

# Forking and invariant measures in NIP theories

by

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**Abstract.** We give an example of an NIP theory  $T$  in which there is a formula that does not fork over  $\emptyset$  but has measure 0 under any global  $\emptyset$ -invariant Keisler measure, and we show that this cannot occur if  $T$  is also first-order amenable. We also comment on some connections with topological dynamics.

**0. Notation.** Throughout,  $T$  will denote a complete theory in a first-order language  $L$ , with a monster model  $\mathfrak{C}$ . We will always be working over  $\emptyset$ . So: (i) if we say a formula or type *forks* (respectively *divides*), we mean that it forks (respectively divides) over  $\emptyset$ , (ii) by *definable* (respectively *type-definable*) we mean definable (respectively type-definable) without parameters, and (iii) if we say two tuples  $a, b$  are *conjugate*, we mean they have the same type over  $\emptyset$ , which we denote by  $a \equiv b$ .

**1. Introduction.** This paper continues the investigation of one of the themes of [2], namely, the relationship between two natural ideals of  $L(\mathfrak{C})$ -formulas in a theory  $T$ : the ideal of formulas which fork, and the ideal of formulas which are given measure 0 under any automorphism-invariant Keisler measure on  $\mathfrak{C}$ . The former is always contained in the latter (see for example [2, Fact 1.2]), and the other inclusion holds if  $T$  is stable (see for example [2, Corollary 1.4]). But the other inclusion need not hold in general, and [2] gives such an example where  $T$  is simple. In this paper we will discuss the case when  $T$  is NIP; for a much more general context, we refer to the introduction of [2].

For the rest of the introduction, let  $\phi(x, b)$  be an  $L(\mathfrak{C})$ -formula that does not fork, and let  $(*)$  denote the statement that  $\mu(\phi(x, b)) > 0$  for some global automorphism-invariant Keisler measure  $\mu$ . It is an early fact from [1]

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(attributed there to [10]) that, if  $T$  is NIP, then  $\phi(x, b)$  is contained in a global type which is Lascar-invariant over  $\emptyset$ ; in particular, if  $\text{dcl}(\emptyset)$  is a model, then  $\phi(x, b)$  is contained in some automorphism-invariant *type*, and so  $(*)$  holds. In [8], Hrushovski and the first author showed further that, if  $T$  is NIP, then  $\phi(x, b)$  is contained in a global type which is KP-invariant over  $\emptyset$ ; by lifting the Haar measure on the KP-Galois group of  $T$ , they have used this to show that, if  $T$  is NIP, then  $(*)$  holds provided  $b \in \text{dcl}(\emptyset)$  (this follows from Fact 2.2 below). Nevertheless, we show here that  $(*)$  need not hold in general when  $T$  is NIP; this is the content of Section 4. We present there a general construction which, starting with a structure  $M$ , produces a new structure  $M_{\text{cyc}}$  in such a way that the forking of formulas of a certain shape in  $M_{\text{cyc}}$  is controlled precisely by forking behavior *and* algebraic behavior in  $M$ . By applying this construction to the binary tree  $M = (2^{<\omega}, \leq)$  we obtain an NIP theory in which there are non-forking formulas that have measure 0 under any automorphism-invariant Keisler measure.

A notion related to the above theme is that of *first-order amenability*, introduced by Hrushovski, Krupiński, and the first author in [7, 6]. We say  $T$  is *amenable* if every type over  $\emptyset$  extends to a global automorphism-invariant Keisler measure. Using the construction from [8] mentioned above, [7] shows that, if  $T$  is NIP, then  $T$  is amenable if and only if  $\emptyset$  is an extension base – i.e. no formula without parameters forks. In Section 2 we observe that  $(*)$  holds when  $T$  is NIP and amenable by adapting an argument of Chernikov and Simon; the result follows very quickly from the general machinery of NIP theories. In contrast, as shown in [2],  $(*)$  need not hold even if  $T$  is simple and amenable, as the example found there *is* amenable.

Finally, we use this paper as an opportunity to give a ‘topological dynamics account’ of some of the behavior of forking and dividing in NIP theories, which we provide in Section 3. The main point is to give appropriate ‘automorphism group analogues’ of some dynamics notions that are well-studied in the context of definable groups, especially in NIP theories.

**2. Amenable case.** Recall from [7] that a theory  $T$  is said to be *first-order amenable*, or just *amenable*, if every type over  $\emptyset$  extends to a global automorphism-invariant Keisler measure. In any theory  $T$ , amenability implies that  $\emptyset$  is an extension base (i.e. no formula without parameters forks), and, using Fact 2.2 below, it is observed in [7] that the converse holds when  $T$  is NIP. We now observe that, if  $T$  is NIP and first-order amenable, then having measure 0 under any automorphism-invariant Keisler measure implies forking. This is the ‘automorphism group analogue’ of Corollary 3.34 in [4] and is proved in an analogous way; the proof is a very quick consequence of results from [8, 5], but requires a number of preliminaries to formulate, which we give below.

## 2.1. Preliminaries

**2.1.1. KP-strong types.** Recall that, for a small tuple of variables  $x$ , two  $x$ -tuples  $b, c \in \mathfrak{C}^x$  are said to be *Kim–Pillay equivalent* (over  $\emptyset$ ), written  $E_{\text{KP}}(b, c)$ , if they lie in the same equivalence class of every bounded type-definable equivalence relation on  $\mathfrak{C}^x$ . For small tuples of variables  $x, y$  of the same arities and respective sorts, the relation  $E_{\text{KP}}(x, y)$  is bounded and type-definable, and the *KP-strong type* of  $b$  is defined to be the equivalence class of  $b$  under  $E_{\text{KP}}$ . The *KP-Galois group*, denoted  $\text{Gal}_{\text{KP}}(T)$ , is the quotient of  $\text{Aut}(\mathfrak{C})$  by the (normal) subgroup of automorphisms  $\sigma$  such that  $E_{\text{KP}}(b, \sigma(b))$  holds for every small tuple  $b$ . Given an automorphism  $\sigma$  of  $\mathfrak{C}$ , we will denote by  $\bar{\sigma}$  its image in  $\text{Gal}_{\text{KP}}(T)$ .

