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On the growth rate of powers of a strongly Kreiss bounded operator on an L^p-space

by

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Abstract. Let T be a strongly Kreiss bounded linear operator on L^p. We obtain a bound on the rate of growth of the norms of the powers of T. The bound is optimal with respect to the polynomial scale. The proof makes use of Fourier multipliers, in particular of the Littlewood–Paley inequalities on arbitrary intervals as initiated by Rubio de Francia and developed by Kislyakov and Parilov.

1. Introduction. Let T be a bounded linear operator on a Banach space X. We study the growth rate of $\|T^n\|$ when $X = L^p(\mathcal{M}, \Sigma, \mu)$, where $(\mathcal{M}, \Sigma, \mu)$ is a σ-finite measure space and T satisfies the so-called strong Kreiss condition. We shall simply write $L^p(\mathcal{M})$ or even L^p when no confusion is possible.

To put our result into a proper context, let us first review several basic statements on the asymptotics of powers of Kreiss operators.

In [11], Kreiss introduced the following conditions:

$$\sigma(T) \subset \overline{\mathbb{D}} \quad \text{and} \quad \|(\lambda - T)^{-1}\| \le \frac{C}{|\lambda| - 1}, \quad |\lambda| > 1, \tag{1.1}$$

where $\mathbb{D}$ denotes the unit disk and $\sigma(T)$ stands for the spectrum of T. Moreover, Kreiss proved that if X is a *finite-dimensional* Banach space, (1.1) is equivalent to the power-boundedness of T, i.e., $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$.

Later, McCarthy [13] considered the following strengthening of (1.1):

$$\sigma(T) \subset \overline{\mathbb{D}} \quad \text{and} \quad \|(\lambda - T)^{-k}\| \le \frac{C}{(|\lambda| - 1)^k}, \quad |\lambda| > 1,\ k \in \mathbb{N}, \tag{1.2}$$

known as the *strong Kreiss condition* (or *iterated Kreiss condition*). The

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operators satisfying (1.2) will be called *strongly Kreiss bounded*. We denote by C_{SK} the smallest constant $C > 0$ such that (1.2) holds.

Lubich and Nevanlinna [12] proved that (1.1) implies $\|T^N\| = O_{N\to\infty}(N)$ and, by a result of Shields (see [18, Theorem 1 and Proposition 3]), this is optimal.

Moreover, Lubich and Nevanlinna also proved (see [12, Theorem 2.1 and Example 2.2]) that (1.2) implies $\|T^N\| = O_{N\to\infty}(\sqrt{N})$, and that this estimate is the best possible for general Banach spaces. In [16, Proposition 1.1], Nevanlinna proved that an operator T satisfies the strong Kreiss condition if and only if there exists $L > 0$ such that

$$\|\mathrm{e}^{zT}\| \le L\mathrm{e}^{|z|}, \qquad z \in \mathbb{C}, \tag{1.3}$$

and it turns out that the smallest constant $L > 0$ such that (1.3) holds is C_{SK} (see [7, before formula (2)]). When X is a Hilbert space, (1.3) implies $\|T^N\| = O_{N\to\infty}(\log^\kappa N)$ for some $\kappa > 0$ (see [4, Theorem 4.5]). Moreover, by [4, Proposition 4.9], for any $\kappa > 0$ and any $1 \le p < \infty$, the bounded operator T_κ on $\ell^p(\mathbb{N})$ defined by

$$T_\kappa((x_n)_{n\in\mathbb{N}}) = \left(\frac{\log^\kappa(n+2)}{\log^\kappa(n+1)} x_{n+1}\right)_{n\in\mathbb{N}} \tag{1.4}$$

is a strongly Kreiss bounded operator T such that for every $N \in \mathbb{N}$,

$$\|T_\kappa^N\| = \frac{1}{\log^\kappa 2} \log^\kappa(N+2).$$

Norm bounds for powers of operators satisfying the Kreiss condition were also obtained in [4]. In [5], these bounds were extended to L^p-spaces, $1 < p < \infty$ (and more generally, to UMD Banach spaces with nontrivial type and/or cotype; see [5] for the definition and properties of UMD Banach spaces). However, the norm estimates for powers of strongly Kreiss bounded operators on L^p-spaces have not been addressed in [5], and it is the purpose of the present paper to study these estimates thoroughly. The next statement is the main result.

THEOREM 1.1. *Let T be a strongly Kreiss bounded operator on $L^p(\mathcal{M})$, $1 < p < \infty$, and let $\tau_p := |1/2 - 1/p|$. There exist constants $C, \kappa > 0$ depending only on p and C_{SK} such that for every $N \in \mathbb{N}$,*

$$\|T^N\| \le CN^{\tau_p} \log^\kappa(N+1). \tag{1.5}$$

REMARKS. When $p = 2$, we recover the optimal result from [4]. The theorem does not cover the cases $p \in \{1, \infty\}$, but in these cases the bound of Lubich and Nevanlinna is the best possible, as we show in Proposition 1.2 below. Finally, let us notice that the case of UMD Banach spaces has been investigated very recently by Deng, Lorist and Veraar [6]. They prove (see

[6, Corollary 3.2]) that any strongly Kreiss bounded operator T on a UMD Banach space satisfies $\|T^N\| = O_{N\to\infty}(N^\alpha)$ for some $\alpha < 1/2$.

It follows from Proposition 1.2 that the exponent τ_p in (1.5) is the best possible. The proposition is a generalization of an example of Lubich and Nevanlinna [12] who considered the case $p = \infty$.

We write $B(X)$ for the space of bounded operators on X.

PROPOSITION 1.2. *There exists an operator V strongly Kreiss bounded on every $\ell^p(\mathbb{Z})$, $1 \le p \le \infty$, such that*

$$C^{-1}N^{|1/2-1/p|} \le \|V^N\|_{B(\ell^p(\mathbb{Z}))} \le CN^{|1/2-1/p|}, \tag{1.6}$$

for a constant $C \ge 1$ and all $N \in \mathbb{N}$.

While Proposition 1.2 proves that the exponent τ_p in (1.5) is sharp, we do not know whether the extra logarithmic terms are needed in general. The example (1.4) proves that one cannot avoid the logarithmic terms if $p = 2$. Hence it is natural to conjecture that this is also the case for $1 < p < \infty$.

Another question of interest is whether one can find an optimal bound for κ (and the implicit constant in the notation O). For $p = 2$, it is possible to provide an explicit bound on these constants, following carefully the arguments in [4, pp. 13–15], but we do not know whether the κ obtained will be optimal. For $p \ne 2$, the constant κ depends on several Fourier multiplier norms in L^p and is hard to estimate an optimal way. On the other hand, some of these constants tend to ∞ as p is getting close to 1 (or to ∞). For instance, the best possible constant R_p in formula (3.1) below tends to ∞ when p is getting close to 1 (or to ∞).

Let us mention that, when T is a *positive* strongly Kreiss bounded operator on $L^p(\mathcal{M})$ and $p \in [1, 4/3) \cup (4, \infty]$, it is possible to improve (1.5) by replacing $N^{|1/2-1/p|}$ with $N^{1/\bar{p}}$, $\bar{p} = \max(p, p/(p-1))$; see Proposition 5.2 below.

We describe the organization of the paper. In Section 2 we state and prove auxiliary results concerning Fourier multipliers in L^p. Section 3 is dedicated to the proof of Theoremm̃efmain-theorem and in Section 4 we prove Proposition 1.2. In Section 5 we provide improved estimates on the growth rate of $\|T^n\|$ for strongly Kreiss bounded operators under extra assumptions on T. In particular, we consider the case where T is a positive operator on $L^p(\mathcal{M})$, $1 \le p \le \infty$, hence including the cases $p \in \{1, \infty\}$.

2. Auxiliary results. Throughout the paper we will denote by $\mathbb{T} = [-\pi, \pi)$ the torus and by λ the Haar–Lebesgue measure on $\mathbb{T}$. We also set $e_n(t) := \mathrm{e}^{int}$ for all $t \in \mathbb{T}$ and $n \in \mathbb{Z}$.

Given a bounded interval $I \subset \mathbb{Z}$, we define an operator M_I on $L^p(\mathbb{T})$ by setting, for every $f \in L^p(\mathbb{T})$,

$$M_I f(t) := \sum_{i\in I} c_i(f)e_i(t), \quad t \in \mathbb{T},$$

where $\sum_{n\in\mathbb{Z}} c_n(f)e_n$ is the formal Fourier series of f.

Recall the definition of the weak L^1-norm $\|\cdot\|_{L^{1,\infty}(\mathbb{T})}$ on $(\mathbb{T},\lambda)$: for every measurable function g on $\mathbb{T}$, we set

$$\|g\|_{L^{1,\infty}(\mathbb{T})} := \sup_{x>0} x\lambda(|g|\geq x) = \sup_{x>0} x\lambda(\{t\in\mathbb{T} : |g(t)|\geq x\}).$$

Proposition 2.1. *Let $1 < p < \infty$ and $p' = \min(2,p)$. There exists $D_p > 0$ such that for every finite collection $(I_\ell)_{1\leq \ell\leq L}$ of mutually disjoint intervals of integers,*

$$\Big\|\Big(\sum_{\ell=1}^{L} |M_{I_\ell} f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{T})} \leq D_p L^{1/p'-1/2}\|f\|_{L^p(\mathbb{T})}, \qquad f\in L^p(\mathbb{T}). \tag{2.1}$$

Furthermore, there exists $D_1 > 0$ such that for every finite collection $(I_\ell)_{1\leq\ell\leq L}$ of mutually disjoint intervals of integers,

$$\Big\|\Big(\sum_{\ell=1}^{L} |M_{I_\ell} f|^2\Big)^{1/2}\Big\|_{L^{1,\infty}(\mathbb{T})} \leq D_1 L^{1/2}\|f\|_{L^1(\mathbb{T})}, \qquad f\in L^1(\mathbb{T}). \tag{2.2}$$

Proof. For $p \geq 2$, (2.1) is the so-called Rubio de Francia inequality [17]. Actually, Rubio de Francia proved the result on the real line. The result for the torus (with extensions) appears in Kislyakov–Parilov [10].

