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The one-weight inequality for the $\mathcal{H}$-harmonic Bergman projection

by

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Abstract. Let $n \geqslant 3$ be an integer. For the Bekollé–Bonami weight ω on the real unit ball $\mathbb{B}_n$, we obtain the following sharp one-weight estimate for the $\mathcal{H}$-harmonic Bergman projection: for $1 < p < \infty$ and $-1 < \alpha < \infty$,

$$\|P_\alpha\|_{L^p(\omega\, d\nu_\alpha) \to L^p(\omega\, d\nu_\alpha)} \leqslant C[\omega]_{p,\alpha}^{\max\{1, \frac{1}{p-1}\}},$$

where $[\omega]_{p,\alpha}$ is the Bekollé–Bonami constant. Our proof is inspired by dyadic harmonic analysis, and the key ingredient involves the discretization of the Bergman kernel for the $\mathcal{H}$-harmonic Bergman spaces.

1. Introduction. Let $n \geqslant 3$ be an integer, and let $\mathbb{B}_n = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball of the Euclidean space $\mathbb{R}^n$ where $|x| = (\sum_{i=1}^n |x_i|^2)^{1/2}$ is the standard Euclidean norm. We denote by $\mathrm{Aut}(\mathbb{B}_n)$ the Möbius transformation group on the unit ball $\mathbb{B}_n$. It is also known [19, Chapter 2] that the Möbius transformation group has the following explicit form:

$$\mathrm{Aut}(\mathbb{B}_n) = \{A \circ \varphi_y : A \in O(n),\, y \in \mathbb{B}_n\},$$

where $O(n)$ denotes the orthogonal group in $\mathbb{R}^n$, and

$$\varphi_y(x) = \frac{y|x-y|^2 + (1-|y|^2)(y-x)}{[x,y]^2},$$

with

$$[x,y]^2 = |x-y|^2 + (1-|y|^2)(1-|x|^2), \qquad \forall x, y \in \mathbb{B}_n.$$

Let $C^2(\mathbb{B}_n)$ be the space of complex-valued functions on the real unit ball $\mathbb{B}_n$ whose second-order partial derivatives are continuous on $\mathbb{B}_n$, and

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consider the standard Laplace operator

$$\varDelta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Then the invariant Laplace operator $\varDelta_h$ is defined to be a linear operator on $C^2(\mathbb{B}_n)$ such that for each $f \in C^2(\mathbb{B}_n)$ and $x \in \mathbb{B}_n$,

$$(\varDelta_h f)(x) = \varDelta(f \circ \varphi_x)(0).$$

Moreover, by using the standard Laplace operator $\varDelta$ and the gradient operator ∇, we have, for each $x \in \mathbb{B}_n$,

$$(\varDelta_h f)(x) = (1-|x|^2)^2 \varDelta f(x) + 2(n-2)(1-|x|^2)\langle x, \nabla f(x)\rangle, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^n$. Recall also the gradient operator ∇ is given by

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

A complex-valued function $f \in C^2(\mathbb{B}_n)$ is said to be *$\mathcal{H}$-harmonic* if for each $x \in \mathbb{B}_n$,

$$(\varDelta_h f)(x) = 0.$$

Let $\mathcal{H}(\mathbb{B}_n)$ be the complex linear space of $\mathcal{H}$-harmonic functions on $\mathbb{B}_n$. If $n = 2$, by the equality (1.1), we have

$$(\varDelta_h f)(x) = (1-|x|^2)^2 \varDelta f(x)$$

for each $x \in \mathbb{B}_n$. In this case, $\mathcal{H}(\mathbb{B}_2)$ coincides with the space of harmonic functions on the unit ball in $\mathbb{R}^2$, i.e. the unit disc.

Let ν be the normalized Lebesgue measure on $\mathbb{B}_n$ such that $\nu(\mathbb{B}_n) = 1$. For $-1 < \alpha < \infty$, the standard weighted measure on the unit ball $\mathbb{B}_n$ is

$$d\nu_\alpha(x) = c_\alpha (1-|x|^2)^\alpha \, d\nu(x),$$

where c_α is the normalizing constant such that $\nu_\alpha(\mathbb{B}_n) = 1$.

For $1 \leqslant p < \infty$ and $-1 < \alpha < \infty$, the *$\mathcal{H}$-harmonic Bergman space $\mathcal{B}_\alpha^p$* is the Banach space

$$\mathcal{B}_\alpha^p = \{f \in \mathcal{H}(\mathbb{B}_n) : \|f\|_{L^p(d\nu_\alpha)} < \infty\},$$

where

$$\|f\|_{L^p(d\nu_\alpha)} = \left[\int_{\mathbb{B}_n} |f(x)|^p \, d\nu_\alpha(x)\right]^{1/p}.$$

By the mean value estimate, the point evaluations $f \mapsto f(x)$ at any $x \in \mathbb{B}_n$ are bounded on each $\mathcal{B}_\alpha^2$, $-1 < \alpha < \infty$ [5]. Therefore, the $\mathcal{H}$-harmonic Bergman space $\mathcal{B}_\alpha^2$ is a reproducing kernel Hilbert space. It follows that for any $x \in \mathbb{B}_n$, the reproducing kernel $\mathcal{R}_\alpha(x,\cdot)$ is the unique function in $\mathcal{B}_\alpha^2$

such that for all $f \in \mathcal{B}^2_\alpha$,

$$f(x) = \langle f(\cdot), \mathcal{R}_\alpha(x, \cdot)\rangle_\alpha = \int\limits_{\mathbb{B}_n} \overline{\mathcal{R}_\alpha(x,y)} f(y)\, d\nu_\alpha(y).$$

One can show that $\mathcal{R}_\alpha(\cdot,\cdot)$ is a real-valued function. Hence, for any $f \in \mathcal{B}^2_\alpha$, we have

$$f(x) = \int\limits_{\mathbb{B}_n} \mathcal{R}_\alpha(x,y) f(y)\, d\nu_\alpha(y), \qquad x \in \mathbb{B}_n.$$

For more details on $\mathcal{H}$-harmonic functions and related function spaces on the unit ball, one can consult [5, 8, 12, 19, 20].

Recall that the *$\mathcal{H}$-harmonic Bergman projection* P_α is the orthogonal projection from $L^2(\,d\nu_\alpha)$ onto $\mathcal{B}^2_\alpha$. Using the reproducing kernel property, we have

$$P_\alpha f(x) = \int\limits_{\mathbb{B}_n} \mathcal{R}_\alpha(x,y) f(y)\, d\nu_\alpha(y), \qquad \forall x \in \mathbb{B}_n.$$

By [22, Theorem 1.1], P_α is bounded on $L^p(d\nu_\alpha)$ for $1 < p < \infty$ and $-1 < \alpha < \infty$.

In this paper, a *weight* ω is a non-negative integrable function on the real unit ball $\mathbb{B}_n$. The weighted Lebesgue space $L^p(\omega\, d\nu_\alpha)$ on $\mathbb{B}_n$ is given by

$$L^p(\omega\, d\nu_\alpha) = \{f : \|f\|_{L^p(\omega\, d\nu_\alpha)} < \infty\},$$

where

$$\|f\|_{L^p(\omega\, d\nu_\alpha)} := \Big[\int\limits_{\mathbb{B}_n} |f(x)|^p \omega(x)\, d\nu_\alpha(x)\Big]^{1/p}.$$

The weighted theory of Bergman projection-type operators on the unit ball, namely, the operators whose integral kernels are reproducing kernels for some holomorphic or harmonic function spaces on the unit ball, has attracted much attention in recent years. In particular, Bekollé and Bonami [2, 3, 4], as well as the ingenious work by Pott and Reguera [15], present complete characterizations of the sharp linear norm estimate in the one-weight inequality for the holomorphic Bergman projections on the unit ball. In [16], Rahm, Tchoundja, and Wick extended these sharp one-weight inequalities to the setting of the Berezin transform on the unit ball. Volberg and Wick [23] first applied non-homogeneous harmonic analysis techniques to weighted inequalities for Bergman projections, obtaining deep applications concerning Carleson measures for function spaces on the unit ball. For further developments on weighted inequalities for holomorphic Bergman projections and their applications, we refer the reader to [1, 7, 9, 10, 18].

Motivated by these developments, it is natural to study the following one-weight inequality for the $\mathcal{H}$-harmonic Bergman projection:

PROBLEM. Let $1 < p < \infty$ and $-1 < \alpha < \infty$. Characterize the weights ω for which the $\mathcal{H}$-harmonic Bergman projection P_α is bounded on $L^p(\omega\, d\nu_\alpha)$.

Observe that the $\mathcal{H}$-harmonic Bergman projection on $\mathbb{B}_2$ coincides with the classical harmonic Bergman projection on the unit disc in $\mathbb{R}^2$. Although the results of Bekollé–Bonami and Pott–Reguera were initially established for the holomorphic Bergman projection, their results extend naturally to the harmonic setting. Consequently, the one-weight inequality for the $\mathcal{H}$-harmonic Bergman projection on $\mathbb{B}_2$ is characterized by the Bekollé–Bonami class of weights. In the case $n \geqslant 3$, it is therefore natural to conjecture that the Bekollé–Bonami weights continue to ensure the boundedness of the $\mathcal{H}$-harmonic Bergman projection on $L^p(\omega\, d\nu_\alpha)$. In this note, we shall confirm this folklore conjecture.