The type of a tuple over any small model determines its KP-strong type. With a bit more argument, it follows that, for any small  $M, N \prec \mathfrak{C}$ , if we let  $\bar{n}$  be an enumeration of  $N$  and let  $S_{\bar{n}}(M)$  denote the space of complete types over  $M$  extending  $\text{tp}(\bar{n}/\emptyset)$ , then the map taking  $\text{tp}(\sigma(\bar{n})/M)$  to  $\bar{\sigma}$  gives a well-defined surjection from  $S_{\bar{n}}(M)$  to  $\text{Gal}_{\text{KP}}(T)$ . The quotient topology on  $\text{Gal}_{\text{KP}}(T)$  induced by this map is independent of the choice of  $M, N$ , and makes  $\text{Gal}_{\text{KP}}(T)$  into a compact Hausdorff topological group. See for example [12] for details <sup>(1)</sup>.

In particular,  $\text{Gal}_{\text{KP}}(T)$  is equipped with a bi-invariant Haar measure of weight 1, i.e. a regular Borel measure  $\eta$  such that  $\eta(\text{Gal}_{\text{KP}}(T)) = 1$  and  $\eta(\bar{\sigma}S\bar{\tau}) = \eta(S)$  for any Borel subset  $S \subseteq \text{Gal}_{\text{KP}}(T)$  and any  $\bar{\sigma}, \bar{\tau} \in \text{Gal}_{\text{KP}}(T)$ .

**2.1.2. Forking in NIP theories.** In this section assume that  $T$  is NIP. Recall that a global type  $p(x) \in S_x(\mathfrak{C})$  is *KP-invariant* if, for any  $L(\mathfrak{C})$ -formula  $\phi(x, b)$  and any  $b'$  with the same KP-strong type as  $b$ , we have  $p(x) \vdash \phi(x, b) \leftrightarrow \phi(x, b')$ . From [8, Proposition 2.11] we have the following:

**FACT 2.1.** *A global type does not fork if and only if it is KP-invariant.*

In particular, given a global non-forking type  $p(x) \in S_x(\mathfrak{C})$  and an  $L(\mathfrak{C})$ -formula  $\phi(x, b)$ , the subset  $S_{p, \phi(x, b)} \subseteq \text{Gal}_{\text{KP}}(T)$  given by  $\{\bar{\sigma} : \sigma p(x) \vdash \phi(x, b)\}$  is well-defined. Again from [8] <sup>(2)</sup>, we have:

**FACT 2.2.** *If  $p(x) \in S_x(\mathfrak{C})$  does not fork, then  $S_{p, \phi(x, b)}$  is Borel for every  $L(\mathfrak{C})$ -formula  $\phi(x, b)$ . In particular, taking  $\mu_p(\phi(x, b)) := \eta(S_{p, \phi(x, b)})$  for each  $L(\mathfrak{C})$ -formula  $\phi(x, b)$  gives a well-defined Keisler measure on  $\mathfrak{C}^x$ , which is automorphism-invariant by the left-invariance of  $\eta$ .*

<sup>(1)</sup> More specifically, see Lemma 2.19 there for the definition of the closed Galois group of  $T$ , and then Theorem 2.21 for the equivalence of the notion with the KP-Galois group as we have defined it.

<sup>(2)</sup> This is proved in [8, Proposition 4.7]. For another writeup of the proof, using more similar language to that which we use in this paper, see remarks after [7, Fact 2.20].

Now, recall that a small subset  $B \subset \mathfrak{C}$  is called an *extension base* if no type over  $B$  forks over  $B$ . We have the following from [3, Theorem 1.2 and Remark 3.33]:

**FACT 2.3.** *Suppose  $B$  is an extension base. Then an  $L(\mathfrak{C})$ -formula forks over  $B$  if and only if it divides over  $B$ . Moreover, given an  $L$ -formula  $\psi(x, y)$ , the set of all  $c \in \mathfrak{C}^y$  such that  $\psi(x, c)$  forks over  $B$  is type-definable over  $B$ .*

We will also need the following, which is Proposition 25 in [5]; it is stated and proved there when working over a model, but precisely the same proof works over any extension base, so we repeat the proof here for clarity without that assumption. For the statement of the  $(p, k)$ -property and the  $(p, k)$ -theorem see [5, Section 0.4].

**FACT 2.4.** *Suppose  $\emptyset$  is an extension base,  $\phi(x, y)$  is an  $L$ -formula, and  $\tau(y)$  is a partial type over  $\emptyset$  such that for all  $b \models \tau$ ,  $\phi(x, b)$  does not fork. Then there are finitely many non-forking global types  $p_1, \dots, p_n \in S_x(\mathfrak{C})$  such that, for every  $b \models \tau$ , there is some  $p_i$  containing  $\phi(x, b)$ .*

*Proof.* Let  $X$  denote the closed subset of  $S_x(\mathfrak{C})$  consisting of those types which do not fork. Given  $b \models \tau$ , let  $F_b = \{q \in X : q \vdash \phi(x, b)\}$ , and let  $\mathcal{F}$  be the family  $\{F_b : b \models \tau\}$ . Since  $T$  is NIP,  $\mathcal{F}$  has finite VC-dimension, and hence finite VC-codimension – say strictly smaller than  $k \in \omega$ . We claim that  $\mathcal{F}$  has the  $(p, k)$ -property for some  $p \in \omega$ . Otherwise, for each  $p \in \omega$ , there is a sequence  $b_1, \dots, b_p$  of realizations of  $\tau$  such that, for any  $s \subseteq [p]$  of size  $k$ , we have  $\bigcap_{i \in s} F_{b_i} = \emptyset$ , so that the formula  $\bigwedge_{i \in s} \phi(x, b_i)$  forks.

