STUDIA MATHEMATICA
Online First version

On complex interpolation for pairs of quasi-Banach function spaces with a non-separable component

by

Moritz Egert and Benjamin W. Kosmala

Abstract. Kalton and Mitrea (1998) characterized complex interpolation spaces of quasi-Banach function spaces as Calderón products if both interpolants are separable. We show that one separability assumption may be omitted and establish a Wolff-reiteration result with one non-separable endpoint space.

1. Introduction. Recent literature uses Kalton and Mitrea's complex interpolation theory for quasi-Banach spaces [15] to prove estimates for solutions to e.g. elliptic boundary value problems in tent spaces [3, 2]. Typically, these spaces are characterized by an integrability parameter $p \in (0, \infty]$ such that $p \in (0, 1)$ yields quasi-Banach spaces, $p \in [1, \infty]$ Banach spaces, and separable spaces are obtained if and only if $p \in (0, \infty)$.

Amongst others, Amenta and Auscher [2] establish interpolation identities for the whole range $p \in (0, \infty]$, using so-called Wolff reiteration [22, Thm. 2] as follows: They show the desired interpolation identity for parameters in $(0, \infty)$, use a duality argument on $[1, \infty)$ to cover parameters in $(1, \infty]$ and finally 'glue' $(0, \infty)$ and $(1, \infty]$ together using Wolff reiteration. Regarding this final gluing procedure, [2, Thm. 2.12] contains a subtle gap, because Wolff reiteration for the complex method is not known for general quasi-Banach spaces.

Huang [10] derives such identities differently, by characterizing interpolation spaces elegantly as appropriate products and then mainly argues on the product side. In their case, for instance in [10, Thm. 4.3], coincidence of products and interpolation spaces is simply taken for granted.

In the setting of quasi-Banach function spaces, the purpose of this paper is thus to establish the identification of complex interpolation spaces as appropriate products and to derive from this the desired Wolff-type result.

2020 *Mathematics Subject Classification*: Primary 46B70; Secondary 46E30.
Key words and phrases: complex interpolation, quasi-Banach function space, Calderón product, A-convexity, separability, reiteration.
Received 24 January 2025; revised 9 September 2025.

DOI: 10.4064/sm250124-5-12

We first expand on Kalton and Mitrea's pioneering work [15, Thm. 3.4] by identifying complex interpolation spaces as Calderón products if only one interpolant is non-separable. Prior to our note, this was only known for sequence spaces [14, Sect. 7]. For Calderón (or Calderón–Lozanovskiĭ) products of various other function spaces we refer for instance to [17]. Our (standard) notation is explained in Sections 2 and 3.

THEOREM 1.1 (Calderón formula). *Let X_0, X_1 be A-convex quasi-Banach function spaces over a separable and σ-finite measure space Ω, one of which is separable. Then $X_0 + X_1$ is A-convex and for every $\theta \in (0,1)$ the spaces $[X_0, X_1]_\theta$ and $X_0^{1-\theta} X_1^\theta$ are separable and agree up to equivalence of quasi-norms.*

We then apply a reiteration result for Calderón products due to Gomez and Milman [9, Thm. 4.13], which provides the desired Wolff reiteration for quasi-Banach function spaces.

THEOREM 1.2. *Let X_0, X_1, X_2 and X_3 be A-convex quasi-Banach function spaces over a separable and σ-finite measure space Ω, where either X_0 or X_3 is separable and $X_0 \cap X_3$ is dense in X_1 and X_2. Further, let $\theta, \eta, \lambda, \mu \in [0,1]$ be parameters subject to the following conditions:*

(i) $0 < \theta < \eta < 1$,
(ii) $\theta = \lambda \eta$,
(iii) $\eta = (1-\mu)\theta + \mu$.

If $X_1 = [X_0, X_2]_\lambda$ and $X_2 = [X_1, X_3]_\mu$, then $X_1 = [X_0, X_3]_\theta$ and $X_2 = [X_0, X_3]_\eta$ hold up to equivalence of quasi-norms.

REMARK 1.3. It remains an interesting open problem to prove Theorem 1.2 for more general A-convex quasi-Banach spaces.

We will recall the complex interpolation method for general quasi-Banach spaces in Section 2 and then focus on quasi-Banach function spaces from Section 3 onward, while also establishing the main tools for the proofs of the Calderón formula and the reiteration result given in Sections 4 and 5.

Concerning notation, all vector spaces are taken over $\mathbb{C}$, $\mathbb{D}$ is the open unit disc in $\mathbb{C}$ and $\mathbb{S} := \{z \in \mathbb{C} \mid \operatorname{Re}(z) \in (0,1)\}$. Quasi-norms on quasi-Banach spaces X will be denoted by $\| \cdot \|_X$, measures of measurable sets E will be denoted by $|E|$ and we let $\theta \in (0,1)$.