Inequality (2.2) appears in [10] (take $g_k = f$ in [10, the first inequality in Section 1.4, p. 6419]); see also [8, Exercise 6.4.1(a), p. 337] for a version on the real line.

Then (2.1) for $1 < p < 2$ follows by applying the Marcinkiewicz interpolation theorem [8, Theorem 1.3.2] to the sublinear operator $f \mapsto (\sum_{\ell=1}^{L} |M_{I_\ell} f|^2)^{1/2}$ with $p_0 = 1$ and $p_1 = 2$. ∎

Corollary 2.2. *Let $1 < p < \infty$. Let $p'' = \max(2,p)$ and recall that $p' = \min(2,p)$. For every finite collection $(I_\ell)_{1\leq\ell\leq L}$ of disjoint and consecutive intervals of integers and every $f \in L^p(\mathbb{T})$, setting $I := \bigcup_{\ell=1}^{L} I_\ell$, we have*

$$\begin{aligned}\|M_I f\|_{L^p(\mathbb{T})} &\leq D_q L^{1/2-1/p''}\Big\|\Big(\sum_{\ell=1}^{L} |M_{I_\ell} f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{T})} \\ &\leq D_q L^{1/2-1/p''}\Big(\sum_{\ell=1}^{L} \|M_{I_\ell} f\|_{L^p(\mathbb{T})}^{p'}\Big)^{1/p'},\end{aligned} \tag{2.3}$$

where $q = p/(p-1)$ and D_q is defined in Proposition 2.1.

Proof. Clearly, it is enough to assume that f is a trigonometric polynomial supported in I. Let $g \in L^q(\mathbb{T})$, where $q = p/(p-1)$. Note that $q' = \min(p/(p-1), 2) = p''/(p''-1)$ and that $1/q' - 1/2 = 1/2 - 1/p''$. Using orthogonality and the Cauchy–Schwarz and Hölder inequalities, we see that

$$\begin{aligned}\Big|\int_{\mathbb{T}} f\bar{g}\, d\lambda\Big| &= \Big|\int_{\mathbb{T}} \sum_{\ell=1}^{L} M_{I_\ell} f M_{-I_\ell} \bar{g}\, d\lambda\Big| \\ &\le \Big\|\Big(\sum_{\ell=1}^{L} |M_{I_\ell} f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{T})} \Big\|\Big(\sum_{\ell=1}^{L} |M_{-I_\ell} \bar{g}|^2\Big)^{1/2}\Big\|_{L^q(\mathbb{T})} \\ &\le D_q L^{1/2-1/p''} \Big\|\Big(\sum_{\ell=1}^{L} |M_{I_\ell} f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{T})} \|\bar{g}\|_{L^q(\mathbb{T})},\end{aligned}$$

where we have also used Proposition 2.1. Then (2.3) follows by taking the supremum over $g \in L^q(\mathbb{T})$, with $\|g\|_{L^q(\mathbb{T})} = 1$. The last estimate follows by using the fact that $x \mapsto x^{p/2}$ is subadditive on $\mathbb{R}_+$ when $p \le 2$, and Minkowski's inequality in $L^{p/2}(\mathbb{T})$ when $p > 2$. ∎

Corollary 2.2 implies the following statement. For the remainder of this paper, we will use, without explicit mention, the isomorphism between $L^p(\mathcal{M} \times \mathcal{N})$ and $L^p(\mathcal{M}, L^p(\mathcal{N}))$ for measure spaces $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$.

COROLLARY 2.3. *Let $1 < p < \infty$. There exists $C_p > 0$ such that for every $N \in \mathbb{N}$ and each $(x_1, \dots, x_{N^2}) \in (L^p(\mathcal{M}))^{N^2}$,*

(2.4)
$$\Big\|\sum_{k=1}^{N^2} e_k x_k\Big\|_{L^p(\mathbb{T}, L^p(\mathcal{M}))} \le C_p N^{1/2-1/p''} \Big(\sum_{n=0}^{N-1} \Big\|\sum_{k=n^2+1}^{(n+1)^2} e_k x_k\Big\|_{L^p(\mathbb{T}, L^p(\mathcal{M}))}^{p'}\Big)^{1/p'}.$$

Proof. Let $N \in \mathbb{N}$ and $I_n := [n^2+1, (n+1)^2)$ for $n \in \{0, \cdots, N-1\}$. By Corollary 2.2, for all $(a_1, \dots, a_{N^2}) \in \mathbb{C}^{N^2}$ and $f \in L^p(\mathbb{T})$,

$$\Big\|\sum_{k=1}^{N^2} a_k e_k\Big\|_{L^p(\mathbb{T})} \le D_q N^{1/2-1/p''} \Big(\sum_{n=0}^{N-1} \Big\|\sum_{k=n^2+1}^{(n+1)^2} a_k e_k\Big\|_{L^p(\mathbb{T})}^{p'}\Big)^{1/p'}. \tag{2.5}$$

Assume that $p > 2$ and apply (2.5) with $a_k = x_k(\omega)$, $\omega \in \mathcal{M}$. Then, using Fubini's theorem, we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{N^2} e_k x_k\Big\|_{L^p(\mathbb{T}, L^p(\mathcal{M}))}^p &= \int_{\mathcal{M}} \Big(\int_{\mathbb{T}} \Big|\sum_{k=1}^{N^2} x_k(\omega) e_k\Big|^p d\lambda\Big) d\mu(\omega) \\ &\le (D_q N^{1/2-1/p''})^p \int_{\mathcal{M}} \Big(\sum_{n=0}^{N-1} \Big\|\sum_{k=n^2+1}^{(n+1)^2} x_k(\omega) e_k\Big\|_{L^p(\mathbb{T})}^2\Big)^{p/2} d\mu(\omega).\end{aligned}$$

We conclude by using Minkowski's inequality in $L^{p/2}(\mathcal{M})$ and Fubini's theorem again.

The case where $1 \le p \le 2$ may be handled similarly, using Fubini's theorem only once.

Hence we get the result with $C_p = D_q$. ∎

We conclude this section by an auxiliary statement which will be used frequently. In order to interpret condition (2.7), we first recall the notion of a sequence of bounded variation.

Definition 2.4. We say that a sequence of complex numbers $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ is *of bounded variation* if $v(\mathbf{a}) := \sum_{n\in\mathbb{Z}} |a_{n+1} - a_n| < \infty$. We then call $v(\mathbf{a})$ the *variation* of $\mathbf{a}$.

In the remainder of the paper, $\sum_{\alpha\le n\le\beta}$ for $\alpha, \beta \in \mathbb{R}$ will denote the sum over $n \in \mathbb{Z}$ such that $\alpha \le n \le \beta$.

Lemma 2.5. *There exists $C_{\exp} > 1$ such that for every $N \in \mathbb{N}$ and every integer $K \in [-6\sqrt{N}, 1]$,*

$$\frac{\mathrm{e}^N}{C_{\exp}\sqrt{N}} \le \frac{N^{N+K}}{(N+K)!} \le \frac{C_{\exp}\mathrm{e}^N}{\sqrt{N}}, \tag{2.6}$$

and

$$\sum_{N+2-2\sqrt{N}\le n\le N} \left| \Big(\sum_{n-4\sqrt{N}\le k\le n} \frac{N^k}{k!} \Big)^{-1} - \Big(\sum_{n+1-4\sqrt{N}\le k\le n+1} \frac{N^k}{k!} \Big)^{-1} \right| \le \frac{C_{\exp}}{\mathrm{e}^N}. \tag{2.7}$$

In particular, the sequences

$$\Big\{ \mathrm{e}^N \Big(\sum_{n-4\sqrt{N}\le k\le n} \frac{N^k}{k!} \Big)^{-1} : N+2-2\sqrt{N} \le n \le N \Big\}$$

are of uniformly bounded variation.

Proof. The upper bound of (2.6) follows from [4, Lemma 3.4] and the lower bound may be proved similarly. Then (2.7) follows from the fact that, for $N + 2 - 2\sqrt{N} \le n \le N$, writing $m := \lfloor n + 1 - 4\sqrt{N} \rfloor$, where $\lfloor\cdot\rfloor$ denotes the integer part, we have

$$\left| \Big(\sum_{n-4\sqrt{N}\le k\le n} \frac{N^k}{k!} \Big)^{-1} - \Big(\sum_{n+1-4\sqrt{N}\le k\le n+1} \frac{N^k}{k!} \Big)^{-1} \right| \le \frac{N^m/m! + N^n/(n+1)!}{\big(\sum_{n+1-4\sqrt{N}\le k\le n} N^k/k!\big)^2} \le \frac{\tilde{C}e^{-N}}{\sqrt{N}},$$

for a constant $\tilde{C} > 0$, where we have used (2.6). ∎

3. Proof of Theorem 1.1. The proof of Theorem 1.1 makes use of Fourier multipliers; see [5, pp. 11–12] for a brief description of Fourier multipliers in UMD Banach spaces (in particular in L^p-spaces). Actually, we only make use of *real-valued* Fourier multipliers in our proofs and then use Fubini's theorem to obtain results for L^p-valued Fourier multipliers.