Now, since  $\emptyset$  is an extension base, the set of  $(c_1, \dots, c_k)$  such that  $\bigwedge_{i \in [k]} \phi(x, c_i)$  forks is type-definable by Fact 2.3. Thus, by compactness, we may find an infinite sequence  $(b_i)_{i \in \omega}$  of realizations of  $\tau(y)$  such that  $\bigwedge_{i \in s} \phi(x, b_i)$  forks for every  $s \subset \omega$  of size  $k$ , and then again by compactness and by Ramsey's theorem we may assume  $(b_i)_{i \in \omega}$  is indiscernible. But this is a contradiction; letting  $q(x)$  be any global non-forking type containing  $\phi(x, b_0)$ , then by Lascar-invariance  $q$  contains  $\phi(x, b_i)$  for each  $i \in \omega$ , so that  $\bigwedge_{i \in [k]} \phi(x, b_i)$  does not fork.

So indeed  $\mathcal{F}$  has the  $(p, k)$ -property for some  $p \in \omega$ . By the  $(p, k)$ -theorem, there is thus some  $n \in \omega$  such that, for every finite subset  $\mathcal{F}_0 \subseteq \mathcal{F}$ , there is an  $n$ -element subset of  $X$  which meets every element of  $\mathcal{F}_0$ . Hence, by compactness, the partial type in variables  $x_1, \dots, x_n$  containing the following formulas is consistent: (i)  $\bigvee_{i \in [n]} \phi(x_i, b)$  for every  $b \models \tau$ , and (ii)  $\neg \psi(x_i, c)$  for every  $i \in [n]$  and every  $\psi(x_i, c) \in L(\mathfrak{C})$  which forks. Taking any global completion and letting  $p_i$  be its restriction to the variable  $x_i$ , the desired result follows. ■

**2.2. Proof of result.** We can now prove the result of this section. As mentioned above, it is proved analogously to Corollary 3.34 in [4], and is a quick corollary of Facts 2.2 and 2.4.

**THEOREM 2.5.** *Suppose that  $T$  is NIP,  $\emptyset$  is an extension base, and  $\phi(x, b)$  is a non-forking  $L(\mathfrak{C})$ -formula. Then there is a global automorphism-invariant Keisler measure giving  $\phi(x, b)$  positive measure.*

*Proof.* By Fact 2.4 applied to the type  $\pi(x) := \text{tp}(b/\emptyset)$ , there are non-forking types  $p_1, \dots, p_n \in S_x(\mathfrak{C})$  such that, for every  $b' \equiv b$ , the formula  $\phi(x, b')$  is contained in some  $p_i$ . In particular, the sets  $S_{p_1, \phi(x, b)}, \dots, S_{p_n, \phi(x, b)}$  defined in Section 2.1.2 cover  $\text{Gal}_{\text{KP}}(T)$ . Hence, for some  $i \in [n]$ ,  $S_{p_i, \phi(x, b)}$  has Haar measure at least  $1/n$ , and then the measure  $\mu_{p_i}$  from Fact 2.2 is automorphism-invariant and gives  $\phi(x, b)$  measure at least  $1/n$ . ■

**3. Connection with dynamics.** Corollary 3.34 in [4] is deduced using a ‘definable group analogue’ of Fact 2.4 <sup>(3)</sup>, which is proved using topological dynamics in [4, Section 3.3]. We observe in this section that an analogous argument also gives a topological dynamics proof of Fact 2.4. The proof of Fact 2.4 from [5] is quite short, so our main intention here is less to give a different proof of it, and more to use the proof as an opportunity to point out the ‘automorphism group analogues’ of some notions from topological dynamics that are well-studied for definable groups.

First let us recall the basic setup. Let  $G$  be a (discrete) group. Then a  $G$ -flow is a non-empty compact Hausdorff space  $X$  together with a left action of  $G$  on  $X$  by homeomorphisms. A *subflow* of  $X$  is a closed  $G$ -invariant subset of  $X$ , i.e. a non-empty closed subset  $Y \subseteq X$  with  $gY = Y$  for all  $g \in G$ . The flow  $X$  is *point-transitive* if, for some  $p \in X$ , the orbit  $Gp$  is a dense subset of  $X$ . An open subset  $U \subseteq X$  is called *generic* if  $X = g_1U \cup \dots \cup g_nU$  for some  $g_1, \dots, g_n \in G$ , and is called *weakly generic* if there is a non-generic open set  $V$  such that  $U \cup V$  is generic. A point  $p \in X$  is called (weakly) generic if every one of its open neighborhoods is (weakly) generic. Finally, a point  $p \in X$  is said to be *almost periodic* if it lies in a minimal subflow of  $X$ . We have the following from [9, Lemma 1.7(3)]:

**FACT 3.1.** *Let  $X$  be a point-transitive  $G$ -flow. Then every almost periodic point of  $X$  is weakly generic.*

There are several flows that arise naturally in model theory. For example, let  $x$  be a tuple of variables. Then  $S_x(\mathfrak{C})$  is naturally a flow for the automorphism group  $G = \text{Aut}(\mathfrak{C})$ , and a subset  $Y \subseteq S_x(\mathfrak{C})$  is a subflow if and only if it is of the form  $S_{\pi(x)}(\mathfrak{C})$ , where  $\pi(x)$  is an automorphism-invariant partial type over  $\mathfrak{C}$  and  $S_{\pi(x)}(\mathfrak{C})$  is the set of complete  $x$ -types over  $\mathfrak{C}$  extending  $\pi(x)$ . Fix such a  $\emptyset$ -invariant partial type  $\pi(x)$ .

