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Mixing rates for linear operators under infinitely divisible measures on Banach spaces

by

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Abstract. We derive rates of convergence for the mixing of operators under infinitely divisible measures in the framework of linear dynamics on Banach spaces. Our approach is based on the characterization of mixing in terms of codifference functionals and control measures, and extends previous results obtained in the Gaussian setting via the use of covariance operators. Explicit mixing rates are obtained for weighted shifts under compound Poisson, α-stable, and tempered α-stable measures.

1. Introduction. The mixing and ergodicity properties of Gaussian processes and dynamical systems under Gaussian measures have been originally studied in [WA57] and [CFS82, Chapter 14, §2 and Theorems 1 and 2], in connection with the spectral properties of unitary transformations, spectral measures and Gaussian covariances.

On the other hand, characterizations of mixing of continuous linear operators $T : E \to E$ invariant on a complex separable Banach space E have been obtained in the framework of linear dynamics under a Gaussian measure μ on a complex Banach space E. Recall (see, e.g., [BM09, Definition 5.23]) that a measure-preserving map T on (E, μ) is *strongly mixing* if either of the following two equivalent conditions is satisfied:

(i) $\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B)$, $A, B \in \mathcal{B}$,

(ii) $\lim_{n\to\infty} I_n(f, g) = 0$, $f, g \in L^2(E, \mu)$,

where $\mathcal{B}$ is the Borel σ-algebra of E and

$$I_n(f,g) := \int_E f(z)g(T^n z)\,\mu(dz) - \int_E f(z)\,\mu(dz)\int_E g(z)\,\mu(dz), \qquad n \geq 0.$$

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Likewise, T is *weakly mixing* with respect to μ if either of the following two equivalent conditions is satisfied:

(i) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\mu(A\cap T^{-k}(B))-\mu(A)\mu(B)|=0,\ A,B\in\mathcal{B},$

(ii) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|I_n(f,g)|=0,\ f,g\in L^2(E,\mu).$

When μ is a Gaussian measure on E, the mixing of linear operators T has been characterized in [BM09, Theorem 5.24] and references therein using the covariance operator $R: E^* \to E$ of μ, defined by

$$\langle Rx^*,y^*\rangle=\int_E\overline{\langle z,x^*\rangle}\langle z,y^*\rangle\,\mu(dz),\qquad x^*,y^*\in E^*,$$

where E^* is the continuous dual of E and $\langle\cdot,\cdot\rangle : E\times E^*\to\mathbb{C}$ denotes the duality product. Such characterizations have recently been extended in [MP24] from Gaussian measures to a wide class of infinitely divisible probability measures μ on real and complex separable Banach spaces E using strong and weak mixing properties of stationary infinitely divisible processes established in [Mar70, RZ96, RZ97, FS13, PV19].

Recall that a probability measure μ on the Banach space E is *infinitely divisible* if for every $n\geq 1$ there exists another probability measure μ_n on E such that

$$\mu=\underbrace{\mu_n\star\cdots\star\mu_n}_{n\,\text{times}}$$

(see e.g. [Lin86, §5.1]), where $\star$ denotes measure convolution. It is known in addition that every infinitely divisible probability measure on a complex Banach space E has a characteristic functional of the form

$$\begin{aligned}(1.1)\qquad &\int_E e^{i\,\mathrm{Re}\,\langle z,x^*\rangle}\,\mu(dz)\\ &=\exp\Big(-\tfrac14\langle Rx^*,x^*\rangle+i\,\mathrm{Re}\,\langle\hat{x},x^*\rangle+\int_E(e^{i\,\mathrm{Re}\,\langle z,x^*\rangle}-1-i\kappa(z)\,\mathrm{Re}\,\langle z,x^*\rangle)\,\lambda(dz)\Big)\end{aligned}$$

for all $x^*\in E^*$ (see e.g. [Ros87, §II.1]), where $\hat{x}\in E$ and

- $R: E^*\to E$ is a conjugate symmetric and positive semidefinite covariance operator,
- λ is a *Lévy measure*, i.e., a measure on E such that $\lambda(\{0\})=0$ and

$$(1.2)\qquad \int_E\min(1,(\mathrm{Re}\,\langle z,x^*\rangle)^2)\,\lambda(dz)<\infty,\qquad x^*\in E^*,$$

- $\kappa(z)$ is a bounded measurable function on E such that $\lim_{z\to0}\kappa(z)=1$ and $\kappa(z)=O(1/\|z\|)$ as $\|z\|\to\infty$, called a *truncation function.*

In this infinitely divisible setting, the characterization result of [MP24] uses the codifference functionals defined by

$$C_\mu^{=}(x^*, y^*)$$
$$:= \log \int_E e^{i \operatorname{Re} \langle z, x^* - y^* \rangle} \mu(dz) - \log \int_E e^{i \operatorname{Re} \langle z, x^* \rangle} \mu(dz) - \log \int_E e^{-i \operatorname{Re} \langle z, y^* \rangle} \mu(dz)$$

and

$$C_\mu^{\neq}(x^*, y^*) := \log \int_E e^{i \operatorname{Re} \langle z, x^* \rangle - i \operatorname{Im} \langle z, y^* \rangle} \mu(dz) - \log \int_E e^{i \operatorname{Re} \langle z, x^* \rangle} \mu(dz) - \log \int_E e^{-i \operatorname{Im} \langle z, y^* \rangle} \mu(dz)$$

for $x^*, y^* \in E^*$.

In Theorem 2.4 below we start by improving on Proposition 2.2 of [MP24], by removing the vanishing support Condition 2.1 imposed therein on the Lévy measure of the pushforwards of μ by linear functionals $x^* \in E^*$. This condition originated in [Mar70, RZ96], and we rely on results of [FS13, PV19] that relaxed it in the framework of stochastic processes. As a result, we characterize the mixing of linear operators T via the asymptotic vanishing of the codifferences,

$$\lim_{n \to \infty} C_\mu^{=}(ax^*, aT^{*n}x^*) = 0 \quad \text{and} \quad \lim_{n \to \infty} C_\mu^{\neq}(ax^*, aT^{*n}x^*) = 0 \tag{1.3}$$

for all x^* and for some $a \neq 0$ depending on $x^* \in E^*$; see Theorem 2.4. In Examples 2.5 and 4.4 we consider measures μ that can be treated by Theorem 2.4, and to which Proposition 2.2 of [MP24] does not apply.

Next, we focus our attention on the speed of mixing via the derivation of decay rates for the codifferences appearing in (1.3). In the setting of Gaussian measures on Hilbert spaces H, covariance decay rates of the form

$$|\langle Rx^*, T^{*n}x^* \rangle| \leq C \frac{\|x^*\|^2}{n^\gamma},$$

where C is a constant independent of x^*, have been obtained in [Dev13], provided that T is σ-spanning (i.e. for every σ-measurable subset $A \subset \mathbb{T}$ such that $\sigma(A) = 1$, the eigenspaces $\ker(T - \lambda I)$, $\lambda \in A$, span a dense subset of H), and T admits a γ-Hölderian eigenvector field for some $\gamma \in (0, 1]$, where σ denotes the normalized Lebesgue measure on the complex unit circle. In the more general setting of Banach spaces, similar covariance decay rates for classes of functions which satisfy a central limit theorem have been described in [Bay15].

In Section 3 we derive bounds on the codifferences $C_\mu^{=}(x^*, T^{*n}x^*)$ and $C_\mu^{\neq}(x^*, T^{*n}x^*)$ which provide quantitative estimates of mixing speed in the infinitely divisible setting. For this, in addition to (1.1), we consider infinitely divisible measures with characteristic functionals of the form

$$\int_E e^{i\,\mathrm{Re}\,\langle z,x^*\rangle}\,\mu(dz) \tag{1.4}$$

$$= \exp\Big(-\tfrac{1}{4}\langle Rx^*,x^*\rangle + \int_E \int_{-\infty}^{\infty} (e^{iu\,\mathrm{Re}\,\langle z,x^*\rangle} - 1 - iu\kappa(u)\,\mathrm{Re}\,\langle z,x^*\rangle)\rho(z,du)\,\xi(dz)\Big)$$

for all $x^* \in E^*$, where

- $\{\rho(z,\cdot)\}_{z\in E}$ is a family of Lévy measures on $\mathbb{R}$,
- ξ is a σ-finite measure on E called a *control measure*,
- κ is the truncation function

$$\kappa(u) := \mathbf{1}_{\{|u|<1\}} + \frac{1}{|u|}\mathbf{1}_{\{|u|\ge 1\}}, \qquad u \in \mathbb{R}.$$

In Section 4, using the control measure bounds of Section 3 we derive mixing rates under compound Poisson measures on $E = \ell^p(\mathbb{N})$, $p \in [1,2)$, which have characteristic functional (1.1) and Lévy measure of the form

$$\lambda(dz) := \sum_{n=0}^{\infty} \delta_{\lambda_n e_n}(dz),$$

where $(e_n)_{n\ge 0}$ denotes the canonical basis of $\ell^p(\mathbb{N})$ and δ_x is the Dirac measure at $x \in E$.

In Section 5 we let μ be an α-stable measure with $\alpha \in (0,2)\setminus\{1\}$, in which case $\langle Rx^*, y^*\rangle = 0$ and the characteristic functional of μ can be written by the Tortrat Theorem [Tor77] as

$$\int_E e^{i\,\mathrm{Re}\,\langle z,x^*\rangle}\mu(dz) \tag{1.5}$$

$$= \exp\left(c_\alpha \int_E \int_{-\infty}^{\infty} (e^{iu\,\mathrm{Re}\,\langle z,x^*\rangle} - 1 - iu\kappa(u)\,\mathrm{Re}\,\langle z,x^*\rangle)\,\frac{du}{|u|^{1+\alpha}}\,\xi(dz)\right)$$

$$= \exp\Big(-\int_E |\mathrm{Re}\,\langle z,x^*\rangle|^\alpha\,\xi(dz)\Big), \quad x^* \in E^*,$$

where ξ is a finite control measure, $\kappa(u)$ is a truncation function, and $c_\alpha > 0$ (see also [RZ96, Sec. 3], [Woy19, p. 6], [LT91, Corollary 5.5], [Lin86, Theorem 6.4.4 and Corollary 7.5.2], [Sat99, Lemma 14.11], or [PP16, Corollary 4.1]).

In particular, in Proposition 5.2 we derive explicit codifference decay rates of the form

$$\sup_{x^*,y^*\in E^*\setminus\{0\}} \frac{|C_\mu^{=,\neq}(x^*,T^{*n}y^*)|}{\|x^*\|^{\alpha/2}\|y^*\|^{\alpha/2}} = O(\eta^{\alpha n/2}),$$

where $C_\mu^{=,\neq}$ denotes $C_\mu^{=}$ or $C_\mu^{\neq}$, for weighted forward shifts T on $E = \ell^p(\mathbb{Z})$ under α-stable measures with $\alpha \in (1/2,2)\setminus\{1\}$ and $p \in (\alpha,2\alpha)\cap[1,2]$, where $\eta \in (0,1)$ is a constant depending on T.

In Corollary 5.3, we extend those results by deriving decay rates for the quantity

$$I_n(f,g) := \int_E f(z)g(T^n z)\,\mu(dz) - \int_E f(z)\,\mu(dz)\int_E g(z)\,\mu(dz), \tag{1.6}$$

where f, g are finite or infinite linear combinations of exponentials, and T is a weighted forward shift as in Proposition 5.2.

In Section 6 we consider the case where μ is a tempered stable measure whose characteristic functional takes the form (1.4) and $\rho(z, du)$ is given by

$$\rho(z,du) = \left(\frac{a_-}{|u|^{1+\alpha}} e^{-\lambda_-|u|}\mathbf{1}_{\mathbb{R}^-}(u) + \frac{a_+}{u^{1+\alpha}} e^{-\lambda_+ u}\mathbf{1}_{\mathbb{R}^+}(u)\right) du,$$

with $a_-, a_+, \lambda_-, \lambda_+ > 0$ and $\alpha \in (0,1)$ (see [KT13] and also [Dev13]). In this setting we derive decay rates of the form

$$\sup_{x^*,y^*\in E^*\setminus\{0\}} \frac{|C_\mu^{=,\neq}(x^*, T^{*n}y^*)|}{\|x^*\|^{p/2}\|y^*\|^{p/2}} = O(n^{-\zeta})$$

for certain backward weighted shift operators on $E = \ell^p(\mathbb{Z})$, where $\zeta \in (0,1)$ is a constant depending on T for codifferences and quantities of the form (1.6); see Proposition 6.4 and Example 6.5.

2. Mixing conditions. The goal of this section is to prove Theorem 2.4 below, which extends the necessary and sufficient conditions for the mixing of linear operators in terms of codifference functionals in [MP24, Proposition 2.2] by removing the technical support Condition 2.1 below.

Recall (see e.g. [App09, Theorem 1.2.14]) that similarly to (1.1), every infinitely divisible random variable on $\mathbb{R}^d$, $d \geq 1$, has a characteristic functional of the form

$$\int_{\mathbb{R}^d} e^{i\langle z,y\rangle_d}\,\mu(dz) = \exp\Big(-\tfrac{1}{2}\langle Ry, y\rangle_d + i\langle y_0, y\rangle_d + \int_{\mathbb{R}^d} (e^{i\langle z,y\rangle_d} - 1 - i\kappa(z)\langle z, y\rangle_d)\,\nu(dz)\Big)$$

for all $y \in \mathbb{R}^d$, where

- $y_0 \in \mathbb{R}^d$ is a fixed vector,
- $\langle\cdot,\cdot\rangle_d$ denotes the Euclidean inner product in $\mathbb{R}^d$,
- $R : \mathbb{R}^d \to \mathbb{R}^d$ is a symmetric and positive semidefinite covariance operator,
- $\kappa(z)$ is a bounded measurable function on $\mathbb{R}^d$ such that $\lim_{z\to 0}\kappa(z) = 1$ and $\kappa(z) = O(1/\|z\|)$ as $\|z\| \to \infty$, called a *truncation function*,
- ν is a Lévy measure on $\mathbb{R}^d$, i.e. a measure that satisfies $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min(1, \langle z, y\rangle_d^2)\,\nu(dz) < \infty, \qquad y \in \mathbb{R}^d.$$

In what follows, we consider an E-valued random variable X with infinitely divisible distribution μ.

