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Bilinearity of the Cauchy exponential difference

by

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Summary. We study the functional equation

$$f(x)f(y) - f(x+y) = \alpha xy,$$

where α is a fixed nonzero real number.

1. Introduction and statement of the results. The four classical functional equations

$$\begin{aligned} &f(x) + f(y) = f(x+y), \quad && f(x)f(y) = f(x+y), \\ &f(x) + f(y) = f(xy), && f(x)f(y) = f(xy) \end{aligned} \tag{1.1}$$

are known as *Cauchy's equations*. These equations and their relatives have been investigated by Legendre, Gauss, and numerous other mathematicians. For detailed information on this subject we refer to Aczél [1, Chapter 2.1] and Kuczma [2, Chapters 5.2, 13].

In this paper, we focus on the second equation in (1.1), called *Cauchy's exponential functional equation.* It is known that $f : \mathbb{R} \to \mathbb{R}$ is a solution of $f(x)f(y) = f(x+y)$ if and only if either $f \equiv 0$ or there exists an additive function $g : \mathbb{R} \to \mathbb{R}$ such that $f = \exp g$. Here, we look for functions f such that the Cauchy difference $f(x)f(y) - f(x+y)$ is equal to the bilinear form αxy, where α is a fixed nonzero real parameter.

First, we study the functional equation

$$f(x)f(y) - f(x+y) = \alpha xy \tag{1.2}$$

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under the assumption that the set of zeros $Z(f) = \{x \in \mathbb{R} : f(x) = 0\}$ is nonempty.

THEOREM 1. *Let $\alpha \neq 0$ be a real number and let f be a solution of* (1.2). *The following conditions are equivalent:*

(i) $Z(f) \neq \emptyset$.
(ii) $\alpha > 0$ *and either* $f(x) = 1 - \sqrt{\alpha}x$ *or* $f(x) = 1 + \sqrt{\alpha}x$ *for* $x \in \mathbb{R}$.

CONJECTURE. If f is a solution of (1.2), then $Z(f) \neq \emptyset$.

Next, we determine all solutions of (1.2) under the assumption that f is differentiable at least at one point.

THEOREM 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at least at one point and let $\alpha \neq 0$ be a real number. The functional equation* (1.2) *is solvable if and only if $\alpha > 0$. In this case,*

$$f(x) = 1 - \sqrt{\alpha}x \quad \text{and} \quad f(x) = 1 + \sqrt{\alpha}x \tag{1.3}$$

are the only solutions of (1.2).

In view of (1.1) and (1.2) it is natural to ask about solutions of the equations

$$f(x) + f(y) - f(x + y) = \alpha xy, \tag{1.4}$$

$$f(x) + f(y) - f(xy) = \alpha xy, \tag{1.5}$$

$$f(x) + f(y) - f(xy) = \alpha xy, \tag{1.6}$$

where $x, y \in \mathbb{R}$ and $\alpha \neq 0$ is a fixed real parameter. These equations are easier to handle than (1.2).

REMARK 1. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.4) for all $x, y \in \mathbb{R}$ if and only if there is an additive function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = -\frac{\alpha}{2}x^2 + \varphi(x), \qquad x \in \mathbb{R}. \tag{1.7}$$

REMARK 2. There is no function $f : \mathbb{R} \to \mathbb{R}$ which satisfies (1.5) for all $x, y \in \mathbb{R}$.

REMARK 3. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy (1.6) for all $x, y \in \mathbb{R}$. We set $g(t) = f(t) + \alpha t$ and obtain

$$f(x)f(y) = g(xy).$$

This means that (1.6) can be reduced to a Pexider equation. Applying [2, Theorem 13.3.8] shows that if f is a continuous solution of (1.6), then $f(x) = ax$, where $a^2 - a = \alpha$.

2. Proofs

Proof of Theorem 1. It suffices to show that (i) implies (ii). From (1.2) with $y = 0$ we get

$$f(x)(f(0) - 1) = 0. \tag{2.1}$$

Since $f \equiv 0$ is no solution, there exists a real number x_0 with $f(x_0) \neq 0$. From (2.1) with $x = x_0$ we find $f(0) = 1$.

Let $x_0 \in Z(f)$. From (1.2) we obtain $-f(x_0 + y) = \alpha x_0 y$. This leads to

$$f(x) = -\alpha x_0 x + \alpha x_0^2.$$

Since $f(0) = 1$, we get $\alpha x_0^2 = 1$. Thus, $\alpha > 0$ and either $f(x) = 1 - \sqrt{\alpha} x$ or $f(x) = 1 + \sqrt{\alpha} x$, for all $x \in \mathbb{R}$. ∎

Proof of Theorem 2. First, we note that if $\alpha > 0$, then the two functions given in (1.3) solve (1.2). Next, we assume that f is a solution of (1.2) and is differentiable at $x_1 \in \mathbb{R}$. We consider two cases.

