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COMMUTATIVITY OF THE RESTRICTED CONNECTED HULL IN BANACH ALGEBRAS

BY

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Abstract. The restricted topology is defined to investigate sets between the spectrum and its hull that arise by filling the holes. In this paper, we study this topology in the context of Banach algebras. We explore the properties of the associated connected hull that are in common or in contrast with the well-known notions of spectrum and exponential spectrum. Further, we deduce sufficient conditions for the commutativity of the connected hull.

1. Introduction. Let $\mathcal{A}$ be a complex Banach algebra with unit $\mathbf{1}$. We write the element $\lambda\mathbf{1}$ as λ, for all $\lambda \in \mathbb{C}$. Let $\mathcal{A}^{-1}$ and $\operatorname{Sing}(\mathcal{A})$ be the sets of all invertible and all singular elements of $\mathcal{A}$, respectively. For any $a \in \mathcal{A}$, the spectrum of a is given by

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{A}^{-1}\}.$$

For any compact set $K \subseteq \mathbb{C}$, the *connected hull* of K, denoted by $\hat{K}$, is the union of K and its holes, where the *holes* of K are the bounded components of $\mathbb{C} \setminus K$.

The spectrum, a compact subset of $\mathbb{C}$ [R91, Theorem 10.13], may contain holes within it, and by the definition of the connected hull, $\widehat{\sigma(a)}$ fills all the holes of the spectrum. Thus, if we have a set $K \subseteq \mathbb{C}$ which has some holes, recognizing this set through a notion of spectrum in a Banach algebra $\mathcal{A}$ becomes an interesting exploration.

One such example is the exponential spectrum $\varepsilon(a)$, which can be identified by using $\operatorname{Exp}(\mathcal{A})$ $(= \operatorname{Comp}_{\mathcal{A}}(\mathbf{1}, \mathcal{A}^{-1}))$, the connected component of $\mathbf{1}$ in $\mathcal{A}^{-1}$. For any $a \in \mathcal{A}$, the exponential spectrum is given by

$$\varepsilon(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin \operatorname{Exp}(\mathcal{A})\}.$$

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We also have the inclusion from [H76, Theorem 1],

$$\partial\varepsilon(a) \subseteq \sigma(a) \subseteq \varepsilon(a) \subseteq \widehat{\sigma(a)} \qquad (a \in \mathcal{A}),$$

where ∂K is the topological boundary of any set K in $\mathbb{C}$. Therefore, $\varepsilon(a)$ is formed by filling some of the holes of $\sigma(a)$ and correspondingly we have $\mathrm{Exp}(\mathcal{A}) \subseteq \mathcal{A}^{-1}$. Hence, other ways of realizing the sets K such that $\sigma(a) \subseteq K \subseteq \widehat{\sigma(a)}$ include expanding $\mathrm{Exp}(\mathcal{A})$ or partitioning $\mathrm{Exp}(\mathcal{A})$ into smaller subsets.

In this setting, Mouton and Harte [MH17] introduced the restricted connected hull $\eta_{\mathcal{B}}(a)$, associated with a closed subalgebra $\mathcal{B}$ of $\mathcal{A}$. For each $a \in \mathcal{B}$, this hull satisfies

$$\varepsilon(a) \subseteq \eta_{\mathcal{B}}(a) \subseteq \widehat{\sigma(a)}$$

and can be identified through a subset of $\mathrm{Exp}(\mathcal{A})$. This restricted connected hull is obtained by constructing a different topology (restricted topology) in $\mathcal{A}$ and is equivalent to the exponential spectrum. Moreover, when $\mathcal{B} = \mathcal{A}$, the restricted connected hull relative to $\mathcal{B}$ equals the exponential spectrum.

Mouton and Harte defined this restricted topology on any additive topological group and probed into certain properties related to this topology. They primarily focused on the topological properties related to the group operation (addition), and later studied this topology in the context of Banach algebras in [MH17, MH24]. In this paper, in Section 2, we investigate further topological properties by exploiting the availability of the multiplicative structure.

In Section 3, we see some of the properties of the restricted connected hull. The spectrum and the exponential spectrum share many properties, but not all. We are interested in knowing the properties that the restricted connected hull $\eta_{\mathcal{B}}(a)$ shares with the spectrum and/or exponential spectrum, as well as those that are not shared. Theorem 3.5 tells us that for all $a \in \mathcal{A}$, $\eta_{\mathcal{B}}(a)$ is closed and for $a \in \mathcal{B}$, $\eta_{\mathcal{B}}(a)$ is compact. Thus, the compactness of the restricted connected hull partially matches the spectrum and the exponential spectrum.

Both spectrum and exponential spectrum are Ransford spectrums corresponding to the Ransford sets $\mathcal{A}^{-1}$ and $\mathrm{Exp}(\mathcal{A})$, respectively. So, if there is a Ransford set $\mathcal{K} \subseteq \mathcal{A}$ whose Ransford spectrum is $\sigma_{\mathcal{K}}$ such that $\sigma(a) \subseteq \sigma_{\mathcal{K}}(a) \subseteq \widehat{\sigma(a)}$, then it can be effectively analyzed. In this regard, Remark 3.7 ensures that the restricted connected hull can be identified by a Ransford set.