For a set E in the unit sphere $\partial\mathbb{B}_n = \{x \in \mathbb{R}^n : |x| = 1\}$, a *Carleson box* $\widehat{E}$ in the unit ball associated with the set E is defined by

$$\widehat{E} := \{z \in \mathbb{B}_n : z/|z| \in E,\, 1 - r < |z| < 1\},$$

where $r = \mathrm{diam}(E)/2$ and $\mathrm{diam}(E) = \max\{|x - y| : x, y \in E\}$.

Let $1 < p < \infty$ and $-1 < \alpha < \infty$. A weight ω on $\mathbb{B}_n$ is said to be a *Bekollé–Bonami $B_{p,\alpha}$ weight* if the Bekollé–Bonami constant $[\omega]_{p,\alpha}$ is finite. Here $[\omega]_{p,\alpha}$ is defined by

$$[\omega]_{p,\alpha} := \sup_{B \subset \partial\mathbb{B}_n} \frac{|\widehat{B}|_{\omega,\alpha}}{|\widehat{B}|_\alpha} \left(\frac{|\widehat{B}|_{\omega^{1-p'},\alpha}}{|\widehat{B}|_\alpha} \right)^{p-1},$$

where $1/p + 1/p' = 1$, $|\widehat{B}|_\alpha = \nu_\alpha(\widehat{B})$, $|\widehat{B}|_{\omega,\alpha} = \int_{\widehat{B}} \omega\, d\nu_\alpha$ and the supremum is taken over all Euclidean balls $B \subset \partial\mathbb{B}_n$.

Our main result is stated as follows

THEOREM 1.1. *Let $1 < p < \infty$ and $-1 < \alpha < \infty$. If ω is a $B_{p,\alpha}$ weight, then the $\mathcal{H}$-harmonic Bergman projection P_α is bounded on $L^p(\omega\, d\nu_\alpha)$. More precisely, there exists a constant $C = C(n, p, \alpha)$ such that*

$$\|P_\alpha\|_{L^p(\omega\, d\nu_\alpha) \to L^p(\omega\, d\nu_\alpha)} \leqslant C[\omega]_{p,\alpha}^{\max\{1, \frac{1}{p-1}\}}.$$

REMARK 1.1. Our strategy of proving Theorem 1.1 is inspired by Pott–Reguera's recent beautiful work on the sharp one-weight estimate for the Bergman projection [15]. However, there is no explicit closed form of the kernel of the $\mathcal{H}$-harmonic Bergman projection. To overcome this difficulty, we construct suitable dyadic positive operators on appropriately chosen Carleson box systems, which allow us to effectively dominate the original operator.

REMARK 1.2. For a general weight ω, one may ask whether the condition $\omega \in B_{p,\alpha}$ is also necessary for the one-weight inequality for the $\mathcal{H}$-harmonic

Bergman projection. A reasonable approach, which we have not yet accomplished, is to obtain sharper estimates of $|\mathcal{R}_\alpha(x_0,y)-\mathcal{R}_\alpha(x,y)|$. Our ongoing obstacle remains the lack of an explicit formula for the reproducing kernel. We would like to thank the anonymous referee for the comments on this question.

The paper is organized as follows. In Section 2, based on certain dyadic Carleson boxes on the unit ball, we provide a discretization of the $\mathcal{H}$-harmonic Bergman projection. The proof of the main result is presented in Section 3, which is divided into two parts. In Section 3.1, we prove the case $p=2$, while in Section 3.2 we deal with the case $p\neq 2$ through Rubio de Francia's extrapolation method. In the final section, by constructing specific examples, we demonstrate that the norm estimate in the one-weight inequality for the $\mathcal{H}$-harmonic Bergman projection is sharp.

2. Carleson systems and the discretization

2.1. The Carleson box on the real unit ball. For two points x and y in $\partial\mathbb{B}_n$, let $\rho(x,y)$ denote the angle between them, defined by

$$\rho(x,y)=\arccos(\langle x,y\rangle).$$

PROPOSITION 2.1. *The function $\rho(x,y)$ defines a metric on $\partial\mathbb{B}_n$ that is equivalent to the standard Euclidean distance on $\partial\mathbb{B}_n$.*

Proof. To verify that $\rho(\cdot,\cdot)$ is a metric on $\partial\mathbb{B}_n$, it suffices to establish the triangle inequality: for any $x,y,z\in\partial\mathbb{B}_n$,

$$\rho(x,y)\leqslant\rho(x,z)+\rho(z,y).$$

By the definition of the function $\rho(\cdot,\cdot)$, the triangle inequality holds if and only if

$$\cos\rho(x,y)\geqslant\cos(\rho(x,z)+\rho(z,y)).$$

Note that $|x|=|y|=|z|=1$, by the Binet–Cauchy identity

$$\langle x,y\rangle=\langle x,z\rangle\langle z,y\rangle+\sum_{1\leqslant i<j\leqslant n}(x_iz_j-x_jz_i)(y_iz_j-y_jz_i),$$

we have

$$\begin{aligned}\cos\rho(x,y)&=\langle x,z\rangle\langle z,y\rangle+\sum_{1\leqslant i<j\leqslant n}(x_iz_j-x_jz_i)(y_iz_j-y_jz_i)\\&=\cos\rho(x,z)\cos\rho(z,y)+\sum_{1\leqslant i<j\leqslant n}(x_iz_j-x_jz_i)(y_iz_j-y_jz_i)\\&\geqslant\cos\rho(x,z)\cos\rho(z,y)\\&\quad-\Big(\sum_{1\leqslant i<j\leqslant n}|x_iz_j-x_jz_i|^2\Big)^{1/2}\Big(\sum_{1\leqslant i<j\leqslant n}|y_iz_j-y_jz_i|^2\Big)^{1/2}.\end{aligned}$$

By using the Binet–Cauchy identity again, we get

$$\sum_{1\leqslant i<j\leqslant n}|x_iz_j-x_jz_i|^2=|x|^2|z|^2-|\langle x,z\rangle|^2=\sin^2\rho(x,z).$$

It follows that

$$\begin{aligned}\cos\rho(x,y)&\geqslant\cos\rho(x,z)\cos\rho(z,y)-\sin\rho(x,z)\sin\rho(z,y)\\&=\cos(\rho(x,z)+\rho(z,y)).\end{aligned}$$

Next, we show that the metric $\rho(\cdot,\cdot)$ on $\partial\mathbb{B}_n$ is equivalent to the Euclidean distance. For any two points $x,y\in\partial\mathbb{B}_n$, we observe that

$$|x-y|^2=|x|^2+|y|^2-2\langle x,y\rangle=2(1-\cos\rho(x,y))=4\left(\sin\frac{\rho(x,y)}{2}\right)^2.$$

Note that for any $0\leqslant\theta\leqslant\pi/2$, we have

$$\frac{2}{\pi}\theta\leqslant\sin\theta\leqslant\theta;$$

and for any $x,y\in\partial\mathbb{B}_n$, we have

$$0\leqslant\rho(x,y)\leqslant\pi.$$

Hence,

$$\frac{2}{\pi}\rho(x,y)\leqslant|x-y|\leqslant\rho(x,y).\ \blacksquare\tag{2.2}$$

Let $x\in\partial\mathbb{B}_n$ and $0<r<1$. We consider a ball $B_\rho(x,r)$ in the unit sphere $\partial\mathbb{B}_n$ that is defined by

$$B_\rho(x,r)=\{y\in\partial\mathbb{B}_n:\rho(x,y)<r\}$$

and its associated Carleson box $\widehat{B}_\rho(x,r)$ in the unit ball,

$$\widehat{B}_\rho(x,r):=\{z\in\mathbb{B}_n:z/|z|\in B_\rho(x,r),\ 1-r<|z|<1\}.$$

By Proposition 2.1, the Bekollé–Bonami constant $[w]_{p,\alpha}$ is equivalent, up to a constant, to the following quantity:

$$\sup_{B_\rho\subset\partial\mathbb{B}_n}\frac{|\widehat{B}_\rho|_{\omega,\alpha}}{|\widehat{B}_\rho|_\alpha}\left(\frac{|\widehat{B}_\rho|_{\omega^{1-p'},\alpha}}{|\widehat{B}_\rho|_\alpha}\right)^{p-1}<\infty,\tag{2.3}$$

where $1/p+1/p'=1$ and the supremum is taken over all balls B_ρ satisfying $B_\rho(x,r)\subset\partial\mathbb{B}_n$.