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<sup>(3)</sup> The analogue in this case is that, in a definably amenable NIP group, for any  $f$ -generic formula  $\phi(x, b)$  there are finitely many global  $f$ -generic types  $p_1, \dots, p_n$  such that every translate of  $\phi(x, b)$  is contained in some  $p_i$ .

Let us say that an  $L(\mathfrak{C})$ -formula  $\phi(x, b)$  is (weakly) generic for  $\pi(x)$  if the clopen subset of  $S_{\pi(x)}(\mathfrak{C})$  corresponding to  $\phi(x, b)$  is (weakly) generic; then  $\phi(x, b)$  is generic for  $\pi(x)$  if and only if  $\pi(x) \vdash \bigvee_{i \in [n]} \phi(x, b_i)$  for some finitely many  $b_1, \dots, b_n$  with  $b_i \equiv b$  for each  $i \in [n]$ , and, by compactness in  $S_{\pi(x)}(\mathfrak{C})$ ,  $\phi(x, b)$  is weakly generic for  $\pi(x)$  if and only if there is an  $L(\mathfrak{C})$ -formula  $\psi(x, c)$  not generic for  $\pi(x)$  such that  $\phi(x, b) \vee \psi(x, c)$  is generic for  $\pi(x)$ .

Now the following lemmas correspond respectively to Proposition 3.33 and Proposition 3.30 in [4]. Note that Lemma 3.2 does not assume  $\emptyset$  is an extension base, and Lemma 3.3 does not assume that  $T$  is NIP.

**LEMMA 3.2.** *Suppose that  $T$  is NIP and  $\tau(y)$  is a partial type over  $\emptyset$  such that  $\phi(x, b)$  does not divide for all  $b \models \tau$ . Then there are finitely many almost periodic types  $p_1, \dots, p_n \in X = S_x(\mathfrak{C})$  such that, for any  $b \models \tau$ , there is  $i \in [n]$  with  $p_i \vdash \phi(x, b)$ .*

*Proof.* Given  $b \models \tau$ , let  $F_b = \{p \in X : p \vdash \phi(x, b)\}$ . Since  $T$  is NIP, the family of subsets  $\mathcal{F} = \{F_b : b \models \tau\}$  has finite VC-codimension, say strictly smaller than  $q \in \omega$ . Now  $\mathcal{F}$  has the  $(p, q)$ -property for some  $p \in \omega$ ; otherwise we can find arbitrary long sequences  $b_1, \dots, b_p$  of realizations of  $\tau$  such that, for any  $s \subseteq [p]$  of size  $q$ , we have  $\bigcap_{i \in s} F_{b_i} = \emptyset$ , so that  $\bigwedge_{i \in s} \phi(x, b_i)$  is inconsistent. But then by Ramsey and compactness we may find an indiscernible sequence  $(b_i)_{i \in \omega}$  of realizations of  $\tau$  such that  $\bigwedge_{i \in s} \phi(x, b_i)$  is inconsistent for every  $s \subset \omega$  of size  $q$ , contradicting that  $\phi(x, b)$  does not divide for all  $b \models \tau$ . So indeed  $\mathcal{F}$  has the  $(p, q)$ -property for some  $p \in \omega$ , and so by compactness and the  $(p, q)$ -theorem we may find finitely many  $p_1, \dots, p_n \in X$  such that, for every  $b \models \tau$ , there is some  $i \in [n]$  with  $p_i \vdash \phi(x, b)$ .

Now consider the flow  $X^n$ , equipped with the product topology and on which  $G$  acts diagonally. Let  $Y$  be the closure in  $X^n$  of the orbit of  $(p_1, \dots, p_n)$ . We claim that, for any  $(q_1, \dots, q_n) \in Y$ , and any  $b \models \tau$ , there is some  $i \in [n]$  with  $q_i \vdash \phi(x, b)$ . To see this, suppose otherwise. Let  $O$  be the clopen subset of  $X$  corresponding to the formula  $\neg\phi(x, b)$ ; then  $O^n$  is an open subset of  $X^n$  containing  $(q_1, \dots, q_n)$ . But  $O^n$  is also disjoint from the orbit of  $(p_1, \dots, p_n)$ ; indeed, if  $\sigma$  is an automorphism, then some  $p_i$  contains  $\phi(x, \sigma^{-1}(b))$ , and so some  $\sigma p_i$  contains  $\phi(x, b)$ . So  $(q_1, \dots, q_n)$  lying in  $O^n$  contradicts the definition of  $Y$ .