Let $K(a)$ be a subset of $\mathbb{C}$ associated with a Banach algebra element $a \in \mathcal{A}$. We say $K(a)$ *commutes* if for all a, $b \in \mathcal{A}$,

$$K(ab) \setminus \{0\} = K(ba) \setminus \{0\}.$$

This commutativity is a well-explored and interesting area of study when the underlying algebra is non-commutative (see [G, GR08, KR17, M92, RS22]). It is well known that if $K(a) = \sigma(a)$, then $K(a)$ commutes [A91, Lemma 3.1.2]. As a direct consequence, $K(a) = \widehat{\sigma(a)}$ also inherits the commutativity property. On the other hand, if $K(a) = \varepsilon(a)$, then although $\sigma(a) \subseteq K(a) \subseteq \widehat{\sigma(a)}$, it does not necessarily commute [KR17, Theorem 1.1]. Motivated by the inclusion $\sigma(a) \subseteq \varepsilon(a) \subseteq \eta_{\mathcal{B}}(a) \subseteq \widehat{\sigma(a)}$, $(a \in \mathcal{B})$, we establish sufficient conditions for $K(a) = \eta_{\mathcal{B}}(a)$ to commute (Theorems 4.1, 4.2).

1.1. Preliminaries. Let X be a topological space and Y be a topological subspace of X. For any $x \in X$, let $\mathrm{Nbd}_X(x)$ be the set of all neighbourhoods of x in X, and $\mathrm{Int}_X(K)$ and $\mathrm{cl}_X(K)$ be the interior and closure of the set K in X, respectively. Let $\mathrm{Comp}_X(x, K) \subseteq X$ be the component of K containing x in X.

Definition 1.1 (Restricted closure). Let $\mathcal{A}$ be an additive topological group and $\mathcal{B}$ be a subgroup of $\mathcal{A}$. Let $K \subseteq \mathcal{A}$. Then the *restricted closure of K in $\mathcal{A}$ relative to $\mathcal{B}$* is defined by

$$\mathrm{cl}^{\mathcal{B}}(K) = \{a \in \mathcal{A} : \forall U \in \mathrm{Nbd}_{\mathcal{B}}(0),\ (a - U) \cap K \neq \emptyset\}.$$

If $K \subseteq \mathcal{B}$, the restricted closure of K in $\mathcal{A}$ relative to $\mathcal{B}$ is the same as the relative closure of K in $\mathcal{B}$.

Theorem 3.5 in [MH17] demonstrates that the restricted closure satisfies the Kuratowski closure axioms and consequently gives rise to a topology on $\mathcal{A}$.

Definition 1.2 (Restricted topology). The topology induced on an additive topological group $\mathcal{A}$ by the restricted closure in $\mathcal{A}$ relative to a subgroup $\mathcal{B}$ is the *restricted topology* or the *$\mathcal{B}$-topology.*

Proposition 1.3 ([MH17, Proposition 3.7]). *If $K \subseteq \mathcal{A}$, then*

$$\mathrm{Int}^{\mathcal{B}}(K) = \{a \in \mathcal{A} : \exists U \in \mathrm{Nbd}_{\mathcal{B}}(0),\ a - U \subseteq K\}.$$

Proposition 1.4 ([MH17, Proposition 3.8]). *If $a \in \mathcal{A}$, then*

$$\mathrm{Nbd}^{\mathcal{B}}(a) = \{U \subseteq \mathcal{A} : a - U \in \mathrm{Nbd}_{\mathcal{B}}(0)\}.$$

Corollary 1.5. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ a subgroup of $\mathcal{A}$. The interior of a set and neighbourhoods in the $\mathcal{B}$-topology can be described as follows:*

(1) *For $K \subseteq \mathcal{A}$, an element $a \in \mathcal{A}$ belongs to $\mathrm{Int}^{\mathcal{B}}(K)$ if and only if there exists $r > 0$ such that $a - (B(0, r) \cap \mathcal{B}) \subseteq K$.*
(2) *For $a \in \mathcal{A}$,*

$$\mathrm{Nbd}^{\mathcal{B}}(a) = \{U \subseteq \mathcal{A} : \exists r > 0 \text{ such that } a - (B(0, r) \cap \mathcal{B}) \subseteq U\}.$$

LEMMA 1.6 ([MH17, Lemma 3.9, Corollaries 3.10, 3.12]). *Let $\mathcal{B}$ and $\mathcal{C}$ be closed subalgebras of $\mathcal{A}$ such that $\mathbf{1} \in \mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$.*

(1) *If $K \subseteq \mathcal{A}$, then $\mathrm{cl}^{\mathcal{C}}(K) \subseteq \mathrm{cl}^{\mathcal{B}}(K)$.*
(2) *The $\mathcal{C}$-topology of $\mathcal{A}$ is stronger than the $\mathcal{B}$-topology of $\mathcal{A}$. Hence, the $\mathcal{B}$-topology is stronger than the norm topology of $\mathcal{A}$.*
(3) *For $K \subseteq \mathcal{A}$,*

$$\mathrm{Int}^{\mathcal{A}}(K) \subseteq \mathrm{Int}^{\mathcal{B}}(K) \subseteq \mathrm{Int}^{\mathcal{C}}(K) \subseteq \mathrm{cl}^{\mathcal{C}}(K) \subseteq \mathrm{cl}^{\mathcal{B}}(K) \subseteq \mathrm{cl}^{\mathcal{A}}(K).$$