Lemma 2.2. *Let $n\geqslant 3$ and $0<r<1$. For a ball $B_\rho(x,r)\subset\partial\mathbb{B}_n$, there exist positive constants $C_1=C_1(n,\alpha)$ and $C_2=C_2(n)$ such that*

$$C_1\frac{c_\alpha}{\alpha+1}r^{n+\alpha}(2-r)^{\alpha+1}\leqslant|\widehat{B}_\rho(x,r)|_\alpha\leqslant C_2\frac{c_\alpha}{\alpha+1}r^{n+\alpha}(2-r)^{\alpha+1}.\tag{2.4}$$

Proof. Given a ball $B_\rho(x,r)\subset\partial\mathbb{B}_n$, by the inequality (2.2),

$$B(x,2r/\pi)\subset B_\rho(x,r)\subset B(x,r).$$

Here, $B(x,r) = \{y \in \partial\mathbb{B}_n : |x-y| < r\}$ denotes the standard Euclidean ball centered at x with radius r. Therefore, there exist positive constants $c_1 = c_1(n)$ and $c_2 = c_2(n)$ such that

$$c_1 r^{n-1} \leqslant \int\limits_{B_\rho(x,r)} d\sigma(\xi) \leqslant c_2 r^{n-1},$$

where $d\sigma$ is the normalized sphere area measure. Hence,

$$\begin{aligned} |\widehat{B}_\rho(x,r)|_\alpha &= c_\alpha \int\limits_{\widehat{B}_\rho(x,r)} (1-|z|^2)^\alpha \, d\nu(z) \\ &= n c_\alpha \int\limits_{B_\rho(x,r)} d\sigma(\xi) \int\limits_{1-r}^{1} t^{n-1}(1-t^2)^\alpha \, dt \\ &\leqslant n c_2 c_\alpha r^{n-1} \int\limits_{1-r}^{1} t^{n-1}(1-t^2)^\alpha \, dt. \end{aligned}$$

Since $1-r < t < 1$ and $n \geqslant 3$, letting $C_2 = nc_2$, we have

$$|\widehat{B}_\rho(x,r)|_\alpha \leqslant C_2 c_\alpha r^{n-1} \int\limits_{1-r}^{1} t(1-t^2)^\alpha \, dt = C_2 \frac{c_\alpha}{\alpha+1} r^{n+\alpha}(2-r)^{\alpha+1}.$$

This establishes the upper bound in (2.4).

Next, we find the lower bound. For $0 < r \leqslant 1/2$, we have

$$1/2 \leqslant 1-r < t < 1.$$

Therefore,

$$\begin{aligned} |\widehat{B}_\rho(x,r)|_\alpha &\geqslant n c_1 c_\alpha r^{n-1} \int\limits_{1-r}^{1} t^{n-1}(1-t^2)^\alpha \, dt \\ &\geqslant \frac{nc_1}{2^{n-2}} c_\alpha r^{n-1} \int\limits_{1-r}^{1} t(1-t^2)^\alpha \, dt \\ &\geqslant \frac{nc_1}{2^{n-2}} \frac{c_\alpha}{\alpha+1} r^{n+\alpha}(2-r)^{\alpha+1}. \end{aligned}$$

For $1/2 \leqslant r < 1$, we obtain

$$\begin{aligned} |\widehat{B}_\rho(x,r)|_\alpha &\geqslant n c_1 c_\alpha r^{n-1} \int\limits_{1/2}^{1} t^{n-1}(1-t^2)^\alpha \, dt \\ &\geqslant \frac{nc_1}{2^{n-2}} c_\alpha r^{n-1} \int\limits_{1/2}^{1} t(1-t^2)^\alpha \, dt \\ &= \frac{nc_1}{2^{n-2}} \left(\frac{3}{4}\right)^{\alpha+1} \frac{c_\alpha}{\alpha+1} r^{n-1}. \end{aligned}$$

Since $0 < 2r - r^2 < 1$, we have

$$\begin{aligned} |\widehat{B}_\rho(x,r)|_\alpha &\geqslant \frac{nc_1}{2^{n-2}}\left(\frac{3}{4}\right)^{\alpha+1}\frac{c_\alpha}{\alpha+1}r^{n-1}(2r-r^2)^{\alpha+1} \\ &= \frac{nc_1}{2^{n-2}}\left(\frac{3}{4}\right)^{\alpha+1}\frac{c_\alpha}{\alpha+1}r^{n+\alpha}(2-r)^{\alpha+1}. \end{aligned}$$

Let

$$C_1 = \min\left\{\frac{nc_1}{2^{n-2}}, \frac{nc_1}{2^{n-2}}\left(\frac{3}{4}\right)^{\alpha+1}\right\} = \frac{nc_1}{2^{n-2}}\left(\frac{3}{4}\right)^{\alpha+1}.$$

Then, for any $x \in \partial\mathbb{B}_n$ and $0 < r < 1$,

$$|\widehat{B}_\rho(x,r)|_\alpha \geqslant C_1\frac{c_\alpha}{\alpha+1}r^{n+\alpha}(2-r)^{\alpha+1}.$$

This is the lower bound in (2.4). ∎

2.2. An example of $B_{2,\alpha}$. By using Lemma 2.2, in this subsection we obtain a precise estimate for the Bekollé–Bonami constant $[\omega]_{2,\alpha}$ for

$$\omega(x) = (1-|x|^2)^s, \quad x \in \mathbb{B}_n,$$

where $-1 < \alpha < \infty$, $0 < \delta < 1$ and $s = (\alpha+1)(1-\delta)$. Such examples play an important role in the sharp estimate for the norm of P_α in Section 4.

Indeed, for any $x \in \partial\mathbb{B}_n$ and $0 < r < 1$, we have

$$|\widehat{B}_\rho(x,r)|_{\omega,\alpha} = \frac{c_\alpha}{c_{\alpha+s}}|\widehat{B}_\rho(x,r)|_{\alpha+s} \quad \text{and} \quad |\widehat{B}_\rho(x,r)|_{\omega^{-1},\alpha} = \frac{c_\alpha}{c_{\alpha-s}}|\widehat{B}_\rho(x,r)|_{\alpha-s}.$$

Observe that

$$\alpha+s = (\alpha+1)(2-\delta)-1 > -1 \quad \text{and} \quad \alpha-s = (\alpha+1)\delta-1 > -1,$$

and the upper estimate in (2.4) holds for any $-1 < \alpha < \infty$. We have

$$\begin{aligned} |\widehat{B}_\rho(x,r)|_{\omega,\alpha} &\leqslant C_2\frac{c_\alpha}{s+\alpha+1}r^{n+s+\alpha}(2-r)^{s+\alpha+1}, \\ |\widehat{B}_\rho(x,r)|_{\omega^{-1},\alpha} &\leqslant C_2\frac{c_\alpha}{-s+\alpha+1}r^{n-s+\alpha}(2-r)^{-s+\alpha+1}. \end{aligned}$$

Moreover, by the lower bound in (2.2),

$$|\widehat{B}_\rho(x,r)|_\alpha \geqslant C_1\frac{c_\alpha}{\alpha+1}r^{n+\alpha}(2-r)^{\alpha+1}.$$

Note also that we can choose C_1, C_2 independent of δ. Hence

$$\begin{aligned} \sup_{B_\rho(x,r)\subset\partial\mathbb{B}_n}\frac{|\widehat{B}_\rho(x,r)|_{\omega,\alpha}|\widehat{B}_\rho(x,r)|_{\omega^{-1},\alpha}}{|\widehat{B}_\rho(x,r)|_\alpha^2} &\leqslant \left(\frac{C_2}{C_1}\right)^2\frac{1}{(s+\alpha+1)(-s+\alpha+1)} \\ &\leqslant \left(\frac{C_2}{C_1(\alpha+1)}\right)^2\frac{1}{\delta}. \end{aligned} \tag{2.5}$$

It follows from (2.3) and (2.5) that there is a constant $c_3 = c_3(n, \alpha)$ such that

$$[\omega]_{2,\alpha} \leqslant c_3/\delta. \tag{2.6}$$

2.3. The Carleson box system on the real unit ball. To discretize the $\mathcal{H}$-harmonic Bergman projection P_α, we need dyadic systems on $(\partial\mathbb{B}_n, \rho)$. A collection of sets

$$\mathscr{D} = \{Q_{k,i} \subset \partial\mathbb{B}_n : 1 \leqslant k < \infty,\ 1 \leqslant i \leqslant M(k)\}$$

is called a *dyadic systems* on $(\partial\mathbb{B}_n, \rho)$ if

(i) for each integer $k \geqslant 1$, $\{Q_{k,i}\}_{i=1}^{M(k)}$ is a disjoint covering of $\partial\mathbb{B}_n$, i.e.,
  - $\partial\mathbb{B}_n = \bigcup_{i=1}^{M(k)} Q_{k,i}$,
  - for all $1 \leqslant i \neq j \leqslant M(k), Q_{k,i} \cap Q_{k,j} = \emptyset$.

(ii) if $1 \leqslant k \leqslant l$, then either $Q_{l,j} \subset Q_{k,i}$ or $Q_{k,i} \cap Q_{l,j} = \emptyset$.