2. Complex interpolation of quasi-Banach spaces. Let (X_0, X_1) be an interpolation couple, namely X_0 and X_1 are quasi-Banach spaces that continuously embed in some common Hausdorff topological vector space Z. Following [14, 15], a function $F : \mathbb{S} \to X_0 + X_1$ is called *analytic* if for every $z_0 \in \mathbb{S}$ there exists an $r \in (0, \operatorname{dist}(z_0, \partial\mathbb{S}))$ such that F has a power series representation $F(z) = \sum_{k=0}^{\infty} f_k(z - z_0)^k$ with convergence in $X_0 + X_1$ for every $z \in B_r(z_0)$. The space of *admissible functions*, denoted by $\mathcal{F}$, contains all bounded and analytic functions which can be continuously extended to $\overline{\mathbb{S}}$ so that for $j = 0, 1$ the traces $t \mapsto F(j + \mathrm{i}t)$ are continuous and bounded functions into X_j. This space is endowed with the quasi-norm

$$\|F\|_{\mathcal{F}} := \max\Big\{\sup_{t\in\mathbb{R}} \|F(\mathrm{i}t)\|_{X_0}, \sup_{t\in\mathbb{R}} \|F(1 + \mathrm{i}t)\|_{X_1}, \sup_{z\in\mathbb{S}} \|F(z)\|_{X_0+X_1}\Big\}.$$

We define the *complex interpolation space* as $[X_0, X_1]_\theta := \{F(\theta) \mid F \in \mathcal{F}\}$ and equip it with the quotient quasi-norm

$$\|f\|_{[X_0,X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}} \mid F(\theta) = f,\, F \in \mathcal{F}\}.$$

Both $\mathcal{F}$ and $[X_0, X_1]_\theta$ are again quasi-Banach spaces, which agree with Calderón's method when the X_j are Banach spaces [4, §2, §3]. There are other complex interpolation methods for quasi-Banach spaces, such as [5, Ch. 3] and [9], which come with their own partly undesired properties (the former does not agree with Calderón's method in the Banach setting [21, Ch. 1.6.7, Rem. 3] and the latter does not in general produce subspaces of $X_0 + X_1$; see [7]). They are not used in this paper.

Remark 2.1. In comparison to [15], we make two harmless but noteworthy changes:

(i) We drop the assumption that $X_0 \cap X_1$ is dense in both X_j, as it is needed neither in our arguments nor in other works in the field [23, 24]. In particular, already [14, 15] contain results for which this density assumption cannot hold. Density is typically used to extend operators T which act on $X_0 \cap X_1$ first to both X_j and then to $[X_0, X_1]_\theta$. However, if T is already defined on both X_j, then it can be extended to $X_0 + X_1$ simply by setting $T(x_0 + x_1) := T_0(x_1) + T_1(x_1)$.

(ii) We drop the requirement of locally uniform convergence of the series expansion of admissible functions on $B_r(z_0)$, because convergence of quasi-Banach-valued power series inside an open disc implies absolute convergence inside the same open disc, from which locally uniform convergence easily follows [18, Lem. 5.10].

3. Quasi-Banach function spaces. From now on, let Ω be a measure space. We denote by $\mathrm{L}^0(\Omega)$ the space of $\mathbb{C}$-valued measurable functions on Ω, where two functions are identified if they agree almost everywhere.

In order to 'do function spaces right', we introduce them as in [20, Ch. 2].

Definition 3.1. A quasi-Banach space $X \subseteq \mathrm{L}^0(\Omega)$ is called a *quasi-Banach function space* or a *quasi-Banach function space over* Ω if

(i) X contains a *weak order unit*, i.e. there exists $f \in X$ such that $f > 0$ a.e.,
(ii) X has the *lattice property*, i.e. if $f \in X$, $g \in \mathrm{L}^0(\Omega)$ with $|g| \leq |f|$ a.e., then $g \in X$ with $\|g\|_X \leq \|f\|_X$.

Let (X_0, X_1) be a couple of quasi-Banach function spaces. It is then admissible for complex interpolation, because the sum X_0+X_1 is well-defined in $\mathrm{L}^0(\Omega)$ and is again a quasi-Banach function space, into which X_0 and X_1 continuously embed. Also the complex interpolation space $[X_0, X_1]_\theta$ is again a quasi-Banach function space: As weak order unit we take the minimum of the weak order units in the X_j, and if $f \in [X_0, X_1]_\theta$ is obtained as $f = F(\theta)$ for some $F \in \mathcal{F}$, then every $g \in \mathrm{L}^0(\Omega)$ with $|g| \leq |f|$ is obtained as $g = G(\theta)$ for $G := (g/f)F$ satisfying $\|G\|_{\mathcal{F}} \leq \|F\|_{\mathcal{F}}$.

