies that $\psi(\Sigma^n)$ is totally geodesic. Hence, $\psi(\Sigma^n)$ is the equator $\mathbb{H}^n$ of hyperbolic space $\mathbb{H}^{n+1}$, which is absurd.

On the other hand, if $H \equiv 0$ on $\Sigma^n \setminus f^{-1}(0)$, then since $\ell_a \geq \varrho_0 > 0$ anywhere on Σ^n, from (3.2) we get

$$\Delta f = \Delta(\ell_a - \varrho_0)^2 = 2(\ell_a - \varrho_0)\Delta(\ell_a) + 2|\nabla \ell_a|^2 \geq 2n(\ell_a - \varrho_0)\ell_a \geq 0 \tag{4.4}$$

on $\Sigma^n \setminus f^{-1}(0)$. Applying Lemma 2.3 to $f = (\ell_a - \varrho_0)^2 \in C^\infty(\Sigma^n)$ and $\nabla f \in \mathfrak{X}(\Sigma^n)$ we get $\Delta f \equiv 0$ on $\Sigma^n \setminus f^{-1}(0)$. Back to (4.4), we see that

$\ell_a - \varrho_0$ vanishes identically on $\Sigma^n \setminus f^{-1}(0)$ and, according to the arguments of the first case, $\psi(\Sigma^n)$ would have to be isometric to the equator $\mathbb{H}^n$ of $\mathbb{H}^{n+1}$, which contradicts $\psi(\Sigma^n) \subset \mathbb{H}_a^+$. ∎

REMARK 4.2. From (4.1) and (4.2), we point out that in Theorem 4.1 we can replace nonnegativity of the mean curvature H of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1} \setminus \mathbb{H}^n$ by the weaker hypothesis: *the Newton transformation* P_1 *given in* (2.2) *is positive semi-definite in the direction* $a^\top$. We only adopted the hypothesis on H because it is more geometric. The same holds in our next results.

In order to characterize the equator $\mathbb{H}^n$ of hyperbolic space $\mathbb{H}^{n+1}$ determined by a spacelike vector $a \in \mathbb{L}^{n+2}$, we will slightly modify the arguments in the proof of Theorem 4.1 to provide the following uniqueness result.

THEOREM 4.3. *Let* $a \in \mathbb{L}^{n+2}$ *be a unit spacelike vector. The only complete noncompact* 1*-minimal hypersurface* $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ *immersed into hyperbolic space* $\mathbb{H}^{n+1}$*, with nonnegative mean curvature* H *and such that its distance* d *from* Σ^n *to the equator* $\mathbb{H}^n$ *converges to zero at infinity, is the equator* $\mathbb{H}^n \looparrowright \mathbb{H}^{n+1}$ *determined by* a*.*

Proof. First, we observe that the equator $\mathbb{H}^n \looparrowright \mathbb{H}^{n+1}$ of hyperbolic space $\mathbb{H}^{n+1}$ satisfies all the given restrictions.

Now, let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ be a complete noncompact 1-minimal hypersurface having nonnegative mean curvature H and such that its distance d from Σ^n to the equator $\mathbb{H}^n$ converges to zero at infinity. So, if $\ell_a = \langle \psi, a \rangle$ is the height function of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$, from Remark 3.2 we know that ℓ_a^2 converges to zero at infinity.

Since $H \geq 0$ on Σ^n, from Lemma 2.2, the Newton transformation P_1 defined in (2.2) is positive semi-definite. Furthermore, since $H_2 \equiv 0$ on Σ^n, from (2.4) and (3.3),

$$\square(\ell_a^2) = 2\ell_a \square(\ell_a) + 2\langle P_1(\nabla \ell_a), \nabla \ell_a \rangle \geq 2n(n-1)H\ell_a^2 \geq 0 \tag{4.5}$$