We shall rely on the following well-known results on L^p-multipliers used in a similar context in [5].

LEMMA 3.1. *Let X be a UMD space and $1 < p < \infty$.*

(i) Riesz's theorem: *There exists $R_p > 0$ such that for every interval $I \subset \mathbb{Z}$ and every finitely supported sequence $(c_i)_{i\in\mathbb{Z}} \subset X$ we have*

$$\Big\|\sum_{i\in I} e_i c_i\Big\|_{L^p(\mathbb{T},X)} \leq R_p \Big\|\sum_{i\in\mathbb{Z}} e_i c_i\Big\|_{L^p(\mathbb{T},X)}. \tag{3.1}$$

(ii) Stechkin's theorem: *There exists $S_p > 0$ such that for every sequence of complex numbers $(a_i)_{i\in\mathbb{Z}}$ with bounded variation and every finitely supported sequence $(c_i)_{i\in\mathbb{Z}} \subset X$,*

$$\Big\|\sum_{i\in\mathbb{Z}} a_i e_i c_i\Big\|_{L^p(\mathbb{T},X)} \leq S_p\big(\|(a_i)_{i\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} + v((a_i)_{i\in\mathbb{Z}})\big)\Big\|\sum_{i\in\mathbb{Z}} e_i c_i\Big\|_{L^p(\mathbb{T},X)}.$$

To prove our main result, we need two key lemmas. The first one shows that if one controls the $L^p(\mathbb{T}, L^p(\mathcal{M}))$-norms of sums of the form $\sum_{N+2-2\sqrt{N}\leq n\leq N} e_n T^n x$ with a growth rate of N^α, $\alpha > 0$, then it is possible to deduce a bound for the $L^p(\mathbb{T}, L^p(\mathcal{M}))$-norms of $\sum_{1\leq n\leq N} e_n T^n x$ with a slightly larger growth rate of $N^{\alpha+\delta_p}$. The second one, conversely, provides a bound on the $L^p(\mathbb{T}, L^p(\mathcal{M}))$-norms of sums $\sum_{N+2-2\sqrt{N}\leq n\leq N} e_n T^n x$ with a growth rate of $N^{\beta/2}$, assuming that the $L^p(\mathbb{T}, L^p(\mathcal{M}))$-norms of $\sum_{1\leq n\leq N} e_n T^n x$ are controlled with a growth rate of N^β, $\beta > 0$, and that T is strongly Kreiss bounded on $L^p(\mathcal{M})$.

We now state and prove the first lemma. Recall that $p' = \min(p, 2)$ and $p'' = \max(p, 2)$.

LEMMA 3.2. *Let T be a bounded operator on $L^p(\mathcal{M})$, $1 < p < \infty$. Assume that there exist $D > 0$ and $0 < \alpha \leq 1$ such that for every $N \in \mathbb{N}$ and every $x \in L^p(\mathcal{M})$,*

$$\Big\|\sum_{N+2-2\sqrt{N}\leq n\leq N} e_n T^n x\Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \leq DN^\alpha \|x\|_{L^p(\mathcal{M})}. \tag{3.2}$$

Set $\delta_p := \frac{1}{2}(2/p' - 1/p)$. Then there exists $E_p^{(1)} > 0$ such that for every $N \in \mathbb{N}$ and every $x \in L^p(\mathcal{M})$, we have

$$\Big\|\sum_{1\leq n\leq N} e_n T^n x\Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \leq DE_p^{(1)} N^{\alpha+\delta_p} \|x\|_{L^p(\mathcal{M})}.$$

Proof. By Corollary 2.3 and the assumption, for every $M \in \mathbb{N}$ and every $x \in L^p(\mathcal{M})$,

$$\Big\| \sum_{1\le n\le M^2} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))}^{p'}$$
$$\le (C_p M^{1/2-1/p''})^{p'} \Big(\sum_{n=0}^{M-1} \Big\| \sum_{k=n^2+1}^{(n+1)^2} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))}^{p'} \Big)$$
$$\le (C_p D)^{p'} M^{p'(1/2-1/p'')} \sum_{n=0}^{M-1} (n+1)^{2p'\alpha} \|x\|_{L^p(\mathcal{M})}$$
$$\le (C_p D)^{p'} M^{p'(1/2-1/p'')+2p'\alpha+1} \|x\|_{L^p(\mathcal{M})}.$$

Let $N \in \mathbb{N}$ and $M := \lfloor \sqrt{N} \rfloor + 1$. By the Riesz theorem (Lemma 3.1(i)), using $M^2 \le 4N$ and $M \le N$, we infer that

$$\Big\| \sum_{1\le n\le N} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \le R_p \Big\| \sum_{1\le n\le M^2} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))}$$
$$\le R_p C_p D (4N)^{(1/2-1/p'')/2+\alpha+1/(2p')} \|x\|_{L^p(\mathcal{M})},$$

and the result follows since

$$(1/2 - 1/p'')/2 + \alpha + 1/(2p') = \alpha + (1/2 + 1/p' - 1/p'')/2 = \alpha + \delta_p. \ \blacksquare$$

We now state and prove our second key lemma in a general setting where we assume that the $L^p(\mathbb{T}, L^p(\mathcal{M}))$-norms of $\sum_{n=1}^N e_n T^n x$ are bounded by the growth rate of $f(N)$ where $f : \mathbb{R}_+ \to (1,\infty)$ is a given function.

LEMMA 3.3. *Let T be a strongly Kreiss bounded operator on $L^p(\mathcal{M})$, $1 < p < \infty$. Assume that there exist a function $f : \mathbb{R}_+ \to (1,\infty)$ and $D > 0$ such that for every $N \in \mathbb{N}$ and every $x \in L^p(\mathcal{M})$,*

$$\Big\| \sum_{1\le n\le N} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \le D f(N) \|x\|_{L^p(\mathcal{M})}. \tag{3.3}$$

Then there exists $E_p^{(2)}$ such that for every $N \in \mathbb{N}$ and every $x \in L^p(\mathcal{M})$,

$$\Big\| \sum_{N+2-2\sqrt{N}\le n\le N} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \le D E_p^{(2)} f(4\sqrt{N}) \|x\|_{L^p(\mathcal{M})}.$$

Proof. The argument is similar to [4, proof of Lemma 4.7].

Let $x \in L^p(\mathcal{M})$. First, since T is strongly Kreiss bounded, using assumption (3.3) we have, for all $N \in \mathbb{N}$,

$$\Big\| \mathrm{e}^{e_1 NT} \sum_{n=1}^{4\sqrt{N}} e_n T^n x \Big\|_{L^p(\mathbb{T}, L^p(\mathcal{M}))} \leq C_{\mathrm{SK}} \mathrm{e}^N \Big\| \sum_{n=1}^{4\sqrt{N}} e_n T^n x \Big\|_{L^p(\mathbb{T}, L^p(\mathcal{M}))} \leq C_{\mathrm{SK}} D \mathrm{e}^N f(4\sqrt{N}) \|x\|_{L^p(\mathcal{M})}. \tag{3.4}$$

Recall that C_{SK} denotes the smallest constant $L > 0$ such that (1.3) holds.

Furthermore, for all $N \in \mathbb{N}$,

$$e^{e_1 NT} \sum_{n=1}^{4\sqrt{N}} e_n T^n x = \sum_{1 \leq n \leq 4\sqrt{N}} e_n T^n x \sum_{0 \leq k \leq n} \frac{N^k}{k!} + \sum_{n \geq 4\sqrt{N}+1} e_n T^n x \sum_{n-4\sqrt{N} \leq k \leq n} \frac{N^k}{k!}.$$

For every N large enough (such that $N + 2 - 2\sqrt{N} \geq 4\sqrt{N}$), using the Riesz theorem (Lemma 3.1(i)) we have

$$R_p \Big\| e^{e_1 NT} \sum_{n=1}^{4\sqrt{N}} e_n T^n x \Big\|_{L^p(\mathbb{T}, L^p(\mathcal{M}))} \geq \Big\| \sum_{N+2-2\sqrt{N} \leq n \leq N} e_n T^n x \sum_{n-4\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \Big\|_{L^p(\mathbb{T}, L^p(\mathcal{M}))}. \tag{3.5}$$

Let

$$a_n^N := \Bigg(\sum_{n-4\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \Bigg)^{-1}, \quad N + 2 - 2\sqrt{N} \leq n \leq N,$$

and define $\mathbf{a} := (a_n)_{n \in \mathbb{Z}}$ by

$$a_n := \begin{cases} a_n^N & \text{if } N + 2 - 2\sqrt{N} \leq n \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.5,

$$\|\mathbf{a}\|_{\ell^\infty(\mathbb{Z})} \leq \Bigg(\sum_{N-4\sqrt{N} \leq k \leq N+2-2\sqrt{N}} \frac{e^N}{C_{\exp}\sqrt{N}} \Bigg)^{-1} \leq \frac{C_{\exp}}{e^N} \text{ and } v(\mathbf{a}) \leq \frac{C_{\exp}}{e^N},$$

with $C_{\exp}$ defined as in that lemma. Then, by Stechkin's theorem (see

Lemma 3.1(ii)),

$$
\text{(3.6)}\qquad 2C_{\exp}S_p\Big\|\sum_{N+2-2\sqrt{N}\le n\le N} e_nT^nx\sum_{n-4\sqrt{N}\le k\le n}\frac{N^k}{k!}\Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \ge e^N\Big\|\sum_{N+2-2\sqrt{N}\le n\le N} e_nT^nx\Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))}.
$$

Combining (3.4)–(3.6), we get the desired result. ∎

From [7, Theorem] and the proof of [15, Theorem 3.1] we obtain the following lemma.