Having pointwise products available will allow us to characterize interpolation spaces via

Definition 3.2. The *Calderón product* $X_0^{1-\theta} X_1^\theta$ of quasi-Banach function spaces X_0 and X_1 over Ω is defined to be the set of all $f \in \mathrm{L}^0(\Omega)$ such that

$$\|f\|_{X_0^{1-\theta} X_1^\theta} := \inf \big\{ \|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^\theta \mid |f| \leq |f_0|^{1-\theta} |f_1|^\theta \text{ a.e., } f_j \in X_j \big\}$$

is finite.

With slight adaptations regarding the quasi-triangle inequality, completeness of Calderón products is proved as in [4, §13.5], making them quasi-Banach function spaces that appear naturally in the context of interpolation.

In order to show that interpolation spaces and Calderón products agree in the Banach case, Calderón used an argument based on approximation via truncations [4, §13.6(i)], which will also be our strategy for the quasi-Banach setting. This requires that truncations give rise to admissible functions and that some sort of pointwise a.e. convergence should be compatible with quasi-norm convergence.

To identify limits of truncations in $X_0^{1-\theta} X_1^\theta$ and $[X_0, X_1]_\theta$ in the common ambient space $X_0 + X_1$ later on, we will need

Proposition 3.3 ([20, Prop. 2.3]). *Let X be a quasi-Banach function space over Ω. If $x_n \to x$ in X, then $x_n \to x$ locally in measure. In particular, convergent sequences in X admit pointwise a.e. converging subsequences, and consequently limits in X are determined by their pointwise a.e. limits.*

Now, we can provide the classical construction of admissible functions with prescribed boundary data on $\mathbb{S}$. The result seems to be well known and we only include an elementary proof for completeness, since in the quasi-Banach case we cannot rely on difference quotients [12, p. 275].

Lemma 3.4. *Let* X_0 *and* X_1 *be quasi-Banach function spaces over* Ω *and* $f_j \in X_j$ *such that for some* $M > 1$ *their non-zero absolute values are contained in* $[M^{-1}, M]$ *almost everywhere. Then* $F : \mathbb{S} \to X_0 + X_1$, $z \mapsto |f_0|^{1-z}|f_1|^z$, *is an admissible function.*

Proof. The proof is in four steps.

Step 1: *F is well-defined.* Let $z \in \mathbb{S}$. Since powers of 0 are not unanimously defined, we need to be careful when defining F. We set $|f_0|^{1-z} := g_z \circ |f_0|$, where

$$g_z : [0,\infty) \to \mathbb{C}, \qquad w \mapsto \begin{cases} w^{1-z} & \text{if } w \neq 0, \\ 0 & \text{else.} \end{cases}$$

Then g_z is measurable and using a similar definition for $|f_1|^z$, defining $F(z) := |f_0|^{1-z}|f_1|^z$ yields a measurable function. By Young's inequality,

$$\big| |f_0|^{1-z}|f_1|^z \big| = |f_0|^{\mathrm{Re}(1-z)}|f_1|^{\mathrm{Re}(z)} \leq \mathrm{Re}(1-z)|f_0| + \mathrm{Re}(z)|f_1|.$$

Thus, $F(z) \in X_0 + X_1$ by the lattice property of $X_0 + X_1$.

Step 2: *F is bounded on* $\mathbb{S}$. The pointwise estimate for F on $\mathbb{S}$ directly translates into a quasi-norm estimate:

$$(1) \quad \|F(z)\|_{X_0+X_1} \leq \big\| \mathrm{Re}(1-z)|f_0| + \mathrm{Re}(z)|f_1| \big\|_{X_0+X_1} \leq \|f_0\|_{X_0} + \|f_1\|_{X_1}.$$

Step 3: *F extends continuously to $\overline{\mathbb{S}}$ so that the traces are bounded into X_j.* To define an extension, we only need to observe that the definition of g_z can be extended to $z \in \overline{\mathbb{S}}$ and still yields a measurable function. Abusing notation, we denote this extension of F to $\overline{\mathbb{S}}$ again by F. Then $|F(j + \mathrm{i}t)| = |f_j|$ and we deduce that $t \mapsto F(j + \mathrm{i}t)$ is a bounded map into X_j.