2. Restricted topology in Banach algebras. Throughout this section, $\mathcal{A}$ will be a complex Banach algebra with unit $\mathbf{1}$ and $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$ such that $\mathbf{1} \in \mathcal{B}$. The set of all invertible elements of $\mathcal{B}$ will be denoted by $\mathcal{B}^{-1}$, while $\mathrm{Sing}(\mathcal{B}) = \mathcal{B} \setminus \mathcal{B}^{-1}$ denotes the set of all singular (non-invertible) elements of $\mathcal{B}$.

A sequence $(a_n) \subseteq \mathcal{A}$ is said to converge to a in the $\mathcal{B}$-topology if for every $U \in \mathrm{Nbd}^{\mathcal{B}}(a)$, there exists $N \in \mathbb{N}$ such that $a_n \in U$ for all $n \geq N$. Since $\mathcal{A}$ is a Banach algebra, we can see that a sequence (a_n) converges to a in the $\mathcal{B}$-topology if for every $r > 0$ there exists $N \in \mathbb{N}$ such that $a_n \in a - B(0,r) \cap \mathcal{B}$ for all $n \geq N$, and we denote it by $a_n \xrightarrow{\mathcal{B}} a$.

Note that when the underlying topological group $\mathcal{A}$ is a Banach algebra, the $\mathcal{B}$-topology is Hausdorff. Hence, the limit of any converging sequence is unique.

LEMMA 2.1. *Let $a \in \mathcal{A}$ and (a_n) be a sequence in $\mathcal{A}$ such that $a_n \xrightarrow{\mathcal{B}} a$. Then*

(1) *for all $b \in \mathcal{B}$, $a_n b \xrightarrow{\mathcal{B}} ab$,*
(2) *for all $c \in \mathcal{A}$, $c - a_n \xrightarrow{\mathcal{B}} c - a$.*

Proof. (1) Let $U \in \mathrm{Nbd}^{\mathcal{B}}(ab)$. Then there exists $r > 0$ such that $ab - (B(0,r) \cap \mathcal{B}) \subseteq U$. Since $a_n \xrightarrow{\mathcal{B}} a$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \in a - (B(0, r/\|b\|) \cap \mathcal{B})$. Hence, $a_n - a \in \mathcal{B}$ and $\|a_n - a\| < r/\|b\|$ for $n \geq N$. Now, for all $n \geq N$, $a_n b - ab = (a_n - a)b \in \mathcal{B}$ and

$$\|a_n b - ab\| = \|(a_n - a)b\| \leq \|a_n - a\| \, \|b\| < r.$$

Hence, for all $n \geq N$, $a_n b \in ab - (B(0,r) \cap \mathcal{B}) \subseteq U$. Thus, $a_n b \xrightarrow{\mathcal{B}} ab$.

(2) Let $U \in \mathrm{Nbd}^{\mathcal{B}}(c - a)$. Then there exists $V \in \mathrm{Nbd}_{\mathcal{A}}(0)$ such that $U = (c-a) - V \cap \mathcal{B}$. It is easy to see that, for all $n \geq N'$, $c - a_n \in (c-a) - V \cap \mathcal{B} = U$. Hence, $c - a_n \xrightarrow{\mathcal{B}} c - a$. ■

EXAMPLE 2.2. Let us consider the algebra $\mathcal{A} = \mathbb{C}^2$. Here, a closed subalgebra $\mathcal{B}$ of $\mathbb{C}^2$ containing $(1,1)$ is $\mathcal{B} = \{(\lambda, \lambda) : \lambda \in \mathbb{C}\}$ (with the usual operations).

Let $a_n = (1/n, 1/n)$. Then $a_n \xrightarrow{\mathcal{B}} (0,0)$. For $b = (2,1)$, we have $a_n b = (2/n, 1/n)$ and $(0,0)b = (0,0)$. One can see that $a_n b$ does not converge to $(0,0)$ in the $\mathcal{B}$-topology. Thus, when $b \notin \mathcal{B}$, the sequence $a_n b$ need not converge to ab in the $\mathcal{B}$-topology.

Remark 2.3. In a normed algebra, the multiplication operation is continuous. On the other hand, the above example shows that the multiplication operation need not be continuous in the $\mathcal{B}$-topology.

Proposition 2.4. *Let $K \subseteq \mathcal{A}$. Then $a \in \mathrm{cl}^{\mathcal{B}}(K)$ if and only if there exists a sequence (a_n) in K such that $a_n \xrightarrow{\mathcal{B}} a$.*

Proof. Let $a \in \mathrm{cl}^{\mathcal{B}}(K)$. Then for all $n \in \mathbb{N}$,

$$[a - (B(0, 1/n) \cap \mathcal{B})] \cap K \neq \emptyset.$$

Let $a_n \in [a - (B(0, 1/n) \cap \mathcal{B})] \cap K$ for all n. Then $a_n - a \in \mathcal{B}$ and $\|a_n - a\| < 1/n$.

