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Some notes on uncountable models of arithmetic

by

Wei Wang

Abstract. Historically, logicians have found it very difficult to understand the structure of uncountable models of arithmetic. I argue that if a model M of arithmetic has an element a with $|[0,a]^M| = \aleph_0$, then it resembles a countable model. As evidence, I present two theorems. The first states that if M is such a model with its standard system contained in some Scott set $\mathcal{S}$ and $X \in \mathcal{S}$, then M admits an elementary extension N with $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$. The second states that if M is such a model and L is a finite lattice such that every countable M_0 has an elementary extension N_0 with $\mathrm{Lt}(N_0/M_0) \cong L$, then M can also be extended this way. An alternative proof of Ehrenfeucht's lemma is also included.

1. Introduction. Models of arithmetic are interesting objects for logicians, possibly because they allow arithmetization of metamathematics. Throughout this article, *models* are always models of arithmetic, which are first-order structures $(M; 0, 1, +, \times, <)$ satisfying the first order axiomatization PA of Peano's arithmetic. In the theory of models of arithmetic, countable models are extensively studied, while the structure of uncountable models is very difficult to understand.

One example is the Scott set problem. Recall that a *Scott set* $\mathcal{S}$ is a non-empty collection of subsets of $\mathbb{N}$ such that

- if both X and Y are in $\mathcal{S}$ and Z is recursive in $X \oplus Y$, then $Z \in \mathcal{S}$;
- if $\mathcal{S}$ contains an infinite binary tree T, then $\mathcal{S}$ also contains an infinite path on T.

Given a model M, a subset of $\mathbb{N}$ is *coded* in M if it is the intersection of some definable subset of M with $\mathbb{N}$. The collection of subsets of $\mathbb{N}$ that are coded in M is called the *standard system* of M, denoted by $\mathrm{SSy}(M)$. Scott [10]

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proved that the standard system of a countable model can be characterized in a computability theoretic way as a Scott set. Knight and Nadel [6] found that Scott's characterization also holds for models of cardinality $\aleph_1$. In recent years, there have been some attempts to understand standard systems of uncountable models in general (e.g., [2, 3, 5]). But the Scott set problem, whether every Scott set is exactly $\mathrm{SSy}(M)$ of some model M, remains open.

Another example concerns cofinal extensions of models. For M a submodel of N, N is called a *cofinal* extension of M (denoted by $M \subseteq_{\mathrm{cof}} N$), if every $b \in N$ is less than some $a \in M$; and N is called an *end* extension of M ($M \subseteq_{\mathrm{end}} N$) if every $b \in N - M$ is greater than every $a \in M$. The famous Mac Dowell–Specker theorem [8] says that every model has an elementary end extension, and Gaifman [4] improved the Mac Dowell–Specker theorem by showing that every model has a minimal elementary end extension, where an elementary extension N of M is *minimal* if $M \prec N$ and there does not exist M' such that $M \prec M' \prec N$. On the other hand, Blass [1] proved that every countable model has a minimal elementary cofinal extension. It is another open question whether every model has a minimal elementary cofinal extension, though Gitman [5] obtained a positive answer for a very special case.

In this article, I shall present some attempts to understand uncountable models. The models considered here are mainly models $M \models \mathrm{PA}$ with a countable infinite initial segment $[0, a]^M$. Section 2 includes an alternative proof of Ehrenfeucht's lemma from the author's unpublished draft [11]. Ehrenfeucht's lemma is well-known and yields an alternative proof of Knight and Nadel's theorem. The alternative proof of Ehrenfeucht's lemma looks more straightforward and may be interesting in its own right. In Section 3, I shall introduce an improvement of Ehrenfeucht's lemma, which also improves some results of Gitman. In Section 4 we focus on interstructure lattices over such M, and prove that if M has a countable infinite initial segment $[0, a]^M$ and L is a finite lattice that can be realized as an interstructure lattice over any countable model, then M has a cofinal elementary extension N with $\mathrm{Lt}(N/M) \cong L$. Detailed definitions of interstructure lattices and the Lt operator will be given at the end of this section. From these results, the reader may find that the models mentioned above are very much like countable models indeed.

The terminology and notations here are quite standard. We refer to [7] for general background knowledge. For the reader's convenience, some conventions and notions important to this article are included below.

A set $X \subseteq \omega$ is identified with its characteristic function. So $X(n) = 1$ if $n \in X$, and $X(n) = 0$ if $n \notin X$.