For continuity on $\overline{\mathbb{S}}$, let $z, z' \in \overline{\mathbb{S}}$ and denote by $\overline{zz'}$ the line segment connecting z and z'. On $A := \mathrm{supp}(f_0) \cap \mathrm{supp}(f_1)$, we obtain pointwise a.e. the estimate

$$\begin{aligned} |F(z) - F(z')| &= \big| |f_0|^{1-z}|f_1|^z - |f_0|^{1-z'}|f_1|^{z'} \big| \\ &= \Big| \int_{\overline{zz'}} \log(|f_1|/|f_0|)|f_0|^{1-\xi}|f_1|^{\xi} \,\mathrm{d}\xi \Big| \\ &\leq |z - z'| \log(|f_1|/|f_0|) \sup_{\xi \in \overline{zz'}} |f_0|^{1-\mathrm{Re}(\xi)}|f_1|^{\mathrm{Re}(\xi)} \\ &\leq |z - z'| \log(M^2)M. \end{aligned}$$

The same estimate holds true on $\Omega \backslash A$. As $\mathbf{1}_A \leq M^{-1}|f_j|$, the lattice property of $X_0 + X_1$ yields $\mathbf{1}_A \in X_0 + X_1$ with

$$\|F(z) - F(z')\|_{X_0+X_1} \leq |z - z'| \log(M^2)M \|\mathbf{1}_A\|_{X_0+X_1},$$

showing continuity of F up to $\overline{\mathbb{S}}$. The continuity of the traces is proved similarly by taking X_j-norms when $z, z' \in j + \mathrm{i}\mathbb{R}$.

Step 4: *F is analytic.* Let $z_0 \in \mathbb{S}$. By the series expansion of the exponential, we deduce that, for all $z \in \mathbb{C}$ pointwise a.e. on A,

$$\begin{aligned} F(z) = |f_0|^{1-z}|f_1|^z &= |f_0|\mathrm{e}^{z_0 \log(|f_1/f_0|)}\mathrm{e}^{(z-z_0)\log(|f_1/f_0|)} \\ &= \sum_{k=0}^{\infty} \frac{|f_0|}{k!}\mathrm{e}^{z_0 \log(|f_1/f_0|)} \log^k(|f_1/f_0|)(z-z_0)^k \\ &=: \sum_{k=0}^{\infty} g_k(z-z_0)^k. \end{aligned}$$

We again denote by g_k the extension of g_k by 0 to Ω. This yields measurable functions and $g_k \in X_0 + X_1$, as $k!|g_k| \leq |f_0|M^2[\log(M^2)]^k$ a.e. and the latter is in X_0. By Proposition 3.3, this is the only candidate for a series expansion in $X_0 + X_1$ and we need to check that the series converges in $X_0 + X_1$. If $z \in \mathbb{S}$, then

$$\begin{aligned} \Big\| \sum_{k=n}^{m} g_k(z-z_0)^k \Big\|_{X_0+X_1} &\leq \Big\| \sum_{k=n}^{m} \frac{|f_0|}{k!} M^2 \log^k(M^2)|z-z_0|^k \Big\|_{X_0+X_1} \\ &\leq M\|\mathbf{1}_A\|_{X_0+X_1} M^2 \sum_{k=n}^{m} \frac{\log^k(M^2)}{k!}|z-z_0|^k, \end{aligned}$$

so the series indeed converges in $X_0 + X_1$ for all $z \in B_{\mathrm{dist}(z_0,\partial\mathbb{S})}(z_0)$. ■

To relate functions to their truncations, we use the following concept.

Definition 3.5. A quasi-Banach function space X over Ω is *order continuous* if for every non-negative sequence $(f_n)_n$ in X with $f_n \searrow 0$ a.e. we have $f_n \to 0$ in X.

We have order continuity if Ω is 'reasonably separable'.

Definition 3.6 ([20, p. 12]). The measure space Ω is called *separable* if there exists a sequence of measurable sets $(A_n)_n$ in Ω such that for every measurable set $E \subseteq \Omega$ with $|E| < \infty$ and every $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $|A_n \triangle E| < \varepsilon$.

In particular, if Ω is a separable metric space which is endowed with its Borel σ-algebra and a regular measure, then Ω is separable in the above sense. (Take for $(A_n)_n$ an enumeration of finite unions of balls with rational radii centered around a dense sequence in Ω.)

Theorem 3.7. *Let X be a quasi-Banach space over a separable and σ-finite measure space Ω. Then X is separable if and only if X is order continuous.*

For the implication yielding order continuity from separability, see [20, Prop. 3.13]. The other direction is more involved: Arguing by contraposition boils down to constructing a discrete copy of $\mathcal{P}(\mathbb{N})$ in X as in [1, Cor. 14.5], which can be adapted to the quasi-Banach setting.

