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Paley problem for plurisubharmonic functions of lower order $\rho > 1$

by

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Abstract. Khabibullin established the best estimate in the Paley problem for a plurisubharmonic (psh) function u of lower order, $0 \leq \rho \leq 1$. For $\rho > 1$, obtaining a sharp estimate has remained an open question. In this work, we solve this problem. We also provide estimates for the types of characteristic functions $T(r, u)$ and $M(r, u)$. Finally, we compare our results with those of Dahlberg for subharmonic functions and show that the latter are not optimal for psh functions of finite lower order $\rho > 1$.

1. Introduction. Given a function $u : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$, let $M(r, u) = \max\{u(z) : |z| = r, z \in \mathbb{C}^n\}$ and $u^+(z) = \max\{u(z), 0\}$. Define

$$T(r, u) = \int_{\mathbb{S}^n} u^+(r\zeta)\, ds_n(\zeta) \tag{1.1}$$

to be the Nevanlinna characteristic of u, where $\mathbb{S}^n$ is the unit sphere in $\mathbb{C}^n$, and s_n is the normalized, rotation-invariant, positive Borel measure on $\mathbb{S}^n$. In the theory of functions of complex variables, it is of interest to estimate the quantity

$$\vartheta(u) = \liminf_{r \to +\infty} \frac{M(r, u)}{T(r, u)}. \tag{1.2}$$

When $u = \log|f|$ with f an entire function in $\mathbb{C}$ of arbitrary order $0 \leq \rho \leq \infty$, Paley [18] made the conjecture

$$\vartheta(\log|f|) = \begin{cases} \dfrac{\pi\rho}{\sin(\pi\rho)}, & 0 \leq \rho \leq 1/2, \\ \pi\rho, & \rho > 1/2. \end{cases} \tag{1.3}$$

The best estimate for $\vartheta(\log|f|)$ was known to Paley for the range $0 \leq \rho \leq 1/2$. The case $\rho > 1/2$ remained unresolved and it became known as the Paley

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problem. This was solved definitely by Govorov [6] in 1967. Petrenko [19] later extended the Paley problem to meromorphic functions of finite lower order. Dahlberg [3] obtained an analogous estimate for $\vartheta(u)$, where u is a subharmonic function of finite lower order in $\mathbb{R}^m, m \geq 2$. Further generalizations in terms of the L^p metric, for $1 \leq p < +\infty$, were also considered by Sodin [21] for $m = 2$. Kondratyuk, Tarasyuk, and Vasyl'kiv [14] obtained respective generalizations for $m > 2$. Different proofs were obtained by Ostrovskiĭ [17] for meromorphic functions and by Essén [5] for subharmonic functions in $\mathbb{C}^n$.

The Paley problem was eventually extended to complex functions of several variables. Khabibullin investigated this problem for meromorphic functions in $\mathbb{C}^n$ with $n > 1$, and subsequently formulated the problem of determining the optimal estimate (1.2) for the class of plurisubharmonic (psh) functions and entire functions of several complex variables [10, 11]. For $n \geq 2$, let

$$P_n(\rho) = \begin{cases} \dfrac{\pi\rho}{\sin(\pi\rho)} \displaystyle\prod_{k=1}^{n-1} \left(1 + \frac{\rho}{2k}\right), & 0 \leq \rho \leq 1/2, \\ \pi\rho \displaystyle\prod_{k=1}^{n-1} \left(1 + \frac{\rho}{2k}\right), & \rho > 1/2. \end{cases}$$

More precisely, Khabibullin showed that for a psh function u in $\mathbb{C}^n$ of finite lower order ρ, we have

$$\vartheta(u) \leq \begin{cases} P_n(\rho), & 0 \leq \rho \leq 1, \\ e^{n-1} P_n(\rho), & \rho > 1. \end{cases} \tag{1.4}$$

The estimate for $0 \leq \rho \leq 1$ is the best possible. In connection with the Paley problem for psh functions, Khabibullin [12] proposed a conjecture—also expressible in three equivalent forms [13]—which, if true, would imply the validity of the estimate $P_n(\rho)$ for the case $\rho > 1$ as well. However, Sharipov [20] constructed the first counterexample to this hypothesis. This negative result was extended in [1] to the case $n = 2$ and any $\rho > 1$. In [2], a counterexample was constructed for the general case $n \geq 2$, $\rho > 1$. Because $\vartheta(u)$ in (1.4) is uniformly bounded by the constant $e^{n-1}P_n(\rho)$ when $\rho > 1$, there must exist the best possible estimate

$$K_n(\rho) \leq e^{n-1} P_n(\rho)$$

such that $\vartheta(u) \leq K_n(\rho)$ for all psh u when $\rho > 1$. We wish to call $K_n(\rho)$ *Khabibullin's constant*. It is the main goal of this work to establish an exact formula for $K_n(\rho)$. In particular, by explicit calculations, we show that $K_2(\rho) < eP_2(\rho)$, verifying that the upper bound given in (1.4) for $\rho > 1$ is not optimal—at least in the case $n = 2$.

This paper is organized as follows. In Section 2, we present basic definitions and results about plurisubharmonic functions. Section 3 is devoted to the construction of a maximizing sequence of nonnegative, increasing functions, which is essential for obtaining the sharp estimate $K_n(\rho)$. The approach involves utilizing sign-changing integrals, as was done in the construction of the general counterexample in [2]. Section 4 presents the main result, Theorem 4.1, which establishes that a plurisubharmonic (psh) function u of lower order $\rho > 1$ satisfies the inequality $\vartheta(u) \leq K_n(\rho)$, where the constant $K_n(\rho)$ is given by (2.14). In addition, we provide estimates for the types of functions $M(r, u)$ and $T(r, u)$ (Theorem 4.2). In Section 5, we compare our results with those of Dahlberg for the class of subharmonic functions. Explicit calculations for $n = 2$ and $n = 3$ show that indeed Dahlberg's estimates are not optimal for psh functions.

2. Plurisubharmonic functions

2.1. Basic facts. A function $u : \mathbb{C}^n \to [-\infty, \infty)$ is called *plurisubharmonic* (psh) if

(1) u is upper semicontinuous and $u \not\equiv -\infty$;
(2) for every $r \geq 0$ and every $z, w \in \mathbb{C}^n$ we have

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta} w)\, d\theta. \tag{2.1}$$

The second condition is equivalent to saying that the mapping $\tau \mapsto u(z+\tau w)$ is subharmonic in $\mathbb{C}$ for any fixed $z, w \in \mathbb{C}^n$. The set of psh functions forms a convex cone in the vector space of semicontinuous functions. If $u_1, u_2, \ldots$ is a decreasing sequence of psh functions, then $u^* = \lim_{n\to\infty} u_n$ is also a psh function ([9, pp. 225–226]).

PROPOSITION 2.1 ([8, Theorem 4.1.11]). *Let $u : \mathbb{C}^n \to [-\infty, \infty)$ be a psh function. Then its Nevanlinna characteristic $T(r, u)$ is an increasing function of r and convex in $\log r$, whenever $n \geq 2$. When $n = 1$, it is only increasing.*

A psh function u is of *lower order* $\rho > 0$ if

$$\liminf_{r\to+\infty} \frac{\log T(r, u)}{\log r} = \rho. \tag{2.2}$$

A function $\rho(r)$ that satisfies the conditions

$$\lim_{r\to+\infty} \rho(r) = \rho \quad \text{and} \quad \lim_{r\to+\infty} r\rho'(r) \log r = 0$$

is known as the *refined order* of u (see e.g. [16, p. 32]). We call

$$\sigma_M = \limsup_{r\to+\infty} \frac{M(r, u)}{r^{\rho(r)}}, \quad \sigma_T = \limsup_{r\to+\infty} \frac{T(r, u)}{r^{\rho(r)}} \tag{2.3}$$

the *types* of the characteristic functions $M(r,u)$ and $T(r,u)$ with respect to $\rho(r)$.

2.2. First results. Let v be a psh function in $\mathbb{C}^n$ such that $v(0)=0$, and for some $\rho>0$,

$$T(r,v)\le r^{\rho}\quad \forall r>0. \tag{2.4}$$

As established in the proof of [11, Theorem 1], the inequality (1.4) follows from the estimate (2.5) proved in [11, Main Lemma]:

$$M(1,v)\le \begin{cases} P_n(\rho), & 0\le\rho\le 1,\\ e^{n-1}P_n(\rho), & \rho>1. \end{cases} \tag{2.5}$$

In view of (2.5), it is desirable to improve the upper bound on $M(1,v)$ when $\rho>1$. Without loss of generality, assume that v is nonnegative. For a fixed $\zeta\in\mathbb{S}^n$, consider the subharmonic (with respect to $w\in\mathbb{C}$) slice function $v_\zeta(w)=v(\zeta w)$. By [11, Lemma 1],

$$v_\zeta(1)\le 4\mu^2\int_0^{+\infty} T(t,v_\zeta)\varphi_\mu(t)\,dt, \tag{2.6}$$

where

$$\mu=\max\left\{\frac{1}{2},\rho\right\},\qquad \varphi_\mu(t)=\frac{t^{2\mu-1}}{(1+t^{2\mu})^2},$$

and

$$T(t,v_\zeta)=\frac{1}{2\pi}\int_0^{2\pi} v_\zeta(te^{i\theta})\,d\theta \tag{2.7}$$

is the *Nevanlinna characteristic* of the function v_ζ. Inequality (2.6) is a direct application of a more general result about subharmonic functions u in $\mathbb{C}$, for which their Laplacian Δu vanishes identically in a neighbourhood of the origin. For further details, see [5, Lemma 4.1].