Fix a model M. In M, there are various ways of coding sets. The preferred coding here is by binary expansion. Thus, if $a = \sum_{i=0}^{k} b_i 2^i$ in M with $b_i < 2$, then $(a)_i = b_i$ and a codes $\{i < k : b_i = 1\}$. As functions can be identified with their graphs, elements of M can also code functions. Sets and functions coded in M are called *M-finite sets* and *M-finite functions* and are usually identified with their codes in M. So an expression like $X \in M$ means that X is coded in M, etc.

By a *type over M*, we mean a set $p(x)$ of formulas such that

- each $\varphi(x) \in p(x)$ has a unique free variable x and possibly parameters from M;
- $p(x)$ is finitely satisfiable in M.

Via the coding power of PA, we can assume without loss of generality that more free variables are allowed in the type. The type $p(x)$ is *complete* if for every $\varphi(x)$ possibly with parameters from M either $p(x) \vdash \varphi(x)$ or $p(x) \vdash \neg\varphi(x)$. We say $p(x)$ is *bounded* if $p(x) \vdash x < b$ for some b from the underlying model M. For an element a in an extension N of M, *a realizes* a type $p(x)$ over M if $N \models \varphi(a)$ for every $\varphi(x) \in p(x)$.

If $M \prec N$ and N is the Skolem hull of $M \cup \{a\}$ for some $a \in N$, then N is occasionally written as $M(a)$ and consists of $F^N(a)$'s where F is a function definable over M by some φ and F^N is the function defined over N using the same φ. Of course, this is due to the existence of definable Skolem functions in PA. If the element a above realizes a complete type $p(x)$ over M, then $M(a)$ is called a *$p(x)$-extension* of M. If $p(x)$ is complete and N_0, N_1 are two $p(x)$-extensions of M, then N_0 and N_1 are isomorphic over M, i.e., there is an isomorphism $\pi : N_0 \to N_1$ and π is the identity on M. The $p(x)$-extensions in this article are always obtained from bounded types $p(x)$. Since definable functions with coded domains are also coded in M, it follows that if $M(a)$ is a $p(x)$-extension of M for some $p(x) \vdash x < b$, then

$$M(a) = \{f(a) : f \in M \models \text{“}f \text{ is a function with domain } [0, b]\text{”}\},$$

where $f(a)$ is certainly evaluated in $M(a)$.

Also recall the *interstructure lattice* $\mathrm{Lt}(N/M)$ of $M \prec N$. For $M \prec N$, $\mathrm{Lt}(N/M)$ consists of all M' such that $M \preceq M' \preceq N$. The lattice $\mathrm{Lt}(N/M)$ is naturally ordered by the elementary submodel relation which makes $\mathrm{Lt}(N/M)$ a complete lattice. The infimum of $\mathcal{A} \subseteq \mathrm{Lt}(N/M)$ simply equals $\bigcap \mathcal{A}$, and the supremum of $\mathcal{A}$ equals the Skolem hull of $\bigcup \mathcal{A}$.

2. An alternative proof of Ehrenfeucht's lemma. As mentioned in the introduction, Knight and Nadel proved that every Scott set of cardinality $\aleph_1$ is the standard system of some model. Besides the original proof of Knight and Nadel, there are several known alternative proofs, and some use the following Theorem 2.1 called *Ehrenfeucht's lemma.* The reader can find a proof of Ehrenfeucht's lemma in Gitman [5], which uses recursive saturation. But the new proof here looks more straightforward and does not need recursive saturation.

THEOREM 2.1 (Ehrenfeucht). *Let $\mathcal{S}$ be a Scott set and M a countable non-standard model of* PA *with* $\mathrm{SSy}(M) \subseteq \mathcal{S}$. *For every $X \in \mathcal{S}$ there exists a countable elementary extension N of M with $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$.*

Proof. Fix $a \in M - \omega$. We shall construct a complete type $p(x)$ over M such that $p(x) \vdash x < 2^a$ and then let N be a $p(x)$-extension of M. As M is countable, N will be countable as well.

To construct $p(x)$, first construct an increasing sequence $(p_i(x) : i \in \omega)$ of types over M, then take $p(x)$ as a completion of $\bigcup_i p_i(x)$ over M. As M is countable, we can fix a list $(f_i : i \in \omega)$ of all $f \in M$ which map $2^a = \{n \in M : n < 2^a\}$ to M. Assume that f_0 is the identity function on 2^a.