Let $U \in \mathrm{Nbd}^{\mathcal{B}}(a)$. Then there exists $N \in \mathbb{N}$ such that $a - B(0, 1/N) \cap \mathcal{B} \subseteq U$. Also, $a_n \in a - (B(0, 1/N) \cap \mathcal{B}) \subseteq U$ for $n \geq N$. Hence, $a_n \xrightarrow{\mathcal{B}} a$.

Conversely, assume there exists a sequence (a_n) in K such that $a_n \xrightarrow{\mathcal{B}} a$. Now, for any $V \in \mathrm{Nbd}^{\mathcal{B}}(a)$ there exists $r > 0$ such that $a - B(0,r) \cap \mathcal{B} \subseteq V$. Since $a_n \xrightarrow{\mathcal{B}} a$, there exists $N \in \mathbb{N}$ such that $a_n \in a - B(0,r) \cap \mathcal{B}$ for all $n \geq N$. Hence, $V \cap K \neq \emptyset$, which implies $a \in \mathrm{cl}^{\mathcal{B}}(K)$. ∎

Remark 2.5. In the above proposition, suppose $\mathcal{B}$ is a closed unital subalgebra of $\mathcal{A}$ such that $\mathrm{Sing}(\mathcal{B}) \subseteq \mathrm{Sing}(\mathcal{A})$ and $a \in \mathcal{B} \cap \mathrm{cl}^{\mathcal{B}}(\mathcal{A}^{-1})$. Then we can construct an invertible sequence (a_n) in $\mathcal{B}$ such that $a_n \xrightarrow{\mathcal{B}} a$.

In [MH17], Mouton and Harte proved that if $U \subset \mathcal{A}$ is closed (respectively open or connected) in the $\mathcal{B}$-topology, then for every $a \in \mathcal{A}$, the translated set $a + U$ is also closed (respectively open or connected).

Lemma 2.6 ([MH17, Lemma 3.17, Corollaries 3.18, 3.19]). *Let $a \in \mathcal{A}$ and $G \subseteq \mathcal{A}$.*

(1) *If G is closed in $\mathcal{A}$ in the $\mathcal{B}$-topology, so is $a - G$.*
(2) *If G is open in $\mathcal{A}$ in the $\mathcal{B}$-topology, so is $a - G$.*
(3) *If G is connected in $\mathcal{A}$ in the $\mathcal{B}$-topology, so is $a - G$.*

Likewise, the following lemmas characterize the circumstances under which aU remains closed, open, or connected, whenever U is closed, open, or connected.

Lemma 2.7. *Let $K \subseteq \mathcal{A}$ and $a \in \mathcal{A}$ such that $a^{-1} \in \mathcal{B}$, then $\mathrm{cl}^{\mathcal{B}}(aK) \subseteq a\,\mathrm{cl}^{\mathcal{B}}(K)$. If $a \in \mathcal{B}$, then $a\,\mathrm{cl}^{\mathcal{B}}(K) \subseteq \mathrm{cl}^{\mathcal{B}}(aK)$. Moreover, if $a \in \mathcal{B}^{-1}$, then $\mathrm{cl}^{\mathcal{B}}(aK) = a\,\mathrm{cl}^{\mathcal{B}}(K)$.*

Proof. Let $b \in \mathrm{cl}^{\mathcal{B}}(aK)$. Then there exists a sequence (b_n) in aK such that $b_n \xrightarrow{\mathcal{B}} b$. Since $b_n \in aK$, $b_n = ac_n$ for some $c_n \in K$. Since $a^{-1} \in \mathcal{B}$, by Lemma 2.1 we have $a^{-1}b_n = c_n \xrightarrow{\mathcal{B}} a^{-1}b$. This implies $a^{-1}b \in \mathrm{cl}^{\mathcal{B}}(K)$ and hence $b \in a\,\mathrm{cl}^{\mathcal{B}}(K)$. Thus, $\mathrm{cl}^{\mathcal{B}}(aK) \subseteq a\,\mathrm{cl}^{\mathcal{B}}(K)$.

Now, let $b \in \mathrm{cl}^{\mathcal{B}}(K)$. Then there exists a sequence (b_n) in K such that $b_n \xrightarrow{\mathcal{B}} b$. Since $a \in \mathcal{B}$, $ab_n \xrightarrow{\mathcal{B}} ab$ and it follows that $ab \in \mathrm{cl}^{\mathcal{B}}(aK)$. Thus, $a\,\mathrm{cl}^{\mathcal{B}}(K) \subseteq \mathrm{cl}^{\mathcal{B}}(aK)$. ∎

LEMMA 2.8. *Let $U \subseteq \mathcal{A}$ be open in the $\mathcal{B}$-topology and $a \in \mathcal{A}$ such that $a^{-1} \in \mathcal{B}$. Then aU, Ua are open in the $\mathcal{B}$-topology.*

Proof. Let $x \in aU$. Then $x = ay$ for some $y \in U$. Since U is open in the $\mathcal{B}$-topology, there exists $r > 0$ such that $y - (B(0,r) \cap \mathcal{B}) \subseteq U$.