LEMMA 2.2 ([15, Proposition 5.1]). *A real-valued function $S(t)$, with $S(0)=0$, is increasing on $[0,+\infty)$ and convex with respect to* $\log t$ *if and only if there exists an increasing function $s(t)$ on $[0,+\infty)$ such that $S(t)$ can be represented as*

$$S(t)=\int_0^t \frac{s(x)}{x}\,dx\quad \textit{for } t\ge 0. \tag{2.8}$$

PROPOSITION 2.3. *Let v be a psh function of finite lower order $\rho>0$ such that $T(t,v)\le t^\rho$ for all $t>0$. Then there exists a nonnegative increasing* *function* $s(\cdot, v_\zeta) : [0, +\infty) \to [0, +\infty)$ *such that*

$$\int_0^1 (1-x)^{n-1}\, \frac{s(tx, v_\zeta)}{x}\, dx \le t^{\rho/2} \tag{2.9}$$

and

$$v_\zeta(1) \le 2\rho \int_0^{+\infty} \frac{s(t, v_\zeta)}{t}\, \frac{1}{1+t^\rho}\, dt. \tag{2.10}$$

Proof. Let v be a psh function of finite lower order $\rho > 0$. For a k-dimensional subspace L_k of $\mathbb{C}^n$, if $T(t, v; L_k)$ denotes the Nevanlinna characteristic of the restriction of v to L_k, then $T(t, v; L_k)$ is psh on L_k (see e.g. [11, p. 312]). In particular, $T(t, v_\zeta) = T(t, v; L_1)$ implies that $T(t, v_\zeta)$ is a psh function. By Proposition 2.1, $T(t, v_\zeta)$ is nondecreasing on $[0, +\infty)$ and convex in $\log t$. An application of Lemma 2.2 to $T(t, v_\zeta)$ yields

$$T(t, v_\zeta) = \int_0^t \frac{s(x, v_\zeta)}{x}\, dx, \quad t \ge 0$$

for some increasing function $s(\cdot, v_\zeta)$. Furthermore, by [11, Lemma 3] we have

$$G^{n-1}[T(\cdot, v_\zeta)](t) \le \frac{1}{2^{n-1}(n-1)!}\, T(t, v) t^{2n-2}, \tag{2.11}$$

where G is the operator

$$G[f](t) = \int_0^t x f(x)\, dx.$$

the change of variables $xt \equiv x$ in (2.11) implies

$$\int_0^1 (1-x^2)^{n-1}\, \frac{s(tx, v_\zeta)}{x}\, dx \le T(t, v).$$

Substituting $x \equiv x^2$ and $2s(t^2x^2, v_\zeta) \equiv s(tx, v_\zeta)$ yields

$$\int_0^1 (1-x)^{n-1}\, \frac{s(tx, v_\zeta)}{x}\, dx \le T(t^{1/2}, v) \le t^{\rho/2}.$$

This proves inequality (2.9). On the other hand, integration by parts

$$\begin{aligned}\int_0^{+\infty} T(t, v_\zeta)\varphi_\mu(t)\, dt &= \int_0^{+\infty} \left(\int_0^t \frac{s(x, v_\zeta)}{x}\, dx \right) \varphi_\mu(t)\, dt \\ &= \frac{1}{2\mu} \int_0^{+\infty} \frac{s(t, v_\zeta)}{t}\, \frac{1}{1+t^{2\mu}}\, dt,\end{aligned}$$

together with $2s(t^2, v_\zeta) \equiv s(t, v_\zeta)$, imply for (2.6) the equivalent estimate

$$v_\zeta(1) \le 2\mu \int_0^{+\infty} \frac{s(t, v_\zeta)}{t} \frac{1}{1+t^\mu}\, dt.$$

Since $\rho > 1$, we have $\mu = \rho$. This proves inequality (2.10). ■

Let $\mathrm{psh}_\rho(\mathbb{C}^n)$ denote the set of nonnegative functions v that are psh in $\mathbb{C}^n$ such that $T(t, v) \le t^\rho$. Let $\mathrm{inc}_\rho(\mathbb{R}_+)$ be the set of nonnegative and increasing functions on $\mathbb{R}_+$ satisfying the integral inequality (2.9). Note that

$$M(1, v) = \max\{v(z) : z \in \mathbb{C}^n,\ |z| = 1\} = \max\{v_\zeta(1) : \zeta \in \mathbb{S}^n\}.$$

As an immediate consequence of Proposition 2.3, we get

$$M(1, v) \le 2\rho \sup_{s \in \mathrm{inc}_\rho(\mathbb{R}_+)} \int_0^{+\infty} \frac{s(t)}{t} \frac{1}{1+t^\rho}\, dt. \tag{2.12}$$

For a given $s \in \mathrm{inc}_\rho(\mathbb{R}_+)$, we denote the integral in (2.12) by $J_\rho(s)$, and define

$$J_\rho = \sup_{s \in \mathrm{inc}_\rho(\mathbb{R}_+)} J_\rho(s). \tag{2.13}$$

Lastly, we define *Khabibullin's constant*, denoted by $K_n(\rho)$, as follows:

$$K_n(\rho) = 2\rho J_\rho. \tag{2.14}$$

2.3. Pólya peaks. An important element in the proof of the main result is the use of Pólya peaks. Let $\rho > 0$. A sequence $r_1, r_2, \dots$ is called a sequence of *Pólya peaks* for a function $T(r)$ if there exist sequences $(\varepsilon_k), (\xi_k), (a_k), (A_k)$ of positive numbers satisfying

(2.15)
$$\lim_{k\to+\infty} \varepsilon_k = \lim_{k\to+\infty} \xi_k = \lim_{k\to+\infty} a_k = 0 \quad \text{and} \quad \lim_{k\to+\infty} a_k r_k = \lim_{k\to+\infty} A_k = +\infty,$$

such that for r satisfying $r_k a_k \le r \le r_k A_k$, the following inequality holds:

$$T(r) \le (1+\xi_k)\left(\frac{r}{r_k}\right)^{\rho+\varepsilon_k} T(r_k). \tag{2.16}$$

Edrei [4] proved that a sequence of Pólya peaks for $T(r)$ always exists provided that $T(r)$, for $r \ge 1$, is an unbounded, nondecreasing, nonnegative continuous function satisfying

$$\liminf_{r\to+\infty} \frac{\log T(r)}{\log r} = \rho < +\infty. \tag{2.17}$$

COROLLARY 2.4. *Let u be a nonnegative psh function in $\mathbb{C}^n$ of lower order ρ. Then the Nevanlinna characteristic $T(r, u)$ of u has a sequence of Pólya peaks of order ρ.*

LEMMA 2.5. *Let u be a nonnegative psh function in $\mathbb{C}^n$ of lower order ρ, and let (r_k) be the sequence of Pólya peaks for the Nevanlinna characteristic $T(r,u)$. Define the sequence (u_k) of functions by*

$$u_k(z) = \frac{u(r_k z)}{T(r_k,u)} \quad \text{for each } k \in \mathbb{N}. \tag{2.18}$$

Then (u_k) is a sequence of psh functions that is uniformly bounded on compact subsets of $\mathbb{C}^n$.

Proof. We begin by noting that for every $k \in \mathbb{N}$, the function u_k is psh in $\mathbb{C}^n$, as $T(r_k,u) > 0$ and $u(r_k z)$ is psh in $\mathbb{C}^n$. Next, for any given nonnegative psh function v in $\mathbb{C}^n$, it is known that

$$M(r,v) \le 3 \cdot 2^{2n-2} T(2r,v) \quad \text{(see e.g. [11, (2.2)])}.$$

We now apply this inequality to each u_k, which yields

$$0 \le u_k(z) = \frac{u(r_k z)}{T(r_k,u)} \le \frac{M(r_k r,u)}{T(r_k,u)} \le 3 \cdot 2^{2n-2} \frac{T(2r_k r,u)}{T(r_k,u)}, \quad |z| \le r.$$

In view of inequality (2.16) for $a_k \le 2r \le A_k$, we obtain

$$\frac{T(2r_k r,u)}{T(r_k,u)} \le (1+\xi_k) r^{\rho+\varepsilon_k} \le (1+\xi) r^{\rho+\varepsilon},$$

for some $\xi, \varepsilon > 0$. Therefore, for $a_k \le 2|z| \le A_k$, we get

$$0 \le u_k(z) \le 3 \cdot 2^{2n-2} (1+\xi) r^{\rho+\varepsilon}.$$

When $2|z| \le a_k$, using the definition of u_k, it follows that

$$\begin{aligned} 0 \le u_k(z) &\le \frac{M(a_k r_k/2,u)}{T(r_k,u)} \\ &\le \frac{M(r_k/2,u)}{T(r_k,u)} \le 3 \cdot 2^{2n-2} \quad \text{for sufficiently large } k. \end{aligned}$$