Our generalization of the Calderón formula is based on

PROPOSITION 3.8. *Let X_0 and X_1 be quasi-Banach function spaces over Ω. If either of them is order continuous, then so is $X_0^{1-\theta}X_1^{\theta}$.*

The proof in [8, Thm. 1.29] relies only on pointwise a.e. estimates and the lattice property and thus works *verbatim* in the quasi-Banach setting.

By Theorem 3.7, the space $X_0^{1-\theta}X_1^{\theta}$ is order continuous (and therefore truncation arguments work) if $X_0^{1-\theta}X_1^{\theta}$ is separable, which is the case if for instance X_0 and X_1 are separable, as is easily seen from the definition of $\|\cdot\|_{X_0^{1-\theta}X_1^{\theta}}$. By Proposition 3.8 and Theorem 3.7, we can reduce to a single separability assumption.

We will need further properties that a quasi-Banach function space may have. The first one is a weak version of the maximum principle.

DEFINITION 3.9. A quasi-Banach function space X over Ω is said to be *A-convex* or *analytically convex* if there exists $C \geq 1$ such that for all polynomials $P : \overline{\mathbb{D}} \to X$ we have $\|P(0)\|_X \leq C \sup_{|z|=1} \|P(z)\|_X$.

REMARK 3.10. Tent spaces as in [2, 10, 6] are covered by our definitions of quasi-Banach function spaces and A-convexity; see e.g. the proof of [10, Thm. 4.3]. Additionally, general Banach spaces are A-convex [14, p. 21].

The second property will yield a different perspective on the proof of the Calderón formula later on.

DEFINITION 3.11. A quasi-Banach function space X over Ω has the *weak Fatou property* if there exists $C \geq 1$ such that for all sequences $(f_n)_n$ of non-negative functions in X that satisfy $\liminf_{n\to\infty} \|f_n\|_X < \infty$, we have $\liminf_{n\to\infty} f_n \in X$ with $\|\liminf_{n\to\infty} f_n\|_X \leq C \liminf_{n\to\infty} \|f_n\|_X$.

Among the different versions of the Fatou property [25, Ch. 15, §65] we choose this one, because it is most reminiscent of Fatou's lemma from integration theory.

4. Proof of the Calderón formula. We recall the Aoki–Rolewicz Theorem [16, Thm. 1.3], which asserts that for every quasi-Banach space X there is an $r \in (0,1]$ and an equivalent quasi-norm which is s-subadditive and continuous for all $s \in (0,r]$. In particular, $\|\sum_{k=0}^{n} x_k\|_X^s \leq C^2 \sum_{k=0}^{n} \|x_k\|_X^s$ whenever $x_0,\dots,x_n \in X$ and $s \in (0,r]$, where $C \geq 1$ is the constant from the quasi-norm equivalence.

For the following proof, we may assume that the equivalent quasi-norms on $[X_0, X_1]_\theta$ and $X_0^{1-\theta} X_1^\theta$ are s-subadditive for the same value $r \in (0,1]$ and switching to them from the original norms will come at the cost of constants that we denote by $C \geq 1$ and $D \geq 1$, respectively. As only the equivalent quasi-norm on $X_0^{1-\theta} X_1^\theta$ shows up explicitly, this one will be denoted by $\| \cdot \|$. We also denote the constant from the quasi-triangle inequality of $[X_0, X_1]_\theta$ by $C_\square$.

Proof of Theorem 1.1. We use ideas from [4, §13.6(i)] to show that the inclusion $X_0^{1-\theta} X_1^\theta \subseteq [X_0, X_1]_\theta$ holds, that it is bounded and that both spaces are separable.

That $X_0 + X_1$ is A-convex with bounded inclusion $[X_0, X_1]_\theta \subseteq X_0^{1-\theta} X_1^\theta$ was proven in [15, Thm. 3.4] without the assumption of separability. To convince the reader that these results indeed do not rely on separability, an overview of their proofs can be found in Section 5. We set $K := 2D^2$ and work with the auxiliary set

$$\begin{aligned}\mathcal{D} := \bigl\{ f \in X_0^{1-\theta} X_1^\theta \bigm| \exists\, M > 1 : |f| = K \|f\|_{X_0^{1-\theta} X_1^\theta} |f_0|^{1-\theta} |f_1|^\theta, \\ f_j \in X_j,\ \|f_j\|_{X_j} \leq 1,\ M^{-1} \leq |f_j| \leq M \text{ a.e. on } \operatorname{supp}(f_j) \bigr\}.\end{aligned}$$

The actual argument is in three steps.