Furthermore, we say the dyadic system $\mathscr{D}$ is *associated to a triple of positive constants* $(\eta, \kappa_0, \kappa_1)$ if there exists a point set

$$\mathcal{P} = \{x_{k,i} \in \partial\mathbb{B}_n : 1 \leqslant k < \infty, 1 \leqslant i \leqslant M(k)\}$$

such that

$$B_\rho(x_{k,i}, \kappa_0\eta^k) \subset Q_{k,i} \subset B_\rho(x_{k,i}, \kappa_1\eta^k), \tag{2.7}$$

for each $k \geqslant 1, 1 \leqslant i \leqslant M(k)$.

By Hytönen and Kairema's result [11, Theorem 4.1], we have the following result on the metric space $(\partial\mathbb{B}_n, \rho)$:

Proposition 2.3. *There is a finite collection of dyadic systems* $\{\mathscr{D}_t\}_{t=1}^N$ *associated to the triple* $(1/96, 1/12, 4)$ *on* $(\partial\mathbb{B}_n, \rho)$ *such that for any ball* $B_\rho \subset \partial\mathbb{B}$*, there exists* $1 \leqslant t \leqslant N$ *and a cube* $Q \in \mathscr{D}_t$ *with*

$$B_\rho \subset Q \quad \textit{and} \quad \operatorname{diam}_\rho(Q) \leqslant C_3 \operatorname{diam}_\rho(B_\rho),$$

where $\operatorname{diam}_\rho(Q) = \max\{\rho(x,y) : x, y \in Q\}$ *and the constants* N *and* C_3 *only depend on the dimension* n*.*

Remark 2.1. For the case $n = 2$, the corresponding dyadic systems in Proposition 2.3 are two shifted systems of the standard dyadic system on the unit circle. One can consult Mei's elegant arguments in [14].

For $1 \leqslant t \leqslant N$ and a cube $Q \in \mathscr{D}_t$, the Carleson box associated to Q is defined by

$$\widehat{Q} := \left\{z \in \mathbb{B}_n : z/|z| \in Q, 1 - \tfrac{1}{2}\operatorname{diam}_\rho(Q) < |z| < 1\right\},$$

then the dyadic system $\mathscr{D}_t$ induces a Carleson box system $\mathscr{Q}_t$ in the unit ball by

$$\mathscr{Q}_t = \{\widehat{Q} : Q \in \mathcal{D}_t\}.$$

In addition, for $0 < \varepsilon < 1$, we define the ε-Carleson box associated to Q by

$$\widehat{Q}_\varepsilon = \big\{z \in \mathbb{B}_n : z/|z| \in Q, 1 - \tfrac{1}{2}\varepsilon \operatorname{diam}_\rho(Q) < |z| < 1\big\}.$$

Lemma 2.4. *Let $1 \leqslant t \leqslant N$ and $k \geqslant 1$. If there exist cubes $Q_{k+1,j}, Q_{k,i} \in \mathscr{D}_t$ with $Q_{k+1,j} \subset Q_{k,i}$, then*

$$\widehat{Q}_{k+1,j} \subset (\widehat{Q}_{k,i})_{1/2}.$$

Proof. Fix an integer $1 \leqslant t \leqslant N$, and recall $\mathscr{D}_t$ is a dyadic system in $(\partial\mathbb{B}_n, \rho)$ associated with the triple $(1/96, 1/12, 4)$. For each positive integer $k \geqslant 1$, by Proposition 2.3, we have

$$\frac{1}{12} \cdot \left(\frac{1}{96}\right)^k \leqslant \frac{1}{2} \operatorname{diam}_\rho(Q_{k,i}) \leqslant 4 \cdot \left(\frac{1}{96}\right)^k.$$

Consequently,

$$\widehat{Q}_{k+1,j} \subset \left\{z \in \mathbb{B}_n : \frac{z}{|z|} \in Q_{k,i},\ 1 - 4 \cdot \left(\frac{1}{96}\right)^{k+1} < |z| < 1\right\}.$$

Noting that

$$\left\{z \in \mathbb{B}_n : \frac{z}{|z|} \in Q_{k,i},\ 1 - \frac{1}{24} \cdot \left(\frac{1}{96}\right)^k < |z| < 1\right\} \subset (\widehat{Q}_{k,i})_{1/2},$$

we conclude that

$$\widehat{Q}_{k+1,j} \subset (\widehat{Q}_{k,i})_{1/2},$$

which completes the proof. ∎

Lemma 2.5. *Let $-1 < \alpha < \infty$. For an integer $1 \leqslant t \leqslant N$ and a cube $Q \in \mathscr{D}_t$, there exists a constant $C_4 = C_4(n, \alpha)$ such that*

$$|\widehat{Q}|_\alpha \leqslant C_4 |\widehat{Q} \setminus \widehat{Q}_{1/2}|_\alpha.$$

Proof. Pick a cube $Q \in \mathscr{D}_t$. We shall find a positive constant C_4 that only depends on n and α such that

$$\frac{|\widehat{Q} \setminus \widehat{Q}_{1/2}|_\alpha}{|\widehat{Q}|_\alpha} \geqslant \frac{1}{C_4} > 0.$$

Let $r = \operatorname{diam}_\rho(Q)/2$ and $h(t) = t^{n-1}(1 - t^2)^\alpha$ on $[0, 1]$. We have

$$\frac{|\widehat{Q} \setminus \widehat{Q}_{1/2}|_\alpha}{|\widehat{Q}|_\alpha} = \frac{n c_\alpha \int_Q d\sigma(\xi) \int_{1-r}^{1-r/2} t^{n-1}(1 - t^2)^\alpha \, dt}{n c_\alpha \int_Q d\sigma(\xi) \int_{1-r}^{1} t^{n-1}(1 - t^2)^\alpha \, dt} = \frac{\int_{1-r}^{1-r/2} h(t)\, dt}{\int_{1-r}^{1} h(t)\, dt}.$$

For $0 < r \leqslant 1$, set

$$g(r) = \frac{\int_{1-r}^{1-r/2} h(t)\, dt}{\int_{1-r}^{1} h(t)\, dt}.$$

Observe that

$$\lim_{r\to 0} g(r) = \lim_{r\to 0} \frac{-\frac{1}{2}h\big(1-\frac{r}{2}\big)+h(1-r)}{h(1-r)}$$

$$= 1 - \lim_{r\to 0} \frac{\big(1-\frac{r}{2}\big)^{n-1}\big(r-\frac{r^2}{4}\big)^{\alpha}}{2(1-r)^{n-1}(2r-r^2)^{\alpha}} = 1-\frac{1}{2}\cdot\bigg(\frac{3}{4}\bigg)^{\alpha} > 0,$$

and

$$g(1) = \frac{\int_0^{1/2} h(t)\,dt}{\int_0^1 h(t)\,dt} > 0.$$

Let $g(0) = \lim_{r\to 0} g(r)$. Then $g(r)$ is continuous on $[0,1]$. Hence, there exists $0 < r_0 < 1$ such that

$$g(r_0) = \min_{0\leqslant r\leqslant 1} g(r).$$

We claim that $g(r_0) > 0$. Otherwise, if $g(r_0) = 0$, then

$$g(r_0) = \frac{\int_{1-r_0}^{1-r_0/2} h(t)\,dt}{\int_{1-r_0}^{1} h(t)\,dt} = 0.$$

Since the function h is positive and integrable on $(0,1)$, we have

$$\int\limits_{1-r_0}^{1-r_0/2} h(t)\,dt = 0.$$

On the other hand, note that

$$h(t) > 0, \quad \forall t \in [1-r_0, 1-r_0/2],$$

yields a contradiction and we get

$$g(r_0) = \min_{0\leqslant r\leqslant 1} g(r) > 0.$$

Letting

$$C_4 = \frac{1}{\min_{0\leqslant r\leqslant 1} g(r)},$$

we get the desired lower bound. ■

Let $1 < p < \infty$ and $-1 < \alpha < \infty$. Let $\mathscr{D}_t$, $t = 1,\dots,N$, be the Carleson dyadic system of Proposition 2.3. A weight ω is said to be a *Bekollé–Bonami weight with respect to the Carleson dyadic system* $\mathscr{D}_t$ if

$$[\omega]_{p,\alpha,\mathscr{D}_t} := \sup_{Q\in\mathscr{D}_t} \frac{|\widehat{Q}|_{\omega,\alpha}}{|\widehat{Q}|_{\alpha}} \bigg(\frac{|\widehat{Q}|_{\omega^{1-p'},\alpha}}{|\widehat{Q}|_{\alpha}}\bigg)^{p-1} < \infty.$$

Definition 2.1. A weight ω on $\mathbb{B}_n$ is called a *Bekollé–Bonami* $B_{p,\alpha,\mathscr{D}}$ *weight associated* $\mathscr{D} = \bigcup_{t=1}^{N} \mathscr{D}_t$ if

$$[\omega]_{p,\alpha,\mathscr{D}} := \max_{t\colon 1\leqslant t\leqslant N} [\omega]_{p,\alpha,\mathscr{D}_t} < \infty.$$