Lemma 3.4. *Let T be a strongly Kreiss bounded operator on a Banach space X. Then there exists $D_{\mathrm{SK}}>0$ depending only on C_{SK} such that*

$$
\Big\|\sum_{1\le n\le N}\lambda^nT^n\Big\|_X \le D_{\mathrm{SK}}N,\qquad \lambda\in\mathbb{C},\ |\lambda|=1.
$$

This statement enables us to establish the following estimate for a strongly Kreiss bounded operator on $L^p(\mathcal{M})$: for every $N\in\mathbb{N}$ and every $x\in L^p(\mathcal{M})$,

$$
\text{(3.7)}\qquad \Big\|\sum_{1\le n\le N} e_nT^n\Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))}\le D_{\mathrm{SK}}N
$$

where D_{SK} is given in the preceding lemma. Subsequently, Lemmas 3.3 and 3.2 can be applied inductively to yield an improvement of estimate (3.7). This improvement is detailed in the next result.

Corollary 3.5. *Let T be a strongly Kreiss bounded operator on $L^p(\mathcal{M})$, $1<p<\infty$. Then there exists $E_p>0$ such that for all integers $N\ge 1$, $K\ge 0$ and all $x\in L^p(\mathcal{M})$,*

$$
\text{(3.8)}\qquad \Big\|\sum_{1\le n\le N} e_nT^nx\Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))}\le D_{\mathrm{SK}}E_p^{K+1}N^{2\delta_p+(1/2-\delta_p)2^{-K}}\|x\|_{L^p(\mathcal{M})}.
$$

Proof. We proceed by induction on K.

By (3.7), T satisfies (3.3) with $D=D_{\mathrm{SK}}$ and $f(N)=N$, hence using Lemma 3.3 we infer that for all $N\in\mathbb{N}$ and $x\in L^p(\mathcal{M})$,

$$
\Big\|\sum_{N+2-2\sqrt{N}\le n\le N} e_nT^nx\Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))}\le D_{\mathrm{SK}}E_p^{(2)}f(4\sqrt{N})\|x\|_{L^p(\mathcal{M})} = 4D_{\mathrm{SK}}E_p^{(2)}N^{1/2}\|x\|_{L^p(\mathcal{M})}.
$$

Hence T satisfies (3.2) with $D=4D_{\mathrm{SK}}E_p^{(2)}$ and $\alpha=1/2$. By Lemma 3.2, for every $N \in \mathbb{N}$ and every $x \in L^p(\mathcal{M})$,

$$\Big\| \sum_{1\le n\le N} e_n T^n x \Big\|_{L^p(\mathcal{M})} \le 4D_{\mathrm{SK}} E_p^{(1)} E_p^{(2)} N^{1/2+\delta_p} \|x\|_{L^p(\mathcal{M})} = D_{\mathrm{SK}} E_p N^{1/2+\delta_p} \|x\|_{L^p(\mathcal{M})},$$

where $E_p = 4E_p^{(1)} E_p^{(2)}$. Thus we obtain (3.8) with $K = 0$.

The induction step from $K-1$ to K can be justified similarly. Assume that T satisfies

$$\Big\| \sum_{1\le n\le N} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \le D_{\mathrm{SK}} E_p^K N^{2\delta_p+(1/2-\delta_p)2^{-(K-1)}} \|x\|_{L^p(\mathcal{M})}.$$

Then T satisfies (3.3) with

$$D = D_{\mathrm{SK}} E_p^K \quad \text{and} \quad f(N) = N^{2\delta_p+(1/2-\delta_p)2^{-(K-1)}},$$

and Lemma 3.3 gives, for every $N \in \mathbb{N}$ and every $x \in L^p(\mathcal{M})$,

$$\Big\| \sum_{1\le n\le N} e_n T^n x \Big\|_{L^p(\mathcal{M})} \le 4D_{\mathrm{SK}} E_p^{(2)} E_p^K N^{\delta_p+(1/2-\delta_p)2^{-K}} \|x\|_{L^p(\mathcal{M})}.$$

Now, T satisfies (3.2) with

$$D = 4D_{\mathrm{SK}} E_p^{(2)} E_p^K \quad \text{and} \quad \alpha = \delta_p + (1/2-\delta_p)2^{-K}.$$

Lemma 3.2 then yields, for every $N \in \mathbb{N}$ and every $x \in L^p(\mathcal{M})$,

$$\Big\| \sum_{1\le n\le N} e_n T^n x \Big\|_{L^p(\mathcal{M})} \le 4D_{\mathrm{SK}} E_p^{(1)} E_p^{(2)} E_p^K N^{\delta_p+(1/2-\delta_p)2^{-K}+\delta_p} \|x\|_{L^p(\mathcal{M})} = D_{\mathrm{SK}} E_p^{K+1} N^{2\delta_p+(1/2-\delta_p)2^{-K}} \|x\|_{L^p(\mathcal{M})}. \ \blacksquare$$

Proof of Theorem 1.1. Without loss of generality, we may and do assume that $N \ge 3$. Let $K \ge 0$ be the integer such that $2^{K+1} \le \log(N+1) \le 2^{K+2}$. Then we have $K+1 \le \log(\log(N+1))/\log 2$ and

$$(E_p)^{K+1} \le \exp\big(\log E_p \log(\log(N+1))/\log 2\big) = \log^{\kappa_p}(N+1). \tag{3.9}$$

with

$$\kappa_p := \frac{\log E_p}{\log 2}.$$

Moreover, since $2^{-(K+1)} \le 2/\log(N+1)$,

$$N^{(1/2-\delta_p)2^{-K}} \le N^{2^{-(K+1)}} \le \mathrm{e}^{2^{-(K+1)}\log(N+1)} \le \mathrm{e}^2. \tag{3.10}$$

Using (3.8)–(3.10), we infer that for every $N \in \mathbb{N}$ and every $x \in L^p(\mathcal{M})$,

$$\Big\| \sum_{1\le n\le N} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \le e^2 D_{\mathrm{SK}} N^{2\delta_p} \log^{\kappa_p}(N+1) \|x\|_{L^p(\mathcal{M})}. \tag{3.11}$$

Since T^* is strongly Kreiss bounded on $L^q(\mathcal{M})$, $q = p/(p-1)$, we obtain a similar estimate for T^*, that is, for every $N \in \mathbb{N}$ and every $x \in L^q(\mathcal{M})$,

$$\Big\| \sum_{1\le n\le N} e_n T^{*n} x^* \Big\|_{L^q(\mathbb{T},L^q(\mathcal{M}))} \le e^2 D_{\mathrm{SK}} N^{2\delta_q} \log^{\kappa_q}(N+1)\|x^*\|_{L^q(\mathcal{M})}. \tag{3.12}$$

Applying once more Lemma 3.3 we obtain

$$\Big\| \sum_{N+2-2\sqrt{N}\le n\le N} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \le 4e^2 D_{\mathrm{SK}} E_p^{(2)} N^{\delta_p} \log^{\kappa_p}(N+1)\|x\|_{L^p(\mathcal{M})}.$$

It follows that for all $x \in L^p(\mathcal{M})$ and $x^* \in L^q(\mathcal{M})$,

$$\begin{aligned}
&(\lfloor 2\sqrt{N}\rfloor - 1)\big|\langle x^*, T^{N+1}x\rangle\big| \\
&\quad = \Big| \int_{\mathbb{T}} \Big\langle \sum_{1\le n\le 2\sqrt{N}-1} e_n T^{*n} x^*, \sum_{1\le m\le 2\sqrt{N}-1} \bar{e}_m T^{N+1-m} x \Big\rangle \, d\lambda(\gamma) \Big| \\
&\quad \le \Big\| \sum_{1\le n\le 2\sqrt{N}-1} e_n T^{*n} x^* \Big\|_{L^q(\mathbb{T},L^p(\mathcal{M}))} \Big\| \sum_{N+2-2\sqrt{N}\le n\le N} e_n T^n x \Big\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \\
&\quad \le 4e^4 E_p^{(2)} N^{\delta_p+\delta_q} \log^{\kappa_p+\kappa_q}(N+1)\|x\|_{L^p(\mathcal{M})}\|x^*\|_{L^q(\mathcal{M})} \\
&\quad = 4e^4 D_{\mathrm{SK}}^2 E_p^{(2)} N^{|1/2-1/p|+1/2} \log^{\kappa}(N+1)\|x\|_{L^p(\mathcal{M})}\|x^*\|_{L^q(\mathcal{M})}
\end{aligned}$$

with

$$\kappa := \kappa_p + \kappa_q.$$

The result follows by taking the supremum over x^* and x with $\|x\|_{L^p(\mathcal{M})} = \|x^*\|_{L^q(\mathcal{M})} = 1$. ∎

4. Optimality of the polynomial rate in Theorem 1.1 . In this section we give an example showing optimality in Theorem 1.1 and we prove Proposition 1.2. This example has been presented by Lubich and Nevanlinna [12] to prove that the bound $\|T^N\| = O_{N\to\infty}(\sqrt{N})$ is the best possible for strongly Kreiss bounded operators T on general Banach spaces. More precisely, their example corresponds to the case where $p = \infty$ in the construction below.

Lubich and Nevanlinna provided only an outline of the proof partly based on a book by Brenner, Thomée and Wahlbin [3], which deals with Fourier analysis on the real line while the example is defined on $\mathbb{Z}$. Hence, we decided to give here a detailed proof independent of [3], at least of its results.