In both cases, u_k is uniformly bounded on the ball $\mathbb{B}_k := \{z \in \mathbb{C}^n : |z| \le A_k/2\}$ for sufficiently large k. Finally, for any compact set $K \subset \mathbb{C}^n$, we find large enough k such that $K \subseteq \mathbb{B}_k$. Thus, the sequence (u_k) is uniformly bounded on compact subsets of $\mathbb{C}^n$. ∎

LEMMA 2.6 ([8, Theorem 4.1.9]). *Let $L^1_{\mathrm{loc}}(\mathbb{C}^n)$ denote the space of locally integrable functions in $\mathbb{C}^n$, and let (v_k) be a sequence of subharmonic functions in $\mathbb{C}^n$ that is uniformly bounded on compact subsets of $\mathbb{C}^n$. If v_k does not converge to $-\infty$ uniformly on every compact subset of $\mathbb{C}^n$, then there exists a subsequence (v_{n_k}) convergent to a subharmonic function v in $L^1_{\mathrm{loc}}(\mathbb{C}^n)$. Moreover,*

$$\limsup_{k\to+\infty} v_{n_k}(z) \le v(z), \qquad z \in \mathbb{C}^n, \tag{2.19}$$

and the two sides are equal and finite almost everywhere.

Remark 2.7. Because a psh function is subharmonic, Lemma 2.5 implies that Lemma 2.6 holds as well for sequences of nonnegative psh functions.

3. Boundedness of J_ρ

3.1. An upper estimate for J_ρ. The main goal in this section is to find an upper estimate for J_ρ as defined in (2.13). Let Q be the operator acting on a function f by the formula

$$Q[f](t) = \int_0^t f(x)\,dx, \qquad t \geq 0.$$

To an operator Q corresponds an operator Q^{-1}, the inverse operator of Q, which acts on a function f by the formula

$$Q^{-1}[f](t) = \frac{df(t)}{dt}.$$

Repeated integration yields the well known Cauchy integral identity

$$Q^k[f](t) = \frac{1}{(k-1)!}\int_0^t (t-x)^{k-1} f(x)\,dx \quad \forall k \in \mathbb{N}.$$

For this operator, we have

$$\int_0^1 (1-u)^{n-1}\,\frac{s(tu)}{u}\,du = \frac{1}{t^{n-1}}\int_0^t (t-u)^{n-1}\,\frac{s(x)}{x}\,du = \frac{(n-1)!}{t^{n-1}}\,Q^n\left[\frac{s(x)}{x}\right](t). \tag{3.1}$$

For $\rho > 1$ let $\varphi_\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be the function

$$\varphi_\rho(t) = \frac{1}{1+t^\rho}, \qquad t \geq 0. \tag{3.2}$$

Lemma 3.1. *For all $k = 0, 1, \ldots,$ we have*

$$|(Q^{-1})^k[\varphi_\rho](t)| = O(t^{\rho-k}) \qquad \text{as } t \to 0, \tag{3.3}$$

$$|(Q^{-1})^k[\varphi_\rho](t)| = O(t^{-\rho-k}) \quad \text{as } t \to +\infty. \tag{3.4}$$

Proof. Relation (3.3) follows from the power series expansion $\varphi_\rho(t) = 1 - t^\rho + t^{2\rho} - \cdots$ as $t \to 0$. Similarly, (3.4) follows from the expansion $\varphi_\rho(t) = t^{-\rho} - t^{-2\rho} + \cdots$ as $t \to +\infty$. ∎

Lemma 3.2. *For any $s \in \mathrm{inc}_\rho(\mathbb{R}_+)$, we have the inequalities*

$$Q^{k+1}\left[\frac{s(x)}{x}\right](t) \leq \frac{1}{k!}t^{\rho/2+k} \quad k = 0, 1, \ldots. \tag{3.5}$$

Proof. This follows immediately from (3.1) and Proposition 2.3. ∎

LEMMA 3.3. *For all $k = 0, 1, \dots,$ we have*

$$\lim_{t\to 0,+\infty} |(Q^{-1})^k[\varphi_\rho](t)| \cdot Q^{k+1}\left[\frac{s(x)}{x}\right](t) = 0. \tag{3.6}$$

Proof. By Lemma 3.2, we have

$$Q^{k+1}\left[\frac{s(x)}{x}\right](t) = O(t^{\rho/2+k}).$$

By Lemma 3.1, we get the asymptotic behaviour:

$$|(Q^{-1})^k[\varphi_\rho](t)| \cdot Q^{k+1}\left[\frac{s(x)}{x}\right](t) = \begin{cases} O(t^{3\rho/2}), & \text{as } t \to 0, \\ O(t^{-\rho/2}), & \text{as } t \to +\infty. \end{cases} \tag{3.7}$$

The limits in (3.6) follow immediately for all $k = 0, 1, \dots$. ■

For $\rho > 1$, let $\psi_\rho : \mathbb{R} \to \mathbb{R}$ be defined by

$$\psi_\rho(t) = (-1)^n (Q^{-1})^n[\varphi_\rho](t), \quad t \geq 0. \tag{3.8}$$

Furthermore, let

$$D_+ = \{t \in \mathbb{R}_+ : \psi_\rho(t) \geq 0\} \quad \text{and} \quad D_- = \{t \in \mathbb{R}_+ : \psi_\rho(t) < 0\}.$$

PROPOSITION 3.4 ([2, Proposition 3.2]). *If $\rho > 1$, then for any $n \geq 2$, $\psi_\rho(t)$ vanishes at most on a finite number of points. In particular $\psi_\rho(t) > 0$ for all large enough $t > 0$.*

Let $0 \leq \tau_1 \leq \tau_2 \leq \dots < +\infty$ be an enumeration of the zeros of ψ_ρ. Define the index set

$$I = \{i \in \mathbb{N} : \psi_\rho(t) < 0, \forall t \in (\tau_{i-1}, \tau_i)\}.$$

Denote

$$\Phi_{\rho,k}(t) = (-1)^{n+k} \frac{\Gamma(\rho/2 + n)}{\Gamma(n)\Gamma(\rho/2 + n - k)} t^{\rho/2+n-k-1} \varphi_\rho^{(n-k)}(t) \tag{3.9}$$

for $k = 0, 1, \dots, n-1$ and

$$\Phi_{\rho,n}(t) = \frac{\Gamma(\rho/2 + n)}{\Gamma(n)\Gamma(\rho/2 + 1)} \arctan t^\rho \qquad \text{for } k = n. \tag{3.10}$$

PROPOSITION 3.5. *For any $s \in \mathrm{inc}_\rho(\mathbb{R}_+)$,*

$$J_\rho(s) \leq \frac{P_n(\rho)}{2\rho} + \sum_{i\in I}\sum_{k=0}^{n} (\Phi_{\rho,k}(\tau_{i-1}) - \Phi_{\rho,k}(\tau_i)). \tag{3.11}$$

Proof. Let $\rho > 1$. Then, for any $s \in \mathrm{inc}_\rho(\mathbb{R}_+)$, by (3.8) and Lemma 3.3, we have

$$J_\rho(s) = \int_0^{+\infty} \frac{s(t)}{t} \varphi_\rho(t)\, dt = \int_0^{+\infty} Q^n\left[\frac{s(x)}{x}\right](t)\psi_\rho(t)\, dt.$$

Since $s \in \mathrm{inc}_\rho(\mathbb{R}_+)$, by Lemma 3.2, we obtain

$$\int_0^{+\infty} Q^n\left[\frac{s(x)}{x}\right](t)\psi_\rho(t)\,dt \le \int_{D_+} Q^n\left[\frac{s(x)}{x}\right](t)\psi_\rho(t)\,dt \le \frac{1}{(n-1)!}\int_{D_+} t^{\rho/2+n-1}\psi_\rho(t)\,dt.$$

The last integral can be written as

$$\frac{1}{(n-1)!}\int_{D_+} t^{\rho/2+n-1}\psi_\rho(t)\,dt = \frac{1}{(n-1)!}\int_0^{+\infty} t^{\rho/2+n-1}\psi_\rho(t)\,dt - \frac{1}{(n-1)!}\int_{D_-} t^{\rho/2+n-1}\psi_\rho(t)\,dt.$$

Integration by parts and Lemma 3.1 imply

$$\frac{1}{(n-1)!}\int_0^{+\infty} t^{\rho/2+n-1}\psi_\rho(t)\,dt = \frac{\rho}{2}\prod_{k=1}^{n-1}\left(1+\frac{\rho}{2k}\right)\int_0^{+\infty}\frac{t^{\rho/2-1}}{1+t^\rho}\,dt = \prod_{k=1}^{n-1}\left(1+\frac{\rho}{2k}\right)\int_0^{+\infty}\frac{dt^{\rho/2}}{1+t^\rho} = \frac{\pi}{2}\prod_{k=1}^{n-1}\left(1+\frac{\rho}{2k}\right) = \frac{P_n(\rho)}{2\rho}.$$