CONDITION 2.1. For any $x^* \in E^*$, the Lévy measure ν_{x^*} of the $\mathbb{R}^2$-valued random variable $(\mathrm{Re}\,\langle X, x^*\rangle, \mathrm{Im}\,\langle X, x^*\rangle)$ satisfies

$$\nu_{x^*}(\mathbb{R} \times 2\pi\mathbb{Z}) = 0 \quad \text{and} \quad \nu_{x^*}(2\pi\mathbb{Z} \times \mathbb{R}) = 0. \tag{2.1}$$

The above assumption originates from a condition appearing in [Mar70], which was used in [RZ96, Theorem 1] to characterize mixing by codifferences. The value of 2π in (2.1) is chosen for consistency with the literature, however, it can be replaced with an arbitrary non-zero constant without affecting Condition 2.1.

DEFINITION 2.2. Given ν a measure on $\mathbb{R}^d$, we define the set $Z_d(\nu)$ as follows. If

$$\nu(\mathbb{R}^{j-1} \times \{2k\pi\} \times \mathbb{R}^{d-j}) = 0 \qquad \text{for all } k \in \mathbb{Z},\ j = 1, \ldots, d,$$

we define $Z_d(\nu) := \mathbb{R} \setminus \{1\}$, else we let

$$Z_d(\nu) := \bigcup_{k\in\mathbb{Z}} \left\{ k\frac{2\pi}{s} : s \in \mathbb{R} \setminus \{0\} \text{ and } \sum_{j=1}^{d} \nu(\mathbb{R}^{j-1} \times \{s\} \times \mathbb{R}^{d-j}) > 0 \right\}.$$

The following result shows that for any Lévy measure ν, the set $\mathbb{R} \setminus Z_d(\nu)$ always contains a non-zero element.

LEMMA 2.3. *For any Lévy measure ν on $\mathbb{R}^d$, the set $Z_d(\nu)$ either is $\mathbb{R}\setminus\{1\}$ or is at most countable.*

Proof. Clearly, we may assume that

$$\sum_{j=1}^{d} \nu(\mathbb{R}^{j-1} \times \{2k\pi\} \times \mathbb{R}^{d-j}) > 0$$

for some $k \in \mathbb{Z}$, otherwise $Z_d(\nu) = \mathbb{R} \setminus \{1\}$ and the proof is complete. Denoting by $B_d(0,1)$ the unit ball and by $\|\cdot\|_d$ the Euclidean norm of $\mathbb{R}^d$, by definition of Lévy measures we have

$$\int_{\mathbb{R}^d} \min(1, \|x\|_d^2)\, \nu(dx) = \int_{B_d(0,1)} \|x\|_d^2\, \nu(dx) + \int_{\mathbb{R}^d \setminus B_d(0,1)} \nu(dx) < \infty.$$

Letting $P_1 : \mathbb{R}^d \to \mathbb{R}$ denote the projection onto the first coordinate in $\mathbb{R}^d$, and denoting by ρ the pushforward of ν by P_1, we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} \min(1,x^2)\,\rho(dx) &= \int_{(-1,1)} x^2\,\rho(dx) + \int_{\mathbb{R}\setminus(-1,1)} \rho(dx) \\
&\le \int_{(-1,1)\times\mathbb{R}^{d-1}} (P_1x)^2\,\nu(dx) + \int_{\mathbb{R}^d\setminus B_d(0,1)} \nu(dx) \\
&= \int_{B_d(0,1)} (P_1x)^2\,\nu(dx) + \int_{((-1,1)\times\mathbb{R}^{d-1})\setminus B_d(0,1)} (P_1x)^2\,\nu(dx) + \int_{\mathbb{R}^d\setminus B_d(0,1)} \nu(dx) \\
&\le \int_{B_d(0,1)} \|x\|_d^2\,\nu(dx) + \int_{((-1,1)\times\mathbb{R}^{d-1})\setminus B_d(0,1)} \nu(dx) + \int_{\mathbb{R}^d\setminus B_d(0,1)} \nu(dx) \\
&< \infty,
\end{aligned}
$$

hence ρ is σ-finite and therefore it has countably many atoms, i.e. there are at most countably many values of $s \in \mathbb{R}$ such that $\nu(\{s\}\times\mathbb{R}^{d-1}) > 0$. Repeating this argument for each coordinate, we find that the cardinality of $Z_d(\nu)$ is at most that of $\mathbb{N}^d$, i.e. countable. ■

The following results are stated for complex Banach spaces, but they also apply to real Banach spaces by ignoring vanishing imaginary components. Theorem 2.4 below allows for the mixing property of T to be checked without imposing Condition 2.1. Recall that a set $D \subset \mathbb{N}$ has *density one* if $\lim_{n\to\infty} |D \cap \{0,1,\dots,n\}|/(n+1) = 1$.

THEOREM 2.4. *Let μ be an infinitely divisible distribution on a complex separable Banach space E. For any $x^* \in E^*$, let ν_{x^*} denote the Lévy measure of* $(\mathrm{Re}\,\langle X, x^*\rangle, \mathrm{Im}\langle X, x^*\rangle)$ *on $\mathbb{R}^2$. Then*

(i) *T is mixing if and only if for each $x^* \in E^*$ we have*

$$\lim_{n\to\infty} C_\mu^{=}(ax^*, aT^{*n}x^*) = 0 \quad \textit{and} \quad \lim_{n\to\infty} C_\mu^{\neq}(ax^*, aT^{*n}x^*) = 0$$

for some non-zero $a \in \mathbb{R}\setminus Z_2(\nu_{x^*})$;

(ii) *T is weakly mixing if and only if for each $x^* \in E^*$ there exists a density one set $D_{x^*} \subset \mathbb{N}$ such that*

$$\lim_{\substack{n\to\infty\\ n\in D_{x^*}}} C_\mu^{=}(ax^*, aT^{*n}x^*) = 0 \quad \textit{and} \quad \lim_{\substack{n\to\infty\\ n\in D_{x^*}}} C_\mu^{\neq}(ax^*, aT^{*n}x^*) = 0$$

for some non-zero $a \in \mathbb{R}\setminus Z_2(\nu_{x^*})$.

In Example 2.5 we present a non-mixing operator inspired by the example on page 282 of [RZ96], which can be treated by Theorem 2.4 while Condition 2.1 is not satisfied. See also Example 4.4 for a mixing operator.

EXAMPLE 2.5. Let E be the (real) sequence space $\ell^p(\mathbb{N})$, $p \ge 1$, take $R := 0$, $\hat{x} := 2\pi e_0$, κ such that $\kappa(\hat{x}) = 1$, and let $\lambda(dz) := \delta_{2\pi e_0}(dz)$ in (1.1). Then, for $x^* \in E^*$ such that $\langle e_0, x^*\rangle = 1$, the Lévy measure $\nu_{x^*} = \delta_{2\pi\langle e_0, x^*\rangle}$

on $\mathbb{R}$ does not satisfy (2.1). However, the application of Theorem 2.4 shows that the identity operator $T = \mathrm{Id}$ is not mixing.

Proof. Let μ be the infinitely divisible distribution on $E = \ell^p(\mathbb{N})$ with Lévy measure $\lambda(dz)$. In this case, X can be defined by $X := 2\pi N e_0$ where N is a standard Poisson random variable, and we have

$$\begin{aligned}\mathbb{E}[e^{ia\langle X,x^*\rangle}] &= \int_E e^{ia\langle z,x^*\rangle}\,\mu(dz) = \exp\Big(\int_E (e^{ia\langle z,x^*\rangle}-1)\,\lambda(dz)\Big)\\ &= \exp(e^{2ia\pi\langle e_0,x^*\rangle}-1),\end{aligned}$$

hence for any $x^* \in E^*$ the random variable $\langle X, x^*\rangle = 2\pi N\langle e_0, x^*\rangle$ has Lévy measure $\nu_{x^*} = \delta_{2\pi\langle e_0,x^*\rangle}$. In this case, we have

$$Z_1(\nu_{x^*}) = \left\{\frac{k}{\langle e_0, x^*\rangle} : k \in \mathbb{Z}\right\}$$

if $\langle e_0, x^*\rangle \notin \mathbb{Z}$, and $Z_1(\nu_{x^*}) = \mathbb{R}\setminus\{1\}$ otherwise. Hence,

$$\begin{aligned}C_\mu^{=}(ax^*, aT^{*n}x^*) = C_\mu^{=}(ax^*, ax^*) &= -2\log\int_E e^{ia\langle z,x^*\rangle}\,\mu(dz)\\ &= -2(e^{2ia\pi\langle e_0,x^*\rangle}-1)\end{aligned}$$

is constant in $n \geq 1$ and does not vanish for any $a \in \mathbb{R}\setminus Z_1(\nu_{x^*})$, therefore Theorem 2.4 shows that T is not mixing. ∎

The proof of Theorem 2.4 is stated at the end of this section by carrying over Theorem 2 of [RZ96] from the stochastic process setting to the framework of linear dynamics, thereby completing the characterization of mixing of infinitely divisible measures on Banach spaces. For this, we need to prove the following multidimensional extension of [RZ96, Theorem 2] on mixing and weak mixing for discrete-time stochastic processes, which removes the support condition assumed in [FS13, Theorem 2.1] and [PV19, Theorem 4.3].

PROPOSITION 2.6. *Let $d \geq 1$, and let $(X_n)_{n\geq 0} = (X_n^{(1)},\dots,X_n^{(d)})_{n\geq 0}$ be a stationary infinitely divisible $\mathbb{R}^d$-valued process. Denote by ν_0 the Lévy measure of X_0. Then*

(a) *$(X_n)_{n\geq 0}$ is mixing if and only if for some non-zero $a \in \mathbb{R}\setminus Z_d(\nu_0)$ we have*

$$\lim_{n\to\infty}\mathbb{E}[e^{ia(X_n^{(j)}-X_0^{(k)})}] = \mathbb{E}[e^{iaX_0^{(j)}}]\mathbb{E}[e^{-iaX_0^{(k)}}]$$

for any $j, k \in \{1,\dots,d\}$;

(b) *$(X_n)_{n\geq 0}$ is weakly mixing if and only if for some non-zero $a \in \mathbb{R}\setminus Z_d(\nu_0)$ and a density one set $D \subset \mathbb{N}$ we have*

$$\lim_{n\to\infty,\, n\in D}\mathbb{E}[e^{ia(X_n^{(j)}-X_0^{(k)})}] = \mathbb{E}[e^{iaX_0^{(j)}}]\mathbb{E}[e^{-iaX_0^{(k)}}]$$

for any $j, k \in \{1,\dots,d\}$.

Proof. If the condition

$$\nu_0(\{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \exists j \in \{1, \dots, d\},\, x_j \in 2\pi\mathbb{Z}\}) = 0 \tag{2.2}$$

holds, the assertion follows from [FS13, Theorem 2.1] in the mixing case and from [PV19, Theorem 4.3] in the weak mixing case, hence we may assume that (2.2) does not hold. Then, as in [RZ96, Theorem 2], we observe that $(X_n)_{n\ge 0}$ is mixing, respectively weak mixing, if and only if $(aX_n)_{n\ge 0}$ is, and furthermore the Lévy measure $\nu_0^a(\cdot)$ of aX_0 is $\nu_0(a^{-1}(\cdot))$. Now, since $a \notin Z_d(\nu_0)$, it follows that

$$\nu_0^a(\{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \exists j \in \{1, \dots, d\},\, x_j \in 2\pi\mathbb{Z}\}) = 0.$$

The conclusion follows for mixing by [FS13, Theorem 2.1] (see also [PV19, Theorem 3.2]) and for weak mixing by [PV19, Theorem 4.3]. ∎

Proof of Theorem 2.4. Let X denote a random variable with distribution μ on E. For any $x^* \in E^*$, let the process $(X_n^{x^*})_{n\ge 0}$ be defined by

$$X_n^{x^*} := (\operatorname{Re}\langle X, T^{*n}x^*\rangle, \operatorname{Im}\langle X, T^{*n}x^*\rangle), \qquad n \ge 0.$$

By [MP24, Lemma 2.1], T is mixing if and only if $(X_n^{x^*})_{n\ge 0}$ is mixing for each $x^* \in E^*$, hence the claim follows from Proposition 2.6 and the relations

$$C_\mu^{=}(ax^*, aT^{*n}x^*) = \log \frac{\mathbb{E}[e^{ia\operatorname{Re}\langle X,x^*\rangle - ia\operatorname{Re}\langle X,T^{*n}x^*\rangle}]}{\mathbb{E}[e^{ia\operatorname{Re}\langle X,x^*\rangle}]\mathbb{E}[e^{-ia\operatorname{Re}\langle X,x^*\rangle}]},$$

$$C_\mu^{\neq}(ax^*, aT^{*n}x^*) = \log \frac{\mathbb{E}[e^{ia\operatorname{Re}\langle X,x^*\rangle - ia\operatorname{Im}\langle X,T^{*n}x^*\rangle}]}{\mathbb{E}[e^{ia\operatorname{Re}\langle X,x^*\rangle}]\mathbb{E}[e^{-ia\operatorname{Im}\langle X,x^*\rangle}]}, \qquad a \in \mathbb{R}. \ ∎$$

3. Codifference bounds. Our first step towards the derivation of codifference decay rates in (1.3) is to derive bounds on codifferences using Lévy and control measures.