We denote by $A(\mathbb{T})$ the space of continuous functions f on $\mathbb{T}$ having an absolutely convergent Fourier series.

For $f \in A(\mathbb{T})$, define $f(R)$ where R is the right shift on any of $\ell^p(\mathbb{Z})$, $1 \le p \le \infty$, by the Hille–Phillips calculus:

$$f(R) := \sum_{n\in\mathbb{Z}} c_n(f) R^n,$$

where $\mathbf{c}_f := (c_n(f))_{n\in\mathbb{Z}}$ is the sequence of Fourier coefficients of f. We use here the convention that $c_n(f) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e_{-n}(t)\,dt$.

Notice that

$$f(R) = T_{\mathbf{c}_f},$$

where $T_{\mathbf{m}}$ for $\mathbf{m} := (m_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ is the convolution operator defined by

$$T_{\mathbf{m}} : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z}), \qquad \mathbf{b} \mapsto \mathbf{m} \star \mathbf{b}.$$

It is known that

$$\|T_{\mathbf{m}}\|_{B(\ell^\infty(\mathbb{Z}))} = \sum_{n\in\mathbb{Z}} |m_n| \tag{4.1}$$

and that for every $1 \le p < \infty$,

$$\|T_{\mathbf{m}}\|_{B(\ell^p(\mathbb{Z}))} \le \|T_{\mathbf{m}}\|_{B(\ell^\infty(\mathbb{Z}))}. \tag{4.2}$$

Moreover, for $1 \le p \le \infty$ and $q = \frac{p}{p-1}$,

$$\|T_{\mathbf{m}}\|_{B(\ell^p(\mathbb{Z}))} = \|T_{\mathbf{m}}\|_{B(\ell^q(\mathbb{Z}))}. \tag{4.3}$$

Now, let us describe the function f that will be used to construct our operator V.

Let a be a complex number with $0 < |a| < 1$ and let $\varphi \in \mathbb{R}$. Set

$$q_{a,\varphi}(z) := \mathrm{e}^{i\varphi}\frac{z-a}{1-\bar{a}z}, \qquad z \in \overline{\mathbb{D}}.$$

The function $q_{a,\varphi}$ is a so-called Möbius transformation mapping bijectively $\overline{\mathbb{D}}$ onto itself, which is analytic from $\mathbb{D}$ to $\mathbb{D}$ and continuous from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$. We associate with it a function from $A(\mathbb{T})$ by setting

$$\psi_{a,\varphi} := q_{a,\varphi} \circ e_1.$$

When no confusion is possible, we simply write ψ and q instead of, respectively, $\psi_{a,\varphi}$ and $q_{a,\varphi}$.

We define the operator V by setting

$$V := \psi(R).$$

We shall now prove that V enjoys all the properties described in Proposition 1.2. The proof will be separated into three sections.

4.1. Proof of the lower bound in (1.6). As mentioned in [12] one can establish the lower bound in (1.6) for the operator V by recasting the problem as that of obtaining a lower bound for the iterates of an operator on $L^p(\mathbb{R})$, using results of [3]. Here, we present a more direct proof that does not rely on [3], except for the use of van der Corput's lemma. Nonetheless, the core arguments are similar to those in [3].

In view of (4.3) (notice that $\tau_q = \tau_p$ if $1/p + 1/q = 1$) it is enough to prove the result for $p \in [1, 2]$. We consider this case now.

Let $f \in A(\mathbb{T})$, to be chosen later. By Parseval's equality and Hölder's inequality with $q = p/(p-1)$, for every $N \in \mathbb{N}$,

$$
\begin{aligned}
(4.4)\qquad \|f\|_{L^2(\mathbb{T})}^2 = \|\psi^N f\|_{L^2(\mathbb{T})}^2 &= 2\pi \sum_{n\in\mathbb{Z}} c_n(\psi^N f)\overline{c_n(\psi^N f)} \\
&\le 2\pi \big\|\big(c_n(\psi^N f)\big)_{n\in\mathbb{Z}}\big\|_{\ell^p(\mathbb{Z})} \big\|\big(c_n(\psi^N f)\big)_{n\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \\
&\le 2\pi \|V^N\|_{B(\ell^p(\mathbb{Z}))} \|(c_n(f))_{n\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \|(c_n(\psi^N f))_{n\in\mathbb{Z}}\|_{\ell^q(\mathbb{Z})}.
\end{aligned}
$$

By Hölder's inequality, for every $N \in \mathbb{N}$ we have

$$
\begin{aligned}
(4.5)\qquad \|(c_n(\psi^N f))_{n\in\mathbb{Z}}\|_{\ell^q(\mathbb{Z})} &\le \|(c_n(\psi^N f))_{n\in\mathbb{Z}}\|_{\ell^2(\mathbb{Z})}^{2/q} \|(c_n(\psi^N f))_{n\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}^{1-2/q} \\
&= \frac{\|f\|_{L^2(\mathbb{T})}^{2/q}}{(2\pi)^{1/q}} \|(c_n(\psi^N f))_{n\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}^{2/p-1}.
\end{aligned}
$$

Combining (4.4) and (4.5) and using $\|(c_n)_{n\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \le \|(c_n)_{n\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})}$, we obtain, for every $N \in \mathbb{N}$,

$$
\|V^N\|_{B(\ell^p(\mathbb{Z}))} \ge \frac{(2\pi)^{1-1/q}\|f\|_{L^2(\mathbb{T})}^{1-2/q}}{\|(c_n(f))_{n\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})}} \|(c_n(\psi^N f))_{n\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}^{1-2/p}.
$$

Let us assume for a moment that there exists $D = D_f$ such that for every $N \in \mathbb{N}$,

$$
(4.6)\qquad \|(c_n(\psi^N f))_{n\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \le \frac{1}{DN^{1/2}}.
$$

Then, for every $N \in \mathbb{N}$,

$$
\|V^N\|_{B(\ell^p(\mathbb{Z}))} \ge \frac{\min(1, 2\pi\|f\|_{L^2(\mathbb{T})})}{\|(c_n(f))_{n\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})}} \min(1, D)N^{1/p-1/2},
$$

which is exactly the desired lower bound in (1.6), provided that f is not the null function.

It remains to prove that we can find a function $f \in A(\mathbb{T})$, $f \not\equiv 0$, that satisfies (4.6).

For every $t \in \mathbb{T}$, set $\sigma(t) := -i\int_0^t \frac{\psi'}{\psi}(u)\,du$. Then σ is twice continuously differentiable on $\mathbb{T}$, and $\psi(t) = \mathrm{e}^{i\sigma(t)}$ for every $t \in \mathbb{T}$. Moreover, since $|\psi| \equiv 1$, σ is real-valued.

Since q is not a rotation, σ'' is not identically equal to 0 and there exist $-\pi < \alpha < \beta < \pi$ and $c > 0$ such that $|\sigma''(t)| \ge c$ for every $t \in [\alpha, \beta]$.

Let f be any function in $A(\mathbb{T})$, not identically equal to 0, such that

$f(t)=0$ if $t\in\mathbb{T}\setminus[\alpha,\beta]$ and $f\in C^1((\alpha,\beta))$. Integrating by parts, we obtain

$$\begin{aligned}2\pi c_n(\psi^N f) &= \int_{-\pi}^{\pi} f(t)\psi^N(t)e^{-int}\,dt = \int_{\alpha}^{\beta} f(t)\psi^N(t)e^{-int}\,dt\\ &= \int_{\alpha}^{\beta} f'(t)\int_{\alpha}^{t}\psi^N(x)e^{-inx}\,dx\,dt = \int_{\alpha}^{\beta} f'(t)\int_{\alpha}^{t} e^{-inx+iN\sigma(x)}\,dx\,dt\end{aligned}$$

for all n and N from $\mathbb{N}$. By van der Corput's lemma (see [3, Lemma 1.5.1]), there exists $c>0$ such that for all $n,N\in\mathbb{N}$ and $t\in[\alpha,\beta]$,

$$\Big|\int_{\alpha}^{t} e^{-inx+iN\sigma(x)}dx\Big|\le 8c^{-1/2}N^{-1/2},$$

so that

$$\sup_{n\in\mathbb{Z}}|c_n(\psi^N f)|\le C\|f'\|_{L^1(\mathbb{T})}N^{-1/2}$$

with $C=4c^{-1/2}/\pi$, which is exactly (4.6).

4.2. Proof that V is strongly Kreiss bounded on $\ell^p(\mathbb{Z})$. By (4.2) it suffices to prove that V is strongly Kreiss bounded on $\ell^\infty(\mathbb{Z})$. The proof of that fact has been outlined by Lubich and Nevanlinna [12]. We propose below a complete and detailed proof, from which the uniformity of the bounds obtained, with respect to the different parameters involved, will be clear.

For $\lambda\in\mathbb{C}$ with $|\lambda|>1$ and for $k\in\mathbb{N}$, consider the function

$$\chi_{\lambda,k}:=\left(\frac{|\lambda|-1}{\lambda-\psi}\right)^k.$$

Since

$$(|\lambda|-1)^k R(\lambda,V)^k=\chi_{\lambda,k}(V),$$

to prove that V is strongly Kreiss bounded on $\ell^\infty(\mathbb{Z})$, it suffices to control the ℓ^1-norm of the sequence of Fourier coefficients $(c_n(\chi_{\lambda,k}))_{n\in\mathbb{Z}}$ uniformly with respect to λ and k.