Successive integration by parts implies the following identity in indefinite form:

$$\int t^{\rho/2+n-1}\psi_\rho(t)\,dt = \frac{\Gamma(\rho/2+n)}{\Gamma(\rho/2+1)}\arctan t^\rho + \sum_{k=0}^{n-1}(-1)^{n+k}\frac{\Gamma(\rho/2+n)}{\Gamma(\rho/2+n-k)}t^{\rho/2+n-k-1}\varphi_\rho^{(n-k)}(t).$$

Then, using (3.9) and (3.10), we can write

$$\frac{1}{(n-1)!}\int_{D_-} t^{\rho/2+n-1}\psi_\rho(t)\,dt = \sum_{i\in I}\sum_{k=0}^{n}(\Phi_{\rho,k}(\tau_i)-\Phi_{\rho,k}(\tau_{i-1})). \tag{3.12}$$

Altogether, for $\rho > 1$ we have

$$J_\rho(s) \le \frac{1}{(n-1)!}\int_{D_+} t^{\rho/2+n-1}\psi_\rho(t)\,dt = \frac{P_n(\rho)}{2\rho} + \sum_{i\in I}\sum_{k=0}^{n}(\Phi_{\rho,k}(\tau_{i-1})-\Phi_{\rho,k}(\tau_i)).$$

This completes the proof. ∎

REMARK 3.6. In view of [2, Theorem 1], there exists $s \in \operatorname{inc}_\rho(\mathbb{R}^+)$ such that

$$J_\rho(s) > \frac{P_n(\rho)}{2\rho},$$

which implies that

$$\sum_{i\in I}\sum_{k=0}^{n}(\Phi_{\rho,k}(\tau_{i-1}) - \Phi_{\rho,k}(\tau_i)) > 0.$$

In particular, it follows that $K_n(\rho) > P_n(\rho)$.

PROPOSITION 3.7. *There does not exist $s \in \operatorname{inc}_\rho(\mathbb{R}_+)$ such that $J_\rho(s) = J_\rho$ for $\rho > 1$.*

Proof. Suppose that there is some $s \in \operatorname{inc}_\rho(\mathbb{R}_+)$ with $J_\rho(s) = J_\rho$. Since $s(x)$ is increasing, Lebesgue's differentiation theorem [7, Theorem 9.3.1] shows that s is differentiable almost everywhere on $\mathbb{R}_+$. Let W be the set of points at which s is not differentiable. Define

$$f(t) = Q^n\left[\frac{s(x)}{x}\right](t), \qquad t \geq 0.$$

By Proposition 3.4, for any $n \geq 2$, we have $D_- \neq \emptyset$ whenever $\rho > 1$, and $\psi_\rho(t)$ vanishes at most on a finite number of points $\tau \in \mathbb{R}_+$. For $\varepsilon \in (0,1)$, let f_0 be defined piecewise as

$$f_0(t) = \begin{cases} f(t)(1-\varepsilon\eta(t)), & t \in D_-, \\ f(t), & t \in D_+, \end{cases} \tag{3.13}$$

where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

(3.14)

$$\eta(t) = \sum_{i\in I}\eta_i(t), \quad \eta_i(t) = \begin{cases} \left[\cos\left(\dfrac{\pi t}{2\tau_{i-1}}\right)\cos\left(\dfrac{\pi t}{2\tau_i}\right)\right]^{2n}, & t \in (\tau_{i-1}, \tau_i], \\ 0, & t \notin (\tau_{i-1}, \tau_i], \end{cases}$$

with the agreement that if $1 \in I$, then

$$\eta_1(t) = \cos^{2n}\left(\frac{\pi t}{2\tau_1}\right) \quad \text{for all } t \in (0, \tau_1],$$

and zero otherwise. The finite sequence $(\tau_i)_{i\in I}$ consists of the zeros of ψ_ρ. Note that η is smooth, nonnegative with support in D_-, and uniformly bounded above by 1. Moreover, by definition of η,

$$\lim_{t\to\tau_i^-}\frac{d^k}{dt^k}[f(t)\eta(t)] = 0 \quad \text{and} \quad \lim_{t\to\tau_{i-1}^+}\frac{d^k}{dt^k}[f(t)\eta(t)] = 0 \tag{3.15}$$

for $k=1,\dots,n+1$ and for all τ_i. To f_0, we can associate s_0 defined piecewise by

(3.16)

$$s_0(t)=t(Q^{-1})^n[f_0](t)=\begin{cases} s(t)-\varepsilon t(Q^{-1})^n[f(\cdot)\eta(\cdot)](t), & t\in D_-\cap(\mathbb{R}_+\setminus W),\\ s(t), & t\in D_+\cap(\mathbb{R}_+\setminus W).\end{cases}$$

In view of [2, Theorem 3.1], it is possible to find an $\varepsilon\in(0,1)$ such that $s_0, s_0'\ge 0$ on $\mathbb{R}_+\setminus W$. And on W, we can arrange the values of s_0 so that it is nonnegative and preserves monotonicity on $\mathbb{R}_+$. Hence, s_0 is a nonnegative increasing function. By Lemma 3.2 and (3.13), we have

$$Q^n\left[\frac{s_0(x)}{x}\right](t)=f_0(t)\le\frac{t^{\rho/2+n-1}}{(n-1)!},$$

and therefore, $s_0\in\mathrm{inc}_\rho(\mathbb{R}_+)$. On the other hand, since W is at most countable (hence of Lebesgue measure zero), it does not contribute to the integral $J_\rho(s_0)$. This, in turn, implies

$$J_\rho(s_0)=\int\limits_{\mathbb{R}_+\setminus W}\frac{s_0(t)}{t}\varphi_\rho(t)\,dt=\int\limits_0^{+\infty}\frac{s_0(t)}{t}\varphi_\rho(t)\,dt,$$

which, together with representation (3.16), yields

$$J_\rho(s_0)=\int\limits_0^{+\infty}\frac{s(t)}{t}\varphi_\rho(t)\,dt-\varepsilon\int\limits_{D_-}(Q^{-1})^n[f(\cdot)\eta(\cdot)](t)\varphi_\rho(t)\,dt.$$

By Proposition 3.4, it follows that $D_-=\bigcup_{i\in I}(\tau_{i-1},\tau_i)$. So we can write $\int_{D_-}=\sum_{i\in I}\int_{\tau_{i-1}}^{\tau_i}$. Integrating by parts in each term of this sum and using the vanishing limits (3.15) implies

$$\begin{aligned}J_\rho(s_0)&=\int\limits_0^{+\infty}\frac{s(t)}{t}\varphi_\rho(t)\,dt-\varepsilon\int\limits_{D_-}f(t)\eta(t)\psi_\rho(t)\,dt\\ &>\int\limits_0^{+\infty}\frac{s(t)}{t}\varphi_\rho(t)\,dt=J_\rho(s).\end{aligned}$$

However, this contradicts the fact that $J_\rho(s)$ is maximal. ■

As an immediate corollary we obtain:

COROLLARY 3.8. *There is no* $v\in\mathrm{psh}_\rho(\mathbb{C}^n)$, $\rho>1$, *such that* $M(1,v)=J_\rho$.

3.2. Constructing a maximizing sequence. In view of Proposition 3.7, it is desirable to construct a sequence $(s_k)_{k\in\mathbb{N}}\subseteq\mathrm{inc}_\rho(\mathbb{R}_+)$ such that $\lim_{k\to+\infty}J_\rho(s_k)=J_\rho$. The proof of the last proposition provides us with a concrete method of constructing such a maximizing sequence. Let $s_0\in\mathrm{inc}_\rho(\mathbb{R}_+)$ and $f_0(t)=Q^n[s_0(x)/x](t)$. By [2, Theorem 3.1], for each

$k \in \mathbb{N}$, we can find a positive constant $\varepsilon_k \in (0,1)$ such that the sequence of functions

$$f_k(t) = \begin{cases} f_{k-1}(t), & t \in D_+, \\ (1-\varepsilon_k \eta(t)) f_{k-1}(t), & t \in D_-, \end{cases} \tag{3.17}$$

yields a corresponding sequence $(s_k)_{k\in\mathbb{N}} \subset \mathrm{inc}_\rho(\mathbb{R}_+)$ defined by the formula $s_k(t) = t(Q^{-1})^n[f_k](t)$. Here, η is the function given by (3.14). We start with

$$s_0(t) = \frac{\rho}{2} \prod_{k=1}^{n-1} \left(1 + \frac{\rho}{2k}\right) t^{\rho/2}, \quad t \geq 0. \tag{3.18}$$