Lemma 2.6. *Let ω be a weight on the unit ball $\mathbb{B}_n$ for $n \geq 2$. Then ω is a $B_{p,\alpha}$ weight if and only if ω is a $B_{p,\alpha,\mathscr{D}}$ weight.*

Proof. Suppose that ω is a $B_{p,\alpha}$ weight. Then $[\omega]_{p,\alpha} < \infty$. For each $1 \leqslant t \leqslant N$, let

$$\mathcal{P}_t = \{x_{k,i} \in \partial\mathbb{B}_n : 1 \leqslant k < \infty,\, 1 \leqslant i \leqslant M(k)\}$$

be the point set associated to the dyadic system $\mathscr{D}_t$ in Proposition 2.3. By (2.7), for any $Q \in \mathscr{D}_t$, there exist an integer $k \geqslant 1$ and a point $x \in \mathcal{P}_t$ such that

$$B_\rho\Big(x, \frac{1}{12 \cdot 96^k}\Big) \subset Q \subset B_\rho\Big(x, \frac{4}{96^k}\Big).$$

Denote $B_1 := B_\rho(x, 1/(12 \cdot 96^k))$ and $B_2 := B_\rho(x, 4/(96^k))$. Then

$$\frac{|\widehat{Q}|_{\omega,\alpha}}{|\widehat{Q}|_\alpha}\bigg(\frac{|\widehat{Q}|_{\omega^{1-p'},\alpha}}{|\widehat{Q}|_\alpha}\bigg)^{p-1} \leqslant \frac{|\widehat{B}_2|_{\omega,\alpha}|\widehat{B}_2|^{p-1}_{\omega^{1-p'},\alpha}}{|\widehat{B}_1|^p_\alpha}.$$

Hence, by (2.3) and Lemma 2.2, there is a constant $c_4 = c_4(n,p,\alpha) > 0$ such that

$$\frac{|\widehat{Q}|_{\omega,\alpha}}{|\widehat{Q}|_\alpha}\bigg(\frac{|\widehat{Q}|_{\omega^{1-p'},\alpha}}{|\widehat{Q}|_\alpha}\bigg)^{p-1} \leqslant c_4\frac{|\widehat{B}_2|_{\omega,\alpha}|\widehat{B}_2|^{p-1}_{\omega^{1-p'},\alpha}}{|\widehat{B}_2|^p_\alpha} \leqslant c_4[w]_{p,\alpha}.$$

Consequently,

$$[\omega]_{p,\alpha,\mathscr{D}} \leqslant c_4[\omega]_{p,\alpha} < \infty. \tag{2.8}$$

Conversly, assume that $[\omega]_{p,\alpha,\mathscr{D}} < \infty$. For each ball $B_\rho \subset \partial\mathbb{B}_n$, by Proposition 2.3, there exist an integer $1 \leqslant t \leqslant N$ and a cube $Q \in \mathscr{D}_t$ such that $B_\rho \subset Q$ and $\operatorname{diam}_\rho(Q) \leqslant C_3 \operatorname{diam}_\rho(B_\rho)$. Then by Lemma 2.2, there is a constant $c_5 = c_5(n,p,\alpha) > 0$ such that

$$\frac{|\widehat{B}_\rho|_{\omega,\alpha}}{|\widehat{B}_\rho|_\alpha}\bigg(\frac{|\widehat{B}_\rho|_{\omega^{1-p'},\alpha}}{|\widehat{B}_\rho|_\alpha}\bigg)^{p-1} \leqslant \frac{|\widehat{Q}|_{\omega,\alpha}|\widehat{Q}|^{p-1}_{\omega^{1-p'},\alpha}}{|\widehat{B}_\rho|^p_\alpha} \leqslant c_5\frac{|\widehat{Q}|_{\omega,\alpha}|\widehat{Q}|^{p-1}_{\omega^{1-p'},\alpha}}{|\widehat{Q}|^p_\alpha}.$$

Hence, by (2.3),

$$[\omega]_{p,\alpha} \leqslant c_5[\omega]_{p,\alpha,\mathscr{D}} < \infty. \blacksquare$$

2.4. The discretization of P_α. Let N be the integer of Proposition 2.3. For each integer $1 \leqslant t \leqslant N$, consider the following positive integral kernel associated to the dyadic system $\mathscr{D}_t$ on $(\partial\mathbb{B}_n, \rho)$:

$$\mathcal{K}^t_\alpha(x,y) = \sum_{Q\in\mathscr{D}_t}\frac{\chi_{\widehat{Q}}(x)\chi_{\widehat{Q}}(y)}{|\widehat{Q}|_\alpha}, \quad \forall x, y \in \mathbb{B}_n, \tag{2.9}$$

where $\chi_{\widehat{Q}}$ is the characteristic function of $\widehat{Q}$ and $|\widehat{Q}|_\alpha = \nu_\alpha(\widehat{Q})$.

Recall that $\mathcal{R}_\alpha$ denotes the integral kernel of P_α. The following lemmas illustrate the relationship between $\mathcal{K}^t_\alpha(x,y)$ and $\mathcal{R}_\alpha(x,y)$.

LEMMA 2.7. *There is a constant $C_5 > 0$ such that for any distinct points $x, y \in \mathbb{B}_n$, there exists a ball $B_\rho(z,r) \subset \partial\mathbb{B}_n$ such that $\widehat{B}_\rho(z,r)$ contains x, y and $[x,y] \geqslant C_5 r$.*

Proof. Pick $x, y \in \mathbb{B}_n$ with $\rho(x,y) = \theta$. Without loss of generality, we assume $|x| \geqslant |y|$. Using the definition of $[x,y]$, we have

$$\begin{aligned}[x,y]^2 &= |x-y|^2 + (1-|x|^2)(1-|y|^2)\\ &= 1 - 2\langle x,y\rangle + |x|^2|y|^2\\ &= \big|1-|x|\,|y|\big|^2 + 2|x|\,|y|(1-\cos\theta)\\ &\geqslant \big|1-|y|\big|^2 + 4|x|\,|y|\sin^2\frac{\theta}{2}\\ &\geqslant \big|1-|y|\big|^2 + \frac{4}{\pi^2}|y|^2\theta^2.\end{aligned}$$

Let $C_5 = 1/\pi$ and

$$r = \max\{1-|y|, \theta\}, \qquad z = x/|x| \in \partial\mathbb{B}_n.$$

We have $x, y \in \widehat{B}_\rho(z,r)$ and $[x,y] \geqslant C_5 r$. ■

LEMMA 2.8. *There exists a constant $C_6 = C_6(n,\alpha) > 0$ such that for any $x, y \in \mathbb{B}_n$, we have*

$$|\mathcal{R}_\alpha(x,y)| \leqslant C_6 \sum_{t=1}^{N} \mathcal{K}_\alpha^t(x,y).$$

Proof. For any $x, y \in \mathbb{B}_n$, by Lemma 2.7 and Proposition 2.3, there exists a ball $B_\rho(z,r) \subset \partial\mathbb{B}_n$, an integer $1 \leqslant t \leqslant N$ and a cube $Q \in \mathscr{D}_t$ such that

$$x, y \in \widehat{B}_\rho(z,r) \subset \widehat{Q}, \qquad \operatorname{diam}_\rho(Q) \leqslant 2C_3 r \leqslant \frac{2C_3}{C_5}[x,y].$$

By Lemma 2.2, there exists a constant $c_6 = c_6(n,\alpha) > 0$ such that

$$|\widehat{Q}|_\alpha \leqslant c_6 [x,y]^{n+\alpha}. \tag{2.10}$$

By [22, Theorem 1.2], the $\mathcal{H}$-harmonic Bergman kernel satisfies

$$|\mathcal{R}_\alpha(x,y)| \leqslant \frac{c_7}{[x,y]^{n+\alpha}}, \qquad \forall x, y \in \mathbb{B}_n,$$

where the constant $c_7 > 0$ depends on n and α. Then for any distinct $x, y \in \mathbb{B}_n$, by (2.10),

$$\begin{aligned}|\mathcal{R}_\alpha(x,y)| &\leqslant \frac{c_7}{[x,y]^{n+\alpha}} \leqslant c_6 c_7 \frac{\chi_{\widehat{Q}}(x)\chi_{\widehat{Q}}(y)}{|\widehat{Q}|_\alpha}\\ &\leqslant c_6 c_7 \sum_{Q\in\mathscr{D}_t} \frac{\chi_{\widehat{Q}}(x)\chi_{\widehat{Q}}(y)}{|\widehat{Q}|_\alpha} \leqslant c_6 c_7 \sum_{t=1}^{N} \mathcal{K}_\alpha^t(x,y).\end{aligned}$$