CLAIM. $ay - (B(0, r/\|a^{-1}\|) \cap \mathcal{B}) \subseteq aU$.

Let $z \in ay - (B(0, r/\|a^{-1}\|) \cap \mathcal{B})$. Then $z = ay - b$ where $b \in \mathcal{B}$ and $\|b\| < r/\|a^{-1}\|$. Now, $a^{-1}z = y - a^{-1}b \in y - (B(0,r) \cap \mathcal{B}) \subseteq U$. This implies $z \in aU$ and hence $ay - (B(0, r/\|a^{-1}\|) \cap \mathcal{B}) \subseteq aU$.

Similarly, we can prove Ua is open in the $\mathcal{B}$-topology. ∎

LEMMA 2.9. *Let $U \subseteq \mathcal{A}$ be connected in the $\mathcal{B}$-topology and $a \in \mathcal{B}$ such that a is invertible in $\mathcal{A}$. Then aU, Ua are connected in the $\mathcal{B}$-topology.*

Proof. Suppose $aU = C \cup D$ where C and D are disjoint open sets in the $\mathcal{B}$-topology. Then $U = a^{-1}C \cup a^{-1}D$. Since $a \in \mathcal{B}$, both $a^{-1}C$ and $a^{-1}D$ are disjoint open sets in the $\mathcal{B}$-topology. This contradicts that U is connected. Hence, aU is connected in the $\mathcal{B}$-topology.

Similarly, we can conclude that Ua is connected. ∎

In Lemma 2.6, it is observed that translating sets by any $a \in \mathcal{A}$ preserves their topological properties. In contrast, Lemmas 2.7–2.9 impose certain restrictions on a. These lemmas need not hold if we relax the conditions, and the following examples support our claim.

EXAMPLE 2.10. Let $\mathcal{A} = \mathbb{C}^2$, $\mathcal{B} = \mathbb{C}$, $K = \{(1/n, 1/n) : n \in \mathbb{N}\}$ and $a = (2,1)$. It is easy to see that $(0,0) \in \mathrm{cl}^{\mathcal{B}}(K)$ and hence $(0,0) \in a\,\mathrm{cl}^{\mathcal{B}}(K)$. But $(0,0) \notin \mathrm{cl}^{\mathcal{B}}(aK)$. Thus, $a\,\mathrm{cl}^{\mathcal{B}}(K) \nsubseteq \mathrm{cl}^{\mathcal{B}}(aK)$.

Similarly, if $K = \{(2/n, 1/n) : n \in \mathbb{N}\}$ and $a = (1/2, 1)$, we can see that $\mathrm{cl}^{\mathcal{B}}(aK) \nsubseteq a\,\mathrm{cl}^{\mathcal{B}}(K)$.

REMARK 2.11. If $\mathcal{A} = \mathbb{C}^n$, we can find a closed subalgebra $\mathcal{B}$ of $\mathcal{A}$ and $K \subseteq \mathcal{A}$ such that $a\,\mathrm{cl}^{\mathcal{B}}(K) \nsubseteq \mathrm{cl}^{\mathcal{B}}(aK)$ and $\mathrm{cl}^{\mathcal{B}}(aK) \nsubseteq a\,\mathrm{cl}^{\mathcal{B}}(K)$.

For instance, let $\mathcal{A} = \mathbb{C}^3$, $\mathcal{B} = \mathbb{C}$ and $K = \{(1/n, 1/n, 1/n) : n \in \mathbb{N}\}$. For $a = (2,1,1)$, we have $a\,\mathrm{cl}^{\mathcal{B}}(K) \nsubseteq \mathrm{cl}^{\mathcal{B}}(aK)$. Similarly, we can choose a suitable $K' \subseteq \mathcal{A}$ with $\mathrm{cl}^{\mathcal{B}}(aK') \nsubseteq a\,\mathrm{cl}^{\mathcal{B}}(K')$.

Example 2.12. Let $\mathcal{A} = (\mathbb{C}^2, \| \|_{\max})$, $\mathcal{B} = \mathbb{C}$ and $U = B(0,1) \cap \mathcal{B} = \{(\lambda, \lambda) : |\lambda| < 1\}$. Then U is open in the $\mathcal{B}$-topology.

Let $a = (0, 1)$. Then $a\,U = \{(0, \lambda) : |\lambda| < 1\}$. Let $(0, \lambda) \in aU$. For any $r > 0$, $(0, \lambda) - B(0, r) \cap \mathcal{B} = \{(-\mu, \lambda - \mu) : |\mu| < r\} \nsubseteq a\,U$. Hence, $a\,U$ is not open in the $\mathcal{B}$-topology.

Example 2.13. Let $\mathcal{A} = C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle centred at 0 in $\mathbb{C}$ and $\mathcal{B} = \mathbb{C}$. Let $U = \{f_r \in \mathcal{A} : f_r(z) = z + r,\, r \in \mathbb{C}\}$. For any $r \in \mathbb{C}$, consider the function $\gamma : [0, 1] \to U$ given by $\gamma(t) = f_{tr}$. This is a path from f_0 to f_r and hence the set U is path-connected. Thus, U is connected in the $\mathcal{B}$-topology.