It is evident that $s_0 \in \mathrm{inc}_\rho(\mathbb{R}_+)$. To s_0 corresponds

$$f_0(t) = \frac{1}{(n-1)!} t^{\rho/2+n-1}, \quad t \geq 0. \tag{3.19}$$

By the recursion formula $f_k(t) = (1-\varepsilon_k \eta(t)) f_{k-1}(t)$, we obtain

$$f_k(t) = \prod_{i=1}^{k} (1-\varepsilon_i \eta(t)) \cdot f_0(t), \quad t \in D_-,\ k = 1, 2, \ldots, \tag{3.20}$$

and correspondingly

$$s_k(t) = t(Q^{-1})^n \Big[\prod_{i=1}^{k} (1-\varepsilon_i \eta) \cdot f_0 \Big](t), \quad t \in D_-,\ k = 1, 2, \ldots. \tag{3.21}$$

Expanding the product $\prod_{i=1}^{k}(1-\varepsilon_i\eta)$ and using linearity of Q^{-1}, we obtain

$$\begin{aligned} s_k(t) = s_0(t) - t(Q^{-1})^n[\eta f_0](t) \sum_{i=1}^{k} \varepsilon_i + t(Q^{-1})^n[\eta^2 f_0](t) \sum_{i<j} \varepsilon_i \varepsilon_j \\ + \cdots + (-1)^k t(Q^{-1})^n[\eta^k f_0](t) \prod_{i=1}^{k} \varepsilon_i. \end{aligned} \tag{3.22}$$

Next, we claim that for sufficiently large k, we can take $\varepsilon_k \sim \alpha/k$ for some constant $\alpha > 0$. From (3.22), the condition $s_k \geq 0$ is equivalent to

$$\begin{aligned} 1 \geq \frac{t(Q^{-1})^n[\eta f_0](t)}{s_0(t)} \sum_{i=1}^{k} \varepsilon_i \\ - \frac{t(Q^{-1})^n[\eta^2 f_0](t)}{s_0(t)} \sum_{i<j} \varepsilon_i \varepsilon_j \\ + \cdots + (-1)^{k+1} \frac{t(Q^{-1})^n[\eta^k f_0](t)}{s_0(t)} \prod_{i=1}^{k} \varepsilon_i. \end{aligned} \tag{3.23}$$

By definitions of f_0 and s_0, it follows that each ratio function in the last expression is continuous. As a consequence, the absolute value of each one attains a certain maximum on the closure of D_-, since the latter set is compact. Additionally, for sufficiently large k, the last term

$$\left|\frac{t(Q^{-1})^n[\eta^k f_0](t)}{s_0(t)}\right| \sim C\frac{k!}{(k-n)!}$$

dominates the whole sum. Here $C > 0$ is a certain constant possibly depending on f_0, s_0, η and n. Without loss of generality, assume that $\varepsilon_k \leq \varepsilon_{k-1} \leq \cdots \leq \varepsilon_1$. Then, for large k,

$$1 \geq C\frac{k!}{(k-n)!}\prod_{i=1}^{k}\varepsilon_i \geq C\frac{k!}{(k-n)!}(\varepsilon_k)^k.$$

Take $\varepsilon_k \sim \alpha/k$ for a certain constant $\alpha > 0$ for all sufficiently large k. In particular, we get $\sum_{k=1}^{N}\varepsilon_k \to +\infty$ as $N \uparrow +\infty$. By standard arguments from the theory of infinite series, it follows that $\prod_{i=1}^{k}(1-\varepsilon_k\eta(t)) \to 0$ as $k \uparrow +\infty$ for each $t \in D_-$. In particular, $\lim_{k\to+\infty} f_k(t) = 0$ for every $t \in D_-$. These observations yield the following result:

PROPOSITION 3.9. *Let s_0, f_0 be given by* (3.18) *and* (3.19), *respectively. Then $(s_k)_{k\in\mathbb{N}} \subset \mathrm{inc}_\rho(\mathbb{R}_+)$, as constructed above, is a maximizing sequence, i.e.,*

$$\lim_{k\to+\infty} J_\rho(s_k) = J_\rho.$$

In particular,

$$J_\rho = \frac{P_n(\rho)}{2\rho} + \sum_{i\in I}\sum_{k=0}^{n}(\Phi_{\rho,k}(\tau_{i-1}) - \Phi_{\rho,k}(\tau_i)). \tag{3.24}$$

Proof. For every $k \in \mathbb{N}$, we have

$$\begin{aligned} J_\rho(s_k) = \int_0^{+\infty}\frac{s_k(t)}{t}\varphi_\rho(t)\,dt &= \int_{D_+}\frac{s_k(t)}{t}\varphi_\rho(t)\,dt + \int_{D_-}\frac{s_k(t)}{t}\varphi_\rho(t)\,dt \\ &= \int_{D_+}\frac{s_k(t)}{t}\varphi_\rho(t)\,dt + \int_{D_-}(Q^{-1})^n[f_k](t)\varphi_\rho(t)\,dt \\ &= \int_{D_+}\frac{s_k(t)}{t}\varphi_\rho(t)\,dt + \int_{D_-}f_k(t)\psi_\rho(t)\,dt. \end{aligned}$$

On the other hand, by construction, $f_0 \geq f_1 \geq f_2 \geq \cdots$, implying in particular that

$$\int_{D_-}f_k(t)|\psi_\rho(t)|\,dt \leq \int_{D_-}f_0(t)|\psi_\rho(t)|\,dt < \int_0^{+\infty}f_0(t)|\psi_\rho(t)|\,dt < +\infty.$$

By Lebesgue's monotone convergence theorem [7, Corollary 6.8.2],

$$\lim_{k\to+\infty}\int_{D_-} f_k(t)\psi_\rho(t)\,dt = \int_{D_-}\lim_{k\to+\infty} f_k(t)\psi_\rho(t)\,dt = 0.$$

Notice that by construction, $s_k = s_0$ on D_+, so we get

$$\begin{aligned}\lim_{k\to+\infty} J_\rho(s_k) &= \lim_{k\to+\infty}\int_{D_+}\frac{s_k(t)}{t}\varphi_\rho(t)\,dt + \lim_{k\to+\infty}\int_{D_-} f_k(t)\psi_\rho(t)\,dt\\ &= \int_{D_+}\frac{s_0(t)}{t}\varphi_\rho(t)\,dt.\end{aligned}$$

The last integral was evaluated in Proposition 3.5 and equals the estimate in (3.11). Since $\lim_{k\to+\infty} J_\rho(s_k) \le J_\rho$, by Proposition 3.5, (3.24) follows. ∎

REMARK 3.10. By (2.5), we have the estimate $M(1,v) \le e^{n-1}P_n(\rho)$ for every $n \ge 2$ and $\rho > 1$. On the other hand, by Proposition 3.9, we obtain $M(1,v) \le 2\rho J_\rho$, where J_ρ is given by the formula (3.24). Since the latter is achieved as the limit of a maximizing sequence, it follows that $2\rho J_\rho \le e^{n-1}P_n(\rho)$.

4. Main result

THEOREM 4.1. *Let $u : \mathbb{C}^n \to [-\infty,+\infty)$ be a psh function of finite lower order $\rho > 1$. Then*

$$\vartheta(u) \le K_n(\rho), \tag{4.1}$$

where $K_n(\rho) = 2\rho J_\rho$, and J_ρ is given by the formula in (3.24).

Proof. We follow the proof of [11, Theorem 1]. Let $u : \mathbb{C}^n \to [-\infty,+\infty)$ be a psh function of finite lower order $\rho > 1$. Without loss of generality, let $u \ge 0$. By Corollary 2.4, it follows that $T(r,u)$ has a sequence of Pólya peaks (r_k) of order ρ. Let (u_k) be defined as in (2.18). By Lemma 2.5, the sequence (u_k) is uniformly bounded on compact subsets of $\mathbb{C}^n$. By Lemma 2.6 and Remark 2.7, there is a subsequence (u_{n_k}) which converges to a nonnegative subharmonic function v in $L^1_{\mathrm{loc}}(\mathbb{C}^n)$. Moreover

$$\limsup_{k\to+\infty} u_{n_k}(z) \le v(z) \quad \text{for all } z \in \mathbb{C}^n.$$

Let $w^*(z) = \limsup_{y\to z} w(y)$ denote the upper semicontinuous regularization of

$$w(z) = \limsup_{k\to+\infty} u_{n_k}(z).$$

Then $w^*(z)$ is upper semicontinuous and w^* is the least function with the property $w^* \ge w$. On the other hand, v is subharmonic and hence upper semicontinuous, implying that

$$\limsup_{k\to+\infty} u_{n_k}(z) = w(z) \le w^*(z) \le v^*(z) = v(z). \tag{4.2}$$

Convergence of (u_{n_k}) to v in L^1 norm implies that $w(z) = v(z)$ almost everywhere (a.e.) on $\mathbb{C}^n$. Indeed, let $K \subset \mathbb{C}^n$ be a compact set. From the triangle inequality, we have

$$\begin{aligned} 0 &\le \int_K |w(z) - v(z)|\, d\lambda(z) \\ &\le \int_K |w(z) - u_{n_k}(z)|\, d\lambda(z) + \int_K |u_{n_k}(z) - v(z)|\, d\lambda(z), \end{aligned}$$

where $d\lambda$ is the usual (real) Lebesgue measure in $\mathbb{C}^n$. Then

$$\begin{aligned} &\int_K |w(z) - v(z)|\, d\lambda \\ &\quad \le \limsup_{k\to+\infty} \int_K |w(z) - u_{n_k}(z)|\, d\lambda + \lim_{k\to+\infty} \int_K |u_{n_k}(z) - v(z)|\, d\lambda \\ &\quad = \limsup_{k\to+\infty} \int_K |w(z) - u_{n_k}(z)|\, d\lambda \\ &\quad \le \int_K \limsup_{k\to+\infty} |w(z) - u_{n_k}(z)|\, d\lambda = 0, \end{aligned}$$

where the passage of the limit inside the integral is justified by (reversed) Fatou's Lemma. This is because $|w(z) - u_{n_k}(z)| \le |w(z)| + C_K \le 2C_K$ for every $k \in \mathbb{N}$, where C_K is a constant possibly depending on K. Therefore, $w(z) = v(z)$ a.e. on K. Since $K \subset \mathbb{C}^n$ is arbitrary, $w(z) = v(z)$ a.e. on $\mathbb{C}^n$. In particular, inequality (4.2) yields $w^*(z) = v(z)$ a.e.

