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p-adic equiangular lines and p-adic van Lint–Seidel relative bound

by

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Summary. We introduce the notion of p-adic equiangular lines and derive the first fundamental relation between common angle, dimension of the space and the number of lines. More precisely, we show that if $\{\tau_j\}_{j=1}^n$ is a collection of p-adic γ-equiangular lines in $\mathbb{Q}_p^d$, then

$$|n|^2 \leq |d| \max\{|n|, \gamma^2\}.$$

We call this inequality the p-adic van Lint–Seidel relative bound. We believe that this complements the fundamental van Lint–Seidel [Indag. Math. 28 (1966)] relative bound for equiangular lines in the p-adic case.

1. Introduction. Let $d \in \mathbb{N}$ and $\gamma \in [0,1]$. Recall that a collection $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{R}^d$ is said to be *γ-equiangular lines* [4, 8] if

$$|\langle \tau_j, \tau_k \rangle| = \gamma, \quad \forall 1 \leq j, k \leq n,], \, j \neq k.$$

A fundamental problem associated with equiangular lines is the following.

PROBLEM 1.1. *Given $d \in \mathbb{N}$ and $\gamma \in [0,1]$, what is the upper bound on n such that there exists a collection $\{\tau_j\}_{j=1}^n$ of γ-equiangular lines in $\mathbb{R}^d$?*

An answer to Problem 1.1, which is the fundamental driving force in the study of equiangular lines, is the following result of van Lint and Seidel.

THEOREM 1.2 (van Lint–Seidel relative bound [8, 9]). *Let $\{\tau_j\}_{j=1}^n$ be γ-equiangular lines in $\mathbb{R}^d$. Then*

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

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In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1-\gamma^2)}{1-d\gamma^2}.$$

While deriving p-adic Welch bounds, the notion of p-adic equiangular lines is hinted at in [7]. In this paper, we make it more rigorous and derive a fundamental relation which complements Theorem 1.2.

2. p-adic equiangular lines. We begin by recalling the notion of p-adic Hilbert space. We refer to [1–3, 5, 6] for more information on such spaces.

Definition 2.1 ([2, 3]). Let $\mathbb{K}$ be a non-Archimedean valued field with valuation $|\cdot|$ and $\mathcal{X}$ be a non-Archimedean Banach space with norm $\|\cdot\|$ over $\mathbb{K}$. We say that $\mathcal{X}$ is a *p-adic Hilbert space* if there is a map (called a *p-adic inner product*) $\langle\cdot,\cdot\rangle : \mathcal{X}\times\mathcal{X} \to \mathbb{K}$ satisfying the following:

(i) If $x \in \mathcal{X}$ is such that $\langle x, y\rangle = 0$ for all $y \in \mathcal{X}$, then $x = 0$.
(ii) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{X}$.
(iii) $\langle \alpha x, y + z\rangle = \alpha(\langle x, y\rangle + \langle x, z\rangle)$ for all $\alpha \in \mathbb{K}$ and all $x, y, z \in \mathcal{X}$.
(iv) $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ for all $x, y \in \mathcal{X}$.

The following is the standard example of a p-adic Hilbert space which we consider in the paper.

Example 2.2 ([5]). Let p be a prime. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ be the standard p-adic Hilbert space equipped with the inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d\rangle := \sum_{j=1}^d a_j b_j, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{Q}_p^d,$$

and the norm

$$\|(x_j)_{j=1}^d\| := \max_{1\leq j\leq d} |x_j|, \quad \forall (x_j)_{j=1}^d \in \mathbb{Q}_p^d.$$

After various trials, we believe that the following is a correct definition of equiangular lines in the p-adic setting.

Definition 2.3. Let p be a prime, $d \in \mathbb{N}$ and $\gamma \geq 0$. A collection $\{\tau_j\}_{j=1}^n$ in $\mathbb{Q}_p^d$ is said to be *p-adic γ-equiangular lines* if the following conditions hold:

(i) $\langle \tau_j, \tau_j\rangle = 1$ for all $1 \leq j \leq n$.
(ii) $|\langle \tau_j, \tau_k\rangle| = \gamma$ for all $1 \leq j, k \leq n$, $j \neq k$.
(iii) The operator

$$S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^n \langle x, \tau_j\rangle \tau_j \in \mathbb{Q}_p^d$$

is similar to a diagonal operator over $\mathbb{Q}_p$ with eigenvalues $\lambda_1, \dots, \lambda_d \in \mathbb{Q}_p$ satisfying

$$\Big|\sum_{j=1}^{d} \lambda_j\Big|^2 \leq |d| \Big|\sum_{j=1}^{d} \lambda_j^2\Big|.$$