Lemma 3.1 (Lévy measure bounds). *Let μ be an infinitely divisible distribution with characteristic functional of the form* (1.1). *For every $p \in [0, 2]$, we have the codifference bounds*

$$|C_\mu^{=}(x^*, y^*)| \le \tfrac{1}{2}|\operatorname{Re}\langle Rx^*, y^*\rangle| + 2^{4-p}\int_E |\operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, y^*\rangle|^{p/2}\,\lambda(dz), \tag{3.1}$$

and

$$|C_\mu^{\neq}(x^*, y^*)| \le \tfrac{1}{2}|\operatorname{Im}\langle Rx^*, y^*\rangle| + 2^{4-p}\int_E |\operatorname{Re}\langle z, x^*\rangle \operatorname{Im}\langle z, y^*\rangle|^{p/2}\,\lambda(dz). \tag{3.2}$$

Proof. The codifference of μ can be rewritten from (1.1) as

$$C_\mu^{=}(x^*, y^*) = \tfrac{1}{2}\operatorname{Re}\langle Rx^*, y^*\rangle + \int_E (e^{i\operatorname{Re}\langle z,x^*\rangle} - 1)(e^{-i\operatorname{Re}\langle z,y^*\rangle} - 1)\,\lambda(dz), \tag{3.3}$$

and

(3.4)
$$cdf2C_\mu^{\neq}(x^*,y^*)=\tfrac{1}{2}\operatorname{Im}\langle Rx^*,y^*\rangle+\int_E(e^{i\operatorname{Re}\langle z,x^*\rangle}-1)(e^{-i\operatorname{Im}\langle z,y^*\rangle}-1)\,\lambda(dz),$$

for $x^*,y^*\in E^*$. Taking the real part in (3.3), we have

$$\begin{aligned}\operatorname{Re}C_\mu^{=}(x^*,y^*)&=\tfrac{1}{2}\operatorname{Re}\langle Rx^*,y^*\rangle\\&\quad+\int_E\big((\cos(\operatorname{Re}\langle z,x^*\rangle)-1)(\cos(\operatorname{Re}\langle z,y^*\rangle)-1)\\&\qquad\qquad+\sin(\operatorname{Re}\langle z,x^*\rangle)\sin(\operatorname{Re}\langle z,y^*\rangle)\big)\,\lambda(dz).\end{aligned}$$

Likewise, taking the imaginary part in (3.3), we obtain

$$\begin{aligned}\operatorname{Im}C_\mu^{=}(x^*,y^*)&=\int_E\big(\sin(\operatorname{Re}\langle z,x^*\rangle)(\cos(\operatorname{Re}\langle z,y^*\rangle)-1)\\&\qquad-\sin(\operatorname{Re}\langle z,y^*\rangle)(\cos(\operatorname{Re}\langle z,x^*\rangle)-1)\big)\,\lambda(dz).\end{aligned}$$

Using the inequalities

$$\max(|\cos x-1|,|\sin x|)\le 2^{\frac{2-p}{2}}|x|^{p/2},\qquad x\in\mathbb{R},\tag{3.5}$$

which are valid for $p\in[0,2]$, it follows from the triangle inequality that

$$|\operatorname{Re}C_\mu^{=}(x^*,y^*)|\le\tfrac{1}{2}|\operatorname{Re}\langle Rx^*,y^*\rangle|+2^{3-p}\int_E|\operatorname{Re}\langle z,x^*\rangle\operatorname{Re}\langle z,y^*\rangle|^{p/2}\,\lambda(dz).$$

Similarly, we obtain

$$|\operatorname{Im}C_\mu^{=}(x^*,y^*)|\le 2^{3-p}\int_E|\operatorname{Re}\langle z,x^*\rangle\operatorname{Re}\langle z,y^*\rangle|\,\lambda(dz),$$

which yields (3.1) by the triangle inequality. Likewise, for (**??**) we have

$$\begin{aligned}\operatorname{Re}C_\mu^{\neq}(x^*,y^*)&=\tfrac{1}{2}\operatorname{Re}\langle Rx^*,y^*\rangle\\&\quad+\int_E\big((\cos(\operatorname{Re}\langle z,x^*\rangle)-1)(\cos(\operatorname{Im}\langle z,y^*\rangle)-1)\\&\qquad\qquad+\sin(\operatorname{Re}\langle z,x^*\rangle)\sin(\operatorname{Im}\langle z,y^*\rangle)\big)\,\lambda(dz),\end{aligned}$$

implying

$$|\operatorname{Re}C_\mu^{\neq}(x^*,y^*)|\le\tfrac{1}{2}|\operatorname{Im}\langle Rx^*,y^*\rangle|+2^{3-p}\int_E|\operatorname{Re}\langle z,x^*\rangle\operatorname{Im}\langle z,y^*\rangle|^{p/2}\,\lambda(dz),$$

and

$$\begin{aligned}\operatorname{Im}C_\mu^{\neq}(x^*,y^*)&=\int_E\big((\sin(\operatorname{Re}\langle z,x^*\rangle)(\cos(\operatorname{Im}\langle z,y^*\rangle)-1)\\&\qquad-\sin(\operatorname{Im}\langle z,y^*\rangle)(\cos(\operatorname{Re}\langle z,x^*\rangle)-1)\big)\,\lambda(dz),\end{aligned}$$

implying

$$|\mathrm{Im}\, C_\mu^{\neq}(x^*,y^*)| \le 2^{3-p} \int_E |\mathrm{Re}\,\langle z,x^*\rangle\, \mathrm{Im}\,\langle z,y^*\rangle|^{p/2}\, \lambda(dz),$$

which yields (3.2). ∎

Next, we consider the case where the characteristic functional of the infinitely divisible measure μ takes the form (1.4). Recall (see [RZ96, §3]) that any random variable X with distribution μ can be represented as

$$X = \int_E z\, \Lambda(dz),$$

where Λ is the infinitely divisible random measure on E defined by its characteristic functional

$$\mathbb{E}[e^{it\Lambda(A)}] = \exp\Big(-\frac{t^2}{4}\int_A \sigma^2(z)\,\xi(dz) + \int_A \int_{-\infty}^{\infty} (e^{iut} - 1 - itu\kappa(u))\rho(z,du)\,\xi(dz)\Big)$$

for measurable $A \subset E$ and $t \in \mathbb{R}$, where $\sigma^2 : E \to [0,\infty)$ is a measurable function.

Lemma 3.2 (Control measure bounds). *Let μ be an infinitely divisible distribution with characteristic functional of the form* (1.4). *For any $p \in [0,2]$ and $c > 0$, we have the codifference bounds*

$$|C_\mu^{=}(x^*,y^*)| \le \tfrac{1}{2}|\mathrm{Re}\,\langle Rx^*,y^*\rangle| + 16\int_E \Big(2^{-p}\int_{-c}^{c} |u|^p|\,\mathrm{Re}\,\langle z,x^*\rangle\,\mathrm{Re}\,\langle z,y^*\rangle|^{p/2}\rho(z,du) + \rho(z,\mathbb{R}\setminus[-c,c])\Big)\,\xi(dz) \tag{3.6}$$

and

$$|C_\mu^{\neq}(x^*,y^*)| \le \tfrac{1}{2}|\mathrm{Im}\,\langle Rx^*,y^*\rangle| + 16\int_E \Big(2^{-p}\int_{-c}^{c} |u|^p|\,\mathrm{Re}\,\langle z,x^*\rangle\,\mathrm{Im}\,\langle z,y^*\rangle|^{p/2}\rho(z,du) + \rho(z,\mathbb{R}\setminus[-c,c])\Big)\,\xi(dz) \tag{3.7}$$

for $x^*, y^* \in E^*$.

Proof. By taking the real part in the relation

$$C_\mu^{=}(x^*,y^*) = \tfrac{1}{2}\,\mathrm{Re}\,\langle Rx^*,y^*\rangle + \int_E\int_{-\infty}^{\infty} (e^{iu\,\mathrm{Re}\,\langle z,x^*\rangle} - 1)(e^{-iu\,\mathrm{Re}\,\langle z,y^*\rangle} - 1)\,\rho(z,du)\,\xi(dz)$$

for $x^*, y^* \in E^*$, we have

$$\begin{aligned}\operatorname{Re} C_\mu^{=}(x^*, y^*) &= \tfrac{1}{2} \operatorname{Re} \langle Rx^*, y^* \rangle \\ &\quad + \int_E \int_{-\infty}^{\infty} \big((\cos(u \operatorname{Re} \langle z, x^* \rangle) - 1)(\cos(u \operatorname{Re} \langle z, y^* \rangle) - 1) \\ &\qquad + \sin(u \operatorname{Re} \langle z, x^* \rangle) \sin(u \operatorname{Re} \langle z, y^* \rangle) \big) \, \rho(z, du) \, \xi(dz).\end{aligned}$$

For any $c > 0$, using (3.5), we have

$$\begin{aligned}&\int_E \int_{-\infty}^{\infty} |(\cos(u \operatorname{Re} \langle z, x^* \rangle) - 1)(\cos(u \operatorname{Re} \langle z, y^* \rangle) - 1)| \, \rho(z, du) \, \xi(dz) \\ &\quad \le \int_E \Big(2^{2-p} \int_{-c}^{c} |u|^p |\operatorname{Re} \langle z, x^* \rangle \operatorname{Re} \langle z, y^* \rangle|^{p/2} \, \rho(z, du) + 4 \int_{\mathbb{R} \setminus [-c,c]} \rho(z, du) \Big) \, \xi(dz).\end{aligned}$$

Likewise,

$$\begin{aligned}&\int_E \int_{-\infty}^{\infty} |\sin(u \operatorname{Re} \langle z, x^* \rangle) \sin(u \operatorname{Re} \langle z, y^* \rangle)| \, \rho(z, du) \, \xi(dz) \\ &\quad \le \int_E \Big(2^{2-p} \int_{-c}^{c} |u|^p |\operatorname{Re} \langle z, x^* \rangle \operatorname{Re} \langle z, y^* \rangle|^{p/2} \, \rho(z, du) + 4 \int_{\mathbb{R} \setminus [-c,c]} \rho(z, du) \Big) \, \xi(dz),\end{aligned}$$

hence by the triangle inequality we have

$$\begin{aligned}&|\operatorname{Re} C_\mu^{=}(x^*, y^*)| \le \tfrac{1}{2} |\operatorname{Re} \langle Rx^*, y^* \rangle| \\ &\quad + 8 \int_E \Big(2^{-p} \int_{-c}^{c} |u|^p | \operatorname{Re} \langle z, x^* \rangle \operatorname{Re} \langle z, y^* \rangle|^{p/2} \, \rho(z, du) + \int_{\mathbb{R} \setminus [-c,c]} \rho(z, du) \Big) \, \xi(dz).\end{aligned}$$

In a similar fashion, we have

$$\begin{aligned}&|\operatorname{Im} C_\mu^{=}(x^*, y^*)| \\ &\quad \le 8 \int_E \Big(2^{-p} \int_{-c}^{c} |u|^p | \operatorname{Re} \langle z, x^* \rangle \operatorname{Re} \langle z, y^* \rangle|^{p/2} \rho(z, du) + \int_{\mathbb{R} \setminus [-c,c]} \rho(z, du) \Big) \, \xi(dz),\end{aligned}$$

which yields (3.6). The bound (3.7) is obtained by application of similar arguments to

$$\begin{aligned}C_\mu^{\neq}(x^*, y^*) &= \tfrac{1}{2} \operatorname{Im} \langle Rx^*, y^* \rangle \\ &\quad + \int_E \int_{-\infty}^{\infty} (e^{iu \operatorname{Re} \langle z, x^* \rangle} - 1)(e^{-iu \operatorname{Im} \langle z, y^* \rangle} - 1) \, \rho(z, du) \, \xi(dz). \quad \blacksquare\end{aligned}$$

4. Compound Poisson measures. In this section, we provide an example of mixing operator T in Proposition 4.3 that can be treated by Theorem 2.4, and to which Proposition 2.2 of [MP24] does not apply: see Example 4.4 below. For this, we consider an infinitely divisible measure μ on $E = \ell^p(\mathbb{N})$ with characteristic functional (1.1) and Lévy measure

$$\sum_{n=0}^{\infty} \delta_{\lambda_n e_n}(dz) \tag{4.1}$$

for appropriate sequences $(\lambda_n)_{n\geq 0}$, i.e. μ is the distribution of an E-valued random variable X whose components in the canonical basis $(e_n)_{n\geq 0}$ of $\ell^p(\mathbb{N})$ are independent Poisson random variables with means $(\lambda_n)_{n\geq 0}$. We first determine precisely the sequences $(\lambda_n)_{n\geq 0}$ for which (4.1) defines a Lévy measure on $\ell^p(\mathbb{N})$, as a consequence of the following auxiliary lemma.

Lemma 4.1. *Let $p \in [1, 2)$ and q be the Hölder conjugate of p. For any complex sequence $(a_n)_{n\geq 0}$, the following statements are equivalent:*

(1) $(a_n)_{n\geq 0} \in \ell^p(\mathbb{N})$.
(2) *For every $(b_n)_{n\geq 0}$ in the real sequence space $\ell^q(\mathbb{N})$, we have*

$$\sum_{n=0}^{\infty} |a_n|^2 |b_n|^2 < \infty.$$

Proof. The implication (1)⇒(2) follows from Hölder's inequality

$$\sum_{n=0}^{\infty} |a_n|\,|b_n| \leq \|(a_n)_{n\geq 0}\|_{\ell^p(\mathbb{N})} \|(b_n)_{n\geq 0}\|_{\ell^q(\mathbb{N})} < \infty$$

and the fact that $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$. The implication (2)⇒(1) follows from the reverse Hölder inequality

$$\infty > \sum_{n=0}^{\infty} |a_n|^2 |b_n|^2 \geq \Big(\sum_{n=0}^{\infty} |a_n|^{2/s}\Big)^{s} \Big(\sum_{n=0}^{\infty} |b_n|^{-2/(s-1)}\Big)^{-(s-1)}$$

applied with $s := 2/p \in [1, 2)$ to any sequence $(b_n)_{n\geq 0} \in \ell^q(\mathbb{N})$ such that $\mathrm{Re}(b_n) \neq 0$ for all $n \geq 0$. ∎

From Lemma 4.1, we obtain the following criterion for Lévy measures in real $\ell^p(\mathbb{N})$.

Lemma 4.2. *Let $p \in [1, 2)$, and let $(\lambda_n)_{n\geq 0}$ be a real sequence. The measure λ defined by*

$$\lambda(dz) := \sum_{n=0}^{\infty} \delta_{\lambda_n e_n}(dz)$$

is a Lévy measure on the real sequence space $\ell^p(\mathbb{N})$ if and only if $(\lambda_n)_{n\geq 0} \in \ell^p(\mathbb{N})$ and $\lambda_n \neq 0$ for all $n \geq 0$.