Now consider any complex line $a + zb \subset \mathbb{C}^n$, where $a, b \in \mathbb{C}^n$ are fixed, and $z \in \mathbb{C}$. By definition of plurisubharmonicity, it follows that $u_{n_k}(a + zb)$ is subharmonic in $z \in \mathbb{C}$. Define

$$w_m(z) = \sup_{k\ge m} u_{n_k}(a + zb)$$

for each $z \in \mathbb{C}$. Then (w_m) is a nonincreasing sequence of functions and $\lim_m w_m(z) = w(a + zb)$ for each $z \in \mathbb{C}$. Let w_m^* denote the upper semicontinuous regularization of w_m. By [9, Theorem 3.2.2], w_m^* is subharmonic. On the other hand, w_m^* is the least function for which $w_m^* \ge w_m$, implying

$$\lim_m w_m^*(z) = w^*(a + zb).$$

Since (w_m^*) is a decreasing sequence of subharmonic functions, the limit $w^*(a+zb)$ is subharmonic in $z \in \mathbb{C}$. Because the complex line $a+zb \subset \mathbb{C}^n$ is arbitrary, we conclude that w^* is psh in $\mathbb{C}^n$. Finally, note that two subharmonic functions that coincide almost everywhere must be equal. Therefore,

$w^*(z) = v(z)$ a.e. on $\mathbb{C}^n$ implies that v is a psh function in $\mathbb{C}^n$. By [14, Theorem 4], the sequence (u_{n_k}) can be chosen so that

$$\lim_{k\to+\infty} \int\limits_{\mathbb{S}_n} |u_{n_k}(r\zeta) - v(r\zeta)|\, ds_n(\zeta) = 0, \quad r > 0.$$

This, in turn, implies

$$\begin{aligned} T(r,v) = \int\limits_{\mathbb{S}_n} v(r\zeta)\, ds_n(\zeta) &\le \int\limits_{\mathbb{S}_n} u_{n_k}(r\zeta)\, ds_n(\zeta) + o(1) \\ &= \int\limits_{\mathbb{S}_n} \frac{u(r_{n_k} r\zeta)}{T(r_{n_k},u)}\, ds_n(\zeta) + o(1) = \frac{T(r_{n_k} r, u)}{T(r_{n_k},u)} + o(1) \end{aligned}$$

as $k \uparrow +\infty$. By (2.16), it follows that $T(r,v) \le r^\rho$, and thus $v \in \mathrm{psh}_\rho(\mathbb{C}^n)$. Moreover, by using the Poisson kernel, as shown in [11, Theorem 1], we obtain

$$\limsup_{k\to+\infty} M(1,u_k) \le M(1,v). \tag{4.3}$$

The definition of (u_k) implies

$$\begin{aligned} M(1,u_k) = M\left(1, \frac{u(r_k z)}{T(r_k,u)}\right) &= \max\left\{ \frac{u(r_k z)}{T(r_k,u)} : |z| = 1,\, z \in \mathbb{C}^n \right\} \\ &= \frac{1}{T(r_k,u)} \max\{u(z) : |z| = r_k,\, z \in \mathbb{C}^n\} \\ &= \frac{M(r_k,u)}{T(r_k,u)}. \end{aligned}$$

Hence, by (4.3), we have

$$\vartheta(u) = \liminf_{r\to+\infty} \frac{M(r,u)}{T(r,u)} \le \limsup_{k\to+\infty} \frac{M(r_k,u)}{T(r_k,u)} \le M(1,v). \tag{4.4}$$

On the other hand, $M(1,v) \le 2\rho J_\rho$. By Proposition 3.5, it follows that

$$\vartheta(u) \le P_n(\rho) + 2\rho \sum_{i\in I} \sum_{k=0}^{n} (\Phi_{\rho,k}(\tau_{i-1}) - \Phi_{\rho,k}(\tau_i)).$$

Now, we demonstrate that this estimate for $\vartheta(u)$ is nonimprovable, i.e., $K_n(\rho) = 2\rho J_\rho$. It suffices to show that there is some psh function u_0 in $\mathbb{C}^n$ of finite lower order $\rho > 1$ such that all inequalities in (4.4) become equalities. To this end, for $\rho > 1$, let

$$E_\rho(z) := \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(1 + j/\rho)}, \quad z \in \mathbb{C}. \tag{4.5}$$

Define $u_0(z) := \log|E_\rho(z_1)|$, where $z := (z_1, \dots, z_n) \in \mathbb{C}^n$. Since E_ρ is an entire function of order ρ, it can be shown by direct calculations that u_0 is a psh function in $\mathbb{C}^n$ of finite lower order ρ. We have the following asymptotic for the characteristic functions of u_0:

$$M(r,u_0)=r^\rho+O(1) \quad\text{and}\quad T(r,u_0)=P_n^{-1}(\rho)r^\rho+O(1). \tag{4.6}$$

This implies that

$$\vartheta(u_0)=\liminf_{r\to+\infty}\frac{M(r,u_0)}{T(r,u_0)}=\lim_{r\to+\infty}\frac{M(r,u_0)}{T(r,u_0)}=P_n(\rho),$$

and in particular, for any Pólya peaks (r_k) of $T(r,u_0)$,

$$\limsup_{k\to+\infty}\frac{M(r_k,u_0)}{T(r_k,u_0)}=\vartheta(u_0).$$

By construction of the sequence $(u_{0,k})$ from given Pólya peaks (r_k), we have

$$u_{0,k}(z)=\frac{u_0(r_k z)}{T(r_k,u_0)}\le\frac{M(r_k r,u_0)}{T(r_k,u_0)},\quad z\in\mathbb{C}^n \text{ with } |z|=r.$$

By the asymptotic formulae (4.6), we obtain

$$u_{0,k}(z)\le\frac{r_k^\rho r^\rho+O(1)}{P_n^{-1}(\rho)r_k^\rho+O(1)},$$

and consequently $\limsup_{k\to+\infty}u_{0,k}(z)\le P_n(\rho)r^\rho$ for every $z\in\mathbb{C}^n$ with $|z|=r$. By the earlier arguments, we have shown that

$$\limsup_{y\to z}\limsup_{k\to+\infty}u_{0,k}(y)=v_0(z)$$

for some psh function $v_0\in\mathrm{psh}_\rho(\mathbb{C}^n)$. In particular, $v_0(z)\le P_n(\rho)|z|^\rho$, and consequently $M(1,v_0)\le P_n(\rho)M(1,|z|^\rho)=P_n(\rho)$. Therefore, $\vartheta(u_0)=M(1,v_0)$. ∎

Theorem 4.2. *The following (best possible) estimates hold:*

$$\sigma_M\le\begin{cases}P_n(\rho)\sigma_T, & 0\le\rho\le1,\\ K_n(\rho)\sigma_T, & \rho>1.\end{cases} \tag{4.7}$$

Proof. By definition (2.3), we have

$$\sigma_M=\limsup_{r\to+\infty}\frac{M(r,u)}{r^{\rho(r)}}\quad\text{and}\quad\sigma_T=\limsup_{r\to+\infty}\frac{T(r,u)}{r^{\rho(r)}}.$$

By Theorem 4.1, for sufficiently small $\varepsilon>0$ and sufficiently large k,

$$\frac{M(r_k,u)}{T(r_k,u)}\le\begin{cases}P_n(\rho)+\varepsilon, & 0\le\rho\le1,\\ K_n(\rho)+\varepsilon, & \rho>1,\end{cases} \tag{4.8}$$

where (r_k) is the sequence of Pólya peaks of $T(r,u)$. Next, we rewrite the ratio as

$$\frac{M(r_k,u)}{T(r_k,u)}=\frac{M(r_k,u)}{r_k^{\rho(r_k)}}\cdot\frac{r_k^{\rho(r_k)}}{T(r_k,u)},$$

and passing to the limit superior in (4.8) yields

$$\limsup_{k\to+\infty}\frac{M(r_k,u)}{r_k^{\rho(r_k)}}\le\begin{cases}P_n(\rho)\limsup\limits_{k\to+\infty}\dfrac{T(r_k,u)}{r_k^{\rho(r_k)}}, & 0\le\rho\le 1,\\ K_n(\rho)\limsup\limits_{k\to+\infty}\dfrac{T(r_k,u)}{r_k^{\rho(r_k)}}, & \rho>1.\end{cases}$$