Let $h(z) = z$ for all $z \in \mathbb{T}$. Then $h\,U = \{hf_r : r \in \mathbb{C}\}$. Consider the sets

$$U_1 = \{hf_r : r \neq 0\}, \qquad U_2 = \{hf_r : r = 0\}.$$

Let $hf_r \in U_1$. Then $hf_r(z) = z^2 + rz$ for all $z \in \mathbb{T}$. For any $\delta > 0$, $(hf_r - B(0, \delta) \cap \mathcal{B}) \cap h\,U = \{hf_r\} \subseteq U_1$. Hence U_1 is open in $h\,U$. Similarly, U_2 is open in $h\,U$. Also we have $h\,U = U_1 \cup U_2$ which implies U_1 and U_2 form a separation for $h\,U$. Thus, $h\,U$ is not connected.

Remark 2.14. In the above example, it is evident that every singleton subset of $h\,U$ is open. Consequently, $h\,U$ is a discrete set in the $\mathcal{B}$-topology.

Theorem 2.15. *Let* $\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ *be the component of* $\mathcal{A}^{-1}$ *containing* $\mathbf{1}$ *in the* $\mathcal{B}$*-topology. For* $a \in \mathcal{B}^{-1}$,

$$a\,\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})\,a^{-1} = \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}).$$

Proof. For $a \in \mathcal{B}^{-1}$, $\mathbf{1} \in a\,\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})a^{-1}$ and by Lemma 2.9, the set $a\,\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})\,a^{-1}$ is connected in the $\mathcal{B}$-topology. Hence,

$$a\,\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})a^{-1} \subseteq \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}).$$

Also

$$\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) = a^{-1}\,(a\ \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})\,a^{-1})\,a \subseteq a^{-1}\ \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})\,a.$$

By replacing a by a^{-1}, we can deduce the desired equality ∎

Theorem 2.16. *Let* $\mathcal{B}$ *be a closed unital subalgebra such that* $\mathrm{Sing}(\mathcal{B}) \subseteq \mathrm{Sing}(\mathcal{A})$ *and* $\mathrm{Exp}(\mathcal{B}) = \mathcal{B}^{-1}$. *Then*

$$\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) = \mathrm{Exp}(\mathcal{B}).$$

Proof. By [MH24, Lemma 2.1], we have

$$\mathcal{B}^{-1} = \mathrm{Exp}(\mathcal{B}) \subseteq \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}).$$

If $\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \neq \mathcal{B}^{-1}$, then $\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \setminus \mathcal{B}^{-1}$ is non-empty.

Claim. $\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \setminus \mathcal{B}^{-1}$ *is open in the* $\mathcal{B}$*-topology.*

Let $a \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \setminus \mathcal{B}^{-1}$. Since $a \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$, there exists $r > 0$ such that $a - B(0, r) \cap \mathcal{B} \subseteq \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$. We have $\mathrm{Sing}(\mathcal{B}) \subseteq$

$\operatorname{Sing}(\mathcal{A})$, which implies $a \in \operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \setminus \mathcal{B}$. Also $\mathcal{B}$ is closed in the norm topology; thus, the distance $d(a, \mathcal{B})$ between a and $\mathcal{B}$ is positive.

Now, let $r' < \min\{r, d(a, \mathcal{B})\}$ and $b \in B(0, r') \cap \mathcal{B}$. Then

$$\|a - (a - b)\| = \|b\| < r'.$$

Therefore, $a - b \in a - B(0, r) \cap \mathcal{B} \subseteq \operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ and $a - b \notin \mathcal{B}$. It follows that $a - b \in \operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \setminus \mathcal{B}^{-1}$, implying $a - B(0, r') \cap \mathcal{B} \subseteq \operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \setminus \mathcal{B}^{-1}$. Thus, $\operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \setminus \mathcal{B}^{-1}$ is open in the $\mathcal{B}$-topology, proving the claim.

Also, $\mathcal{B}^{-1}$ is open in the $\mathcal{B}$-topology. Hence, $\operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ is not connected, which is a contradiction. Thus, $\operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) = \mathcal{B}^{-1} = \operatorname{Exp}(\mathcal{B})$. ∎

3. Restricted connected hull

Definition 3.1. For an element $a \in \mathcal{A}$, the *restricted connected hull* is defined as

$$\eta_{\mathcal{B}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})\}.$$

Remark 3.2. For any subset K of a topological space $\mathcal{A}$ and $t \notin K$, we can analogously define the restricted connected hull $\eta_t^{\mathcal{B}}(K)$ relative to t of K, which can be found in [MH17]. However, it is important to note that the resulting restricted connected hull may not be a subset of $\mathbb{C}$, but rather a subset of $\mathcal{A}$.