In order to have $X \in \mathrm{SSy}(N)$, we shall include in $p_0(x)$ formulas asserting that x codes X. So, let

$$\begin{aligned} p_0(x) &= \{x < 2^a\} \cup \{(x)_n = X(n) : n \in \omega\} \\ &= \{x < 2^a\} \cup \{(f_0(x))_n = X(n) : n \in \omega\}. \end{aligned}$$

As $a > \omega$, $p_0(x)$ is finitely realizable in M. Also note that $p_0(x)$ is recursive in X and hence in $\mathcal{S}$ (modulo coding), and if N is a $p_0(x)$-extension of M, then $X \in \mathrm{SSy}(N)$.

To ensure $\mathrm{SSy}(N) \subseteq \mathcal{S}$, it suffices to include in $p(x)$ formulas asserting that each $f_i(x)$ codes an element of $\mathcal{S}$. This is done inductively as follows. Suppose that for $k \in \omega$ we have the following data:

- $X_0, \ldots, X_k \subseteq \omega$ such that $X_0 = X$ and each X_i is in $\mathcal{S}$;
- a type over M given by

$$p_k(x) = \{x < 2^a\} \cup \{(f_i(x))_n = X_i(n) : i \leq k,\, n \in \omega\}.$$

Note that $p_k(x)$ is recursive in $\bigoplus_{i \leq k} X_i$ and hence in $\mathcal{S}$ (modulo coding), and that if $N = M(b)$ with b realizing $p_k(x)$, then $f_i(b)$ codes X_i for all $i \leq k$.

Let T be the set of tuples $\vec{\sigma} = (\sigma_i : i \leq k+1)$ such that σ_i's are finite binary sequences of equal length and in M the following set is not empty:

$$W(\vec{\sigma}) = \{c < 2^a : \forall i \leq k+1,\, n < |\sigma_i|\, ((f_i(c))_n = \sigma_i(n))\}.$$

So T is in $\mathrm{SSy}(M)$.

Fix $m \in \omega$. For each $i \leq k$, let σ_i be the initial segment of X_i of length m. As $p_k(x)$ is finitely realizable in M, there exists $c \in M$ such that $c < 2^a$ and $(f_i(c))_n = \sigma_i(n)$ for each $i \leq k$ and $n < m$. Define a binary sequence σ_{k+1} of length m by letting $\sigma_{k+1}(n) = (f_i(c))_n$ for $n < m$. Then for this tuple $\vec{\sigma} = (\sigma_i : i \leq k+1)$, the set $W(\vec{\sigma})$ contains c and thus is not empty. So $\vec{\sigma} \in T$. This shows that T is infinite.

Let T' be the set of $\tau \in 2^{<\omega}$ such that if τ_i is the initial segment of X_i of length $|\tau|$, then $(\tau_0, \ldots, \tau_k, \tau) \in T$. By the above paragraph, T' is an infinite binary tree recursive in $\bigoplus_{i\le k} X_i \oplus T$ and thus in $\mathcal{S}$. So by the definition of Scott set, $\mathcal{S}$ contains an infinite path of T', denoted by X_{k+1}.

Hence the following set is a type over M:

$$p_{k+1}(x) = p_k(x) \cup \{(f_{k+1}(x))_n = X_{k+1}(n) : n \in \omega\},$$

and $p_{k+1}(x)$ is recursive in $\mathcal{S}$ (modulo coding).

Finally, let $p(x)$ be a complete type over M containing $\bigcup_k p_k(x)$. Fix b realizing $p(x)$. Then b codes X and each $f_i(b)$ codes X_i which is in $\mathcal{S}$. As $N = M(b) = \{f_i(b) : i \in \omega\}$, N is the desired model.

This ends the proof of Theorem 2.1. ■

3. An improvement of Ehrenfeucht's lemma. In [5], Gitman established several improvements of Ehrenfeucht's lemma. Some of Gitman's improvements related to this section are summarized below.

THEOREM 3.1 (Gitman [5]). *Let $\mathcal{S}$ be a Scott set and $M \models \mathrm{PA}$ with $\mathrm{SSy}(M) \subseteq \mathcal{S}$. Suppose that any of the following conditions applies:*

(1) *$\mathcal{S}$ is arithmetically closed and $\mathrm{SSy}(M)$ is countable;*
(2) *there is a countable $M_0 \preceq_{\mathrm{end}} M \models \mathrm{PA}$.*

Then for every $X \in \mathcal{S}$ there exists N such that $M \preceq N$ and $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$.

We prove the following theorem, which clearly implies the second result of Gitman listed above.