On the other hand, we have

$$\limsup_{k\to+\infty}\frac{T(r_k,u)}{r_k^{\rho(r_k)}}=\limsup_{r\to+\infty}\frac{T(r,u)}{r^{\rho(r)}}=\sigma_T$$

and

$$\limsup_{k\to+\infty}\frac{M(r_k,u)}{r_k^{\rho(r_k)}}=\limsup_{r\to+\infty}\frac{M(r,u)}{r^{\rho(r)}}=\sigma_M. \ \blacksquare$$

5. Comparison with Dahlberg's estimates for subharmonic functions

5.1. Dahlberg's estimate. Since the set of psh functions in $\mathbb{C}^n$ is a proper subset of the broader class of subharmonic functions in $\mathbb{C}^n$, it is of interest to compare the estimate $K_n(\rho)$ for psh functions of lower order $\rho>1$ with the corresponding estimate for subharmonic functions of the same lower order. However, since $K_n(\rho)$ is sharp for the class of psh functions, it follows that $K_n(\rho)$ must be no greater than any nonimprovable estimate for subharmonic functions. We verify this by explicitly comparing $K_n(\rho)$ with the estimate obtained by Dahlberg [3] for subharmonic functions in $\mathbb{R}^m$, where $m\ge 3$. For $\rho>0$, let the Gegenbauer functions C_ρ^γ be defined as the solutions to the differential equation

$$(1-x^2)\frac{d^2f}{dx^2}-(2\gamma+1)x\frac{df}{dx}+\rho(\rho+2\gamma)f=0,\qquad -1<x<1, \tag{5.1}$$

with the normalization condition

$$\lim_{x\uparrow 1}C_\rho^\gamma(x)=C_\rho^\gamma(1)=\frac{\Gamma(\rho+2\gamma)}{\Gamma(2\gamma)\Gamma(\rho+1)}. \tag{5.2}$$

Let $a_\rho=\sup\{t: C_\rho^{(m-2)/2}(t)=0\}$ and define u_ρ in $\mathbb{R}^m$, where $m\ge 3$, by

$$u_\rho(x)=\begin{cases}0, & x_1\le a_\rho r,\\ r^\rho C_\rho^{\frac{m-2}{2}}(x_1/r), & x_1>a_\rho r,\end{cases} \tag{5.3}$$

where $x:=(x_1,\dots,x_m)$ and $r=|x|$. Note that u_ρ is subharmonic in $\mathbb{R}^m$. Dahlberg [3] then showed that for a subharmonic function u in $\mathbb{R}^m$, where $m\ge 3$, we have

$$\vartheta(u)\le\vartheta(u_\rho). \tag{5.4}$$

Identifying $\mathbb{C}^n \cong \mathbb{R}^{2n}$, the inequality (5.4) also holds for functions u that are subharmonic in $\mathbb{C}^n$. Now, in the definition of u_ρ, we have

$$u_\rho(x) = r^\rho C_\rho^{n-1}(x_1/r) \quad \text{whenever } x_1 < a_\rho r \text{ and } x := (x_1, \dots, x_{2n}), \qquad r = |x|.$$

We now calculate the characteristic functions $M(r, u_\rho)$ and $T(r, u_\rho)$:

$$\begin{aligned} M(r, u_\rho) &= \max\{u_\rho(x) : x \in \mathbb{R}^{2n},\ |x| = r\} \\ &= \max\{u_\rho(x) : x \in \mathbb{R}^{2n},\ |x| = r,\ x_1 > a_\rho r\} \\ &= \max\{r^\rho C_\rho^{n-1}(x_1/r) : x \in \mathbb{R}^{2n},\ |x| = r\} \\ &= r^\rho C_\rho^{n-1}(1). \end{aligned}$$

Next, applying the formula for Nevanlinna characteristic (1.1), now on the unit sphere in $\mathbb{R}^{2n}$, we get

$$T(r, u_\rho) = \int_{\mathbb{S}_{2n-1}} u_\rho^+(r\zeta)\, ds_{2n-1}(\zeta) = r^\rho \int_{\mathbb{S}_{2n-1}} C_\rho^{n-1}(\zeta_1) \cdot \chi_{\{\zeta_1 > a_\rho\}}\, ds_{2n-1}(\zeta),$$

where $\zeta := (\zeta_1, \dots, \zeta_{2n}) \in \mathbb{S}_{2n-1}$ and $\chi_{\{\zeta_1 > a_\rho\}} = 1$ on $\{\zeta \in \mathbb{S}_{2n-1} : \zeta_1 > a_\rho\}$ and vanishes elsewhere. Rewriting the last integral in terms of spherical coordinates implies

$$\begin{aligned} T(r, u_\rho) &= \frac{r^\rho}{A(\mathbb{S}_{2n-1})} \int_0^\pi \cdots \int_0^\pi \int_0^{\theta^*} C_\rho^{n-1}(\cos\theta_1) \\ &\qquad \times \Big(\prod_{i=1}^{2n-2} \sin^{2n-(i+1)} \theta_i \Big)\, d\theta_1 \cdots d\theta_{2n-1} \\ &= \frac{r^\rho}{A(\mathbb{S}_{2n-1})} \Big(\int_0^{\theta^*} C_\rho^{n-1}(\cos\theta_1) \sin^{2n-2}\theta_1\, d\theta_1 \Big) \\ &\qquad \times \Big(\int_0^\pi \cdots \int_0^\pi \sin^{2n-3}\theta_2 \cdots \sin\theta_{2n-2}\, d\theta_2 \cdots d\theta_{2n-1} \Big) \\ &= \frac{A(\mathbb{S}_{2n-2})}{A(\mathbb{S}_{2n-1})} r^\rho \int_0^{\theta^*} C_\rho^{n-1}(\cos\theta_1) \sin^{2n-2}\theta_1\, d\theta_1, \end{aligned}$$

where $A(\mathbb{S}_k)$ is the area of the unit sphere in $\mathbb{R}^{k+1}$ and $\theta^* = \arccos a_\rho$. As a result we obtain the ratio

$$\frac{M(r, u_\rho)}{T(r, u_\rho)} = C_\rho^{n-1}(1) \frac{A(\mathbb{S}_{2n-1})}{A(\mathbb{S}_{2n-2})} \cdot \Big(\int_0^{\theta^*} C_\rho^{n-1}(\cos\theta) \sin^{2n-2}\theta\, d\theta \Big)^{-1}.$$

Since the above quantity is independent of r, we conclude that

$$\vartheta(u_\rho) = C_\rho^{n-1}(1) \frac{A(\mathbb{S}_{2n-1})}{A(\mathbb{S}_{2n-2})} \cdot \Big(\int_0^{\theta^*} C_\rho^{n-1}(\cos\theta) \sin^{2n-2}\theta\, d\theta \Big)^{-1}. \tag{5.5}$$

To evaluate the last integral, we make use of a recursion formula for Gegenbauer functions [11] given by

$$C_\rho^{n-1}(t) = \frac{1}{2^{n-1}(n-2)!}\,\frac{d^{n-2}}{dt^{n-2}}C^1_{\rho+n-2}(t), \tag{5.6}$$

which yields

$$\begin{aligned}
&\int_0^{\theta^*} C_\rho^{n-1}(\cos\theta)\sin^{2n-2}\theta\,d\theta\\
&\qquad = \frac{1}{2^{n-1}(n-2)!}\int_0^{\theta^*}\frac{d^{n-2}}{d(\cos\theta)^{n-2}}\,C^1_{\rho+n-2}(\cos\theta)\sin^{2n-2}\theta\,d\theta\\
&\qquad = \frac{(-1)^{n-2}}{2^{n-1}(n-2)!}\int_0^{\theta^*}\frac{d^{n-2}}{d\theta^{n-2}}\,C^1_{\rho+n-2}(\cos\theta)\sin^{n}\theta\,d\theta.
\end{aligned}$$

It is possible to find an explicit formula for $C^1_{\rho+n-2}(\cos\theta)$. Upon solving the differential equation (5.1) for $\gamma = 1$ and $\rho \equiv \rho+n-2$, we get the expression

$$\begin{aligned}
C^1_{\rho+n-2}(\cos\theta) &= \frac{1}{\cos\theta}\int_0^{\theta}\cos((\rho+n-1)t)\,dt\\
&= \frac{\sin((\rho+n-1)\theta)}{(\rho+n-1)\sin\theta}, \quad 0<\theta<\pi.
\end{aligned} \tag{5.7}$$

For illustration, we consider the cases $n\in\{2,3\}$.