Lemma 3.3 ([MH17, Lemma 6.8]). *Let $\mathcal{B}$ and $\mathcal{C}$ be closed subalgebras of $\mathcal{A}$ such that $\mathbf{1} \in \mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$. If $a \in \mathcal{A}$, then*

$$\sigma(a) \subseteq \eta_{\mathcal{A}}(a) = \varepsilon(a) \subseteq \eta_{\mathcal{B}}(a) \subseteq \eta_{\mathcal{C}}(a).$$

Lemma 3.4. *Let $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$ such that $\mathbf{1} \in \mathcal{B}$. For $a \in \mathcal{A}$, if $a' = bab^{-1}$ for some $b \in \mathcal{B}^{-1}$, then $\eta_{\mathcal{B}}(a') = \eta_{\mathcal{B}}(a)$.*

Proof. Let $\lambda \in \eta_{\mathcal{B}}(a)$. Then

$$\begin{aligned} a - \lambda \notin \operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) &\implies b(a - \lambda)b^{-1} \notin b(\operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}))b^{-1} \\ &\implies bab^{-1} - \lambda \notin \operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}). \end{aligned}$$

Thus, $\eta_{\mathcal{B}}(a) \subseteq \eta_{\mathcal{B}}(a')$. Similarly, we can prove $\eta_{\mathcal{B}}(a') \subseteq \eta_{\mathcal{B}}(a)$. ∎

Theorem 3.5. *If $a \in \mathcal{A}$, then $\eta_{\mathcal{B}}(a)$ is closed in $\mathbb{C}$. Moreover, if $a \in \mathcal{B}$, then $\eta_{\mathcal{B}}(a)$ is compact.*

Proof. Let $a \in \mathcal{A}$. Then the restricted connected hull of a is

$$\begin{aligned} \eta_{\mathcal{B}}(a) &= \{\lambda \in \mathbb{C} : a - \lambda \notin \operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})\} \\ &= \{\lambda \in \mathbb{C} : a - \lambda \in \operatorname{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})^c\}. \end{aligned}$$

Consider the function $F : \mathbb{C} \to \mathcal{A}$ given by $F(\lambda) = a - \lambda$.

Claim. *F is continuous from $\mathbb{C}$ to $\mathcal{A}$ with respect to the $\mathcal{B}$-topology.*

Let $D \subseteq \mathbb{C}$. It is enough to show that $F(cl_{\mathbb{C}}(D)) \subseteq \mathrm{cl}^{\mathcal{B}}(F(D))$. Let $t \in \mathrm{cl}_{\mathbb{C}}(D)$ and $U \in \mathrm{Nbd}^{\mathcal{B}}(F(t))$. Then $U = F(t) - V \cap \mathcal{B}$ for some $V \in \mathrm{Nbd}_{\mathcal{A}}(0)$.

Since F is continuous in the norm topology, $F(\mathrm{cl}_{\mathbb{C}}(D)) \subseteq \mathrm{cl}_{\mathcal{A}}(F(D))$. Hence, $F(t) \in \mathrm{cl}_{\mathcal{A}}(F(D))$ and $(F(t) - V) \cap F(D) \neq \emptyset$. Choose $F(\alpha) \in (F(t) - V) \cap F(D)$. Then $F(t) - F(\alpha) \in V$. Now,

$$F(t) - F(\alpha) = a - t - (a - \alpha) = \alpha - t \in \mathcal{B}.$$

Thus, $F(t) - F(\alpha) \in V \cap \mathcal{B}$. Hence, $F(\alpha) \in U \cap F(D)$. It follows that $F(t) \in \mathrm{cl}^{\mathcal{B}}(F(D))$. Hence, F is continuous, proving the claim.

We observe that $\eta_{\mathcal{B}}(a) = F^{-1}(\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})^c)$, and hence $\eta_{\mathcal{B}}(a)$ is closed in $\mathbb{C}$.

Moreover, if $a \in \mathcal{B}$, then by [MH24, Theorem 2.2],

$$\eta_{\mathcal{B}}(a) \subseteq \widehat{\sigma(a)} \subseteq B(0, \|a\|).$$

Thus, $\eta_{\mathcal{B}}(a)$ is bounded, which implies $\eta_{\mathcal{B}}(a)$ is compact whenever $a \in \mathcal{B}$. ∎

Remark 3.6. From [MH17, Theorem 6.7], it follows that $\eta_{\mathbb{C}}(a) = \mathbb{C}$ whenever $a \in \mathcal{A} \setminus \mathbb{C}$. Hence, if $a \notin \mathcal{B}$, $\eta_{\mathcal{B}}(a)$ need not be compact in $\mathbb{C}$.

Remark 3.7 (Restricted connected hull as Ransford spectrum). If $\Omega = \mathcal{B} \cap \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$, then Ω is an open subset of $\mathcal{B}$. Moreover, by [MH24, Corollary 3.3], $\lambda\Omega \subseteq \Omega$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Hence, Ω is a Ransford set in the algebra $\mathcal{B}$. In addition, the map $a \mapsto \eta_{\mathcal{B}}(a)$ is upper semicontinuous on $\mathcal{B}$ as $\eta_{\mathcal{B}}(\cdot)$ is a Ransford spectrum.