By Zorn's lemma, let  $Z$  be a minimal subflow of  $Y$ . Then the image of  $Z$  under any of the projection maps  $X^n \rightarrow X$  is a minimal subflow of  $X$ . In particular, picking any  $(q_1, \dots, q_n) \in Z$ , then the types  $q_1, \dots, q_n$  have the desired properties. ■

**LEMMA 3.3.** *Suppose that forking and dividing coincide (over  $\emptyset$ ), and  $\pi(x)$  is an automorphism-invariant partial type over  $\mathfrak{C}$ . Then a weakly generic formula for  $\pi(x)$  does not fork.*

*Proof.* Suppose that  $\phi(x, b)$  forks and  $\phi(x, b) \vee \psi(x, c)$  is generic for  $\pi(x)$ . We will show that  $\psi(x, c)$  is generic for  $\pi(x)$ , which will give the desired result. Since  $\phi(x, b) \vee \psi(x, c)$  is generic for  $\pi(x)$ , there are  $(b_1, c_1), \dots, (b_n, c_n)$  with  $(b_i, c_i) \equiv (b, c)$  for each  $i \in [n]$  and  $\pi(x) \vdash \bigvee_{i \in [n]} (\phi(x, b_i) \vee \psi(x, c_i))$ . In other words, we have  $\pi(x) \vdash \zeta(x, \bar{b}) \vee \xi(x, \bar{c})$ , where

$$\zeta(x, y_1, \dots, y_n) \equiv \bigvee_{i \in [n]} \phi(x, y_i), \quad \xi(x, z_1, \dots, z_n) \equiv \bigvee_{i \in [n]} \psi(x, z_i),$$

and  $\bar{b} = (b_1, \dots, b_n), \bar{c} = (c_1, \dots, c_n)$ . Now, the formulas which fork form an ideal, and  $\phi(x, b_i)$  forks for each  $i$ , so  $\zeta(x, \bar{b})$  forks as well. Since forking and dividing coincide, there is thus an indiscernible sequence  $(\bar{b}_k = (b_{k1}, \dots, b_{kn}))_{k \in \omega}$  such that  $\bar{b}_0 = \bar{b}$  and the family  $(\zeta(x, \bar{b}_k) : k \in \omega)$  of formulas is inconsistent.

Since  $\bar{b} \equiv \bar{b}_k$  for each  $k$ , we may find  $(\bar{c}_k : k \in \omega)$  such that  $(\bar{b}_k, \bar{c}_k) \equiv (\bar{b}, \bar{c})$  for each  $k$ . In particular,  $\pi(x) \vdash \zeta(x, \bar{b}_k) \vee \xi(x, \bar{c}_k)$  for each  $k$ . On the other hand, for some  $q \in \omega$ , we have  $\bigwedge_{k \in [q]} \zeta(x, \bar{b}_k)$  inconsistent, whence  $\pi(x) \vdash \bigvee_{k \in [q]} \xi(x, \bar{c}_k)$ , i.e.  $\pi(x) \vdash \bigvee_{k \in [q]} \bigvee_{i \in [n]} \psi(x, c_{ki})$ . Since  $\bar{c} \equiv \bar{c}_k$  for each  $k \in \omega$ , it follows that  $c_{ki} \equiv c$  for all  $k, i \in \omega$ , and hence indeed  $\psi(x, c)$  is generic for  $\pi(x)$ . ■

Now Fact 2.4 follows from Lemmas 3.2 and 3.3. Indeed, suppose  $T$  is NIP and  $\emptyset$  is an extension base,  $\phi(x, y)$  is a formula, and  $\tau(y)$  is a partial type over  $\emptyset$  such that  $\phi(x, b)$  does not fork for all  $b \models \tau$ . By Lemma 3.2, there are almost periodic types  $p_1, \dots, p_n \in S_x(\mathfrak{C})$  such that, for any  $b \models \tau$ , the formula  $\phi(x, b)$  lies in some  $p_i$ , so it suffices to show that  $p_i$  is non-forking for each  $i \in [n]$ .

To see this, let  $Y_i$  be the (unique) minimal subflow of  $S_x(\mathfrak{C})$  containing  $p_i$ , and let  $\pi_i(x)$  be an automorphism-invariant partial type over  $\mathfrak{C}$  such that  $Y_i = S_{\pi_i(x)}(\mathfrak{C})$ . Since  $Y_i$  is a minimal flow, by Fact 3.1 any point of  $Y_i$  is weakly generic in  $Y_i$ , and in particular  $p_i$  is. But  $\emptyset$  is an extension base, so forking and dividing coincide by Fact 2.3, and now the result follows from Lemma 3.3.

**4. Counterexample.** In this section we show that Theorem 2.5 can fail if the amenability hypothesis is dropped. First, in Section 4.1, we give a general construction, which produces from a structure  $M$  a new structure  $M_{\text{cyc}}$  such that the forking of formulas of a certain shape is controlled precisely by forking behavior *and* algebraic behavior in  $M$ . Then in Section 4.2 we make some basic observations about the model theory of the binary tree  $M = (2^{<\omega}, \leq)$ , and then in Section 4.3 we show that the structure  $M_{\text{cyc}}$  associated to it has the desired behavior.

**4.1. General construction.** Let  $L$  be a first-order language, and assume for notational convenience that it is one-sorted. We define a new language  $L_{\text{cyc}}$ , which has two sorts  $E$  and  $B$ , a function symbol  $\pi : E \rightarrow B$ , a ternary relation symbol  $\text{cyc}$  on the sort  $E$ , and all the symbols of  $L$ , taken as symbols on the sort  $B$ . Now, given an  $L$ -structure  $M$ , we define a corresponding  $L_{\text{cyc}}$ -structure  $M_{\text{cyc}}$  by setting  $E(M_{\text{cyc}}) = S^1 \times M$  and  $B(M_{\text{cyc}}) = M$ , letting  $\pi : S^1 \times M \rightarrow M$  be the projection map, and taking  $\text{cyc}$  to be the circular order on each fiber of  $\pi$  and to induce no other relations. (So that  $\text{cyc}(x, y, z) \rightarrow \pi(x) = \pi(y) = \pi(z)$ .)