5.2. Calculation of $K_n(\rho)$ for $n = 2$. For small values of n, it is possible to get explicit expression for Khabibullin's constant $K_n(\rho)$. In particular, when $n = 2$, routine calculations show that

$$\psi_\rho(t) = \frac{\rho t^{\rho-2}((\rho+1)t^\rho - (\rho-1))}{(1+t^\rho)^3},$$

and evidently $\psi_\rho(t) < 0$ for $t\in[0,\tau)$ and $\psi_\rho(t) > 0$ for $t>\tau$, where τ is the nonzero solution of $\psi_\rho(t) = 0$, given by the formula

$$\tau(\rho) = \left(\frac{\rho-1}{\rho+1}\right)^{1/\rho}.$$

In this case, $D_- = (0,\tau)$ and by formulas (3.9), (3.10), and (3.12), we obtain

$$\begin{aligned}
\int_0^{\tau} t^{\rho/2+1}\psi_\rho(t)\,dt = &-\frac{(\rho+1)^2}{4\rho}\left(\frac{\rho-1}{\rho+1}\right)^{3/2}\\
&-\left(\frac{\rho}{2}+1\right)\left[\frac{\rho+1}{2\rho}\left(\frac{\rho-1}{\rho+1}\right)^{1/2} - \arctan\frac{\rho-1}{\rho+1}\right],
\end{aligned}$$

whenever $\rho > 1$. Hence, for $n = 2$ and $\rho > 1$, we have

$$K_2(\rho) = \pi\rho\left(\frac{\rho}{2}+1\right) + \frac{(\rho+1)^2}{2}\left(\frac{\rho-1}{\rho+1}\right)^{3/2} \tag{5.8}$$

$$+ 2\rho\left(\frac{\rho}{2}+1\right)\left[\frac{\rho+1}{2\rho}\left(\frac{\rho-1}{\rho+1}\right)^{1/2} - \arctan\frac{\rho-1}{\rho+1}\right]. \tag{5.9}$$

Note that $\lim_{\rho\downarrow 1} K_2(\rho) = 3\pi/2$, which coincides with the sharp estimate when $\rho = 1$. Hence, $K_2(\rho)$ is continuous at $\rho = 1$. Moreover, we get an asymptotic behaviour of $K_2(\rho)$ given by

$$K_2(\rho) = \pi\rho\left(\frac{\rho}{2}+1\right) + o(\rho) \quad \text{as} \quad \rho \downarrow 1$$

and

$$K_2(\rho) \sim \pi\rho\left(\frac{\rho}{2}+1\right) + \frac{\rho^2}{2} + \rho^2\left(\frac{1}{2}-\frac{\pi}{4}\right) < \pi\rho\left(\frac{\rho}{2}+1\right) + \frac{\rho^2}{4} \text{ as } \rho \uparrow +\infty.$$

In both cases, as ρ approaches 1 from above or as ρ gets arbitrarily large, the estimate shows once more that it is smaller than $eP_2(\rho)$. Indeed, by calculations we verify that $K_2(\rho) < eP_2(\rho)$ for all $\rho \geq 1$. Calculating $K_n(\rho)$ explicitly gets complicated when $n \geq 3$ because of the form of $\psi_\rho(t)$. It is known that

$$\psi_\rho(t) = (-1)^n \frac{p(t^\rho)}{t^n(1+t^\rho)^{n+1}} \quad \text{for} \quad t \geq 0,\ n \geq 2,$$

where p is a polynomial in t^ρ of the form $p(x) = c_{n,n-1}x^n + c_{n,n-2}x^{n-1} + \cdots + c_{n,1}x^2 + c_{n,0}x$. The coefficients $c_{n,k}$ are polynomials in ρ, satisfying a certain recurrence relation (see e.g. [2, Proposition 3.1]), which can be helpful in studying the zeros of $p(x)$. These, in turn, determine the zeros τ of the function ψ_ρ.

5.3. Comparison for $n = 2$. When $n = 2$, we have

$$C_\rho^1(\cos\theta) = \frac{\sin((\rho+1)\theta)}{(\rho+1)\sin\theta},$$

and the smallest $\theta > 0$ for which $C_\rho^1(\cos\theta) = 0$ is $\theta^* = \pi/(\rho+1)$. Then

$$\begin{aligned} 2(\rho+1)\int_0^{\theta^*} C_\rho^1(\cos\theta)\sin^2\theta\,d\theta &= \int_0^{\pi/(\rho+1)} \sin((\rho+1)\theta)\sin\theta\,d\theta \\ &= \frac{1}{\rho}\sin\frac{\pi\rho}{\rho+1} - \frac{1}{\rho+2}\sin\frac{\pi(\rho+2)}{\rho+1}. \end{aligned}$$

By the normalization formula (5.2), we get $C_\rho^1(1) = \rho+1$. From the formula for the area of the unit sphere, $A(\mathbb{S}_{k-1}) = 2\pi^{k/2}/\Gamma(k/2)$, we find that

$A(\mathbb{S}_3) = 2\pi^2$, $A(\mathbb{S}_2) = 4\pi$. Therefore, substituting in (5.5) yields

$$\begin{aligned}\vartheta(u_\rho) &= \pi(\rho+1)^2 \cdot \left(\frac{1}{\rho}\sin\frac{\pi\rho}{\rho+1} - \frac{1}{\rho+2}\sin\frac{\pi(\rho+2)}{\rho+1}\right)^{-1} \\ &= \pi\rho\left(1+\frac{\rho}{2}\right)\frac{\rho+1}{\sin(\pi/(\rho+1))}.\end{aligned}$$

Elementary calculations show that

$$\frac{\rho+1}{\sin(\pi/(\rho+1))} \geq \frac{3}{\sin(\pi/3)} = 2\sqrt{3} > e, \quad \rho \geq 2.$$

Therefore,

$$\vartheta(u_\rho) > \pi\rho\left(1+\frac{\rho}{2}\right)\cdot e = eP_2(\rho), \quad \rho \geq 2.$$

In view of Remark 3.10 and Theorem 4.1, we conclude that $\vartheta(u_\rho) > K_2(\rho)$ for all $\rho \geq 2$. For the range $1 < \rho < 2$, one can compare this directly with the explicit estimate for $K_2(\rho)$ given in (5.8).

5.4. Comparison for $n = 3$. When $n = 3$, from the recursion formula (5.6) it follows that

$$\begin{aligned}C_\rho^2(\cos\theta) &= -\frac{1}{4\sin\theta}\cdot\frac{d}{d\theta}C_{\rho+1}^1(\cos\theta) \\ &= -\frac{1}{4}\left(\frac{\cos((\rho+2)\theta)}{\sin^2\theta} - \frac{1}{\rho+2}\frac{\sin((\rho+2)\theta)\cos\theta}{\sin^3\theta}\right) \\ &= \frac{1}{4}\left((\rho+3)\frac{\sin((\rho+1)\theta)}{\sin^3\theta} - (\rho+1)\frac{\sin((\rho+3)\theta)}{\sin^3\theta}\right).\end{aligned}$$

Now, evaluating the integral

$$\begin{aligned}&\int_0^{\theta^*} C_\rho^2(\cos\theta)\sin^4\theta\,d\theta \\ &= \frac{\rho+3}{4}\int_0^{\theta^*}\sin((\rho+1)\theta)\sin\theta\,d\theta - \frac{\rho+1}{4}\int_0^{\theta^*}\sin((\rho+3)\theta)\sin\theta\,d\theta \\ &= \frac{1}{8}\left((\rho+3)\frac{\sin(\rho\theta^*)}{\rho} - 2\sin((\rho+2)\theta^*) + (\rho+1)\frac{\sin((\rho+4)\theta^*)}{\rho+4}\right).\end{aligned}$$

Thus, we obtain the estimate

$$\vartheta(u_\rho) = 3\pi\rho\left(1+\frac{\rho}{2}\right)\left(1+\frac{\rho}{4}\right)\frac{(\rho+1)(\rho+3)}{m(\rho,\theta^*)} = 3P_3(\rho)\frac{(\rho+1)(\rho+3)}{m(\rho,\theta^*)},$$

where

$$\begin{aligned}m(\rho,\theta^*) := (\rho+3)(\rho+4)\sin(\rho\theta^*) &- 2\rho(\rho+4)\sin((\rho+2)\theta^*) \\ &+ \rho(\rho+1)\sin((\rho+4)\theta^*).\end{aligned}$$

A rough approximation for θ^* is $\pi/(\rho+2)$ for large enough ρ. Substituting this into $m(\rho,\theta^*)$ yields

$$m(\rho,\theta^*) \approx ((\rho+3)(\rho+4)-\rho(\rho+1))\sin\left(\frac{\rho}{\rho+2}\pi\right) = 6(\rho+2)\sin\left(\frac{\rho}{\rho+2}\pi\right).$$

Hence, we have

$$\vartheta(u_\rho) = 3P_3(\rho)\frac{(\rho+1)(\rho+3)}{m(\rho,\theta^*)} \approx P_3(\rho)\frac{\rho+1}{2\cdot\sin(\rho\pi/(\rho+2))} > e^2P_3(\rho)$$

for sufficiently large ρ. Once again, from Remark 3.10 and Theorem 4.1, we conclude that for ρ large enough, $\vartheta(u_\rho) > K_3(\rho)$. For small values of ρ, one has to explicitly compute $K_3(\rho)$ and make comparisons. However, as n increases, this becomes increasingly laborious.

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