on Σ^n. Let us consider $Y = P_1(\nabla \ell_a^2) \in \mathfrak{X}(\Sigma^n)$. Since $\square$ admits the divergence form (2.3), from (4.5) we find that $\operatorname{div}(Y) \geq 0$ on Σ^n. In addition, we also have $\langle \nabla \ell_a^2, Y \rangle \geq 0$ on Σ^n. Thus, since ℓ_a^2 converges to zero at infinity, we can apply Lemma 2.3 to guarantee that $\operatorname{div}(Y) \equiv 0$ on $\Sigma^n \setminus f^{-1}(0)$. Consequently, from (4.5), $H\ell_a^2 \equiv 0$ on $\Sigma^n \setminus (\ell_a^2)^{-1}(0)$. So, either $\ell_a \equiv 0$ or $H\ell_a \equiv 0$ on $\Sigma^n \setminus (\ell_a^2)^{-1}(0)$.

In the first case, since we already have $\ell_a \equiv 0$ on the level set

$$(\ell_a^2)^{-1}(0) = \{p \in \Sigma^n : \ell_a^2 = 0\},$$

ℓ_a vanishes identically on Σ^n, and therefore from Remark 3.2 we conclude that $\psi(\Sigma^n)$ is the equator $\mathbb{H}^n$ of $\mathbb{H}^{n+1}$ determined by $a \in \mathbb{L}^{n+2}$.

On the other hand, if $H\ell_a \equiv 0$ on $\Sigma^n \setminus (\ell_a^2)^{-1}(0)$, from (3.2) we have

$$\begin{aligned} \Delta(\ell_a^2) &= 2\ell_a\Delta(\ell_a) + 2\langle\nabla\ell_a, \nabla\ell_a\rangle \\ &= 2nHf_a\ell_a + 2n\ell_a^2 + 2|\nabla\ell_a|^2 \geq 2n\ell_a^2 \geq 0 \end{aligned} \tag{4.6}$$

on $\Sigma^n \setminus (\ell_a^2)^{-1}(0)$. We can again apply Lemma 2.3 to obtain $\Delta(\ell_a^2) \equiv 0$ on $\Sigma^n \setminus \ell_a^{-1}(0)$. Back to (4.6), we see that ℓ_a vanishes identically on $\Sigma^n \setminus (\ell_a^2)^{-1}(0)$, and therefore the result is obtained by arguing as in the first case described above. ■

Remark 4.4. Barros, Aquino and de Lima [BAL14a, Theorem 2] showed that the only complete noncompact 1-minimal hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ with bounded nonnegative mean curvature H, which is between two hyperspheres determined by a spacelike vector $a \in \mathbb{L}^{n+2}$ and such that the norm of the projection $a^\top$ of a onto Σ^n is Lebesgue integrable, is the equator $\mathbb{H}^n \looparrowright \mathbb{H}^{n+1}$ determined by a. Consequently, Theorem 4.3 can be understood as a kind of extension of this result.

Taking into account (2.1), we can write Theorems 4.1 and 4.3 in terms of normalized scalar curvature R of a hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$.

Corollary 4.5. *Let $\mathbb{H}^n$ be the equator of hyperbolic space $\mathbb{H}^{n+1}$ determined by a unit spacelike vector $a \in \mathbb{L}^{n+2}$. There does not exist a complete noncompact hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1} \setminus \mathbb{H}^n$ having nonnegative mean curvature H, with constant normalized scalar curvature $R = -1$ and such that the distance d from Σ^n to the equator $\mathbb{H}^n$ of $\mathbb{H}^{n+1}$ converges to $\varrho_0 \in (0, +\infty)$ at infinity, with $d \geq \varrho_0$ on Σ^n.*

Corollary 4.6. *Let $a \in \mathbb{L}^{n+2}$ be a unit spacelike vector. The only complete noncompact hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ immersed into hyperbolic space $\mathbb{H}^{n+1}$, with nonnegative mean curvature H, having constant normalized scalar curvature $R = -1$ and such that its distance from Σ^n to the equator $\mathbb{H}^n$ converges to zero at infinity, is the equator $\mathbb{H}^n \looparrowright \mathbb{H}^{n+1}$ determined by a.*

Acknowledgements. The author is partially supported by CNPq, Brazil, grant 304891/2021-5. The author would like to thank the referee for their valuable suggestions and useful comments which improved the paper.

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