Step 1: *The bounded inclusion $\mathcal{D} \subseteq [X_0, X_1]_\theta$.* If $f \in \mathcal{D}$, then

$$F : \mathbb{S} \to X_0 + X_1, \qquad z \mapsto \frac{f}{|f|} |f_0|^{1-z} |f_1|^z,$$

is admissible by Lemma 3.4. In view of (1), the bound $\|F\|_{\mathcal{F}} \leq 2$ is immediate, and as $F(\theta) = f (K \|f\|_{X_0^{1-\theta} X_1^\theta})^{-1}$, we obtain $f \in [X_0, X_1]_\theta$ and

$$\|f\|_{[X_0,X_1]_\theta} \leq \|F\|_{\mathcal{F}} K \|f\|_{X_0^{1-\theta} X_1^\theta} \leq 2K \|f\|_{X_0^{1-\theta} X_1^\theta}. \tag{2}$$

Step 2: *$\mathcal{D}$ is dense in $X_0^{1-\theta} X_1^\theta$.* By assumption, either X_0 or X_1 is separable, and hence order continuous by Theorem 3.7. Using Proposition 3.8, we obtain order continuity of $X_0^{1-\theta} X_1^\theta$ and then separability, again by Theorem 3.7.

Let $f \in X_0^{1-\theta} X_1^\theta$. Of course, we can assume $f \neq 0$. Then there exist $f_j \in X_j \setminus \{0\}$ such that

$$|f| \leq |f_0|^{1-\theta} |f_1|^\theta, \quad \|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^\theta \leq \tfrac{3}{2} \|f\|_{X_0^{1-\theta} X_1^\theta}.$$

Replacing f_0 by $|f / |f_1|^\theta|^{1/1-\theta} \leq |f_0|$, we may assume that $|f| = |f_0|^{1-\theta} |f_1|^\theta$. For $n \in \mathbb{N}$ we introduce $E_n := \{\omega \in \Omega \mid 1/n \leq |f_0(\omega)|, |f_1(\omega)| \leq n\}$ and define truncations via $g_n := f \mathbf{1}_{E_n}$. Then $g_n \in X_0^{1-\theta} X_1^\theta$ and order continuity yields $g_n \to f$ in $X_0^{1-\theta} X_1^\theta$ due to $|f - g_n| \searrow 0$.

In order to show $g_n \in \mathcal{D}$, we observe that $g_n \to f$ in $X_0^{1-\theta} X_1^\theta$ implies $\|g_n\| \to \|f\|$ by continuity of $\| \cdot \|$ and so we have $\|f\| \leq \frac{4}{3} \|g_n\|$ for all large enough n. Making n even larger, we can also assume $\|f_j\mathbf{1}_{E_n}\|_{X_j} > 0$ and define $h_{i,n} := f_i\mathbf{1}_{E_n}\|f_i\mathbf{1}_{E_n}\|^{-1}$ for those n. We have $h_{i,n} \in X_i$ and obtain

$$\begin{aligned}
|g_n| &= \|f_0\mathbf{1}_{E_n}\|_{X_0}^{1-\theta}\|f_1\mathbf{1}_{E_n}\|_{X_1}^{\theta}|h_{0,n}|^{\theta}|h_{1,n}|^{1-\theta} \\
&\le \tfrac{3}{2}\|f\|_{X_0^{1-\theta}X_1^{\theta}}|h_{0,n}|^{1-\theta}|h_{1,n}|^{\theta} \le \tfrac{3}{2}\,D\|f\|\,|h_{0,n}|^{1-\theta}|h_{1,n}|^{\theta} \\
&\le 2D\|g_n\|\,|h_{0,n}|^{1-\theta}|h_{1,n}|^{\theta} \le 2D^2\|g_n\|_{X_0^{1-\theta}X_1^{\theta}}|h_{0,n}|^{1-\theta}|h_{1,n}|^{\theta} \\
&= K\|g_n\|_{X_0^{1-\theta}X_1^{\theta}}|h_{0,n}|^{1-\theta}|h_{1,n}|^{\theta}.
\end{aligned}$$

Replacing $h_{0,n}$ by $|g_n(K\|f_n\|_{X_0^{1-\theta}X_1^{\theta}}|h_{1,n}|^{\theta})^{-1}|^{1/(1-\theta)} \le |h_{0,n}|$ achieves equality and we see that every g_n belongs to $\mathcal{D}$ by making a suitable choice of $M > 1$.

STEP 3: $X_0^{1-\theta}X_1^{\theta} \subseteq [X_0, X_1]_\theta$ *follows by density.* This is not immediate, as $\mathcal{D}$ might not be a linear subspace. So, let $f \in X_0^{1-\theta}X_1^{\theta}$. Inductively, we shall construct a sequence $(f_n)_n$ in $\mathcal{D}$ such that

$$\Big\|f - \sum_{k=0}^{n} f_k\Big\|^r \le \rho^{n+1}\|f\|^r, \tag{3}$$

$$\|f_n\|^r \le \rho^{n+1}\|f\|^r, \tag{4}$$

for some $\rho \in (0,1)$.