Note that  $M_{\text{cyc}}$  is interpretable in the two-sorted structure  $(S^1, M)$ ; in particular, if  $M$  is NIP, then  $M_{\text{cyc}}$  will be too. Now, let  $\mathfrak{C}$  be a saturated model of  $\text{Th}(M_{\text{cyc}})$ . By uniqueness of saturated models,  $\mathfrak{C}$  can be identified with the projection  $\bar{S}^1 \times \bar{M} \rightarrow \bar{M}$ , where  $\bar{S}^1, \bar{M}$  are saturated models of the circular order and of  $\text{Th}(M)$ , respectively. Note that the automorphisms of  $\mathfrak{C}$  are in bijection with pairs  $(\alpha, (f_e)_{e \in \bar{M}})$ , where  $\alpha$  is an automorphism of  $\bar{M}$  and  $f_e$  is an automorphism of  $\bar{S}^1$  for each  $e \in \bar{M}$ ; the associated automorphism of  $\mathfrak{C}$  acts on the  $E$ -sort by taking  $(u, e)$  to  $(f_e(u), \alpha(e))$ .

In the section we will focus on  $L_{\text{cyc}}(\bar{M})$ -formulas of the form  $\phi(\pi(x), b)$ , where  $\phi(z, b)$  is an  $L(\bar{M})$ -formula. We will show that  $\phi(\pi(x), b)$  forks in  $\mathfrak{C}$  if and only if  $\phi(z, b)$  implies a disjunction of an algebraic  $L$ -formula and a forking  $L(\bar{M})$ -formula; first we give the backward direction:

**LEMMA 4.1.** *Let  $\theta(z)$  be an algebraic  $L$ -formula and  $\psi(z, d)$  an  $L(\bar{M})$ -formula that forks in  $\bar{M}$ . Then the formula  $\theta(\pi(x)) \vee \psi(\pi(x), d)$  forks in  $\mathfrak{C}$ .*

*Proof.* It suffices to show that  $\theta(\pi(x))$  and  $\psi(\pi(x), d)$  each fork in  $\mathfrak{C}$ . To see that  $\theta(\pi(x))$  forks in  $\mathfrak{C}$ , let  $b_1, \dots, b_n$  be the realizations of  $\theta(z)$ ; now  $\theta(\pi(x))$  implies the disjunction  $\bigvee_{i \in [n]} (\pi(x) = b_i)$ . But each formula  $\pi(x) = b_i$  forks in  $\mathfrak{C}$ , since the induced structure on the fiber  $\bar{S}^1 \times \{b_i\}$  is just the circular order, and any formula in the circular order forks. So we need only show that  $\psi(\pi(x), d)$  forks in  $\mathfrak{C}$ .

By hypothesis,  $\psi(z, d)$  implies a finite disjunction  $\bigvee_{i \in [n]} \xi_i(z, e)$ , where each  $\xi_i(z, e)$  is an  $L(\bar{M})$ -formula that divides in  $\bar{M}$ , and so it suffices to show that each  $\xi_i(\pi(x), e)$  divides in  $\mathfrak{C}$ . But if  $(e_j)_{j \in \omega}$  is an indiscernible sequence from  $\bar{M}$  with  $e_0 = e$  and  $(\xi_i(z, e_j) : j \in \omega)$  inconsistent, then  $(\xi_i(\pi(x), e_j) : j \in \omega)$  is also inconsistent, and  $(e_j)_{j \in \omega}$  is still an indiscernible sequence in  $\mathfrak{C}$ , so that indeed  $\xi_i(\pi(x), e)$  divides in  $\mathfrak{C}$ . ■

In the rest of the section we will show the forward direction of the claim above; first we need some auxiliary observations about types in  $\mathfrak{C}$ .

**LEMMA 4.2.** *Let  $(c_1, \dots, c_n)$  and  $(c'_1, \dots, c'_n)$  be tuples from  $E(\mathfrak{C}) = \bar{S}^1 \times \bar{M}$  and  $a, a'$  elements of  $E(\mathfrak{C})$  such that*

- (i)  $\pi(a) \neq \pi(c_i)$  and  $\pi(a') \neq \pi(c'_i)$  for each  $i$ ,
- (ii)  $(\pi(a), \pi(\bar{c}))$  and  $(\pi(a'), \pi(\bar{c}'))$  have the same type in  $\bar{M}$ ,
- (iii)  $\bar{c}$  and  $\bar{c}'$  have the same type in  $\mathfrak{C}$ .

Then  $(a, \bar{c})$  and  $(a', \bar{c}')$  have the same type in  $\mathfrak{C}$ .

*Proof.* By (iii), let  $(\alpha, (f_e)_{e \in \bar{M}})$  be the data associated to an automorphism of  $\mathfrak{C}$  taking  $\bar{c}$  to  $\bar{c}'$ . Also, by (ii), let  $\beta$  be an automorphism of  $\bar{M}$  taking  $(\pi(a), \pi(\bar{c}))$  to  $(\pi(a'), \pi(\bar{c}'))$ . Finally, using the fact that any two elements of  $\bar{S}^1$  have the same type, let  $f$  be an automorphism of  $\bar{S}^1$  taking the first coordinate of  $a$  to the first coordinate of  $a'$ . Now, for  $e \in \bar{M}$ , define an automorphism  $g_e$  of  $\bar{S}^1$  by cases:

$$g_e = \begin{cases} f & \text{if } e = \pi(a), \\ f_e & \text{if } e = \pi(c_i) \text{ for some } i, \\ \text{id}_{\bar{S}^1} & \text{otherwise.} \end{cases}$$

By condition (i), this is well-defined, and the automorphism of  $\mathfrak{C}$  associated to  $(\beta, (g_e)_{e \in \bar{M}})$  then takes  $(a, \bar{c})$  to  $(a', \bar{c}')$ , as needed. ■