Write $\lambda=|\lambda|\mathrm{e}^{i\alpha}$ for some $\alpha\in\mathbb{R}$ and let $\tau\in\mathbb{R}$ be such that $\psi(\tau)=\mathrm{e}^{i\alpha}$.

Clearly, it is enough to control the ℓ^1-norm of the Fourier coefficients of

$$\mathbb{T}\ni t\mapsto\left(\frac{|\lambda|-1}{|\lambda|-\mathrm{e}^{-i\alpha}\psi(t+\tau)}\right)^k,$$

uniformly with respect to $k\in\mathbb{N}$ and $\lambda\in\mathbb{C}$, $|\lambda|>1$.

Notice that

$$\mathrm{e}^{-i\alpha}\psi_{a,\varphi}(\cdot+\tau)=\psi_{a\mathrm{e}^{-i\tau},\varphi-\alpha}.$$

In particular, it is sufficient to assume that λ is a real number satisfying $\lambda>1$ and that $\psi(0)=1$. Moreover, since a and $a\mathrm{e}^{-i\tau}$ have the same modulus, for our purpose it is enough to obtain a bound depending only on $|a|$, and uniform with respect to $\varphi \in \mathbb{R}$.

In view of our assumption, in particular, we have

$$\frac{\mathrm{e}^{i\varphi}}{1-\bar{a}} = \frac{1}{1-a}. \tag{4.7}$$

For brevity, we will write χ_k instead of $\chi_{\lambda,k}$. All variables indexed by k will be understood to depend implicitly on both k and λ.

The proof makes use of the following three lemmas, which we state now and will prove in the Appendix. The first one is a standard result in the spirit of Carlson's inequality.

Lemma 4.1. *Let $h \in A(\mathbb{T})$ be such that $h' \in L^2(\mathbb{T})$, where h' stands for the derivative of h in the sense of distributions. For every real number $x \geq 1$, we have*

$$\sum_{n\in\mathbb{Z}} |c_n(h)| \leq \sqrt{2}\,(x+1)\|h\|_{L^2(\mathbb{T})} + \frac{\sqrt{2}}{x}\|h'\|_{L^2(\mathbb{T})}. \tag{4.8}$$

The next lemma was stated in [12, Example 2.2].

Lemma 4.2. *Let $\psi_{a,\varphi}$ be as above. Then there exists $\gamma > 0$ depending only on $|a|$ such that for all $t \in \mathbb{T}$ and $\lambda > 1$,*

$$\left|\frac{\lambda-1}{\lambda-\psi_{a,\varphi}(t)}\right| \leq \frac{1}{(1+\gamma t^2/\mu)^{1/2}}, \tag{4.9}$$

where $\mu = \min(\lambda-1, (\lambda-1)^2)$.

Lemma 4.3. *For all $k \in \mathbb{N}$, $\lambda > 1$, $\varphi \in \mathbb{R}$ and $a \in \mathbb{D}\setminus\{0\}$, there exist a C^1 function $\theta_k = \theta_{k,\lambda}$ on $\mathbb{T}$ and a constant $C > 0$ depending only on $|a|$ such that for all $t \in \mathbb{T}$,*

$$|\theta_k(t)| \leq 1, \tag{4.10}$$

$$|(\theta_k\chi_k)'(t)| \leq \frac{Ck|t|}{\mu(1+\gamma t^2/\mu)^{k/2}}, \tag{4.11}$$

$$\sum_{n\in\mathbb{Z}} \left|c_n\left(\frac{1}{\theta_k}\right)\right| \leq 2. \tag{4.12}$$

Remark 4.4. Here, by C^1, we mean that θ may be extended to a 2π-periodic C^1 function on the whole real line.

We are now ready to prove that V is strongly Kreiss bounded on $\ell^\infty(\mathbb{Z})$.

We proceed to show that there exists $C > 0$ depending only on $|a|$ such that for every $k \in \mathbb{N}$ and every real number $\lambda > 1$,

$$\sum_{n\in\mathbb{Z}} |c_n(\chi_k)| \leq C. \tag{4.13}$$

First of all, using (4.12), we have

$$\sum_{n\in\mathbb{Z}}|c_n(\chi_k)| = \sum_{n\in\mathbb{Z}}\left|c_n\Big(\frac{1}{\theta_k}\theta_k\chi_k\Big)\right| = \sum_{n\in\mathbb{Z}}\left|\sum_{r\in\mathbb{Z}}c_r\Big(\frac{1}{\theta_k}\Big)c_{n-r}(\theta_k\chi_k)\right| \tag{4.14}$$
$$\le \Big(\sum_{n\in\mathbb{Z}}\left|c_n\Big(\frac{1}{\theta_k}\Big)\right|\Big)\Big(\sum_{n\in\mathbb{Z}}|c_n(\theta_k\chi_k)|\Big) \le 2\sum_{n\in\mathbb{Z}}|c_n(\theta_k\chi_k)|.$$

Moreover, by (4.9) and (4.10), using the change of variable $u=\sqrt{k/\mu}t$, we see that

$$\int_{-\pi}^{\pi}|\theta_k\chi_k|^2(t)\,dt \le \int_{-\pi}^{\pi}\frac{dt}{(1+\gamma t^2/\mu)^k} \le \Big(\frac{\mu}{k}\Big)^{1/2}\int_{-\infty}^{+\infty}\frac{du}{(1+\gamma u^2/k)^k} \tag{4.15}$$
$$\le \Big(\frac{\mu}{k}\Big)^{1/2}\int_{-\infty}^{+\infty}\frac{du}{(1+\gamma u^2)} \le C\Big(\frac{\mu}{k}\Big)^{1/2},$$

since for every $x>0$, $y\mapsto(1+x/y)^y$ is increasing on $[1,+\infty)$.

On the other hand, $|\theta_k\chi_k|\le 1$, so that

$$\int_{-\pi}^{\pi}|\theta_k\chi_k|^2(t)\,dt \le \min\Big(2\pi, C\Big(\frac{\mu}{k}\Big)^{1/2}\Big). \tag{4.16}$$

We shall now estimate the L^2-norm of the derivative of $\theta_k\chi_k$. Proceeding as above, using (4.11), we obtain

$$\int_{-\pi}^{\pi}|(\theta_k\chi_k)'|^2(t)\,dt \le \frac{4D^2k^2}{\mu^2}\Big(\frac{\mu}{k+1}\Big)^{3/2}\int_{-\infty}^{+\infty}\frac{u^2du}{(1+\frac{\gamma u^2}{k+1})^{k+1}} \le \frac{C\sqrt{k}}{\sqrt{\mu}}. \tag{4.17}$$

If $C(\mu/k)^{1/2}<2\pi$, we apply (4.8) with $x^2=k/(C^2\mu)$ and obtain

$$\sum_{n\in\mathbb{Z}}|c_n(\theta_k\chi_k)| \le \sqrt{2}\Big(\sqrt{\frac{k}{C^2\mu}}+1\Big)\times C\Big(\frac{\mu}{k}\Big)^{1/2}+\sqrt{\frac{C^2\mu}{k}}\times\frac{C\sqrt{k}}{\sqrt{\mu}} \tag{4.18}$$
$$\le \sqrt{2}\Big(1+C\Big(\frac{\mu}{k}\Big)^{1/2}\Big)+C^2 \le \sqrt{2}\,(1+2\pi)+C^2.$$

If $C(\mu/k)^{1/2}\ge 2\pi$, we apply (4.8) with $x=1$ and obtain

$$\sum_{n\in\mathbb{Z}}|c_n(\theta_k\chi_k)| \le 4\sqrt{2}\,\pi+\frac{C\sqrt{k}}{\sqrt{\mu}} \le 4\sqrt{2}\,\pi+\frac{C^2}{2\pi}. \tag{4.19}$$

Combining (4.18) and (4.19), we infer that there exists $C>0$ such that

$$\sum_{n\in\mathbb{Z}}|c_n(\theta_k\chi_k)| \le C,$$

which, by (4.14), implies (4.13). Hence V is strongly Kreiss bounded on every $\ell^p(\mathbb{Z})$, $1\le p\le\infty$.

4.3. Proof of the upper bound in (1.6). Since V is strongly Kreiss bounded on $\ell^\infty(\mathbb{Z})$, by a result of Lubich and Nevanlinna [12] recalled in the Introduction, there exists $C > 0$ such that for every $N \in \mathbb{N}$,

$$\|V^N\|_{B(\ell^\infty(\mathbb{Z}))} \le CN^{1/2}.$$

Now, V is an isometry on $\ell^2(\mathbb{Z})$, hence, by the Riesz–Thorin interpolation theorem applied to V^N (for every $N \in \mathbb{N}$), we infer that the upper bound in (1.6) holds for every $p \in [2, \infty]$. The case where $p \in [1, 2)$ then follows from (4.3).

5. Some particular strongly Kreiss bounded operators. We consider here the case of *positive* strongly Kreiss bounded operators on $L^p(\mathcal{M})$ (we no longer require μ to be σ-finite) or *absolutely* strongly Kreiss bounded operators (on any Banach space). The proof makes use of an idea from [1] on obtaining norm bounds for powers of Kreiss operators.

Let us begin with a general result.