Proof. Let $q > 2$ denote the Hölder conjugate of p. We note that

$$\int_{\ell^p(\mathbb{N})} \min(1, \langle z, x^* \rangle^2)\, \lambda(dz) = \sum_{n=0}^{\infty} \min(1, \lambda_n^2 \langle e_n, x^* \rangle^2)$$

is finite for all $x^* \in \ell^q(\mathbb{N})$ if and only if

$$\sum_{n=0}^{\infty} \lambda_n^2 \langle e_n, x^* \rangle^2$$

is finite for any $x^* \in \ell^q(\mathbb{N})$, i.e. if and only if

$$\sum_{n=0}^{\infty} \lambda_n^2 b_n^2 < \infty$$

for any $(b_n)_{n\geq 0}$ in the real sequence space $\ell^q(\mathbb{N})$. The claim follows from (1.2), Lemma 4.1 and the fact that $\lambda(\{0\}) = 0$ if and only if $\lambda_n \neq 0$ for all $n \geq 0$. ∎

We now turn to the main result of this section.

PROPOSITION 4.3. *Let $p \in [1, 2)$. Let $(\omega_n)_{n\geq 0}$ be a positive weight sequence such that the sequence $(\lambda_n)_{n\geq 0}$ defined by*

$$\lambda_0 > 0 \quad \textit{and} \quad \lambda_n := \lambda_0 \prod_{l=0}^{n-1} \frac{1}{\omega_l}, \qquad n \geq 1,$$

satisfies $(\lambda_n)_{n\geq 0} \in \ell^p(\mathbb{N})$. Let μ be the compound Poisson measure on the real space $E := \ell^p(\mathbb{N})$ with characteristic functional (1.1) given by

$$R := 0, \qquad \hat{x} := \sum_{n=0}^{\infty} \lambda_n e_n \quad \textit{and} \quad \kappa(z) := \mathbf{1}_{\{\|z\|_{\ell^p(\mathbb{N})} \leq \max_{n\geq 0} \lambda_n\}}, \ z \in \ell^p(\mathbb{N}),$$

and the Lévy measure

$$\lambda(dz) := \sum_{n=0}^{\infty} \delta_{\lambda_n e_n}(dz). \tag{4.2}$$

Consider the weighted backward shift operator T defined by $Te_0 := 0$ and

$$Te_{n+1} := \omega_n e_n, \qquad n \geq 0.$$

The following are true:

(1) *μ is an invariant measure for T.*
(2) *We have*

$$\sup_{x^*, y^* \in E^* \setminus \{0\}} \frac{|C_\mu^{=}(x^*, T^{*n} y^*)|}{\|x^*\|^{p/2} \|y^*\|^{p/2}} \leq 2^{4-p} \sum_{l=0}^{\infty} (\lambda_l \lambda_{l+n})^{p/2}, \qquad x^*, y^* \in E^*, \ n \geq 0.$$

In particular, by Theorem 2.4, T is mixing provided that

$$\lim_{n\to\infty}\sum_{l=0}^{\infty}(\lambda_l\lambda_{l+n})^{p/2}=0.$$

Proof. (1) Note that (4.2) defines a Lévy measure by Lemma 4.2, thus μ is well-defined. To show that μ is an invariant measure for T, we use (1.1) and the equalities

$$\begin{aligned}
&\int_E \big(e^{i\,\mathrm{Re}\,\langle Tz,x^*\rangle}-1-i\kappa(z)\,\mathrm{Re}\,\langle Tz,x^*\rangle\big)\,\lambda(dz)\\
&\qquad=\sum_{n=0}^{\infty}\int_E \big(e^{i\,\mathrm{Re}\,\langle Tz,x^*\rangle}-1-i\kappa(z)\,\mathrm{Re}\,\langle Tz,x^*\rangle\big)\,\delta_{\lambda_n e_n}(dz)\\
&\qquad=\sum_{n=1}^{\infty}\big(e^{i\,\mathrm{Re}\,\langle \lambda_n Te_n,x^*\rangle}-1-i\,\mathrm{Re}\,\langle \lambda_n Te_n,x^*\rangle\big)\\
&\qquad=\sum_{n=0}^{\infty}\big(e^{i\,\mathrm{Re}\,\langle \lambda_{n+1} Te_{n+1},x^*\rangle}-1-i\,\mathrm{Re}\,\langle \lambda_{n+1} Te_{n+1},x^*\rangle\big)\\
&\qquad=\sum_{n=0}^{\infty}\big(e^{i\,\mathrm{Re}\,\langle \lambda_n e_n,x^*\rangle}-1-i\,\mathrm{Re}\,\langle \lambda_n e_n,x^*\rangle\big)\\
&\qquad=\int_E \big(e^{i\,\mathrm{Re}\,\langle z,x^*\rangle}-1-i\kappa(z)\,\mathrm{Re}\,\langle z,x^*\rangle\big)\,\lambda(dz),
\end{aligned}$$

hence $\int_E e^{i\,\mathrm{Re}\,\langle Tz,x^*\rangle}\,\mu(dz)=\int_E e^{i\,\mathrm{Re}\,\langle z,x^*\rangle}\,\mu(dz)$ for any $x^*\in E^*$, proving invariance.

(2) As the characteristic functional of μ also takes the form (1.4) with $\xi(dz)=\sum_{n=0}^{\infty}\delta_{e_n}(dz)$ and $\rho(e_n,du)=\delta_{\lambda_n}(du)$, $n\geq 0$, we note that by (3.6) in Lemma 3.2 with $p\in[1,2)$, for any $c>\max_{n\geq 0}\lambda_n$ we have

$$\begin{aligned}
&|C_\mu^{=}(x^*,T^{*n}y^*)|\\
&\qquad\leq 16\sum_{l=0}^{\infty}\Big(2^{-p}\int_{-c}^{c}|u|^p|\langle e_l,x^*\rangle\langle e_l,T^{*n}y^*\rangle|^{p/2}\,\rho(e_l,du)+\rho(e_l,\mathbb{R}\setminus[-c,c])\Big)\\
&\qquad=2^{4-p}\sum_{l=0}^{\infty}\lambda_l^p|\langle e_l,x^*\rangle\langle e_l,T^{*n}y^*\rangle|^{p/2}\\
&\qquad=2^{4-p}\sum_{l=0}^{\infty}\lambda_{l+n}^p|\langle e_{l+n},x^*\rangle\langle e_l,y^*\rangle\omega_{l+n-1}\cdots\omega_l|^{p/2}\\
&\qquad\leq 2^{4-p}\|x^*\|^{p/2}\|y^*\|^{p/2}\sum_{l=0}^{\infty}\lambda_{l+n}^p\prod_{i=l}^{l+n-1}\omega_{l+n}^{p/2}
\end{aligned}$$

$$
\begin{aligned}
&= 2^{4-p}\|x^*\|^{p/2}\|y^*\|^{p/2}\sum_{l=0}^{\infty}\lambda_{l+n}^p\prod_{j=l}^{l+n-1}\left(\frac{\lambda_j}{\lambda_{j+1}}\right)^{p/2}\\
&= 2^{4-p}\|x^*\|^{p/2}\|y^*\|^{p/2}\sum_{l=0}^{\infty}\lambda_{l+n}^p\left(\frac{\lambda_l}{\lambda_{l+n}}\right)^{p/2}\\
&= 2^{4-p}\|x^*\|^{p/2}\|y^*\|^{p/2}\sum_{l=0}^{\infty}(\lambda_l\lambda_{l+n})^{p/2}, \quad x^*\in E^*.
\end{aligned}
$$

In order to deduce the claim from Theorem 2.4 it suffices to note that $x^*\in E^*$ is arbitrary and $\mathbb{R}\setminus Z_1(\nu_{x^*})$ is never empty by Lemma 2.3. ∎

In the framework of Proposition 4.3, the random variable $\langle X, x^*\rangle$ has the distribution of $\sum_{n\geq 0}\lambda_n N_n e_n$, where $(N_n)_{n\geq 0}$ is a sequence of independent standard Poisson random variables. The Lévy measure of $\langle X, x^*\rangle$ is

$$
\nu_{x^*} = \sum_{n\geq 0}\delta_{\lambda_n\langle e_n, x^*\rangle}
$$

on $\mathbb{R}$, and we have

$$
Z_1(\nu_{x^*}) = \left\{\frac{2\pi k}{\lambda_n\langle e_n, x^*\rangle} : n\geq 0,\ k\in\mathbb{Z}\right\}.
$$

In particular, for $x^*\in E^*$ such that $\lambda_0\langle e_0, x^*\rangle = 2\pi$, ν_{x^*} does not satisfy Condition 2.1 and Proposition 2.2 of [MP24] does not apply, as in the next example.

Example 4.4. Let $p\in[1,2)$, $\gamma>1$, and let T be the bounded weighted backward shift operator defined by

$$
Te_0 := 0, \quad Te_1 = e_0, \quad Te_{n+1} := \left(1+\frac{1}{n}\right)^{\gamma/p} e_n, \qquad n\geq 1,
$$

and consider the compound Poisson measure μ on the (real) sequence space $E=\ell^p(\mathbb{N})$ with Lévy measure (4.2), where $\lambda_n := \lambda_0/(n+1)^{\gamma/p}$, $n\geq 0$, for some $\lambda_0>0$. Then μ is an invariant measure for T, and T is mixing with respect to μ, with the rate

$$
\begin{aligned}
&\sup_{x^*,y^*\in E^*\setminus\{0\}}\frac{|C_\mu^{=}(x^*, T^{*n}y^*)|}{\|x^*\|^{p/2}\|y^*\|^{p/2}}\\
&\qquad\leq\begin{cases} 2^{4-p}\lambda_0^p\mathrm{B}(1-\gamma/2,\gamma-1)n^{-(\gamma-1)}, & 1<\gamma<2,\\ 2^{4-p}\lambda_0^p\mathrm{B}(\varepsilon/2,1-\varepsilon)n^{-(1-\varepsilon)}, & \gamma\geq 2,\end{cases}
\end{aligned}
$$

for $n\geq 1$, and any $\varepsilon\in(0,1)$ when $\gamma\geq 2$, where $\mathrm{B}(\cdot,\cdot)$ denotes the beta function.

Proof. For any $\gamma \in (1,2)$, we have

$$\begin{aligned}(4.3)\qquad \sum_{l=0}^{\infty}(\lambda_l\lambda_{l+n})^{p/2} &= \sum_{l=0}^{\infty}\frac{\lambda_0^p}{(l+n+1)^{\gamma/2}(l+1)^{\gamma/2}}\\ &\le \int_0^{\infty}\frac{\lambda_0^p}{(x+n)^{\gamma/2}x^{\gamma/2}}\,dx = \lambda_0^p\int_0^{\infty}\frac{n^{1-\gamma}}{(x+1)^{\gamma/2}x^{\gamma/2}}\,dx\\ &= 2^{4-p}\lambda_0^p\mathrm{B}(1-\gamma/2,\gamma-1)n^{-(\gamma-1)},\end{aligned}$$

and we complete the proof by applying Proposition 4.3. In the case $\gamma \ge 2$, we observe similarly that for any $\varepsilon \in (0,1)$ we have

$$\begin{aligned}(4.4)\qquad \sum_{l=0}^{\infty}(\lambda_l\lambda_{l+n})^{p/2} &= \sum_{l=0}^{\infty}\frac{\lambda_0^p}{(l+n+1)^{\gamma/2}(l+1)^{\gamma/2}}\\ &\le \sum_{l=0}^{\infty}\frac{\lambda_0^p}{(l+n+1)^{(2-\varepsilon)/2}(l+1)^{(2-\varepsilon)/2}}\\ &\le \lambda_0^p\mathrm{B}(\varepsilon/2,1-\varepsilon)n^{-(1-\varepsilon)},\end{aligned}$$

and in both cases the claim follows from Proposition 4.3. ∎

5. Stable measures. In this section, we consider the case where μ is an α-stable distribution, $\alpha \in (0,2)\setminus\{1\}$ with characteristic functional (1.5), i.e. $\sigma^2 \equiv 0$, and $\rho(z,du)$ in (1.4) is

$$\rho(z,du) = \frac{du}{|u|^{1+\alpha}}, \qquad u \ne 0;$$

see the discussion following [RZ96, Theorem 4]. In particular, in Proposition 5.2 we will derive mixing rates for a family of weighted shifts that leave α-stable measures invariant on the sequence space $\ell^p(\mathbb{N})$, $p \in [1,2]$. In Corollary 5.3 we also derive decay rates for quantities of the form (1.6) for finite and infinite linear combinations of exponentials.

LEMMA 5.1. *Let μ be an α-stable distribution with control measure ξ on E, $\alpha \in (0,2)\setminus\{1\}$. For any $p \in (\alpha,2]$ and $c > 0$, we have the codifference bounds*

$$|C_\mu^{=}(x^*,y^*)| \le \frac{2^{5-p}c^{p-\alpha}}{p-\alpha}\int_E |\mathrm{Re}\,\langle z,x^*\rangle\,\mathrm{Re}\,\langle z,y^*\rangle|^{p/2}\,\xi(dz) + \frac{32}{\alpha c^\alpha}\xi(E),$$

$$|C_\mu^{\ne}(x^*,y^*)| \le \frac{2^{5-p}c^{p-\alpha}}{p-\alpha}\int_E |\mathrm{Re}\,\langle z,x^*\rangle\,\mathrm{Im}\,\langle z,y^*\rangle|^{p/2}\,\xi(dz) + \frac{32}{\alpha c^\alpha}\xi(E),$$

for $x^*, y^* \in E^*$.