Letting $C_6 = c_6 c_7$ completes the proof. ■

3. Proof of the main theorem

3.1. The case $p=2$. The main result in this section is the following theorem:

THEOREM 3.1. *Let $-1<\alpha<\infty$. If ω is a $B_{2,\alpha}$ weight, then*

$$P_\alpha : L^2(\omega\, d\nu_\alpha) \to L^2(\omega\, d\nu_\alpha)$$

is bounded. Moreover, there exists a constant $C_7 = C_7(n,\alpha) > 0$ such that

$$\|P_\alpha\|_{L^2(\omega\, d\nu_\alpha)\to L^2(\omega\, d\nu_\alpha)} \leqslant C_7[\omega]_{2,\alpha}.$$

Proof. Assume that ω is a $B_{2,\alpha}$ weight. Let $\{\mathscr{D}_t\}_{t=1}^N$ be the collection of dyadic systems obtained in Proposition 2.3. By (2.8),

$$[\omega]_{2,\alpha,\mathscr{D}} \leqslant c_4[\omega]_{2,\alpha} < \infty.$$

Using Sawyer's powerful duality trick [17], we find that

$$P_\alpha : L^2(\omega\, d\nu_\alpha) \to L^2(\omega\, d\nu_\alpha)$$

is bounded if and only if so is

$$P_{\omega^{-1},\alpha} : L^2(\omega^{-1}\, d\nu_\alpha) \to L^2(\omega\, d\nu_\alpha),$$

where

$$P_{\omega^{-1},\alpha} f(x) = \int\limits_{\mathbb{B}_n} f(y)\mathcal{R}_\alpha(x,y)\omega^{-1}(y)\, d\nu_\alpha(y)$$

for $f \in L^2(\omega^{-1}\, d\nu_\alpha)$. Moreover,

$$\|P_\alpha\|_{L^2(\omega\, d\nu_\alpha)\to L^2(\omega\, d\nu_\alpha)} = \|P_{\omega^{-1},\alpha}\|_{L^2(\omega^{-1}\, d\nu_\alpha)\to L^2(\omega\, d\nu_\alpha)}.$$

For $1 \leqslant t \leqslant N$, we define the integral operator on the unit ball $\mathbb{B}_n$ induced by the discrete kernel (2.9),

$$\begin{aligned} T^t_{\omega,\alpha} f(x) &:= \int\limits_{\mathbb{B}_n} f(y)\mathcal{K}^t_\alpha(x,y)\omega(y)\, d\nu_\alpha(y) \\ &= \sum_{Q\in\mathscr{D}_t} \frac{1}{|\widehat{Q}|_\alpha} \int\limits_{\widehat{Q}} f(y)\omega(y)\, d\nu_\alpha(y)\, \chi_{\widehat{Q}}(x). \end{aligned}$$

By Lemma 2.8, we have

$$\begin{aligned} |P_{\omega^{-1},\alpha} f(x)| &\leqslant \int\limits_{\mathbb{B}_n} |f(y)|\, |\mathcal{R}_\alpha(x,y)|\omega^{-1}(y)\, d\nu_\alpha(y) \\ &\leqslant C_6 \sum_{t=1}^N \int\limits_{\mathbb{B}_n} |f(y)|\mathcal{K}^t_\alpha(x,y)\omega^{-1}(y)\, d\nu_\alpha(y) \\ &= C_6 \sum_{t=1}^N (T^t_{\omega^{-1},\alpha}|f|)(x). \end{aligned}$$

Hence, it suffices to prove that there exists a constant $c_8 > 0$ such that

$$\|T^t_{\omega^{-1},\alpha} f\|_{L^2(\omega\, d\nu_\alpha)} \leqslant c_8 \|f\|_{L^2(\omega^{-1}\, d\nu_\alpha)}$$

for all $0 \leqslant f \in L^2(\omega^{-1}\, d\nu_\alpha)$ and each $1 \leqslant t \leqslant N$.

Note that for a fixed dyadic system $\mathscr{D}_t$, $T^t_{\omega^{-1},\alpha} f \geqslant 0$ when $f \geqslant 0$. Consequently,

$$\begin{aligned}\|T^t_{\omega^{-1},\alpha} f\|_{L^2(\omega\, d\nu_\alpha)} \\ &= \sup_{0\leqslant g \in L^2(\omega\, d\nu_\alpha), \|g\|_{L^2(\omega\, d\nu_\alpha)}=1} \int_{\mathbb{B}_n} (T^t_{\omega^{-1},\alpha} f)(x) g(x) \omega(x)\, d\nu_\alpha(x).\end{aligned}$$

Now we consider the right side of the last equality. For $0 \leqslant g \in L^2(\omega\, d\nu_\alpha)$, we have

$$\begin{aligned}&\int_{\mathbb{B}_n} (T^t_{\omega^{-1},\alpha} f)(x) g(x) \omega(x)\, d\nu_\alpha(x) \\ &\quad= \sum_{Q\in\mathscr{D}_t} \frac{1}{|\widehat{Q}|_\alpha} \Big(\int_{\widehat{Q}} f(x)\omega^{-1}(x)\, d\nu_\alpha(x)\Big)\Big(\int_{\widehat{Q}} g(x)\omega(x)\, d\nu_\alpha(x)\Big) \\ &\quad= \sum_{Q\in\mathscr{D}_t} \frac{|\widehat{Q}|_\alpha}{|\widehat{Q}|_{\omega^{-1},\alpha}|\widehat{Q}|_{\omega,\alpha}} \frac{|\widehat{Q}|_{\omega^{-1},\alpha}|\widehat{Q}|_{\omega,\alpha}}{|\widehat{Q}|^2_\alpha} \Big(\int_{\widehat{Q}} f(x)\omega^{-1}(x)\, d\nu_\alpha(x)\Big) \\ &\qquad\qquad \times \Big(\int_{\widehat{Q}} g(x)\omega(x)\, d\nu_\alpha(x)\Big) \\ &\quad\leqslant [\omega]_{2,\alpha,\mathscr{D}} \sum_{Q\in\mathscr{D}_t} |\widehat{Q}|_\alpha \bigg(\frac{1}{|\widehat{Q}|_{\omega^{-1},\alpha}} \int_{\widehat{Q}} f(x)\omega^{-1}(x)\, d\nu_\alpha(x)\bigg) \\ &\qquad\qquad \times \bigg(\frac{1}{|\widehat{Q}|_{\omega,\alpha}} \int_{\widehat{Q}} g(x)\omega(x)\, d\nu_\alpha(x)\bigg).\end{aligned}$$

By Lemma 2.5, we have

$$|\widehat{Q}|_\alpha \leqslant C_4 |\widehat{Q} \setminus \widehat{Q}_{1/2}|_\alpha.$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{B}_n} (T^t_{\omega^{-1},\alpha} f)(x) g(x) \omega(x)\, d\nu_\alpha(x) \\ &\quad\leqslant C_4[\omega]_{2,\alpha,\mathscr{D}} \sum_{Q\in\mathscr{D}_t} |\widehat{Q} \setminus \widehat{Q}_{1/2}|_\alpha \bigg(\frac{1}{|\widehat{Q}|_{\omega^{-1},\alpha}} \int_{\widehat{Q}} f(x)\omega^{-1}(x)\, d\nu_\alpha(x)\bigg) \\ &\qquad\qquad \times \bigg(\frac{1}{|\widehat{Q}|_{\omega,\alpha}} \int_{\widehat{Q}} g(x)\omega(x)\, d\nu_\alpha(x)\bigg) \\ &\quad= C_4[\omega]_{2,\alpha,\mathscr{D}} \sum_{Q\in\mathscr{D}_t} \int_{\widehat{Q}\setminus\widehat{Q}_{1/2}} \bigg(\frac{1}{|\widehat{Q}|_{\omega^{-1},\alpha}} \int_{\widehat{Q}} f(x)\omega^{-1}(x)\, d\nu_\alpha(x)\bigg) \\ &\qquad\qquad \times \bigg(\frac{1}{|\widehat{Q}|_{\omega,\alpha}} \int_{\widehat{Q}} g(x)\omega(x)\, d\nu_\alpha(x)\bigg)\, d\nu_\alpha.\end{aligned}$$

Note that Lemma 2.4 implies $\{\widehat{Q}\setminus\widehat{Q}_{1/2}\}_{Q\in\mathscr{D}_t}$ are disjoint. Consequently,

$$\begin{aligned}
&\int_{\mathbb{B}_n}(T^t_{\omega^{-1},\alpha}f)(x)g(x)\omega(x)\,d\nu_\alpha(x)\\
&\quad\leqslant C_4[\omega]_{2,\alpha,\mathscr{D}}\int_{\mathbb{B}_n}\bigg(\frac{1}{|\widehat{Q}|_{\omega^{-1},\alpha}}\int_{\widehat{Q}}f(x)\omega^{-1}(x)\,d\nu_\alpha(x)\bigg)\\
&\qquad\qquad\times\bigg(\frac{1}{|\widehat{Q}|_{\omega,\alpha}}\int_{\widehat{Q}}g(x)\omega(x)\,d\nu_\alpha(x)\bigg)\,d\nu_\alpha\\
&\quad\leqslant C_4[\omega]_{2,\alpha,\mathscr{D}}\int_{\mathbb{B}_n}(M^t_{\omega^{-1},\alpha}f)(M^t_{\omega,\alpha}g)\omega^{-1/2}\omega^{1/2}\,d\nu_\alpha\\
&\quad\leqslant C_4[\omega]_{2,\alpha,\mathscr{D}}\|M^t_{\omega^{-1},\alpha}f\|_{L^2(\omega^{-1}\,d\nu_\alpha)}\|M^t_{\omega,\alpha}g\|_{L^2(\omega\,d\nu_\alpha)}.
\end{aligned}$$