To set up Definition 2.3, we are mainly motivated by the following observations:

(1) Recall that a basis $\{\tau_j\}_{j=1}^n$ for $\mathbb{Q}_p^d$ is said to be orthonormal if $\langle \tau_j, \tau_k\rangle = \delta_{j,k}$ for all $1 \leq j, k \leq n$. We then naturally have

$$\sum_{j=1}^{n} \langle x, \tau_j\rangle \tau_j = x, \qquad \forall x \in \mathbb{Q}_p^d.$$

(2) In analogy to tight frames in Hilbert spaces, a generalization of orthonormal bases is the following: A collection $\{\tau_j\}_{j=1}^n$ in $\mathbb{Q}_p^d$ is said to be a *p-adic tight frame* for $\mathbb{Q}_p^d$ if there is a nonzero element $b \in \mathbb{Q}_p$ such that

$$\sum_{j=1}^{n} \langle x, \tau_j\rangle \tau_j = bx, \qquad \forall x \in \mathbb{Q}_p^d.$$

Note that the operator corresponding to a tight frame is a scalar operator and is diagonalizable. Since equiangular lines correspond to tight frames in both real and complex Hilbert spaces, and symmetric operators like those given by the formula in condition (iii) are diagonizable, it is reasonable to impose diagonalizability in the definition of p-adic equiangular lines.

The most general version of p-adic equiangular lines can be obtained by considering Definition 2.3 without condition (iii). However, we have been unable to study this level of generality.

The result of this paper is the following p-adic version of Theorem 1.2.

THEOREM 2.4 (p-adic van Lint–Seidel relative bound). *Let p be a prime, $d \in \mathbb{N}$ and $\gamma \geq 0$. If $\{\tau_j\}_{j=1}^n$ is a collection of p-adic γ-equiangular lines in $\mathbb{Q}_p^d$, then*

$$|n|^2 \leq |d| \max\{|n|, \gamma^2\}.$$

In particular, we have the following:

(i) *If $|n| \leq \gamma^2$, then*

$$|n|^2 \leq |d|\gamma^2.$$

(ii) *If $|n| \geq \gamma^2$, then*

$$|n| \leq |d|.$$

Proof. We see that

$$\mathrm{Tr}(S_\tau) = \sum_{j=1}^{n} \langle \tau_j, \tau_j \rangle,$$

$$\mathrm{Tr}(S_\tau^2) = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \tau_j, \tau_k \rangle^2.$$

Using Definition 2.3, we get

$$\begin{aligned}
|n|^2 &= \left|\sum_{j=1}^{n} \langle \tau_j, \tau_j \rangle\right|^2 = |\mathrm{Tr}(S_\tau)|^2 = \left|\sum_{j=1}^{d} \lambda_j\right|^2 \leq |d| \left|\sum_{j=1}^{d} \lambda_j^2\right| \\
&= |d| \left|\sum_{j=1}^{n} \sum_{k=1}^{n} \langle \tau_j, \tau_k \rangle^2\right| = |d| \left|\sum_{j=1}^{n} \langle \tau_j, \tau_j \rangle^2 + \sum_{1\leq j,k\leq n,\, j\neq k} \langle \tau_j, \tau_k \rangle^2\right| \\
&= |d| \left|\sum_{j=1}^{n} 1 + \sum_{1\leq j,k\leq n,\, j\neq k} \langle \tau_j, \tau_k \rangle^2\right| = |d| \left| n + \sum_{1\leq j,k\leq n,\, j\neq k} \langle \tau_j, \tau_k \rangle^2\right| \\
&\leq |d| \max \left\{ |n|, \left| \sum_{1\leq j,k\leq n,\, j\neq k} \langle \tau_j, \tau_k \rangle^2 \right| \right\} \\
&\leq |d| \max \left\{ |n|, \max_{1\leq j,k\leq n,\, j\neq k} |\langle \tau_j, \tau_k \rangle|^2 \right\} = |d| \max \{|n|, \gamma^2\}. \blacksquare
\end{aligned}$$

COROLLARY 2.5. *Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying the following:*

(i) $\langle \tau_j, \tau_j \rangle = 1$ *for all* $1 \leq j \leq n$.
(ii) *There exists a* $\gamma \geq 0$ *such that* $|\langle \tau_j, \tau_k \rangle| = \gamma$ *for all* $1 \leq j, k \leq n$, $j \neq k$.
(iii) *There exists a nonzero element* $b \in \mathbb{Q}_p$ *such that*

$$bx = \sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j, \qquad \forall x \in \mathbb{Q}_p^d.$$

Then

$$|n|^2 \leq |d| \max \{|n|, \gamma^2\}.$$

A careful inspection of the proof of Theorem 2.4 gives the following general p-adic Welch bound.