Case 1: $x_1 = 0$. Let $y \neq 0$. Equation (1.2) can be written as

$$f(x)\frac{f(y) - f(0)}{y} - \alpha x = \frac{f(x+y) - f(x)}{y}.$$

Since f is differentiable at 0, we obtain

$$\begin{aligned}\lim_{y \to 0}\left(f(x)\frac{f(y) - f(0)}{y} - \alpha x\right) &= f(x)f'(0) - \alpha x \\ &= \lim_{y \to 0}\frac{f(x+y) - f(x)}{y} = f'(x)\end{aligned}$$

for every x. Thus, we have the linear first-order differential equation

$$f'(x) - f'(0)f(x) = -\alpha x. \tag{2.2}$$

The general solution of (2.2) is given by

$$f(x) = \left(b - \alpha \int_a^x t z(t)\, dt\right)\frac{1}{z(x)},$$

where $a, b \in \mathbb{R}$ and

$$z(x) = \exp(-f'(0)(x - a)).$$

We assume that $f'(0) = 0$. From (2.2) we conclude that $f'(x) = -\alpha x$. Thus, $f(x) = 1 - \alpha x^2/2$. Using (1.2) gives

$$0 = f(x)f(y) - f(x+y) - \alpha xy = \frac{\alpha^2 x^2 y^2}{4}.$$

This contradiction yields $f'(0) \neq 0$.

Let $c = -f'(0)$. Then

$$\int_a^x tz(t)\,dt = \frac{e^{-ca}}{c^2}\big((cx-1)e^{cx} - (ca-1)e^{ca}\big).$$

It follows that

$$f(x) = \lambda e^{-cx} + \frac{\alpha}{c^2}(1-cx), \qquad \lambda = \left(b + \frac{\alpha(ca-1)}{c^2}\right)e^{ca}. \tag{2.3}$$

We have

$$f(x)\frac{f(y)}{y} - \frac{f(x+y)}{x+y}\,\frac{x+y}{y} = \alpha x. \tag{2.4}$$

Using the limit relations

$$\lim_{t\to\infty}\frac{f(t)}{t} = -\frac{\alpha}{c} \quad \text{if } c>0,$$
$$\lim_{t\to-\infty}\frac{f(t)}{t} = -\frac{\alpha}{c} \quad \text{if } c<0,$$

we deduce from (2.4) that

$$f(x)\frac{-\alpha}{c} - \frac{-\alpha}{c} = \alpha x,$$

that is,

$$f(x) = 1 - cx. \tag{2.5}$$

From (2.3) and (2.5) we conclude that $\lambda = 0$ and $c^2 = \alpha$. Thus, $\alpha > 0$. Finally, we obtain

$$f(x) = 1 - \sqrt{\alpha}x \quad \text{and} \quad f(x) = 1 + \sqrt{\alpha}x.$$

CASE 2: $x_1 \neq 0$. We consider two subcases.

CASE 2.1: $x_1 \in Z(f)$. Thus, $Z(f) \neq \emptyset$. Applying Theorem 1 leads to $\alpha > 0$ and to (1.3).

CASE 2.2: $x_1 \notin Z(f)$. Then $f(x_1) \neq 0$. Using $f(0) = 1$ and (1.2) yields, for $y \neq 0$,

$$\frac{f(y)-f(0)}{y} = \frac{1}{f(x_1)}\left(\alpha x_1 + \frac{f(x_1+y)-f(x_1)}{y}\right). \tag{2.6}$$

Since f is differentiable at x_1, we conclude from (2.6) that f is differentiable at 0. Using Case 1 we find $\alpha > 0$ and that f is given by (1.3).

The proof of Theorem 2 is complete. ■

Proof of Remark 1. We assume that f is a solution of (1.4). Using

$$\frac{\alpha}{2}x^2 + \frac{\alpha}{2}y^2 - \frac{\alpha}{2}(x+y)^2 = -\alpha xy,$$

we find that the function

$$\varphi(x) = f(x) + \frac{\alpha}{2}x^2$$

satisfies

$$\varphi(x) + \varphi(y) = \varphi(x + y), \qquad x, y \in \mathbb{R}.$$

Thus,

$$f(x) = -\frac{\alpha}{2}x^2 + \varphi(x),$$

where φ is subadditive. Since f as defined in (1.7) satisfies (1.4), the proof is complete. ∎

Proof of Remark 2. We assume that there exists a solution f of (1.5). Setting $y = 0$ gives $f \equiv 0$, which implies that $\alpha = 0$; a contradiction. ∎

References

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