Here $M^t_{\omega,\alpha}$ is the maximal operator associated to the dyadic system $\mathscr{D}_t$ with respect to the measure $\omega\,d\nu_\alpha$, i.e.,

$$M^t_{\omega,\alpha}f(x)=\sup_{Q\in\mathscr{D}_t}\frac{\chi_{\widehat{Q}}(x)}{|\widehat{Q}|_{\omega,\alpha}}\int_{\widehat{Q}}|f|\omega\,d\nu_\alpha.$$

Recall that $\mathscr{D}_t$ is a dyadic system; by the classical result [21, Theorem 5.7] on the maximal operator for a dyadic system, there exists a constant c_9, independent of the weight ω and the dyadic system $\mathscr{D}_t$, such that

$$\|M^t_{\omega,\alpha}f\|_{L^2(\omega\,d\nu_\alpha)}\leqslant c_9\|f\|_{L^2(\omega\,d\nu_\alpha)},\quad\forall f\in L^2(\omega\,d\nu_\alpha).$$

We conclude that

$$\int_{\mathbb{B}_n}(T^t_{\omega^{-1},\alpha}f)(x)g(x)\omega(x)\,d\nu_\alpha(x)\leqslant C_4c_9^2[\omega]_{2,\alpha,\mathscr{D}}\|f\|_{L^2(\omega^{-1}\,d\nu_\alpha)}\|g\|_{L^2(\omega\,d\nu_\alpha)}.$$

Letting $C_7=C_6Nc_4C_4c_9^2$ gives

$$\|P_\alpha\|_{L^2(\omega\,d\nu_\alpha)\to L^2(\omega\,d\nu_\alpha)}\leqslant C_7[\omega]_{2,\alpha}.\ \blacksquare$$

3.2. The extrapolation method for $p\neq 2$. The main goal of this subsection is to obtain a weighted L^p-estimate for P_α. First of all, we shall adapt Rubio de Francia's extrapolation method to prove the case $p>2$ [6]. In addition, the case $1<p<2$ follows from the following simple duality argument.

LEMMA 3.2. *Let $1<p,p'<\infty$ with $1/p+1/p'=1$. Then*

(i) $\|P_\alpha\|_{L^p(\omega\,d\nu_\alpha)\to L^p(\omega\,d\nu_\alpha)}=\|P_\alpha\|_{L^{p'}(\omega^{1-p'}\,d\nu_\alpha)\to L^{p'}(\omega^{1-p'}\,d\nu_\alpha)}$;

(ii) $[\omega]_{p,\alpha}^{\frac{1}{p-1}}=[\omega^{1-p'}]_{p',\alpha}$.

The remaining part of Theorem 1.1 (the case $p>2$) follows from the following general known result on the Bekollé–Bonami weight ω on the real unit ball $\mathbb{B}_n$. For completeness and the reader's convenience, we include a proof here.

LEMMA 3.3. *Let R be a bounded operator on $L^2(\omega\,d\nu_\alpha)$. If there exists a constant C_8 independent of the weight ω such that*

$$\|R\|_{L^2(\omega\,d\nu_\alpha)\to L^2(\omega\,d\nu_\alpha)} \leqslant C_8[\omega]_{2,\alpha} \quad \text{for all } \omega \in B_{2,\alpha},$$

then there exists a constant C_9 independent of ω such that

$$\|R\|_{L^p(\omega\,d\nu_\alpha)\to L^p(\omega\,d\nu_\alpha)} \leqslant C_9[\omega]_{p,\alpha} \quad \text{for all } \omega \in B_{p,\alpha} \text{ and } p > 2.$$

Proof. For $p > 2$, let

$$\phi(p) = \frac{p-2}{p-1}.$$

Then $p'/\phi(p) = p/(p-2)$ is the dual number of $p/2$. By the standard Rubio de Francia's extrapolation, we only need to find an operator

$$D : L^{p'/\phi(p)}(\omega\,d\nu_\alpha) \to L^{p'/\phi(p)}(\omega\,d\nu_\alpha)$$

such that for all

$$0 \leqslant h \in L^{p'/\phi(p)}(\omega\,d\nu_\alpha) \quad \text{with} \quad \|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)} = 1,$$

D has the following properties:

(I) $h \leqslant D(h)$;
(II) $\|D(h)\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)} \leqslant 2$;
(III) $[D(h)\omega]_{2,\alpha} \leqslant C_{10}[\omega]_{p,\alpha}$, where the constant C_{10} is independent of ω and h.

Suppose that such an operator D exists. Observe that for all $f \in L^p(\omega\,d\nu_\alpha)$, we have

$$\|Rf\|^2_{L^p(\omega\,d\nu_\alpha)} = \sup_{\|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)}=1,\, h\geqslant 0} \int_{\mathbb{B}_n} |Rf|^2 h\omega\,d\nu_\alpha.$$

Then

$$\begin{aligned}
\|Rf\|^2_{L^p(\omega\,d\nu_\alpha)} &\leqslant \sup_{\|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)}=1,\, h\geqslant 0} \int_{\mathbb{B}_n} |Rf|^2 D(h)\omega\,d\nu_\alpha \\
&\leqslant C_8^2 \sup_{\|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)}=1,\, h\geqslant 0} \Big([D(h)\omega]^2_{2,\alpha} \int_{\mathbb{B}_n} |f|^2 D(h)\omega\,d\nu_\alpha\Big) \\
&\leqslant C_8^2 C_{10}^2 [\omega]^2_{p,\alpha} \sup_{\|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)}=1,\, h\geqslant 0} \big(\|f\|^2_{L^p(\omega\,d\nu_\alpha)} \|D(h)\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)}\big) \\
&\leqslant 2C_8^2 C_{10}^2 [\omega]^2_{p,\alpha} \|f\|^2_{L^p(\omega\,d\nu_\alpha)}.
\end{aligned}$$

This completes the proof up to the existence of an operator D. ∎

Next, we show that the following maximal operator on the real unit ball $\mathbb{B}_n$ satisfies the properties required for the operator D.

Let M_α be the maximal operator with respect to the measure $d\nu_\alpha$, i.e.,

$$M_\alpha f(x) = \sup_{B \subset \partial\mathbb{B}_n} \frac{\chi_{\widehat{B}}(x)}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} |f| \, d\nu_\alpha,$$

where the supremum runs over all balls $B \subset \partial\mathbb{B}_n$.

Lemma 3.4. *For $\omega \in B_{p,\alpha}$, we have*

$$\|M_\alpha\|_{L^p(\omega\, d\nu_\alpha) \to L^p(\omega\, d\nu_\alpha)} \leqslant C_{11}[\omega]_{p,\alpha}^{\frac{1}{p-1}},$$

where C_{11} is a constant dependent on n and α.

One can see Lerner's work [13] for a simple proof. Since $p > 2$ and $[\omega]_{p,\alpha} \geqslant 1$, we have

$$\|M_\alpha\|_{L^p(\omega\, d\nu_\alpha) \to L^p(\omega\, d\nu_\alpha)} \leqslant C_{11}[\omega]_{p,\alpha}^{\frac{1}{p-1}} \leqslant C_{11}[\omega]_{p,\alpha}. \tag{3.11}$$

For $0 \leqslant h \in L^{p'/\phi(p)}(\omega\, d\nu_\alpha)$, denote

$$S_{\omega,\alpha}(h) = \left(\frac{M_\alpha(h^{1/\phi(p)}\omega)}{\omega} \right)^{\phi(p)}.$$

Lemma 3.5. *For all $0 \leqslant h \in L^{p'/\phi(p)}(\omega\, d\nu_\alpha)$ and weight ω, we have*

$$\left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} h\omega \, d\nu_\alpha \right) \left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} (S_{\omega,\alpha}(h)\omega)^{-1} \, d\nu_\alpha \right) \leqslant [\omega]_{p,\alpha}^{\frac{1}{p-1}}.$$

Proof. Observe that for $x \in \widehat{B}$,

$$M_\alpha(h^{1/\phi(p)}\omega)(x) \geqslant \frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} h^{1/\phi(p)}\omega \, d\nu_\alpha.$$