4. Commutativity of the restricted connected hull. In [M92], Murphy posed the question of whether the exponential spectrum has the commutativity property. Klaja and Ransford provided an example in [KR17] showing that this property does not hold in general. An analogous question arises for the restricted connected hull: does it commute?

The answer is **No!** Since $\eta_{\mathcal{A}}(a) = \varepsilon(a)$, the same example constructed by Klaja and Ransford [KR17] demonstrates that $\eta_{\mathcal{B}}(ab) \setminus \{0\} \neq \eta_{\mathcal{B}}(ba) \setminus \{0\}$. However, there are instances where the restricted connected hull commutes. The following two theorems present such cases.

Theorem 4.1. *Let $\mathcal{B}$ be a closed unital subalgebra of $\mathcal{A}$ such that* $\mathrm{Sing}(\mathcal{B}) \subseteq \mathrm{Sing}(\mathcal{A})$. *Let $a, b \in \mathcal{B}$. Then*

$$\eta_{\mathcal{B}}(ab) \setminus \{0\} = \eta_{\mathcal{B}}(ba) \setminus \{0\}$$

provided either a or b is in the restricted closure of $\mathcal{A}^{-1}$.

Proof. Without loss of generality, let us assume $a \in \mathrm{cl}^{\mathcal{B}}(\mathcal{A}^{-1})$. It follows from [MH24, Theorem 3.4] that it is enough to show that

$$1 - ab \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \iff 1 - ba \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}).$$

Assume that $1 - ab \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$. Since $a \in \mathcal{B} \cap \mathrm{cl}^{\mathcal{B}}(\mathcal{A}^{-1})$ and $\mathrm{Sing}(\mathcal{B}) \subseteq \mathrm{Sing}(\mathcal{A})$, by Proposition 2.4 and Remark 2.5 there exists a sequence $(a_n) \subseteq \mathcal{B}^{-1}$ such that $a_n \xrightarrow{\mathcal{B}} a$. It follows that $1 - a_n b \xrightarrow{\mathcal{B}} 1 - ab$. Since $\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ is open in the $\mathcal{B}$-topology,

$$1 - a_n b \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \quad \text{for all } n \geq N.$$

Thus,

$$1 - b a_n = a_n^{-1}(1 - a_n b) a_n \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1}) \quad \text{for } n \geq N.$$

Since $\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ is closed in $\mathcal{A}^{-1}$, $1 - ba \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$. ■

One may naturally ask whether the converse holds. However, commutativity can still occur even when a and b do not belong to $\mathrm{cl}^{\mathcal{B}}(\mathcal{A}^{-1})$. The following theorem shows that the converse need not always hold.

THEOREM 4.2. *Let $\mathcal{B}$ be a closed unital subalgebra of $\mathcal{A}$. Let $a, b \in \mathcal{B}$ and suppose that either a or b belongs to $\mathrm{cl}^{\mathcal{B}}(Z(\mathcal{B})\mathcal{B}^{-1})$. Then*

$$\eta_{\mathcal{B}}(ab) \setminus \{0\} = \eta_{\mathcal{B}}(ba) \setminus \{0\}$$

where $Z(\mathcal{B}) = \{a \in \mathcal{A} : \forall x \in \mathcal{B},\ ax = xa\}$.

Proof. Without loss of generality, let us assume $a \in \mathrm{cl}^{\mathcal{B}}(Z(\mathcal{B})\mathcal{B}^{-1})$.

CASE 1: $a \in Z(\mathcal{B})\mathcal{B}^{-1}$. Then $a = zx$ where $z \in Z(\mathcal{B})$ and $x \in \mathcal{B}^{-1}$. Now,

$$1 - ab = 1 - zxb = x(1 - bzx)x^{-1} = x(1 - ba)x^{-1}.$$

Hence $1 - ab \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ if and only if $1 - ba \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$.

CASE 2: $a \in \mathrm{cl}^{\mathcal{B}}(Z(\mathcal{B})\mathcal{B}^{-1}) \setminus Z(\mathcal{B})\mathcal{B}^{-1}$. Then there exists a sequence (a_n) in $Z(\mathcal{B})\mathcal{B}^{-1}$ such that $a_n \xrightarrow{\mathcal{B}} a$.

Now assume that $1 - ab \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$. Since $\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ is open, $1 - a_n b \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ for n sufficiently large. By Case 1, $1 - b a_n \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ for n sufficiently large. Since $\mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$ is closed in $\mathcal{A}^{-1}$, $1 - ba \in \mathrm{Comp}^{\mathcal{B}}(\mathbf{1}, \mathcal{A}^{-1})$. ■

We have observed that the restricted connected hull does not commute in general. In the example presented at the beginning of this section, the chosen subalgebra $\mathcal{B}$ is the whole algebra $\mathcal{A}$, leading to the non-commutativity of the restricted connected hull. Thus, the choice of the closed unital subalgebra $\mathcal{B}$ of $\mathcal{A}$ influences the commutativity of the restricted connected hull. This leaves the question: For which closed unital subalgebra $\mathcal{B}$ of $\mathcal{A}$, can we ensure the commutativity of the restricted connected hull?

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