For $n = 0$, let $c, \varepsilon > 0$ be parameters to be chosen momentarily. By the previous step, there is an $f_0 \in \mathcal{D}$ such that $\|cf - f_0\|^r \le \varepsilon\|f\|^r$. This function also satisfies

$$\begin{aligned}
\|f - f_0\|^r &\le \|cf - f_0\|^r + \|(1-c)f\|^r \le (\varepsilon + (1-c)^r)\|f\|^r, \\
\|f_0\|^r &\le \|f_0 - cf\|^r + \|cf\|^r \le (\varepsilon + c^r)\|f\|^r.
\end{aligned}$$

Fixing $c = 1/2$ and ε so that $\rho := \varepsilon + c^r < 1$ yields the claim for $n = 0$. Induction over n is performed similarly by replacing f with $f - \sum_{k=0}^{n} f_k$ and f_0 with f_{n+1}.

Since $\rho \in (0,1)$, estimate (3) implies that $\sum_{k=0}^{n} f_k \xrightarrow{n\to\infty} f$ in $X_0^{1-\theta}X_1^{\theta}$. By (2) and (4) we also obtain

$$\begin{aligned}
\|f_n\|_{[X_0,X_1]_\theta}^r &\le (2K)^r\|f_n\|_{X_0^{1-\theta}X_1^{\theta}}^r \le (2K)^r D^r\|f_n\|^r \\
&\le (2K)^r D^r\rho^{n+1}\|f\|^r \le (2K)^r D^{2r}\rho^{n+1}\|f\|_{X_0^{1-\theta}X_1^{\theta}}^r,
\end{aligned}$$

which we use to show that $(\sum_{k=0}^{n} f_k)_n$ is a Cauchy sequence in $[X_0, X_1]_\theta$:

$$\begin{aligned}
\Big\|\sum_{k=m}^{n} f_k\Big\|_{[X_0,X_1]_\theta}^r &\le C^{2r}\sum_{k=m}^{n}\|f_k\|_{[X_0,X_1]_\theta}^r \\
&\le C^{2r}(2K)^r D^{2r}\sum_{k=m}^{n}\rho^{k+1}\|f\|_{X_0^{1-\theta}X_1^{\theta}}^r.
\end{aligned}$$

Thus, the limit $\widetilde{f}$ in $[X_0, X_1]_\theta$ satisfies

$$\begin{aligned}
&\|\widetilde{f}\|_{[X_0,X_1]_\theta} \\
&\quad \le C_\Box \Big(\Big\| \widetilde{f} - \sum_{k=0}^{n} f_k \Big\|_{[X_0,X_1]_\theta} + \Big\| \sum_{k=0}^{n} f_k \Big\|_{[X_0,X_1]_\theta} \Big) \\
&\quad \le C_\Box \Big(\Big\| \widetilde{f} - \sum_{k=0}^{n} f_k \Big\|_{[X_0,X_1]_\theta} + C^2(2K)D^2 \Big(\sum_{k=0}^{n} \rho^{k+1} \Big)^{1/r} \|f\|_{X_0^{1-\theta}X_1^\theta} \Big)
\end{aligned}$$

and so by passing to the limit, we obtain

$$\|\widetilde{f}\|_{[X_0,X_1]_\theta} \le C_\Box C^2(2K)D^2 \Big(\frac{\rho}{1-\rho} \Big)^{1/r} \|f\|_{X_0^{1-\theta}X_1^\theta}. \tag{5}$$

We have shown that as $n \to \infty$, the partial sums above converge to f in $X_0^{1-\theta}X_1^\theta$ and to $\widetilde{f}$ in $[X_0, X_1]_\theta$. Since both spaces continuously embed into $X_0 + X_1$ (by definition and Young's inequality, respectively), the limits can be identified by virtue of Proposition 3.3. Thus, (5) yields the claim. ∎

Our second main result follows immediately:

Proof of Theorem 1.2. If X_0 is separable, then by Theorem 1.1 also X_1 is separable with $X_1 = [X_0, X_2]_\lambda = X_0^{1-\lambda}X_2^\lambda$. Since $X_2 = [X_1, X_3]_\mu$, separability of X_1 yields $X_2 = X_1^{1-\mu}X_3^\mu$. The same conclusion can be achieved when X_3 is separable. Now, [9, Thm. 4.13] applies and yields $X_1 = X_0^{1-\theta}X_3^\theta$ and $X_2 = X_0^{1-\eta}X_3^\eta$. Using Theorem 1.1 again to convert Calderón products back to interpolation spaces, we finally obtain $X_1 = [X_0, X_3]_\theta$ and $X_2 = [X_0, X_3]_\eta$. ∎

Our proof of the inclusion $X_0^{1-\theta}X_1^\theta \subseteq [X_0, X_1]_\theta$ uses separability of the Calderón product in a crucial way. We can simplify the argument whenever more structure of the interpolation space is known a priori.