Proof. By Lemma 3.2, for $p \in (\alpha, 2]$ we have

$$
\begin{aligned}
&|C_\mu^{=}(x^*, y^*)| \\
&\leq 16 \int_E \left(2^{-p} |\mathrm{Re}\,\langle z, x^* \rangle \,\mathrm{Re}\,\langle z, y^* \rangle|^{p/2} \int_{-c}^{c} \frac{1}{|u|^{1+\alpha-p}}\, du + \int_{\mathbb{R}\setminus[-c,c]} \frac{1}{|u|^{1+\alpha}}\, du \right) \xi(dz) \\
&= 16 \int_E \left(\frac{2^{1-p} c^{p-\alpha}}{p-\alpha} |\mathrm{Re}\,\langle z, x^* \rangle \,\mathrm{Re}\,\langle z, y^* \rangle|^{p/2} + \frac{2}{\alpha c^\alpha} \right) \xi(dz) \\
&= \frac{2^{5-p} c^{p-\alpha}}{p-\alpha} \int_E |\mathrm{Re}\,\langle z, x^* \rangle \,\mathrm{Re}\,\langle z, y^* \rangle|^{p/2}\, \xi(dz) + \frac{32}{\alpha c^\alpha} \xi(E).
\end{aligned}
$$

The proof for $C_\mu^{\neq}(x^*, y^*)$ is similar. ■

In the next proposition, for $p \in (\alpha, 2] \cap [1, 2]$ (so that $\ell^p(\mathbb{Z})$ is a Banach space and Lemma 5.1 applies), we provide rates for the mixing of weighted forward shift operators on $\ell^p(\mathbb{Z})$ considered in [MP24, Proposition 4.2]. In what follows, we write $f(n) = O_y(g(n))$ when

$$
|f(n)| \leq C_y |g(n)|, \quad n \geq 0,
$$

for $C_y > 0$ a constant possibly depending on y and independent of $n \geq 0$. Also, as above, we let $(e_n)_{n\geq 0}$ denote the canonical basis of $\ell^p(\mathbb{N})$.

Proposition 5.2. *Let $\alpha \in (0, 2) \setminus \{1\}$ and $p \geq 1$. Let $(\omega_n)_{n\in\mathbb{Z}}$ be a positive weight sequence such that the sequence $(k_n)_{n\in\mathbb{Z}}$ defined by*

$$
k_n := k_0 \mathbf{1}_{\{n\leq -1\}} \prod_{l=n+1}^{0} \frac{1}{\omega_l} + k_0 \mathbf{1}_{\{n\geq 0\}} \prod_{l=1}^{n} \omega_l
$$

belongs to $\ell^\alpha(\mathbb{Z})$. Let μ be the α-stable measure on $E = \ell^p(\mathbb{Z})$, $p \geq 1$, with characteristic functional (1.5) *and control measure*

$$
\xi(dz) := \frac{1}{2} \sum_{n=-\infty}^{\infty} k_n^\alpha \big(\delta_{e_n}(dz) + \delta_{ie_n}(dz) \big),
$$

and consider the weighted forward shift operator T on E defined by

$$
Te_n := \omega_{n+1} e_{n+1}, \quad n \in \mathbb{Z}.
$$

The following are true:

(1) *μ is an invariant measure for T.*
(2) *Assume that $p \in (\alpha, 2] \cap [1, 2]$ and there exist $q_- \geq 0$, $q_+ \geq 1$ such that*

$$
\eta_- := \sup_{l\leq -q_-} \frac{1}{\omega_l} < 1 \quad \text{and} \quad \eta_+ := \sup_{l\geq q_+} \omega_l < 1
$$

with $\eta_+^{p/2} \neq \eta_-^{\alpha-p/2}$. *Then*

$$\sup_{x^*,y^*\in E^*\setminus\{0\}} \frac{|C_\mu^{=,\neq}(x^*, T^{*n}y^*)|}{\|x^*\|^{\alpha/2}\|y^*\|^{\alpha/2}} = O_{\mu,T}(\max(\eta_-^{2\alpha/p-1}, \eta_+)^{\alpha n/2}), \ n \geq 0. \tag{5.1}$$

In particular, T *is mixing when* $\alpha \in (1/2, 2)\setminus\{1\}$ *and* $p \in (\alpha, 2\alpha) \cap [1, 2]$.

Proof. (1) The condition $\sum_{n=-\infty}^{\infty} |k_n|^\alpha < \infty$ ensures that ξ is finite as a control measure. To show that μ as an invariant measure for T, we use (1.5) and the equalities

$$\begin{aligned}
\int_E |\mathrm{Re}\,\langle Tz, x^*\rangle|^\alpha\, \xi(dz) &= \sum_{n=-\infty}^{\infty} k_n^\alpha |\mathrm{Re}\,\langle Te_n, x^*\rangle|^\alpha \\
&= \sum_{n=-\infty}^{\infty} k_n^\alpha \omega_{n+1}^\alpha |\mathrm{Re}\,\langle e_{n+1}, x^*\rangle|^\alpha \\
&= \sum_{n=-\infty}^{\infty} k_{n+1}^\alpha |\mathrm{Re}\,\langle e_{n+1}, x^*\rangle|^\alpha = \int_E |\mathrm{Re}\,\langle z, x^*\rangle|^\alpha\, \xi(dz),
\end{aligned}$$

hence $\int_E e^{i\,\mathrm{Re}\langle Tz,x^*\rangle}\,\mu(dz) = \int_E e^{i\,\mathrm{Re}\langle z,x^*\rangle}\,\mu(dz)$ for all $x^* \in E^*$, proving invariance.

(2) Without loss of generality, we may assume $q_- = 0$ and $q_+ = 1$. We note that the right-hand side term in Lemma 5.1 can be bounded as

$$\begin{aligned}
\int_E |\mathrm{Re}\,\langle z, x^*\rangle\, \mathrm{Re}\,\langle z, T^{*n}y^*\rangle|^{p/2}\, \xi(dz) &\leq \|x^*\|^{p/2} \int_E \|z\|^{p/2} |\mathrm{Re}\,\langle z, T^{*n}y^*\rangle|^{p/2}\, \xi(dz) \\
&= \|x^*\|^{p/2} \int_E \|z\|^{p/2} |\mathrm{Re}\,\langle z, T^{*n}y^*\rangle|^{p/2}\, \xi(dz) \\
&= \|x^*\|^{p/2} \int_E \|z\|^{p/2} |\mathrm{Re}\,\langle T^n z, y^*\rangle|^{p/2}\, \xi(dz) \\
&\leq \|x^*\|^{p/2} \|y^*\|^{p/2} \int_E \|z\|^{p/2} \|T^n z\|^{p/2}\, \xi(dz)
\end{aligned}$$

for $x^*, y^* \in E^*$, $n \geq 0$. We have

$$\begin{aligned}
\int_E \|z\|^{p/2} \|T^n z\|^{p/2} \xi(dz) &= \frac{1}{2} \sum_{l=-\infty}^{\infty} k_l^\alpha \int_E \|z\|^{p/2} \|T^n z\|^{p/2}\, (\delta_{e_l}(dz) + \delta_{ie_l}(dz)) \\
&= \sum_{l=-\infty}^{\infty} k_l^\alpha \|T^n e_l\|^{p/2} = \sum_{l=-\infty}^{\infty} k_l^\alpha \prod_{j=l+1}^{l+n} \omega_j^{p/2}.
\end{aligned}$$

We split the above series into three components.

- If $l \geq 0$, then
$$\prod_{j=l+1}^{l+n} \omega_j^{p/2} \leq \eta_+^{pn/2},$$
and so
$$\sum_{l=0}^{\infty} k_l^{\alpha} \prod_{j=l+1}^{l+n} \omega_j^{p/2} \leq k_0^{\alpha} \sum_{l=0}^{\infty} \eta_+^{\alpha l} \eta_+^{pn/2} = \frac{k_0^{\alpha}}{1-\eta_+^{\alpha}} \eta_+^{pn/2}.$$
- If $-n < l \leq -1$, then
$$k_l^{\alpha} \prod_{j=l+1}^{l+n} \omega_j^{p/2} = k_0^{\alpha} \prod_{j=l+1}^{0} \frac{1}{\omega_j^{\alpha-p/2}} \prod_{j=1}^{l+n} \omega_j^{p/2} \leq k_0^{\alpha} \eta_-^{(\alpha-p/2)|l|} \eta_+^{(l+n)p/2},$$
and so
$$\begin{aligned}\sum_{l=-n+1}^{-1} k_l^{\alpha} \prod_{j=l+1}^{l+n} \omega_j^{p/2} &\leq k_0^{\alpha} \sum_{l=-n+1}^{-1} \eta_-^{-(\alpha-p/2)|l|} \eta_+^{(l+n)p/2} \\ &= k_0^{\alpha} \eta_+^{pn/2} \sum_{l=-n+1}^{-1} \eta_-^{-(\alpha-p/2)l} \eta_+^{pl/2} \leq k_0^{\alpha} \eta_+^{pn/2} \sum_{l=1}^{n+1} \left(\frac{\eta_-^{\alpha-p/2}}{\eta_+^{p/2}} \right)^l \\ &= k_0^{\alpha} \eta_+^{pn/2} \left(\frac{\eta_-^{\alpha-p/2}}{\eta_+^{p/2}} \right) \frac{1-\left(\eta_-^{\alpha-p/2}/\eta_+^{p/2}\right)^{n+1}}{1-\left(\eta_-^{\alpha-p/2}/\eta_+^{p/2}\right)} \\ &= \frac{k_0^{\alpha}\eta_-^{\alpha-p/2}}{\eta_+^{p/2}-\eta_-^{\alpha-p/2}} \eta_+^{pn/2} - \frac{k_0^{\alpha}\eta_-^{\alpha-p/2}}{\eta_+^{p/2}-\eta_-^{\alpha-p/2}} \left(\frac{\eta_-^{\alpha-p/2}}{\eta_+^{p/2}} \right) \eta_-^{(\alpha-p/2)n}.\end{aligned}$$
- If $l \leq -n$, then
$$k_l^{\alpha} \prod_{j=l+1}^{l+n} \omega_j^{p/2} = k_0^{\alpha} \prod_{j=l+1}^{0} \frac{1}{\omega_j^{\alpha-p/2}} \prod_{j=l+n+1}^{0} \frac{1}{\omega_j^{p/2}} \leq k_0^{\alpha} \eta_-^{(\alpha-p/2)|l|} \eta_-^{p|l+n|/2},$$
and so
$$\begin{aligned}\sum_{l=-\infty}^{n} k_l^{\alpha} \prod_{j=l+1}^{l+n} \omega_j^{p/2} &\leq k_0^{\alpha} \sum_{l=n}^{\infty} \eta_-^{(\alpha-p/2)} \eta_-^{p(l+n)/2} = k_0^{\alpha} \eta_-^{pn/2} \sum_{l=n}^{\infty} \eta_-^{\alpha l} \\ &= \frac{k_0^{\alpha}}{1-\eta_-^{\alpha}} \eta_-^{(\alpha+p/2)n}.\end{aligned}$$

Hence, we have
$$\begin{aligned}&\int_E |\mathrm{Re}\,\langle z, x^* \rangle \,\mathrm{Re}\,\langle z, T^{*n} y^* \rangle|^{p/2}\, \xi(dz) \\ &\qquad \leq k_0^{\alpha} \|x^*\|^{p/2} \|y^*\|^{p/2} (K_1 \eta_+^{pn/2} + K_2 \eta_-^{(\alpha+p/2)n} + K_3 \eta_-^{(\alpha-p/2)n})\end{aligned}$$

for $x^*, y^* \in E^*$, where

$$K_1 := \frac{1}{1-\eta_+^\alpha} + \frac{\eta_-^{\alpha-p/2}}{\eta_+^{p/2} - \eta_-^{\alpha-p/2}}, \qquad K_2 := \frac{1}{1-\eta_-^\alpha},$$
$$K_3 := -\frac{\eta_-^{\alpha-p/2}}{\eta_+^{p/2} - \eta_-^{\alpha-p/2}}\left(\frac{\eta_-^{\alpha-p/2}}{\eta_+^{p/2}}\right)$$

are constants independent of $n \geq 1$. Observe that since $\eta_- < 1$, we have $\eta_-^{(\alpha+p/2)n} = O(\eta_-^{(\alpha-p/2)n})$. If $\eta_+^{p/2} > \eta_-^{\alpha-p/2}$ then $K_3 < 0$ and

$$\int_E |\mathrm{Re}\,\langle z, x^*\rangle\, \mathrm{Re}\,\langle z, T^{*n}y^*\rangle|^{p/2}\,\xi(dz) = \|x^*\|^{p/2}\|y^*\|^{p/2} O_{\mu,T}(\eta_+^{pn/2}). \tag{5.2}$$

Conversely, if $\eta_+^{p/2} < \eta_-^{\alpha-p/2}$ then $K_3 > 0$. We have $\eta_+^{pn/2} = O(\eta_-^{(\alpha-p/2)n})$, and thus

$$\int_E |\mathrm{Re}\,\langle z, x^*\rangle\, \mathrm{Re}\,\langle z, T^{*n}y^*\rangle|^{p/2}\,\xi(dz) = \|x^*\|^{p/2}\|y^*\|^{p/2} O_{\mu,T}(\eta_-^{(\alpha-p/2)n}). \tag{5.3}$$

We now bound the codifferences. If $\eta_+^{p/2} > \eta_-^{\alpha-p/2}$, since $p \in (\alpha, 2]$, by Lemma 5.1 and (5.2) for any $c > 0$ we have

$$|C_\mu^{=}(x^*, T^{*n}y^*)| = \|x^*\|^{p/2}\|y^*\|^{p/2} O_{\mu,T}(c^{p-\alpha}\eta_+^{pn/2}) + O_\mu(c^{-\alpha}).$$

In particular, letting $c_n := \|x^*\|^{-1/2}\|y^*\|^{-1/2}\eta_+^{-n/2}$, $n \geq 1$, putting $c = c_n$ we get

$$|C_\mu^{=}(x^*, T^{*n}y^*)| = \|x^*\|^{\alpha/2}\|y^*\|^{\alpha/2} O_{\mu,T}(\eta_+^{\alpha n/2}).$$

In the case $\eta_+^{p/2} < \eta_-^{\alpha-p/2}$, a similar argument using (5.3) with $c_n := \|x^*\|^{-1/2}\|y^*\|^{-1/2}\eta_-^{(1/2-\alpha/p)n}$, $n \geq 1$, gives

$$|C_\mu^{=}(x^*, T^{*n}y^*)| = \|x^*\|^{\alpha/2}\|y^*\|^{\alpha/2} O_{\mu,T}(\eta_-^{\alpha(\alpha/p-1/2)n}).$$