LEMMA 5.1. *Let $1 \le p \le 2$, $C, D \ge 1$ and $\alpha > 0$. Let T be a bounded operator on a Banach space X such that for every $N \in \mathbb{N}$ and every $x \in X$,*

$$\sum_{N+2-2\sqrt{N} \le n \le N} \|T^n x\|^p \le C^p N^{p/2} \|x\|, \tag{5.1}$$

$$\|T^N\| \le DN^\alpha. \tag{5.2}$$

Then there exist $E > 0$ and $\kappa > 0$ such that for every $N \in \mathbb{N}$,

$$\|T^N\| \le EN^{1/q} \log^\kappa(N+1). \tag{5.3}$$

Proof. We start with the following observation: for all $x \in X$ and $x^* \in X^*$,

$$\begin{aligned}(\lfloor 2\sqrt{N}\rfloor - 1)|\langle x^*, T^{N+1}x\rangle|^p &\le \sum_{1\le n\le 2\sqrt{N}-1} |\langle T^{*n}x^*, T^{N+1-n}x\rangle|^p_{X^*,X} \\ &\le \|x^*\|^p \max_{1\le n\le 2\sqrt{N}-1} \|T^{*n}\|^p \sum_{N+2-2\sqrt{N}\le n\le N} \|T^n x\|^p.\end{aligned}$$

Taking the supremum over x, x^* of norm 1, and using $\|T^{*n}\| = \|T^n\|$ for every $N \in \mathbb{N}$, we get

$$\begin{aligned}\|T^N\| &\le (\lfloor 2\sqrt{N}\rfloor - 1)^{-1/p} \max_{1\le n\le 2\sqrt{N}-1} \|T^n\| \sup_{\|x\|\le 1} \Big(\sum_{N+2-2\sqrt{N}\le n\le N} \|T^n x\|^p\Big)^{1/p} \\ &\le N^{-1/(2p)} \max_{1\le n\le 2\sqrt{N}} \|T^n\| \sup_{\|x\|\le 1} \Big(\sum_{N+2-2\sqrt{N}\le n\le N} \|T^n x\|^p\Big)^{1/p}.\end{aligned}$$

Using assumption (5.1), for all $N \in \mathbb{N}$, we obtain

$$\|T^N\| \le CN^{1/(2q)} \max_{1\le n\le 2\sqrt{N}} \|T^n\|. \tag{5.4}$$

By induction, we then find that for all $N, K \in \mathbb{N}$,

$$\|T^N\| \le D(2^\alpha C)^K N^{2^{-K}\alpha+(1-2^{-K})/q}.$$

Indeed, combining (5.4) with (5.2) shows that for all $N \in \mathbb{N}$,

$$\|T^N\| \le DC2^\alpha N^{\alpha/2+1/2q}, \tag{5.5}$$

which gives the case $K = 1$. For the inductive step from $K - 1$ to K, we combine the inductive hypothesis with (5.4). Finally, we conclude as in the proof of Theorem 1.1 to obtain (5.3). ∎

From Lemma 5.1 we deduce a better bound than the one obtained in Theorem 1.1 for positive strongly Kreiss bounded operators on $L^p(\mathcal{M})$ when $p \in [1, 4/3) \cup (4, +\infty)$ (the cases where $p = 1$ and $p = \infty$ are discussed below).

PROPOSITION 5.2. *Let T be a positive operator that is strongly Kreiss bounded on $L^p(\mathcal{M})$, $1 \le p < \infty$. Then there exist $C, \kappa > 0$ such that for every $N \in \mathbb{N}$,*

$$\|T^N\| \le CN^{1/\bar{p}} \log^\kappa(N + 1),$$

where $\bar{p} = \max(p, p/(p - 1))$.

The proof is straightforward using the following lemma and the fact that every strongly Kreiss bounded operator satisfies (5.2) for $\alpha = 1/2$.

LEMMA 5.3. *Let $1 \le p < \infty$. Then any positive strongly Kreiss bounded operator T on $L^p(\mathcal{M})$ satisfies* (5.1) *for every $x \in L^p(\mathcal{M})$.*

Proof. Using the fact that $\|\cdot\|_{\ell^p} \le \|\cdot\|_{\ell^1}$, the positivity of T, the Krivine calculus [19, Theorem 1.d.1] and Lemma 2.5, for all $N \in \mathbb{N}$ and all nonnegative $x \in L^p(\mathcal{M})$ we obtain

$$\begin{aligned}\Big(\sum_{n\ge 0} \frac{N^n T^n x}{n!}\Big)^p &\ge \sum_{n\ge 0} \Big(\frac{N^n T^n x}{n!}\Big)^p \\ &\ge \sum_{N+2-2\sqrt{N}\le n\le N} \Big(\frac{N^n T^n x}{n!}\Big)^p \\ &\ge \frac{C_{\exp}^p e^{pN}}{N^{p/2}} \sum_{N+2-2\sqrt{N}\le n\le N} (T^n x)^p.\end{aligned}$$

Integrating with respect to μ and using the strong Kreiss boundedness of T, we deduce that for all $N \in \mathbb{N}$ and all nonnegative $x \in L^p(\mathcal{M})$,

$$C_{\mathrm{SK}} e^{pN} \|x\|^p \ge \|e^{NT} x\|^p = \int_{\mathcal{M}} \Big(\sum_{n\ge 0} \frac{N^n T^n x}{n!}\Big)^p d\mu$$

$$\geq \frac{C_{\exp}^p e^{pN}}{N^{p/2}} \int_{\mathcal{M}} \sum_{N+2-2\sqrt{N}\leq n\leq N} (T^n x)^p d\mu$$
$$= \frac{C_{\exp}^p e^{pN}}{N^{p/2}} \sum_{N+2-2\sqrt{N}\leq n\leq N} \|T^n x\|^p.$$

Recall that C_{SK} denotes the smallest constant $L > 0$ such that (1.3) holds. This implies that (5.1) is true for all $x \in L^p(\mathcal{M})$ with $C = C_{\mathrm{SK}}/C_{\exp}$. ∎

We turn now to the special case of absolutely strongly Kreiss bounded operators. Let T be a bounded operator on X. We say that T is *absolutely strongly Kreiss bounded* if there exists $C > 0$ such that

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} \|T^n x\| \leq C e^r \|x\|, \quad r > 0,\ x \in X.$$

These operators were first introduced in [4, remarks, p. 17]. Such operators are clearly strongly Kreiss bounded (see [4, Proposition 4.10]). The operator T defined in (1.4) provides an example of an operator which is absolutely strongly Kreiss and not power-bounded. Moreover, T^* is strongly Kreiss bounded but cannot be absolutely strongly Kreiss, since in this case T would be power-bounded (see [4, Proposition 2.1]). By Lemma 2.5, for every $x \in X$, the operator T satisfies (5.1) with $p = 1$. We can then apply Lemma 5.1 to show that $\|T^N\|$ has a logarithmic bound.

Proposition 5.4. *Let T be an absolutely strongly Kreiss bounded operator on X. Then there exist $C, \kappa > 0$ depending only on C_{SK} such that for every $N \in \mathbb{N}$,*

$$\|T^N\| \leq C \log^{\kappa}(N+1). \tag{5.6}$$

Remark 5.5. When $X = L^p(\mathcal{M})$, $1 \leq p < \infty$, the bound (5.6) is sharp. Indeed, according to [4, Remark 1, p. 16], the operators T_κ, $\kappa > 0$, defined by (1.4), are absolutely strongly Kreiss bounded on $\ell^p(\mathbb{Z})$ and satisfy

$$\|T_\kappa^N\| = \frac{\log^\kappa(N+1)}{\log^\kappa 2}$$

for all $N \in \mathbb{N}$ and $\kappa > 0$.

We now discuss the case of positive strongly Kreiss operator on (AL)- and (AM)-spaces. We refer to [14, Chapter 2] for more details. A Banach lattice X is an *(AL)-space* if the norm is additive on the positive cone on X, that is,

$$\|x + y\| = \|x\| + \|y\|, \quad x, y \in X_+. \tag{5.7}$$

A Banach lattice X is an *(AM)-space* if the norm on X satisfies

$$\|\sup(x, y)\| = \sup(\|x\|, \|y\|), \quad x, y \in X_+.$$

If X is an (AM)-space, then X^* is an (AL)-space. It is known that an (AL)-space is isometrically isomorphic to some L^1-space and that an (AM)-space is isometrically isomorphic to some $C(K)$ where K is a compact space.

PROPOSITION 5.6. *Let T be a positive strongly Kreiss bounded operator on an (AL)-space or an (AM)-space. Then there exist $C, \kappa > 0$ such that for every $N \in \mathbb{N}$,*

$$\|T^N\| \leq C \log^{\kappa}(N+1). \tag{5.8}$$

REMARK 5.7. Since $L^\infty(\mathcal{M})$ is an (AM)-space, Proposition 5.2 remains valid also for the case $p = \infty$ by the above result.

Proof of Proposition 5.6. If T is a positive strongly Kreiss bounded operator on an (AL)-space, then by (5.7), it is straightforward that T is absolutely strongly Kreiss bounded, and we can then conclude invoking Lemma 5.1.