THEOREM 3.2. *Let $\mathcal{S}$ be a Scott set and $M \models \mathrm{PA}$ with $\mathrm{SSy}(M) \subseteq \mathcal{S}$. If there exists $a \in M$ with $|[0,a]^M| = \aleph_0$, then for every $X \in \mathcal{S}$ there exists N such that $M \preceq N$ and $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$.*

Proof. By overspill, pick $a_0 \in M$ such that $\mathbb{N} < a_0 < a_0^{a_0} < a$, and let M_0 be a countable elementary submodel of M containing $[0,a]^M$. By the proof of Ehrenfeucht's lemma in the last section, there exists a complete type $p_0(x)$ over M_0 such that $p_0(x) \vdash x < a_0$, and if $M_0(b)$ is a $p_0(x)$-extension of M_0, then b codes X and $\mathrm{SSy}(M_0(b)) \subseteq \mathcal{S}$. Now let $p(x)$ be a complete type over M extending $p_0(x)$, and let $N = M(b)$ where b is a realization of $p(x)$. Then

$$N = \{f(b) : f \in M \text{ is a function from } a_0 \text{ to } M\}.$$

In M, let $\ell = \max\{i : 2^i < a_0\}$. For each f as above, define g in M by

$$g : a_0 \to a_0, \quad g(c) = \text{ the remainder of } f(c) \text{ divided by } 2^\ell.$$

By the choice of a_0 and M_0, $g \in M_0$. Moreover,

$$\{n \in \mathbb{N} : (f(b))_n = 1\} = \{i \in \mathbb{N} : (g(b))_n = 1\} \in \mathrm{SSy}(M_0(b)) \subseteq \mathcal{S}. \ ■$$

To see that Theorem 3.2 is a non-trivial improvement of Theorem 3.1(2), note that a model M as in Theorem 3.2 may have *no* countable elementary initial segments. This is probably well-known. But for completeness, an easy proof is included here.

PROPOSITION 3.3. *There exists an uncountable N such that N contains a countable infinite $[0,a]^N$, but N does not have any countable $M \prec_{\mathrm{end}} N$.*

Proof. Gödel's Incompleteness Theorem implies the existence of a countable $N_0 \models \mathrm{PA}$ and a parameter-free Σ_0-formula $\varphi(x)$ such that $N_0 \models \exists x\, \varphi(x)$ but the least $b \in N_0$ satisfying $\varphi(x)$ is non-standard. Then *no* $M \subset [0,b]^{N_0}$ satisfying PA can be an elementary submodel of N_0. Let I be a cut of N_0 such that $\mathbb{N} \subset_{\mathrm{end}} I < b$ and I is closed under multiplication. By [7, Corollary 2.1.17], there exists an uncountable N such that $N_0 \prec N$,

$$I = \{a \in N : \exists c \in I\, (a < c)\},$$

and $[0,c]^N$ is uncountable for every $c > I$. Let $a \in I - \mathbb{N}$. Clearly, N and a are as desired. ■

4. Finite interstructure lattices. The interstructure lattice $\mathrm{Lt}(N/M)$ is an extensively studied object. An important but incomplete topic in the theory of models of arithmetic is the characterization of lattices which are isomorphic to $\mathrm{Lt}(N/M)$ for some or all M's. Schmerl's theorem [9], which is a milestone in this topic, implies that for countable non-standard models M the finite lattices isomorphic to some $\mathrm{Lt}(N/M)$ depend only on the theory of M. For more background and development of this topic, the reader is referred to [9] and [7, Chapter 4].

THEOREM 4.1. *Let L be a finite lattice such that every countable $M \models \mathrm{PA}$ has an elementary extension N with $\mathrm{Lt}(N/M) \cong L$. Then for any $M \models \mathrm{PA}$ containing a countable infinite $[0,a]^M$, there exists N such that $\mathrm{Lt}(N/M) \cong L$.*

Note that the model M in the above theorem has minimal cofinal extension as shown by Gitman (see [5, Theorem 5.7] and the follow-up remark). So the theorem here can be considered as an improvement of another result of Gitman.

Proof of Theorem 4.1. Let $a_0 \in M$ be such that $\mathbb{N} < a_0 < a_0^{a_0} < a$ and let M_0 be a countable elementary submodel of M containing $[0,a]^M$. Fix a countably saturated elementary extension $\mathbb{M}$ of M. By Schmerl's [9, Theorem 4.2], there exist $b_0, \ldots, b_n < a_0$ in $\mathbb{M}$, such that $M_0 \prec_{\mathrm{cof}} N_0 = M_0(b_0) \prec \mathbb{M}$, $\mathrm{Lt}(N_0/M_0) \cong L$ and

$$\{N' : M_0 \prec N' \preceq N_0\} = \{M_0(b_i) : i \le n\}.$$

Let $N = M(b_0)$ as computed in $\mathbb{M}$.