Likewise, a similar argument establishes the corresponding inequality for $|C_\mu^{\neq}(x^*, T^{*n}y^*)|$, hence we have

$$|C_\mu^{=,\neq}(x^*, T^{*n}y^*)| = \begin{cases} \|x^*\|^{\alpha/2}\|y^*\|^{\alpha/2} O_{\mu,T}(\eta_+^{\alpha n/2}) & \text{if } \eta_+^{p/2} > \eta_-^{\alpha-p/2}, \\ \|x^*\|^{\alpha/2}\|y^*\|^{\alpha/2} O_{\mu,T}(\eta_-^{\alpha(\alpha/p-1/2)n}) & \text{if } \eta_+^{p/2} < \eta_-^{\alpha-p/2}, \end{cases}$$

which can be rewritten as

$$|C_\mu^{=,\neq}(x^*, T^{*n}y^*)| = \|x^*\|^{\alpha/2}\|y^*\|^{\alpha/2} O_{\mu,T}(\max(\eta_-^{2\alpha/p-1}, \eta_+)^{\alpha n/2}),$$

giving (5.1). Mixing in the case of $\alpha > 1/2$ and $p \in (\alpha, 2\alpha) \cap [1, 2]$ follows from Theorem 2.4 and the decay of codifferences. ∎

Proposition 5.2 also applies to symmetric control measures of the form

$$\xi(dz) := \frac{1}{4} \sum_{n=-\infty}^{\infty} k_n^\alpha \big(\delta_{e_n}(dz) + \delta_{-e_n}(dz) + \delta_{ie_n}(dz) + \delta_{-ie_n}(dz) \big).$$

We now extend Proposition 5.2 by determining decay rates for the quantity

$$I_n(f,g) := \int_E f(z) g(T^n z)\, \mu(dz) - \int_E f(z)\, \mu(dz) \int_E g(z)\, \mu(dz), \qquad n \geq 0,$$

for a family of functions f, g in $L^2(E,\mu)$ when T is mixing. Table 1 displays the equivalent codifferences

$$\begin{aligned} C_\mu^{\phi,\psi}(x^*, y^*) := {} & \log \int_E e^{i\phi(\langle z, x^* \rangle) + i\psi(\langle z, y^* \rangle)}\, \mu(dz) \\ & - \log \int_E e^{i\phi(\langle z, x^* \rangle)}\, \mu(dz) - \log \int_E e^{i\psi(\langle z, y^* \rangle)}\, \mu(dz) \end{aligned}$$

for $x^*, y^* \in E^*$, for different choices of functions ϕ, ψ.

Table 1. Function-codifference triples

$\phi(\cdot)$	$\psi(\cdot)$	$C_\mu^{\phi,\psi}(x^*, y^*)$
$\mathrm{Re}(\cdot)$	$-\mathrm{Re}(\cdot)$	$C_\mu^{=}(x^*, y^*)$
$-\mathrm{Re}(\cdot)$	$\mathrm{Re}(\cdot)$	$C_\mu^{=}(-x^*, -y^*)$
$\mathrm{Im}(\cdot)$	$-\mathrm{Im}(\cdot)$	$C_\mu^{=}(-ix^*, -iy^*)$
$-\mathrm{Im}(\cdot)$	$\mathrm{Im}(\cdot)$	$C_\mu^{=}(ix^*, iy^*)$
$\mathrm{Re}(\cdot)$	$-\mathrm{Im}(\cdot)$	$C_\mu^{\neq}(x^*, y^*)$
$-\mathrm{Re}(\cdot)$	$\mathrm{Im}(\cdot)$	$C_\mu^{\neq}(-x^*, -y^*)$
$\mathrm{Im}(\cdot)$	$\mathrm{Re}(\cdot)$	$C_\mu^{\neq}(-ix^*, -iy^*)$
$-\mathrm{Im}(\cdot)$	$-\mathrm{Re}(\cdot)$	$C_\mu^{\neq}(ix^*, iy^*)$
$-\mathrm{Im}(\cdot)$	$\mathrm{Re}(\cdot)$	$C_\mu^{\neq}(x^*, y^*)$
$\mathrm{Im}(\cdot)$	$-\mathrm{Re}(\cdot)$	$C_\mu^{\neq}(-x^*, -y^*)$
$\mathrm{Re}(\cdot)$	$\mathrm{Im}(\cdot)$	$C_\mu^{\neq}(-ix^*, -iy^*)$
$-\mathrm{Re}(\cdot)$	$-\mathrm{Im}(\cdot)$	$C_\mu^{\neq}(ix^*, iy^*)$

First, we observe that the estimate (5.1) holds for the twelve codifference quantities in Table 1. Next, we derive bounds on $I_n(f,g)$ for f and g in a class of functions defined by infinite series.

Corollary 5.3. *Let μ be the α-stable measure on $E = \ell^p(\mathbb{Z})$ for $\alpha \in (1/2, 2) \setminus \{1\}$ and $p \in (\alpha, 2\alpha) \cap [1,2]$, let T be the mixing weighted forward shift operator defined in Proposition 5.2, let (ϕ, ψ) be a pair of functions given in Table 1, and let $(a_j)_{j\in\mathbb{N}}$ and $(b_l)_{l\in\mathbb{N}}$ be complex $\ell^1(\mathbb{N})$ sequences such that*

$$\sum_{j=0}^{\infty} |a_j|\, \|T\|^{jp/2} < \infty \quad \text{and} \quad \sum_{l=0}^{\infty} |b_l|\, \|T\|^{lp/2} < \infty.$$

For any x^*, y^*, the functions

$$f(z) := \sum_{j=0}^{\infty} a_j e^{i\phi(\langle z, T^{*j}x^*\rangle)} \quad \text{and} \quad g(z) := \sum_{l=0}^{\infty} b_l e^{i\psi(\langle z, T^{*l}y^*\rangle)}$$

are well-defined in $L^2(E,\mu)$, and we have

$$|I_n(f,g)| = O_{f,g,\mu,T}(\max(\eta_-^{2\alpha/p-1}, \eta_+)^{\alpha n/2}), \qquad n \geq 0. \tag{5.4}$$

Proof. For any pair (ϕ,ψ) of functions in Table 1, we have

$$\begin{aligned}
&|I_n(e^{i\operatorname{Re}\phi(\langle\cdot,x^*\rangle)}, e^{-i\operatorname{Re}\psi(\langle\cdot,y^*\rangle)})| \\
&\quad = |\exp(C_\mu^{\phi,\psi}(x^*,y^*)) - 1| \left| \int_E e^{i\operatorname{Re}\langle z,x^*\rangle}\, \mu(dz) \int_E e^{-i\operatorname{Re}\langle z,y^*\rangle}\, \mu(dz) \right| \\
&\quad \leq |\exp(C_\mu^{\phi,\psi}(x^*,y^*)) - 1| \\
&\quad \leq |C_\mu^{\phi,\psi}(x^*,y^*)| \exp(|C_\mu^{\phi,\psi}(x^*,y^*)|).
\end{aligned}$$

If $0 \leq n \leq j - l$, then

$$\begin{aligned}
\exp(|C_\mu^{\phi,\psi}(T^{*j}x^*, T^{*(n+l)}y^*)|) &= \exp(|C_\mu^{\phi,\psi}(T^{*(j-n-l)}x^*, y^*)|) \\
&\leq \exp\Big(\max_{n\geq 0} |C_\mu^{\phi,\psi}(T^{*n}x^*, y^*)|\Big),
\end{aligned}$$

and if $n \geq \max(0, j-l)$, then

$$\begin{aligned}
\exp(|C_\mu^{\phi,\psi}(T^{*j}x^*, T^{*(n+l)}y^*)|) &= \exp(|C_\mu^{\phi,\psi}(x^*, T^{*(n+l-j)}y^*)|) \\
&\leq \exp\Big(\max_{n\geq 0} |C_\mu^{\phi,\psi}(x^*, T^{*n}y^*)|\Big),
\end{aligned}$$

where the maxima are finite by (5.1). Hence, letting

$$f_{j,x^*}(z) := e^{i\phi(\langle z, T^{*j}x^*\rangle)} \quad \text{and} \quad g_{l,y^*}(z) := e^{i\psi(\langle z, T^{*l}y^*\rangle)}, \qquad j,l \geq 0,$$

by Proposition 5.2 we have

$$\begin{aligned}
&|I_n(f_{j,x^*}, g_{l,y^*})| \\
&\leq |C_\mu^{\phi,\psi}(T^{*j}x^*, T^{*(n+l)}y^*)| \\
&\quad \times \exp\Big(\max\Big(\max_{n\geq 0} |C_\mu^{\phi,\psi}(T^{*n}x^*, y^*)|, \max_{n\geq 0} |C_\mu^{\phi,\psi}(x^*, T^{*n}y^*)|\Big)\Big) \\
&\leq \|T^{*j}x^*\|^{\alpha/2} \|T^{*l}y^*\|^{\alpha/2} O_{x^*,y^*,\mu,T}(\max(\eta_-^{2\alpha/p-1}, \eta_+)^{\alpha n/2}), \qquad j,l \geq 0.
\end{aligned}$$

Since $(a_j)_{j\geq 0} \in \ell^1(\mathbb{N})$ and $(b_l)_{l\geq 0} \in \ell^1(\mathbb{N})$, we have $f, g \in L^2(E,\mu)$, the series $\sum_{j=0}^{\infty} f_{j,x^*}(z)$ and $\sum_{l=0}^{\infty} g_{l,y^*}(z)$ converge absolutely for every $z \in E$, and

$$\begin{aligned}|I_n(f,g)| &\le \sum_{j,l=0}^{\infty} |a_j|\,|b_l|\,|I_n(f_{j,x^*},g_{l,y^*})| \\ &\le \|x^*\|^{\alpha/2}\|y^*\|^{\alpha/2} \\ &\quad \times \sum_{j,l=0}^{\infty} |a_j|\,|b_l|\,\|T^j\|^{\alpha/2}\|T^l\|^{\alpha/2} O_{x^*,y^*,\mu,T}(\max(\eta_-^{2\alpha/p-1},\eta_+)^{\alpha n/2}) \\ &\le \|x^*\|^{\alpha/2}\|y^*\|^{\alpha/2} O_{x^*,y^*,\mu,T}(\max(\eta_-^{2\alpha/p-1},\eta_+)^{\alpha n/2}) \\ &\quad \times \sum_{j,l=0}^{\infty} |a_j|\,|b_l|\,\|T^j\|^{\alpha/2}\|T^l\|^{\alpha/2}. \ \blacksquare\end{aligned}$$

Since Table 1 contains every possible pair $(\phi,\psi) \in \{\pm\operatorname{Re}(\cdot), \pm\operatorname{Im}(\cdot)\}^2$ where $\phi \ne \psi$, the rate obtained in (5.4) also applies to f and g given by

$$f(z) := \sum_{\phi\in\Phi}\sum_{j=0}^{\infty} a_{\phi,j} e^{i\phi(\langle z, T^{*j}x^*\rangle)} \quad\text{and}\quad g(z) := \sum_{\psi\in\Psi}\sum_{l=0}^{\infty} b_{\psi,l} e^{i\psi(\langle z, T^{*l}y^*\rangle)},$$

where Φ,Ψ are any disjoint non-empty subsets of $\{\pm\operatorname{Re}(\cdot), \pm\operatorname{Im}(\cdot)\}$, and $(a_{\phi,j})_{j\in\mathbb{N},\,\phi\in\Phi}$, $(b_{\psi,l})_{l\in\mathbb{N},\,\psi\in\Psi}$ are complex $\ell^1(\mathbb{N})$ sequences such that

$$\sum_{j=0}^{\infty} |a_{\phi,j}|\,\|T\|^{jp/2} < \infty \quad\text{and}\quad \sum_{l=0}^{\infty} |b_{\psi,l}|\,\|T\|^{lp/2} < \infty, \qquad \phi\in\Phi,\ \psi\in\Psi.$$

6. Tempered stable measures. In this section, we consider the case where μ is the distribution of an $\ell^p(\mathbb{N})$-valued random variable of the form

$$\sum_{n=0}^{\infty} k_n(\theta_{1,n} + i\theta_{2,n})e_n, \tag{6.1}$$

where k_n is an appropriately chosen positive sequence, and $\theta_{1,n}$ and $\theta_{2,n}$ are independent and identically distributed copies of a real-valued tempered stable random variable θ, centered at zero, with two-sided index of stability $\alpha \in (0,1)$ and characteristic function

$$\mathbb{E}[e^{it\theta}] = \exp\Big(\int_{\mathbb{R}} (e^{itx} - 1 - i\kappa(x)tx)\,\lambda(dx)\Big), \qquad t\in\mathbb{R},$$

where

$$\kappa(x) = \mathbf{1}_{\{|x|<1\}} + \frac{1}{|x|}\mathbf{1}_{\{|x|\ge 1\}}, \qquad x\in\mathbb{R},$$

and

$$\lambda(dx) = a_-\frac{e^{-\lambda_-|x|}}{|x|^{1+\alpha}}\mathbf{1}_{\mathbb{R}^-}(x)\,dx + a_+\frac{e^{-\lambda_+ x}}{x^{1+\alpha}}\mathbf{1}_{\mathbb{R}^+}(x)\,dx, \tag{6.2}$$

with $a_-, a_+, \lambda_-, \lambda_+ > 0$ (see for example [Kop95, KT13]). We first derive criteria under which the series (6.1) is well-defined, i.e. is almost surely $\ell^p(\mathbb{N})$-valued, as in [Sch70, MP24].

Proposition 6.1. *Let $\alpha \in (0,1)$ and $(k_n)_{n\geq 0} \in \ell^\alpha(\mathbb{N})$. Then*

(1) *the series $\sum_{n=0}^\infty |k_n\theta_{1,n}|^p$ converges almost surely for all $p \in (\alpha, \infty)$;*

(2) *the series $\sum_{n=0}^\infty |k_n\theta_{1,n} + ik_n\theta_{2,n}|^p$ converges almost surely for all $p \in [1,\infty)$.*

Proof. To prove the first statement, we use the three-series theorem of Kolmogorov (see e.g. [Dur10]). For notational simplicity write $\theta_n = \theta_{1,n}$. Let

$$\theta_n' = \begin{cases} \theta_n, & |\theta_n| < 1/|k_n|, \\ 0, & |\theta_n| \geq 1/|k_n|, \end{cases}$$

and let $f(x)$ denote the common probability density of θ_n, $n \geq 0$.