Hence, by Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} h\omega \, d\nu_\alpha \right) \left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} (S_{\omega,\alpha}(h)\omega)^{-1} \, d\nu_\alpha \right) \\
&\qquad = \left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} h\omega \, d\nu_\alpha \right) \left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} (M_\alpha(h^{1/\phi(p)}\omega))^{-\phi(p)} \omega^{1-p'} \, d\nu_\alpha \right) \\
&\qquad \leqslant \left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} h^{1/\phi(p)}\omega \, d\nu_\alpha \right)^{\phi(p)} \left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} \omega \, d\nu_\alpha \right)^{\frac{1}{p-1}} \\
&\qquad \times \left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} h^{1/\phi(p)}\omega \, d\nu_\alpha \right)^{-\phi(p)} \left(\frac{1}{|\widehat{B}|_\alpha} \int\limits_{\widehat{B}} \omega^{1-p'} \, d\nu_\alpha \right) \\
&\qquad \leqslant [\omega]_{p,\alpha}^{\frac{1}{p-1}}. \quad \blacksquare
\end{aligned}$$

Furthermore, by (3.11), we have

$$\begin{aligned}\|S_{\omega,\alpha}(h)\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)} &= \Big(\int\limits_{\mathbb{B}_n} |M_\alpha(h^{1/\phi(p)}\omega)|^{p'}\omega^{1-p'}\,d\nu_\alpha\Big)^{\phi(p)/p'}\\ &\leqslant \big(\|M_\alpha\|_{L^{p'}(\omega^{1-p'}\,d\nu_\alpha)\to L^{p'}(\omega^{1-p'}\,d\nu_\alpha)}\|h^{1/\phi(p)}\omega\|_{L^{p'}(\omega^{1-p'}\,d\nu_\alpha)}\big)^{\phi(p)}\\ &\leqslant \|M_\alpha\|^{\phi(p)}_{L^p(\omega\,d\nu_\alpha)\to L^p(\omega\,d\nu_\alpha)}\|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)}\\ &\leqslant C_{11}^{\phi(p)}[\omega]^{\phi(p)}_{p,\alpha}\|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)}.\end{aligned}$$

Let

$$A = \sup_{\|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)}=1}\|S_{\omega,\alpha}(h)\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)} \leqslant C_{11}^{\phi(p)}[\omega]^{\phi(p)}_{p,\alpha}. \tag{3.12}$$

We define an operator

$$D(h) = \sum_{k=0}^{\infty}\frac{1}{2^k}\,\frac{(S_{\omega,\alpha})^k(h)}{A^k},$$

where $(S_{\omega,\alpha})^k$ is the k-fold composition of $S_{\omega,\alpha}$, and $(S_{\omega,\alpha})^0$ is the identity operator. It is easy to see $h \leqslant D(h)$. Now we prove another result that we need.

LEMMA 3.6. *For all $h \geqslant 0, \|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)} = 1$, we have*

$$[D(h)\omega]_{2,\alpha} \leqslant 2C_{11}^{\phi(p)}[\omega]_{p,\alpha}.$$

Proof. We make two observations:

(1) $\|D(h)\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)} \leqslant 2\|h\|_{L^{p'/\phi(p)}(\omega\,d\nu_\alpha)}$;
(2) $S_{\omega,\alpha}(D(h)) \leqslant 2AD(h)$.

Hence, by Lemma 3.5 and (3.12),

$$\begin{aligned}&\Big(\frac{1}{|\widehat{B}|_\alpha}\int\limits_{\widehat{B}} D(h)\omega\,d\nu_\alpha\Big)\Big(\frac{1}{|\widehat{B}|_\alpha}\int\limits_{\widehat{B}}(D(h)\omega)^{-1}\,d\nu_\alpha\Big)\\ &\qquad\leqslant 2A\Big(\frac{1}{|\widehat{B}|_\alpha}\int\limits_{\widehat{B}} D(h)\omega\,d\nu_\alpha\Big)\Big(\frac{1}{|\widehat{B}|_\alpha}\int\limits_{\widehat{B}}(S_{\omega,\alpha}(D(h))\omega)^{-1}\,d\nu_\alpha\Big)\\ &\qquad\leqslant 2A[\omega]^{\frac{1}{p-1}}_{p,\alpha} \leqslant 2C_{11}^{\phi(p)}[\omega]_{p,\alpha}. \ \blacksquare\end{aligned}$$

4. The sharp linear estimate for P_α. By Theorem 1.1, if ω is a $B_{2,\alpha}$ weight, then there is a constant C such that

$$\|P_\alpha\|_{L^2(\omega\,d\nu_\alpha)\to L^2(\omega\,d\nu_\alpha)} \leqslant C[\omega]_{2,\alpha}.$$

It follows that the norm $\|P_\alpha\|_{L^2(\omega\, d\nu_\alpha)\to L^2(\omega\, d\nu_\alpha)}$ has linear growth with respect to the Bekollé–Bonami constant. In this section, we shall show that this linear growth is sharp.

For $n \geqslant 3$, let $-1 < \alpha < \infty$ and $0 < \delta < 1$. Consider the weight on the unit ball $\mathbb{B}_n$ defined in Section 2.2,

$$\omega(x) = (1 - |x|^2)^s$$

with $s = (\alpha + 1)(1 - \delta)$.

By (2.6), there is a constant $c_3 > 0$ such that the Bekollé–Bonami constant has an upper estimate:

$$[\omega]_{2,\alpha} \leqslant c_3/\delta.$$

Let $r_0 = 1$, $e_1 = (1, 0, \ldots, 0)$ and $B_0 = B_\rho(e_1, r_0)$. We consider a function

$$f(x) = (1 - |x|^2)^{-s}\chi_{\widehat{B}_0}(x).$$

Then Lemma 2.2 yields the norm estimate

$$\|f\|^2_{L^2(\omega\, d\nu_\alpha)} = \frac{c_\alpha}{c_{\alpha-s}}|\widehat{B}_0|_{\alpha-s} \leqslant c_{10}/\delta,$$

where $c_{10} = C_2 c_\alpha/(\alpha + 1)$.

For $\gamma > 0$ and $\xi \in \partial\mathbb{B}_n$, we define the non-tangential approach region with vertex ξ by

$$\Omega_\gamma(\xi) := \{y \in \mathbb{B}_n : 0 < |\langle y, \xi\rangle| < 1,\ |y|^2 - |\langle y, \xi\rangle|^2 < \gamma^2(1 - |\langle y, \xi\rangle|)^2\}.$$

In fact, $\Omega_\gamma(\xi)$ is a truncated circular cone in $\mathbb{B}_n$ with vertex at ξ.

Lemma 4.1 ([22, Proposition 6.2]). *For $\alpha > -1$ and $\gamma < 1/2$, there exists a constant $c_{11} = c_{11}(n, \alpha, \gamma) > 0$ such that for all $y \in \mathbb{B}_n$ and $x \in \Omega_\gamma(y/|y|)$, we have*

$$\mathcal{R}_\alpha(x, y) \geqslant \frac{c_{11}}{(1 - |\langle x, y\rangle|)^{n+\alpha}}.$$

Note that $\sin(\arctan(\gamma)) = \gamma/\sqrt{1 + \gamma^2}$. Then there exists a half-ball

$$B^+_\xi\left(0, \frac{\gamma}{\sqrt{1+\gamma^2}}\right) := \left\{y \in \mathbb{B}_n : |y| < \frac{\gamma}{\sqrt{1+\gamma^2}},\ \operatorname{Re}\langle y, \xi\rangle > 0\right\}$$

contained in $\Omega_\gamma(\xi)$. Let

$$G = \left\{x \in \mathbb{B}_n : \rho(x/|x|, e_1) < \pi/2 - r_0,\ |x| < \gamma/\sqrt{1 + \gamma^2}\right\}.$$

Then

$$G \subset \bigcap_{\xi \in B_0} \Omega_\gamma(\xi).$$

For some fixed $\gamma < 1/2$, by Lemma 4.1, there exists a constant $c_{12} = c_{12}(n, \alpha, \gamma) > 0$ such that for all $x \in G$ and $y \in \widehat{B}_0$,

$$\mathcal{R}_\alpha(x, y) \geqslant c_{12}.$$

Then Lemma 2.2 yields the lower bound, when $x \in G$,

$$\begin{aligned} P_\alpha(f)(x) &= \int_{\mathbb{B}_n} \mathcal{R}_\alpha(x,y) f(y)\, d\nu_\alpha(y) \\ &\geqslant c_{12} \frac{c_\alpha}{c_{\alpha-s}} |\widehat{B}_0|_{\alpha-s} \geqslant \frac{c_{13}}{\delta}. \end{aligned}$$

Here $c_{13} = C_1 c_\alpha c_{12}/(\alpha+1)$.

Therefore, we have

$$\begin{aligned} \|P_\alpha(f)\|^2_{L^2(\omega\, d\nu_\alpha)} &\geqslant \int_G |P_\alpha(f)(x)|^2 \omega(x)\, d\nu_\alpha(x) \\ &\geqslant \frac{c_{13}|G|_\alpha}{\delta^2} \geqslant \frac{c_{13}|G|_\alpha}{c_3 c_{10}} [\omega]^2_{2,\alpha} \|f\|^2_{L^2(\omega dA_\alpha)}. \end{aligned}$$

This completes the proof.

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