THEOREM 2.6 (General p-adic Welch bound). *Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying the following:*

(i) $\langle \tau_j, \tau_j \rangle = 1$ *for all* $1 \leq j \leq n$.
(ii) *The operator*

$$S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j \in \mathbb{Q}_p^d$$

is similar to a diagonal operator over $\mathbb{Q}_p$ with eigenvalues $\lambda_1, \dots, \lambda_d \in \mathbb{Q}_p$ satisfying

$$\Big|\sum_{j=1}^{d} \lambda_j\Big|^2 \le |d|\Big|\sum_{j=1}^{d} \lambda_j^2\Big|.$$

Then

$$|n|^2 \le |d| \max_{1\le j,k\le n,\, j\neq k} \{|n|, |\langle \tau_j, \tau_k\rangle|^2\}.$$

We can generalize Definition 2.3 in the following way.

Definition 2.7. Let p be a prime, $d \in \mathbb{N}$, $\gamma \ge 0$ and $a \in \mathbb{Q}_p$ be nonzero. A collection $\{\tau_j\}_{j=1}^n$ in $\mathbb{Q}_p^d$ is said to be *p-adic (γ, a)-equiangular lines* if the following conditions hold:

(i) $\langle \tau_j, \tau_j\rangle = a$ for all $1 \le j \le n$.
(ii) $|\langle \tau_j, \tau_k\rangle| = \gamma$ for all $1 \le j, k \le n$, $j \neq k$.
(iii) The operator

$$S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^{n} \langle x, \tau_j\rangle \tau_j \in \mathbb{Q}_p^d$$

is similar to a diagonal operator over $\mathbb{Q}_p$ with eigenvalues $\lambda_1, \dots, \lambda_d \in \mathbb{Q}_p$ satisfying

$$\Big|\sum_{j=1}^{d} \lambda_j\Big|^2 \le |d|\Big|\sum_{j=1}^{d} \lambda_j^2\Big|.$$

Note that division by the norm of an element is not allowed in a p-adic Hilbert space. Thus we cannot reduce Definition 2.7 to Definition 2.3 (unlike in the real case). By modifying the proof of Theorem 2.4, we easily get the following results.

Theorem 2.8. *If $\{\tau_j\}_{j=1}^n$ is a collection of p-adic (γ, a)-equiangular lines in $\mathbb{Q}_p^d$, then*

$$|n|^2 \le |d| \max\left\{|n|, \frac{\gamma^2}{|a^2|}\right\}.$$

In particular, we have the following.

(i) *If $|a^2 n| \le \gamma^2$, then*

$$|n|^2 \le |d|\frac{\gamma^2}{|a^2|}.$$

(ii) *If $|a^2 n| \ge \gamma^2$, then*

$$|n| \le |d|.$$

Corollary 2.9. *Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying the following:*

(i) *There exists a nonzero element* $a \in \mathbb{Q}_p$ *such that* $\langle \tau_j, \tau_j \rangle = a$ *for all* $1 \leq j \leq n$.
(ii) *There exists a* $\gamma \geq 0$ *such that* $|\langle \tau_j, \tau_k \rangle| = \gamma$ *for all* $1 \leq j, k \leq n$, $j \neq k$.
(iii) *There exists a nonzero element* $b \in \mathbb{Q}_p$ *such that*

$$bx = \sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j, \qquad \forall x \in \mathbb{Q}_p^d.$$

Then

$$|n|^2 \leq |d| \max \left\{ |n|, \frac{\gamma^2}{|a^2|} \right\}.$$

THEOREM 2.10. *Let* $\{\tau_j\}_{j=1}^n$ *be a collection in* $\mathbb{Q}_p^d$ *satisfying the following:*

(i) *There exists a nonzero element* $a \in \mathbb{Q}_p$ *such that* $\langle \tau_j, \tau_j \rangle = a$ *for all* $1 \leq j \leq n$.
(ii) *The operator*

$$S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j \in \mathbb{Q}_p^d$$

is similar to a diagonal operator over $\mathbb{Q}_p$ *with eigenvalues* $\lambda_1, \dots, \lambda_d \in \mathbb{Q}_p$ *satisfying*

$$\left| \sum_{j=1}^{d} \lambda_j \right|^2 \leq |d| \left| \sum_{j=1}^{d} \lambda_j^2 \right|.$$

Then

$$|n|^2 \leq |d| \max_{1 \leq j,k \leq n,\, j \neq k} \left\{ |n|, \frac{|\langle \tau_j, \tau_k \rangle|^2}{|a^2|} \right\}.$$

Note that there is a universal bound for equiangular lines known as Gerzon bound.