- The first series condition requires showing that

$$\begin{aligned} \sum_{n=0}^\infty \mathbb{P}(|k_n\theta_n| > 1) &= \sum_{n=0}^\infty \Big(\int\limits_{1/|k_n|}^\infty f(x)\,dx + \int\limits_{-\infty}^{-1/|k_n|} f(x)\,dx \Big) \\ &= \sum_{n=0}^\infty \int\limits_{1/|k_n|}^\infty f(x)\,dx + \sum_{n=0}^\infty \int\limits_{-\infty}^{-1/|k_n|} f(x)\,dx \\ &< \infty. \end{aligned}$$

  We show finiteness of the first series; the second is similar. From [KT13, Theorem 7.10] we have

$$f(x) \sim Cx^{-1-\alpha}e^{-\lambda_+ x}$$

  as $x \to \infty$, where

$$C = a_+ \exp(-a_+\Gamma(-\alpha)(\lambda_+)^\alpha + a_-\Gamma(-\alpha)[(\lambda_+ + \lambda_-)^\alpha - (\lambda_-)^\alpha])$$

  is a positive constant. The claim follows from

$$\int\limits_{1/|k_n|}^\infty f(x)\,dx \sim C \int\limits_{1/|k_n|}^\infty \frac{e^{-\lambda_+ x}}{x^{1+\alpha}}\,dx \leq C \int\limits_{1/|k_n|}^\infty \frac{1}{x^{1+\alpha}}\,dx = \frac{C}{\alpha}|k_n|^\alpha$$

  and the fact that $(k_n)_{n\geq 0} \in \ell^\alpha(\mathbb{N})$.

- The second series condition requires showing that

$$\begin{aligned} \sum_{n=0}^\infty \mathbb{E}[|k_n\theta_n'|^p] &= \sum_{n=0}^\infty |k_n|^p \Big(\int\limits_{1/|k_n|}^\infty x^p f(x)\,dx + \int\limits_{-\infty}^{-1/|k_n|} x^p f(x)\,dx \Big) \\ &= \sum_{n=0}^\infty |k_n|^p \int\limits_{1/|k_n|}^\infty x^p f(x)\,dx + \sum_{n=0}^\infty |k_n|^p \int\limits_{-\infty}^{-1/|k_n|} x^p f(x)\,dx \\ &< \infty. \end{aligned}$$

Using the asymptotics $f(x) \sim Cx^{-1-\alpha}e^{-\lambda_+ x}$ as $x \to \infty$ since $p > \alpha$ we have

$$\int_1^{k/|k_n|} x^p f(x)\,dx \sim C \int_0^{1/|k_n|} \frac{e^{-\lambda_+ x}}{x^{1+\alpha-p}}\,dx \le C \int_0^{1/|k_n|} \frac{1}{x^{1+\alpha-p}}\,dx = \frac{C}{\alpha-p}|k_n|^{\alpha-p},$$

hence, since $(k_n)_{n\ge 0} \in \ell^\alpha(\mathbb{N})$, we have

$$\sum_{n=0}^{\infty} \mathbb{E}[|k_n\theta'_n|^p] \le \sum_{n=0}^{\infty} \left(|k_n|^p \cdot \frac{C}{\alpha-p}|k_n|^{\alpha-p} \right) < \infty.$$

- The third series condition requires showing that

$$\begin{aligned} \sum_{n=0}^{\infty} \mathrm{Var}[|k_n\theta'_n|^p] &= \sum_{n=0}^{\infty} (\mathbb{E}[|k_n\theta'_n|^{2p}] - \mathbb{E}[|k_n\theta'_n|^p]^2) \\ &= \sum_{n=0}^{\infty} \mathbb{E}[|k_n\theta'_n|^{2p}] - \sum_{n=0}^{\infty} \mathbb{E}[|k_n\theta'_n|^p]^2 \\ &< \infty. \end{aligned}$$

  Observe that

$$\sum_{n=0}^{\infty} \mathbb{E}[|k_n\theta'_n|^{2p}] < \infty$$

  by applying the second series condition argument with $2p > \alpha$. Since variance is non-negative, and the second series has only non-negative terms, it follows that this condition also holds.

Finally, the second statement follows from the first by

$$|a+ib|^p = \left(\sqrt{a^2+b^2}\right)^p \le (|a|+|b|)^p \le 2^{p-1}(|a|^p + |b|^p), \qquad p \ge 1. \blacksquare$$

By Proposition 6.1, (6.1) defines a probability measure μ with $\mu(\ell^p(\mathbb{N})) = 1$ on the space $\ell^p(\mathbb{N})$ of complex sequences. In Lemma 6.2, we determine the representation of the characteristic function of μ of the form (1.4).

LEMMA 6.2. *Let $\alpha \in (0,1)$, $p \in [1,2]$, and let $(k_n)_{n\ge 0} \in \ell^\alpha(\mathbb{N})$ be a positive sequence. Then the distribution μ on $\ell^p(\mathbb{N})$ of the random series* (6.1) *has characteristic functional* (1.4) *with $R \equiv 0$, control measure*

$$\xi(dz) := \sum_{n=0}^{\infty} k_n^\alpha \big(\delta_{e_n}(dz) + \delta_{ie_n}(dz)\big), \tag{6.3}$$

and Lévy measures

$$\rho(e_n, du) = \rho(ie_n, du) = a_- \frac{e^{-\lambda_-|u|/k_n}}{|u|^{1+\alpha}} \mathbf{1}_{\mathbb{R}^-}(u)\,du + a_+ \frac{e^{-\lambda_+ u/k_n}}{u^{1+\alpha}} \mathbf{1}_{\mathbb{R}^+}(u)\,du.$$

Proof. From [KT13, Lemma 4.1], since θ is tempered stable on $\mathbb{R}$ with Lévy measure (6.2), $k_n\theta$ is tempered stable on $\mathbb{R}$ with Lévy measure

$$\lambda_n(du) := k_n^\alpha a_- \frac{e^{-\lambda_-|u|/k_n}}{|u|^{1+\alpha}} \mathbf{1}_{\mathbb{R}^-}(u)\,du + k_n^\alpha a_+ \frac{e^{-\lambda_+ u/k_n}}{u^{1+\alpha}} \mathbf{1}_{\mathbb{R}^+}(u)\,du.$$

Next, by independence of the sequences $(\theta_{1,n})_{n\geq 0}$, $(\theta_{2,n})_{n\geq 0}$, we have

$$\begin{aligned}
\int_{\ell^p(\mathbb{N})} \exp(i\operatorname{Re}\langle z, x^*\rangle)\,\mu(dz) &= \mathbb{E}\Big[\exp\Big(i\operatorname{Re}\Big\langle \sum_{n=0}^\infty k_n(\theta_{1,n}+i\theta_{2,n})e_n, x^*\Big\rangle\Big)\Big] \\
&= \prod_{n=0}^\infty \big(\mathbb{E}[\exp(i\operatorname{Re}\langle k_n\theta_{1,n}e_n, x^*\rangle)]\mathbb{E}[\exp(i\operatorname{Re}\langle k_n\theta_{2,n}ie_n, x^*\rangle)]\big) \\
&= \prod_{n=0}^\infty \big(\mathbb{E}[\exp(i\operatorname{Re}\langle e_n, x^*\rangle k_n\theta_{1,n})]\mathbb{E}[\exp(i\operatorname{Re}\langle ie_n, x^*\rangle k_n\theta_{2,n})]\big) \\
&= \prod_{n=0}^\infty \bigg(\exp\bigg(k_n^\alpha \int_{\mathbb{R}} (e^{iu\operatorname{Re}\langle e_n, x^*\rangle} - 1 - iu\kappa(u)\operatorname{Re}\langle e_n, x^*\rangle)\frac{\lambda_n(du)}{k_n^\alpha}\bigg) \\
&\qquad\times \exp\bigg(k_n^\alpha \int_{\mathbb{R}} (e^{iu\operatorname{Re}\langle ie_n, x^*\rangle} - 1 - iu\kappa(u)\operatorname{Re}\langle ie_n, x^*\rangle)\frac{\lambda_n(du)}{k_n^\alpha}\bigg)\bigg) \\
&= \exp\bigg(\int_{\ell^p(\mathbb{N})}\int_{\mathbb{R}} (e^{iu\operatorname{Re}\langle z, x^*\rangle} - 1 - iu\kappa(u)\operatorname{Re}\langle z, x^*\rangle) \\
&\qquad\times \sum_{n=0}^\infty \lambda_n(du)(\delta_{e_n}(dz) + \delta_{ie_n}(dz))\bigg),
\end{aligned}$$

which is of the form (1.4) with $\rho(e_n, du) = \rho(ie_n, du) = k_n^{-\alpha}\lambda_n(du)$, $n \geq 0$, and ξ given by (6.3), which is finite since $(k_n)_{n\geq 0} \in \ell^\alpha(\mathbb{N})$. ∎

We now present codifference bounds in the tempered stable setting.

LEMMA 6.3. *Let $\alpha \in (0,1)$ and suppose that $(k_n)_{n\geq 0} \in \ell^\alpha(\mathbb{N})$ is a positive sequence. Let $p \in [1,2]$. Let μ be the distribution of (6.1) on $\ell^p(\mathbb{N})$. Then the codifference bounds*

$$\begin{aligned}
|C_\mu^{=}(x^*, y^*)| &\leq 2^{4-p}(a_-\lambda_-^{\alpha-p} + a_+\lambda_+^{\alpha-p})\Gamma(p-\alpha) \\
&\times \Big(\sum_{n=0}^\infty k_n^p|\operatorname{Re}\langle e_n, x^*\rangle \operatorname{Re}\langle e_n, y^*\rangle|^{p/2} + \sum_{n=0}^\infty k_n^p|\operatorname{Im}\langle e_n, x^*\rangle \operatorname{Im}\langle e_n, y^*\rangle|^{p/2}\Big)
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
|C_\mu^{\neq}(x^*, y^*)| &\leq 2^{4-p}(a_-\lambda_-^{\alpha-p} + a_+\lambda_+^{\alpha-p})\Gamma(p-\alpha) \\
&\times \Big(\sum_{n=0}^\infty k_n^p|\operatorname{Re}\langle e_n, x^*\rangle \operatorname{Im}\langle e_n, y^*\rangle|^{p/2} + \sum_{n=0}^\infty k_n^p|\operatorname{Im}\langle e_n, x^*\rangle \operatorname{Re}\langle e_n, y^*\rangle|^{p/2}\Big)
\end{aligned} \tag{6.5}$$

hold for any $x^*, y^* \in (\ell^p(\mathbb{N}))^*$.

Proof. Using the control measure representation from Lemma 6.2, we find that Lemma 3.2 implies that for any $c > 0$ we have

$$
\begin{aligned}
(6.6)\qquad & |C_\mu^{=}(x^*, y^*)| \\
&\le \underbrace{2^{4-p}\sum_{n=0}^{\infty} k_n^\alpha |\mathrm{Re}\,\langle e_n, x^*\rangle\, \mathrm{Re}\,\langle e_n, y^*\rangle|^{p/2} I_c(e_n)}_{S_1(c)} + \underbrace{16\sum_{n=0}^{\infty} k_n^\alpha I'_c(e_n)}_{S_2(c)} \\
&\quad + \underbrace{2^{4-p}\sum_{n=0}^{\infty} k_n^\alpha |\mathrm{Im}\,\langle e_n, x^*\rangle\, \mathrm{Im}\,\langle e_n, y^*\rangle|^{p/2} I_c(ie_n)}_{S_3(c)} + \underbrace{16\sum_{n=0}^{\infty} k_n^\alpha I'_c(ie_n)}_{S_2(c)},
\end{aligned}
$$

where

$$
\begin{aligned}
I_c(e_n) = I_c(ie_n) &= a_- \int_{-c}^{0} \frac{e^{-\lambda_-|u|/k_n}}{|u|^{1+\alpha-p}}\, du + a_+ \int_0^c \frac{e^{-\lambda_+ u k_n}}{u^{1+\alpha-p}}\, du \\
&\le a_- \int_{-\infty}^{0} \frac{e^{-\lambda_-|u|/k_n}}{|u|^{1+\alpha-p}}\, du + a_+ \int_0^\infty \frac{e^{-\lambda_+ u/k_n}}{u^{1+\alpha-p}}\, du \\
&= a_- \left(\frac{\lambda_-}{k_n}\right)^{\alpha-p} \int_0^\infty \frac{e^{-x}}{x^{1+\alpha-p}}\, dx + a_+ \left(\frac{\lambda_+}{k_n}\right)^{\alpha-p} \int_0^\infty \frac{e^{-x}}{x^{1+\alpha-p}}\, dx \\
&= \frac{a_-\lambda_-^{\alpha-p} + a_+\lambda_+^{\alpha-p}}{k_n^{\alpha-p}} \Gamma(p-\alpha),
\end{aligned}
$$

and

$$
\begin{aligned}
I'_c(e_n) = I'_c(ie_n) &= a_- \int_{-\infty}^{-c} \frac{e^{-\lambda_-|u|/k_n}}{|u|^{1+\alpha}}\, du + a_+ \int_c^\infty \frac{e^{-\lambda_+ u/k_n}}{u^{1+\alpha}}\, du \\
&\le 2(a_- + a_+) \int_c^\infty \frac{1}{u^{1+\alpha}}\, du = 2\frac{a_- + a_+}{\alpha c^\alpha}.
\end{aligned}
$$

This gives the bounds

$$
S_1(c) \le (a_-\lambda_-^{\alpha-p} + a_+\lambda_+^{\alpha-p})\Gamma(p-\alpha) \sum_{n=0}^{\infty} k_n^p |\mathrm{Re}\,\langle e_n, x^*\rangle\, \mathrm{Re}\,\langle e_n, y^*\rangle|^{p/2},
$$

$$
S_3(c) \le (a_-\lambda_-^{\alpha-p} + a_+\lambda_+^{\alpha-p})\Gamma(p-\alpha) \sum_{n=0}^{\infty} k_n^p |\mathrm{Im}\,\langle e_n, x^*\rangle\, \mathrm{Im}\,\langle e_n, y^*\rangle|^{p/2},
$$

independently of $c > 0$, and

$$
S_2(c) \le 2\frac{a_- + a_+}{\alpha c^\alpha} \sum_{n=0}^{\infty} k_n^\alpha < \infty.
$$

Now taking the limit as $c \to \infty$ in (6.6) yields (6.4). The proof for (6.5) is similar. ∎

We are now in a position to provide a class of tempered stable measures which admits a mixing operator, and to determine its mixing rate.