If T is a positive strongly Kreiss bounded operator on an (AM)-space, then T^* is a positive strongly Kreiss bounded on an (AL)-space. Then this case is reduced to the first part of the proof. ∎

QUESTION 5.8. In view of the considerations in Section 5, it is natural to ask whether the bound in Proposition 5.2 can be improved to obtain a logarithmic bound. More precisely, for a positive strongly Kreiss bounded operator T on $L^p(\mathcal{M})$ with $1 < p < \infty$, are there $C, \kappa > 0$ such that for every $N \in \mathbb{N}$,

$$\|T^N\| \leq C \log^{\kappa}(N+1)?$$

6. Appendix

6.1. Proof of Lemma 4.1. Let $x \geq 1$ be a real number. Set $m := \lfloor x^2 \rfloor$. By the Cauchy–Schwarz inequality, we have

$$\sum_{|n| \leq m+1} |c_n(h)| \leq \sqrt{2m+3}\Big(\sum_{|n| \leq m+1} |c_n(h)|^2\Big)^{1/2} \leq \sqrt{2m+3}\,\|h\|_{L^2(\mathbb{T})},$$

$$\sum_{|n| \geq m+2} |c_n(h)| \leq \bigg(\sum_{|n| \geq m+2} \frac{1}{|n|(|n|-1)}\bigg)^{1/2} \bigg(\sum_{|n| \geq m+2} n^2 |c_n(h)|^2\bigg)^{1/2}$$
$$\leq \sqrt{\frac{2}{m+1}}\,\|h'\|_{L^2(\mathbb{T})},$$

and the result follows. ∎

6.2. Proof of Lemma 4.2. We first prove that there exists $C > 0$ with

$$\left|\frac{\lambda - 1}{\lambda - q(z)}\right| \leq \frac{1}{\big(1 + \frac{C|1-z|^2}{\mu}\big)^{1/2}}, \qquad |z| = 1. \tag{6.1}$$

The result will follow since, if we write $z = \mathrm{e}^{it}$ with $t \in \mathbb{T}$, then $\psi(t) = q(\mathrm{e}^{it})$ and $|1 - z| = 2|\sin(t/2)| \geq 2|t|/\pi$.

Notice that (6.1) is equivalent to

$$\frac{\mu}{|1-z|^2}\left(\left|\frac{\lambda - q(z)}{\lambda - 1}\right|^2 - 1\right) \geq C, \qquad |z| = 1.$$

Now, for every $z \in \mathbb{C}$ with $|z| = 1$,

$$\begin{aligned}\left|\frac{\lambda - q(z)}{\lambda - 1}\right|^2 - 1 &= \frac{(\lambda-1)^2 + |1-q(z)|^2 + 2(\lambda-1)\big(1 - \mathrm{Re}(q(z))\big)}{(\lambda-1)^2} - 1 \\ &= \frac{2\big(1 - \mathrm{Re}(q(z))\big)}{\lambda - 1} + \frac{|1-q(z)|^2}{(\lambda-1)^2} \\ &\geq \frac{1}{\mu}\min\big(2(1 - \mathrm{Re}(q(z)), |1-q(z)|^2\big) = \frac{1}{\mu}|1-q(z)|^2,\end{aligned}$$

where the last equality comes from

$$|1-z|^2 = 2(1 - \mathrm{Re}(z)), \qquad |z| = 1.$$

Hence, we just have to prove that

$$\frac{|1-q(z)|^2}{|1-z|^2} \geq C, \qquad |z| = 1.$$

However, since q is a bijection of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, this is equivalent to proving

$$\frac{|1-z|^2}{|1-q^{-1}(z)|^2} \geq C, \qquad |z| = 1. \tag{6.2}$$

It is well-known that for all $z \in \overline{\mathbb{D}}$, $q^{-1}(z) = \mathrm{e}^{i\varphi}\frac{z+a}{1+\bar{a}z}$. Since $q^{-1}(1) = 1 = \mathrm{e}^{i\varphi}\frac{1+a}{1+\bar{a}}$, we obtain

$$1 - q^{-1}(z) = 1 - \frac{1+\bar{a}}{1+a}\,\frac{z+a}{1+\bar{a}z} = \frac{(1-|a|^2)(1-z)}{(1+a)(1+\bar{a}z)}, \qquad z \in \overline{\mathbb{D}}.$$

Using the estimate

$$|1 + \bar{a}z| \geq 1 - |a|, \qquad |z| = 1,$$

we deduce

$$|1 - q^{-1}(z)|^2 = \frac{(1-|a|^2)^2|1-z|^2}{|1+\bar{a}|^2|1+\bar{a}z|^2} \leq \frac{(1+|a|)^2}{(1-|a|)^2}|1-z|^2, \qquad |z| = 1.$$

Therefore, (6.2) holds with $C = \frac{(1-|a|)^2}{(1+|a|)^2}$, and (4.9) holds with $\gamma = 4C/\pi^2 = 4\frac{(1-|a|)^2}{(1+|a|)^2}/\pi^2$. ∎

6.3. Proof of Lemma 4.3. Notice that for every $t \in \mathbb{T}$,

$$\psi'(t) = \mathrm{e}^{i\varphi} \frac{i(1-|a|^2)\mathrm{e}^{it}}{(1-\bar{a}\mathrm{e}^{it})^2}, \tag{6.3}$$

$$\psi''(t) = -\mathrm{e}^{i(t+\varphi)}(1-|a|^2)\frac{1+\bar{a}\mathrm{e}^{it}}{(1-\bar{a}\mathrm{e}^{it})^3}. \tag{6.4}$$

Hence, using (4.7), we get

$$\psi'(0) = i\frac{1-|a|^2}{|1-a|^2}. \tag{6.5}$$

Let $\alpha := \frac{1-|a|^2}{|1-a|^2}$. Fix $k \in \mathbb{N}$ and $\lambda > 1$. Set $m_k = m_{k,\lambda} := \lfloor k\alpha/(\lambda-1)\rfloor$, and

$$\theta_k = \theta_{k,\lambda} := e_{-m_k}\left[1 - \frac{1}{4}\left(\frac{k\alpha}{\lambda-1} - m_k\right) + \frac{1}{4}\left(\frac{k\alpha}{\lambda-1} - m_k\right)e_{-4}\right].$$

We shall see that θ_k satisfies all the required properties.

Clearly, for every $t \in \mathbb{T}$, we have $|\theta_k(t)| \le 1 = \theta_k(0)$ and therefore (4.10) holds.

Let us prove (4.11). Setting $u_k := e_{m_k}\theta_k$, we have

$$\begin{aligned} \theta_k' &= -im_k\theta_k - ie_{-m_k}\left(\frac{k\alpha}{\lambda-1} - m_k\right)e_{-4} \\ &= -ie_{-m_k}\left(m_k u_k(t) + \left(\frac{k\alpha}{\lambda-1} - m_k\right)e_{-4}\right). \end{aligned} \tag{6.6}$$

Hence,

$$\theta_k'(0) = -ik\frac{\alpha}{\lambda-1}. \tag{6.7}$$

Moreover,

$$\begin{aligned} (\theta_k\chi_k)' &= \frac{(\lambda-1)^k}{(\lambda-\psi)^{k+1}}\big(\theta_k'(\lambda-\psi) + k\theta_k\psi'\big) \\ &= \frac{(\lambda-1)^k}{(\lambda-\psi)^{k+1}}e_{-m_k}\left(ku_k\psi' - i\left(m_k u_k + \left(\frac{k\alpha}{\lambda-1} - m_k\right)e_{-4}\right)(\lambda-\psi)\right) \\ &= \frac{(\lambda-1)^k e_{-m_k}}{(\lambda-\psi)^k} w_k, \end{aligned}$$

where

$$w_k := \frac{ku_k\psi' - i\big(m_k u_k + \big(\frac{k\alpha}{\lambda-1} - m_k\big)e_{-4}\big)(\lambda-\psi)}{\lambda-\psi}.$$

Using (6.7) and the fact that $\psi'(0) = i\alpha$, we have

$$w_k(0) = k\theta_k(0)\psi'(0) - im_k - i\left(\frac{k\alpha}{\lambda-1} - m_k\right)(\lambda - \psi(0)) = 0. \tag{6.8}$$

Moreover, setting $g_k := u_k\psi'$, we have

$$w_k' = im_k\, u_k' + u_k'' + k\frac{g'(\lambda-\psi) + g_k\psi'^2}{(\lambda-\psi)^2}.$$

Notice that $|u_k'| \le 1$ and $|u_k''| \le 4$. From (6.3) and (6.4), it follows that for every $t \in \mathbb{T}$,

$$|\psi'(t)| \le \frac{1-|a|^2}{(1-|a|)^2} \quad \text{and} \quad |\psi''(t)| \le \frac{(1+|a|)^2}{(1-|a|)^2}.$$

In particular, we infer that

$$|w_k'| \le C\left(m_k + 1 + k\left(\frac{1}{\lambda-1} + \frac{1}{(\lambda-1)^2}\right)\right) \le D\left(m_k + \frac{k}{\mu}\right) \le 2D\frac{k}{\mu},$$

for some constants $C, D > 0$ depending only on a (and not on $k \in \mathbb{N}$ or $\lambda > 1$). Combining this bound with (6.8), we see that for every $t \in \mathbb{T}$,

$$|w_k(t)| \le 2D\frac{k}{\mu}|t|. \tag{6.9}$$

Then (4.11) follows from (6.9) and (4.9).

It remains to prove (4.12). We set $r_k = k\alpha/(\lambda-1) - m_k$. For every $k \in \mathbb{N}$ and $\lambda > 1$, we have

$$\frac{1}{\theta_k(t)} = \frac{e_{m_k}}{1-\frac{r_k}{4}} \sum_{n\in\mathbb{N}} \left(-\frac{r_k}{4(1-r_k/4)}\right)^n e_{-4n}.$$

Therefore,

$$\sum_{n\in\mathbb{Z}} \left|c_n\left(\frac{1}{\theta_k}\right)\right| \le \frac{1}{1-r_k/4} \sum_{n\in\mathbb{N}} \left(\frac{r_k}{4(1-r_k/4)}\right)^n$$
$$= \frac{4}{4-r_k} \times \frac{1}{1-r_k/4-r_k} = \frac{2}{2-r_k} \le 2. \quad ■$$

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