THEOREM 2.11 (Gerzon universal bound [10]). *Let* $\{\tau_j\}_{j=1}^n$ *be a collection of* γ*-equiangular lines in* $\mathbb{R}^d$. *Then*

$$n \leq \frac{d(d+1)}{2}.$$

We have been unable to derive a p-adic version of Theorem 2.11. However, we derive the following Gerzon bound under some additional mild conditions.

THEOREM 2.12. *Let* $\{\tau_j\}_{j=1}^n$ *be a collection in* $\mathbb{Q}_p^d$ *satisfying the following:*

(i) *There exists a nonzero element* $a \in \mathbb{Q}_p$ *such that* $\langle \tau_j, \tau_j \rangle = a$ *for all* $1 \leq j \leq n$.
(ii) *There is an element* $b \in \mathbb{Q}_p$ *such that* $\langle \tau_j, \tau_k \rangle^2 = b$ *for all* $1 \leq j, k \leq n$, $j \neq k$.
(iii) $a^2 \neq b$.

Then

$$n \leq \frac{d(d+1)}{2}.$$

Proof. For $1 \leq j \leq n$, define

$$\tau_j \otimes \tau_j : \mathbb{Q}_p^d \ni x \mapsto (\tau_j \otimes \tau_j)(x) := \langle x, \tau_j \rangle \tau_j \in \mathbb{Q}_p^d.$$

We wish to show that the collection $\{\tau_j \otimes \tau_j\}_{j=1}^n$ is linearly independent over $\mathbb{Q}_p$. Let $c_1, \dots, c_n \in \mathbb{Q}_p$ be such that

$$\sum_{j=1}^n c_j (\tau_j \otimes \tau_j) = 0.$$

Let $1 \leq k \leq n$ be fixed. Then the previous equation gives

$$\sum_{j=1}^n c_j (\tau_j \otimes \tau_j)(\tau_k \otimes \tau_k) = 0.$$

By taking trace we get

$$\begin{aligned} 0 &= \sum_{j=1}^n c_j \operatorname{Tr}((\tau_j \otimes \tau_j)(\tau_k \otimes \tau_k)) = \sum_{j=1}^n c_j \langle \tau_j, \tau_k \rangle^2 \\ &= \sum_{j=1,\, j\neq k}^n c_j \langle \tau_j, \tau_k \rangle^2 + c_k \langle \tau_k, \tau_k \rangle^2 = \sum_{j=1,\, j\neq k}^n c_j b + c_k a^2 \\ &= b\Big(\sum_{j=1}^n c_j - c_k\Big) + c_k a^2 = b\Big(\sum_{j=1}^n c_j\Big) + (a^2 - b) c_k. \end{aligned}$$

Therefore,

$$c_k = \frac{b}{b - a^2} \sum_{j=1}^n c_j =: c, \qquad \forall 1 \leq k \leq n.$$

Finally, we get

$$\begin{aligned} 0 &= \operatorname{Tr}\Big(\sum_{j=1}^n c_j (\tau_j \otimes \tau_j)\Big) = \operatorname{Tr}\Big(\sum_{j=1}^n c (\tau_j \otimes \tau_j)\Big) \\ &= \sum_{j=1}^n c \operatorname{Tr}(\tau_j \otimes \tau_j) = \sum_{j=1}^n c \langle \tau_j, \tau_j \rangle = can. \end{aligned}$$

Since $a \neq 0$, we have $c = 0$. Therefore, $\{\tau_j \otimes \tau_j\}_{j=1}^n$ is linearly independent. Since $\{\tau_j \otimes \tau_j\}_{j=1}^n$ is given by a symmetric n by n matrix over $\mathbb{Q}_p$ (with respect to the standard basis) and the dimension of the vector space of symmetric n by n matrices over $\mathbb{Q}_p$ is $d(d+1)/2$, we must have $n \leq d(d+1)/2$. ∎

It is clear that throughout the paper we can replace $\mathbb{Q}_p$ by any non-Archimedean field.

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