**COROLLARY 4.3.** *Suppose  $C \subseteq E(\mathfrak{C}) = \bar{S}^1 \times \bar{M}$  and  $a \in E(\mathfrak{C})$  are such that (i)  $\pi(a) \notin \pi(C)$ , and (ii)  $\text{tp}(\pi(a)/\pi(C))$  does not divide in  $\bar{M}$ . Then  $\text{tp}(a/C)$  does not divide in  $\mathfrak{C}$ .*

*Proof.* Let  $(\bar{c}_i)_{i \in \omega}$  be an indiscernible sequence, where  $\bar{c}_0$  is a finite tuple from  $C$ . We wish to find  $a'$  such that  $(a', \bar{c}_i)$  and  $(a, \bar{c}_0)$  have the same type in  $\mathfrak{C}$  for every  $i \in \omega$ . Since  $\text{tp}(\pi(a)/\pi(C))$  does not divide, we may (picking a first coordinate arbitrarily) find some  $a' \in E(\mathfrak{C})$  such that  $(\pi(a'), \pi(\bar{c}_i))$  and  $(\pi(a), \pi(\bar{c}_0))$  have the same type in  $\bar{M}$  for every  $i \in \omega$ . But, since  $\pi(a) \notin \pi(C)$ , this gives the desired result by Lemma 4.2. ■

Now we can give the desired characterization.

**LEMMA 4.4.** *Let  $\phi(z, b)$  be an  $L(\bar{M})$ -formula. Then  $\phi(\pi(x), b)$  forks in  $\mathfrak{C}$  if and only if there is an  $L$ -formula  $\theta(z)$  and an  $L(\bar{M})$ -formula  $\psi(z, d)$  such that  $\theta(z)$  is algebraic,  $\psi(z, d)$  forks in  $\bar{M}$ , and  $\phi(z, b) \vdash \theta(z) \vee \psi(z, d)$ .*

*Proof.* The backward direction is Lemma 4.1, so we show the forward direction. Suppose for contradiction that  $\phi(\pi(x), b)$  forks in  $\mathfrak{C}$  but that the desired conclusion does not hold. Then there are some finite  $C \subset E(\mathfrak{C})$  and  $D \subset B(\mathfrak{C}) = \bar{M}$  such that, for any  $a \in \bar{S}^1 \times \bar{M}$  realizing  $\phi(\pi(x), b)$ ,  $\text{tp}(a/C, D)$  divides. Expanding  $C$  if necessary, we may assume without loss of generality that  $\pi(C) \supseteq D$ , so that  $\text{tp}(a/C)$  divides for any  $a \models \phi(\pi(x), b)$ .

Now, by hypothesis,  $\phi(z, b)$  is consistent with  $\neg\theta(z) \wedge \neg\psi(z, d)$  for every algebraic  $L$ -formula  $\theta(z)$  and every  $L(\pi(C))$ -formula  $\psi(z, d)$  that forks in  $\bar{M}$ . Formulas of the last two kinds form an ideal, so by compactness (and picking an arbitrary first coordinate) we may find some  $a \in E(\mathfrak{C})$  such that

- (1)  $\pi(a) \models \phi(z, b)$ ,
- (2)  $\text{tp}(\pi(a)/\emptyset)$  is non-algebraic,
- (3)  $\text{tp}(\pi(a)/\pi(C))$  does not fork in  $\bar{M}$ .

Conditions (2) and (3) together imply that  $\pi(a) \notin \pi(C)$ . By (3) and Corollary 4.3, this implies that  $\text{tp}(a/C)$  does not divide in  $\mathfrak{C}$ . By (1), this contradicts the choice of  $C$ . ■

**4.2. Binary tree.** Let  $L = \{\leq\}$  be the language of trees and let  $M$  be the tree  $(2^{<\omega}, \leq)$ . Note that  $M$  is NIP, and indeed dp-minimal, since any tree is; see [11]. We leave it to the reader to show that the theory of  $M$  has QE after adding a constant for the root, and function symbols for the meet function  $\wedge$  and the predecessor function.

Let  $\bar{M}$  be a saturated model of  $\text{Th}(M)$ . We have  $\text{acl}(\emptyset) = M$ , so that  $M$  is the unique minimal elementary substructure of  $\bar{M}$ . By a back-and-forth argument, or by QE, any two elements  $b, b' \in \bar{M} \setminus M$  have the same type over  $\emptyset$ , and the non-algebraic 1-types over  $M$  correspond to paths through  $2^{<\omega}$ : given  $f \in 2^\omega$ , there is a unique 1-type over  $M$  determined by  $\{x > f|_n : n \in \omega\}$ , and conversely any unrealized 1-type over  $M$  is of that form.

Now, suppose  $I = (b_i)_{i \in \omega}$  is any non-constant indiscernible sequence of elements of  $\bar{M}$ ; then, since  $M = \text{acl}(\emptyset)$ ,  $I$  is indiscernible over  $M$ , and so all the elements of  $I$  determine the same path through  $M$ . In particular, for every  $n \in \omega$ , we have  $b_1 \wedge \cdots \wedge b_n \notin M$ .