We claim that $\mathrm{Lt}(N/M) \cong L$ as well.

If $M_0(b_i) \subseteq M_0(b_j)$, then there exists an M_0-finite $f : a_0 \to a_0$ such that $b_i = f(b_j)$, thus $M(b_i) \subseteq M(b_j)$. On the other hand, if $M(b_i) \subseteq M(b_j)$, then there exists an M-finite $f : a_0 \to a_0$ such that $f(b_j) = b_i$. Since $a_0^{a_0} < a$ and $[0, a]^M \subset M_0$, we have $f \in M_0$ and thus $b_i \in M_0(b_j)$. So $M_0(b_i) \subseteq M_0(b_j)$.

It remains to show that

$$\{N' : M \prec N' \preceq N\} = \{M(b_i) : i \le n\}.$$

Take an arbitrary N' with $M \prec N' \preceq N$, and assume that $N' \neq M(b_j)$ for any j. Let $M(b_i)$ be minimal among $M(b_j)$'s containing N'. Define

$$B = \{j \le n : M_0 \prec M_0(b_j) \prec M_0(b_i)\}.$$

For each $j \in B$, there is $c_j \in N' - M(b_j)$. A finite iteration of pairing yields $c \in N' - \bigcup_{j\in B} M(b_j)$. This and the application of Schmerl's theorem at the beginning are the only places in this proof where we use the finiteness of L. Let $f : a_0 \to M$ be an M-finite function with $c = f(b_i)$. In M, define

$$\bar{f} : a_0 \to a_0, \quad \bar{f}(x) = \min\{y < a_0 : f(y) = f(x)\}.$$

As $a_0^{a_0} < a$ and $[0, a]^M \subset M_0$, $\bar{f}$ is M_0-finite. Let $\bar{c} = \bar{f}(b_i) \in M_0(b_i)$. Note that $\bar{c} < a_0$. There are two cases.

(1) $M_0(\bar{c}) = M_0(b_i)$. Then there is an M_0-finite $\bar{g} : a_0 \to a_0$ with $\bar{g}(\bar{c}) = b_i$. Thus, $\bar{g} \circ \bar{f}(b_i) = b_i$. Let

$$X = \{x \in M_0 : a_0 > x = \bar{g} \circ \bar{f}(x)\}.$$

Then X is M_0-finite, $N \models b_i \in X$ and

$$M_0 \models \bar{f} \text{ is injective on } X.$$

By the definition of $\bar{f}$,

$$M \models f \text{ is injective on } X.$$

So in M there is an inverse of f on X denoted by f^{-1}. It follows that $b_i = f^{-1}(c)$ in N, and thus $M(b_i) \subseteq N' \subseteq M(b_i)$.

(2) $M_0(\bar{c}) \subsetneq M_0(b_j)$ for some $j \in B$. Let $h : a_0 \to a_0$ be an M_0-finite function with $b_j = h(b_i)$, and let $\bar{g} : a_0 \to a_0$ be another M_0-finite function with $\bar{g}(b_j) = \bar{c}$. Thus $\bar{g} \circ h(b_i) = \bar{f}(b_i)$. Let

$$X = \{x \in M_0 : x < a_0 \text{ and } \bar{g} \circ h(x) = \bar{f}(x)\}.$$

Then X is M_0-finite, $N \models b_i \in X$ and

$$M_0 \models \forall x, y \in X\, (h(x) = h(y) \to \bar{f}(x) = \bar{f}(y)).$$

By the definition of $\bar{f}$,

$$M \models \forall x, y \in X\, (h(x) = h(y) \to f(x) = f(y)).$$

Let $h(X)$ be the M_0-finite set consisting of $h(x)$ for $x \in X$. Then $N \models b_j \in h(X)$. In M, define $g : h(X) \to M$ as

$$g(z) = f(x) \quad \text{where } x = \min\{y < a_0 : h(y) = z\}.$$

In N, $g(b_j) = c \in M(b_j)$, contradicting the choice of c. ∎

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Wei Wang
Institute of Logic and Cognition and Department of Philosophy
Sun Yat-Sen University
Guangzhou, P. R. China
E-mail: wwang.cn@gmail.com
wangw68@mail.sysu.edu.cn