PROPOSITION 6.4. *Let $\alpha \in (0,1)$, $p \in [1,2]$, and let $(\omega_n)_{n\geq 0}$ be a bounded positive weight sequence. In the framework of (6.1), assume that the sequence $(k_n)_{n\geq 0}$ defined by $k_0 > 0$ and*

$$k_n := k_0 \prod_{l=0}^{n-1} \frac{1}{\omega_l}, \qquad n \geq 1,$$

satisfies $(k_n)_{n\geq 0} \in \ell^\alpha(\mathbb{N})$. Let μ be the distribution of (6.1) on the Banach space $\ell^p(\mathbb{N})$, and consider the bounded weighted backward shift operator on $\ell^p(\mathbb{N})$ defined by

$$Te_0 := 0, \quad Te_{n+1} := w_n e_n, \qquad n \geq 0.$$

The following are true:

(1) *μ is an invariant measure for T.*
(2) *We have*

$$\sup_{x^*,y^* \in (\ell^p(\mathbb{N}))^* \setminus \{0\}} \frac{|C_\mu^{=,\neq}(x^*, T^{*n}y^*)|}{\|x^*\|^{p/2}\|y^*\|^{p/2}} \leq 2^{5-p}(a_-\lambda_-^{\alpha-p} + a_+\lambda_+^{\alpha-p})\Gamma(p-\alpha)\sum_{l=0}^{\infty} k_l^{p/2} k_{l+n}^{p/2}, \tag{6.7}$$

for $n \geq 0$. In particular, by Theorem 2.4, T is mixing provided that

$$\lim_{n\to\infty} \sum_{l=0}^{\infty} k_l^{p/2} k_{l+n}^{p/2} = 0.$$

Proof. (1) Let the random variable X be represented as in (6.1). Then

$$\begin{aligned} TX &= \sum_{n=0}^{\infty} k_n(\theta_{1,n} + i\theta_{2,n})Te_n = \sum_{n=1}^{\infty} k_n\omega_{n-1}(\theta_{1,n} + i\theta_{2,n})e_{n-1} \\ &= \sum_{n=0}^{\infty} k_n(\theta_{1,n+1} + i\theta_{2,n+1})e_n \stackrel{d}{=} \sum_{n=0}^{\infty} k_n(\theta_{1,n} + i\theta_{2,n})e_n = X, \end{aligned}$$

where the equality in distribution follows since the $\theta_{1,n}$'s and $\theta_{2,n}$'s are independent identical copies of the same random variable. The distribution of TX is also μ, thus μ is an invariant measure for T.

(2) We may represent μ with characteristic functional (1.4) by Lemma 6.2, and thus the bounds of Lemma 6.3 apply. For $C_\mu^{=}(x^*, T^{*n}y^*)$, we have

$$\begin{aligned}\sum_{l=0}^{\infty} k_l^p |\mathrm{Re}\,\langle e_n, x^*\rangle\,\mathrm{Re}\,\langle e_n, T^{*n}y^*\rangle|^{p/2} \\
&= \sum_{l=0}^{\infty} k_{l+n}^p |\mathrm{Re}\,\langle e_{l+n}, x^*\rangle\,\mathrm{Re}\,\langle e_l, y^*\rangle \omega_l \dots \omega_{l+n-1}|^{p/2} \\
&\le \|x^*\|^{p/2}\|y^*\|^{p/2} \sum_{l=0}^{\infty} k_{l+n}^p \prod_{j=l}^{l+n-1} \omega_j^{p/2} \\
&= \|x^*\|^{p/2}\|y^*\|^{p/2} \sum_{l=0}^{\infty} k_{l+n}^p \prod_{j=l}^{l+n-1} \left(\frac{k_j}{k_{j+1}}\right)^{p/2} \\
&= \|x^*\|^{p/2}\|y^*\|^{p/2} \sum_{l=0}^{\infty} k_{l+n}^p \left(\frac{k_l}{k_{l+n}}\right)^{p/2} \\
&= \|x^*\|^{p/2}\|y^*\|^{p/2} \sum_{l=0}^{\infty} k_l^{p/2} k_{l+n}^{p/2},\end{aligned}$$

and likewise

$$\sum_{l=0}^{\infty} k_l^p |\mathrm{Im}\,\langle e_n, x^*\rangle\,\mathrm{Im}\,\langle e_n, T^{*n}y^*\rangle|^{p/2} \le \|x^*\|^{p/2}\|y^*\|^{p/2} \sum_{l=0}^{\infty} k_l^{p/2} k_{l+n}^{p/2}.$$

Similarly, the same bounds hold for $C_\mu^{\neq}(x^*, T^{*n}y^*)$. The claim follows from Lemma 6.3. In particular, by Theorem 2.4, T is mixing on $\ell^p(\mathbb{N})$ when (6.7) goes to zero as $n \to \infty$. ∎

As with Proposition 5.2, Lemma 6.2, and thus Proposition 6.4, can be applied to the symmetrized control measure

$$\xi(dz) := \frac{1}{2} \sum_{n=0}^{\infty} k_n^\alpha \big(\delta_{e_n}(dz) + \delta_{-e_n}(dz) + \delta_{ie_n}(dz) + \delta_{-ie_n}(dz)\big).$$

By controlling the quantity

$$\sum_{j=0}^{\infty} k_j^{p/2} k_{j+l+n}^{p/2} = \sum_{j=0}^{\infty} \frac{1}{(j+l+n+1)^{\gamma/2}(j+1)^{\gamma/2}}$$

as in (4.3)–(4.4) of Example 4.4, Proposition 6.4 yields the following result.

Example 6.5. Let $\alpha \in (0,1)$, $p \in [1,2]$, and $\gamma > 1$. In the context of Proposition 6.4, let T be the bounded weighted backward shift operator on $E = \ell^p(\mathbb{N})$ defined by

$$Te_0 := 0, \qquad Te_{n+1} := \left(1 + \frac{1}{n+1}\right)^{\gamma/p} e_n, \qquad n \ge 0,$$

i.e. the weight sequence is $\omega_n = ((n+2)/(n+1))^{\gamma/p}$, $n \ge 0$. Define the coefficients of (6.1) by

$$k_0 > 0, \qquad k_n = k_0 \prod_{l=0}^{n-1} \frac{1}{\omega_l},$$

and denote by μ the distribution of (6.1). Then μ is an invariant measure for T, and T is mixing with respect to μ, with the rate

$$\sup_{x^*,y^* \in \ell^p(\mathbb{N}))^* \setminus \{0\}} \frac{|C_\mu^{=}(x^*, T^{*n} y^*)|}{\|x^*\|^{p/2} \|y^*\|^{p/2}} \le$$
$$\begin{cases} 2^{5-p}(a_- \lambda_-^{\alpha-p} + a_+ \lambda_+^{\alpha-p}) \Gamma(p-\alpha) k_0^p \mathrm{B}(1-\gamma/2, \gamma-1) n^{-(\gamma-1)}, & 1 < \gamma < 2, \\ 2^{5-p}(a_- \lambda_-^{\alpha-p} + a_+ \lambda_+^{\alpha-p}) \Gamma(p-\alpha) k_0^p \mathrm{B}(\varepsilon/2, 1-\varepsilon) n^{-(1-\varepsilon)}, & \gamma \ge 2, \end{cases}$$

for $n \ge 1$, and for any $\varepsilon \in (0,1)$ when $\gamma \ge 2$.

We now obtain convergence rates for more general functions in the tempered stable setting.

Corollary 6.6. *In the context of Proposition* 6.4, *let* $(a_j)_{j\in\mathbb{N}}$ *and* $(b_l)_{l\in\mathbb{N}}$ *be two complex* $\ell^1(\mathbb{N})$ *sequences such that*

$$\sum_{j=0}^{\infty} |a_j| \, \|T\|^{jp/2} < \infty.$$

For any pair (ϕ, ψ) *chosen in Table* 1, *the functions*

$$f(z) := \sum_{j=0}^{\infty} a_j e^{i\phi(\langle z, T^{*j} x^* \rangle)} \quad \text{and} \quad g(z) := \sum_{l=0}^{\infty} b_l e^{i\psi(\langle z, T^{*l} y^* \rangle)} \tag{6.8}$$

are well-defined in $L^2(E,\mu)$, *and if* T *is mixing then*

$$|I_n(f,g)| = O_{f,g,\mu,T}\Big(\sum_{l=0}^{\infty} |b_l| \sum_{j=0}^{\infty} k_j^{p/2} k_{j+l+n}^{p/2} \Big), \qquad n \ge 0. \tag{6.9}$$

Proof. As in Corollary 5.3, we have $f, g \in L^2(E,\mu)$. We have the estimate

$$\sum_{i=0}^{\infty} k_i^p |\mathrm{Re}\, \langle e_i, T^{*j} x^* \rangle \, \mathrm{Re}\, \langle e_i, T^{*l} y^* \rangle|^{p/2}$$
$$= \sum_{i=\max(0,j-l)}^{\infty} k_{i+l}^p |\mathrm{Re}\, \langle e_{i+l-j}, x^* \rangle \omega_{i+l-j} \dots \omega_{i+l-1} \, \mathrm{Re}\, \langle e_i, y^* \rangle \omega_i \dots \omega_{i+l-1}|^{p/2}$$
$$\le \|T\|^{jp/2} \|x^*\|^{p/2} \|y^*\|^{p/2} \sum_{i=\max(0,j-l)}^{\infty} k_{i+l}^p \prod_{s=i}^{i+l-1} \omega_s^{p/2}$$
$$= \|T\|^{jp/2} \|x^*\|^{p/2} \|y^*\|^{p/2} \sum_{i=\max(0,j-l)}^{\infty} k_i^{p/2} k_{i+l}^{p/2}, \quad j, l \ge 0.$$

The same estimate holds if we replace one or both of the real parts with the imaginary part. Hence, letting

$$f_{j,x^*}(z) := e^{i\phi(\langle z, T^{*j}x^*\rangle)}, \quad g_{l,y^*}(z) := e^{i\psi(\langle z, T^{*l}y^*\rangle)}, \quad j,l \geq 0,$$

by Lemma 6.3 we have

$$\begin{aligned}|I_n(f_{j,x^*}, g_{l,y^*})| &\leq |C_\mu^{\phi,\psi}(T^{*j}x^*, T^{*(n+l)}y^*)| \\ &\qquad \times \exp\Big(\max\Big(\max_{n\geq 0}|C_\mu^{\phi,\psi}(T^{*n}x^*, y^*)|, \max_{n\geq 0}|C_\mu^{\phi,\psi}(x^*, T^{*n}y^*)|\Big)\Big) \\ &= \|T\|^{jp/2} O_{x^*,y^*,\mu,T}\Big(\sum_{i=\max(0,j-l-n)}^{\infty} k_i^{p/2} k_{i+l+n}^{p/2}\Big)\end{aligned}$$

and

$$\begin{aligned}|I_n(f_{x^*}, g_{y^*})| &\leq \sum_{j,l=0}^{\infty} |a_j|\,|b_l|\,|I_n(f_j, g_l)| \\ &= O_{x^*,y^*,\mu,T}\Big(\sum_{j,l=0}^{\infty} |a_j|\,|b_l|\,\|T\|^{jp/2} \sum_{i=\max(0,j-l-n)}^{\infty} k_i^{p/2} k_{i+l+n}^{p/2}\Big),\end{aligned}$$

and (6.9) follows from the series convergence assumption. ∎

Once again, by controlling the quantity

$$\sum_{j=0}^{\infty} k_j^{p/2} k_{j+l+n}^{p/2} = \sum_{j=0}^{\infty} \frac{1}{(j+l+n+1)^{\gamma/2}(j+1)^{\gamma/2}},$$

as in (4.3)–(4.4) of Example 4.4, we obtain the following from Corollary 6.6.

Example 6.7. In the context of Example 6.5, if

$$\sum_{j=0}^{\infty} |a_j|\,\|T\|^{jp/2} < \infty,$$

then T admits the mixing rate

$$|I_n(f,g)| = \begin{cases} O_{f,g,\mu,T}(n^{-(\gamma-1)}), & 1 < \gamma < 2, \\ O_{f,g,\mu,T}(n^{-(1-\varepsilon)}), & \gamma \geq 2, \end{cases}$$

for $n \geq 1$, with any $\varepsilon \in (0,1)$ when $\gamma \geq 2$, where f, g are of the form (6.8).

Finally, we also note that the rate (6.9) holds for mixed exponential functions of the form

$$f(z) := \sum_{\phi\in\Phi}\sum_{j=0}^{\infty} a_{\phi,j} e^{i\phi(\langle z, T^{*j}x^*\rangle)} \quad \text{and} \quad g(z) := \sum_{\psi\in\Psi}\sum_{l=0}^{\infty} b_{\psi,l} e^{i\psi(\langle z, T^{*l}y^*\rangle)},$$

where Φ, Ψ are any disjoint non-empty subsets of $\{\pm\operatorname{Re}(\cdot), \pm\operatorname{Im}(\cdot)\}$, and $(a_{\phi,j})_{j\in\mathbb{N},\,\phi\in\Phi}$, $(b_{\psi,l})_{l\in\mathbb{N},\,\psi\in\Psi}$ are complex $\ell^1(\mathbb{N})$ sequences such that

$$\sum_{j=0}^{\infty} |a_{\phi,j}|\, \|T\|^{jp/2} < \infty, \qquad \phi \in \Phi.$$

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