**LEMMA 4.5.** *Let  $b \in \bar{M} \setminus M$ . Then the formula  $\phi(z, b) \equiv z \leq b$  does not imply a disjunction of an algebraic formula without parameters and an  $L(\bar{M})$ -formula that forks.*

*Proof.* It suffices to show that for every algebraic  $\theta(z)$  without parameters,  $\phi(z, b) \wedge \neg\theta(z)$  does not fork over  $\emptyset$ . So suppose otherwise; then  $\phi(z, b) \wedge \neg\theta(z)$  implies a finite disjunction of formulas that each divide over  $\emptyset$ . Since  $M = \text{acl}(\emptyset)$ , a formula divides over  $\emptyset$  if and only if it divides over  $M$ , and hence  $\phi(z, b) \wedge \neg\theta(z)$  implies a finite disjunction of formulas that each divide over  $M$ ; in other words,  $\phi(z, b) \wedge \neg\theta(z)$  forks over  $M$ . But  $\text{Th}(M)$  is NIP, so by Fact 2.3 forking and dividing coincide over models, and hence  $\phi(z, b) \wedge \neg\theta(z)$  divides over  $M$ . Let  $(b_i)_{i \in \omega}$  be an indiscernible sequence witnessing this; then for some  $n \in \omega$  we have  $\phi(z, b_1) \wedge \cdots \wedge \phi(z, b_n) \vdash \theta(z)$ . In particular, we have  $b_1 \wedge \cdots \wedge b_n \models \theta(z)$ . But  $\theta$  is algebraic, so this forces  $b_1 \wedge \cdots \wedge b_n \in M$ , contradicting the remark above this lemma. ■

**4.3. Main example.** Now we can give the main example. Let  $M$  be the tree from Section 4.2. Let  $\mathfrak{C}$  be a saturated model of  $\text{Th}(M_{\text{cyc}})$ , with  $E$ -sort  $\bar{S}^1 \times \bar{M}$ . Pick any  $b \in \bar{M} \setminus M$ , and let  $\phi(z, b) \equiv z \leq b$ .

LEMMA 4.6. *The formula  $\phi(\pi(x), b)$  does not fork.*

*Proof.* This follows from Lemmas 4.4 and 4.5. ■

LEMMA 4.7. *Let  $\mu$  be an automorphism-invariant Keisler measure on the  $E$ -sort of  $\bar{M}$ . Then  $\mu(\phi(\pi(x), b)) = 0$ .*

*Proof.* Suppose otherwise that  $\mu(\phi(\pi(x), b)) = \varepsilon > 0$ . Recalling that any two elements of  $\bar{M} \setminus M$  have the same type over  $\emptyset$ , we thus have  $\mu(\phi(\pi(x), b')) = \varepsilon$  for every  $b' \in \bar{M} \setminus M$ . Let  $n \in \mathbb{N}$  be such that  $n\varepsilon > 1$ , let  $m \in \mathbb{N}$  be such that  $2^m > n$ , and let  $P_m(z) \equiv \exists^{\leq m} u (u \leq z)$  be the formula of  $\bar{M}$  expressing that  $z$  is of height at most  $m$ . Since  $P_m(z)$  is algebraic, by Lemma 4.1 the formula  $P_m(\pi(x))$  forks over  $\emptyset$ , and so by  $\emptyset$ -invariance of  $\mu$  we have  $\mu(P_m(\pi(x))) = 0$ .

On the other hand, since  $2^m > n$ , we may find *distinct* elements  $c_1, \dots, c_n \in M$  of height precisely  $m$ . Pick any  $b_i \in \bar{M} \setminus M$  with  $b_i \geq c_i$ ; then  $b_i \wedge b_j \models P_m(z)$  for each  $i \neq j$ . Hence  $\phi(z, b_i) \wedge \phi(z, b_j) \vdash P_m(z)$  for all  $i \neq j$ ; since  $P_m(\pi(x))$  has measure 0 under  $\mu$ , also

$$\mu(\phi(\pi(x), b_i) \wedge \phi(\pi(x), b_j)) = 0$$

for all  $i \neq j$ . But then

$$\mu\left(\bigvee_{i \in [n]} \phi(\pi(x), b_i)\right) = \sum_{i \in [n]} \mu(\phi(\pi(x), b_i)) = n\varepsilon > 1,$$

a contradiction. ■

Thus  $\text{Th}(M_{\text{cyc}})$  is an NIP theory for which, in a saturated model, there are non-forking formulas given measure 0 by any automorphism-invariant Keisler measure; this gives the desired example.

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## References

- [1] H. Adler, *Introduction to theories without the independence property*, preprint (2008).
- [2] A. Chernikov, E. Hrushovski, A. Kruckman, K. Krupiński, S. Moconja, A. Pillay and N. Ramsey, *Invariant measures in simple and in small theories*, J. Math. Logic 23 (2023), art. 2250025, 37 pp.
- [3] A. Chernikov and I. Kaplan, *Forking and dividing in  $NTP_2$  theories*, J. Symbolic Logic 77 (2012), 1–20.
- [4] A. Chernikov and P. Simon, *Definably amenable NIP groups*, J. Amer. Math. Soc. 31 (2018), 609–641.

- [5] A. Chernikov and P. Simon, *Externally definable sets and dependent pairs II*, Trans. Amer. Math. Soc. 367 (2015), 5217–5235.
- [6] E. Hrushovski, K. Krupiński and A. Pillay, *Amenability, connected components, and definable actions*, Selecta Math. 28 (2022), art. 16, 56 pp.
- [7] E. Hrushovski, K. Krupiński and A. Pillay, *On first order amenability*, arXiv:2004.08306v3 (2020).
- [8] E. Hrushovski and A. Pillay, *On NIP and invariant measures*, J. Eur. Math. Soc. 13 (2011), 1005–1061.
- [9] L. Newelski, *Topological dynamics of definable group actions*, J. Symbolic Logic 74 (2009), 50–72.
- [10] S. Shelah, *Dependent first order theories, continued*, Israel J. Math. 173 (2009), 1–60.
- [11] P. Simon, *On dp-minimal ordered structures*, J. Symbolic Logic 76 (2011), 448–460.
- [12] M. Ziegler, *Introduction to the Lascar group*, in: Tits Buildings and the Model Theory of Groups (Würzburg, 2000), Cambridge Univ. Press, 2002, 279–298.

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