COROLLARY 4.1. *Let X_0, X_1 be A-convex quasi-Banach function spaces over a σ-finite measure space Ω and let $[X_0, X_1]_\theta$ satisfy the weak Fatou property. Then $X_0 + X_1$ is A-convex and the spaces $[X_0, X_1]_\theta$ and $X_0^{1-\theta}X_1^\theta$ agree up to equivalence of quasi-norms.*

Proof. We took inspiration from the proof of the corollary in [19, p. 242]. In our proof of $X_0^{1-\theta}X_1^\theta \subseteq [X_0, X_1]_\theta$ above, separability is only used to prove that truncations also converge in $X_0^{1-\theta}X_1^\theta$. If $[X_0, X_1]_\theta$ satisfies the Fatou property, we can establish this inclusion differently.

As a preliminary result, we note that $f \in \mathrm{L}^0(\Omega)$ belongs to some quasi-Banach function space X if and only if $|f| \in X$ and in this case we have $\|f\|_X = \big\| |f| \big\|_X$. This follows from the lattice property, since $|f| \le \big| |f| \big|$ and

$\big||f|\big| \leq |f|$. Thus, to show that $X_0^{1-\theta}X_1^{\theta} \subseteq [X_0, X_1]_\theta$ and that the corresponding quasi-norm estimates hold, it suffices to consider absolute values.

Let $f \in X_0^{1-\theta}X_1^{\theta}$. Again, we may assume $f \neq 0$. Then there exist $f_j \in X_j \setminus \{0\}$ such that

$$|f| \leq |f_0|^{1-\theta}|f_1|^{\theta}, \quad \|f_0\|_{X_0}^{1-\theta}\|f_1\|_{X_1}^{\theta} \leq 2\|f\|_{X_0^{1-\theta}X_1^{\theta}}$$

and as in Step 2 of the proof of Theorem 1.1 we may assume that $|f| = |f_0|^{1-\theta}|f_1|^{\theta}$. For $n \in \mathbb{N}$ set $E_n := \{\omega \in \Omega \mid 1/n \leq |f_0(\omega)|, |f_1(\omega)| \leq n\}$ and define $g_n := f\mathbf{1}_{E_n}$. The maps

$$F_n : \mathbb{S} \to X_0 + X_1, \qquad z \mapsto \left|\frac{f_0\mathbf{1}_{E_n}}{\|f_0\mathbf{1}_{E_n}\|_{X_0}}\right|^{1-z}\left|\frac{f_1\mathbf{1}_{E_n}}{\|f_1\mathbf{1}_{E_n}\|_{X_1}}\right|^{z},$$

are well-defined for large enough n and admissible by Lemma 3.4, and so $|g_n| \in [X_0, X_1]_\theta$ with

$$\begin{aligned}\big\||g_n|\big\|_{[X_0,X_1]_\theta} &\leq \|F_n\|_{\mathcal{F}}\|f_0\mathbf{1}_{E_n}\|_{X_0}^{1-\theta}\|f_1\mathbf{1}_{E_n}\|_{X_1}^{\theta}\\ &\leq 2\|f_0\|_{X_0}^{1-\theta}\|f_1\|_{X_1}^{\theta} \leq 4\|f\|_{X_0^{1-\theta}X_1^{\theta}},\end{aligned}$$

where we have used (1) and the lattice property in the second step. Since $|f| = \liminf_n |g_n|$ with $\liminf_n \big\||g_n|\big\|_{[X_0,X_1]_\theta} < \infty$, the weak Fatou property of $[X_0, X_1]_\theta$ now implies that $|f| \in [X_0, X_1]_\theta$ with

$$\big\||f|\big\|_{[X_0,X_1]_\theta} \leq C \liminf_{n\to\infty} \big\||g_n|\big\|_{[X_0,X_1]_\theta} \leq 4C\|f\|_{X_0^{1-\theta}X_1^{\theta}},$$

where $C \geq 1$ is the constant from the weak Fatou property of $[X_0, X_1]_\theta$. By the preliminary remark, we see that $f \in [X_0, X_1]_\theta$ with $\|f\|_{[X_0,X_1]_\theta} \leq 4C\|f\|_{X_0^{1-\theta}X_1^{\theta}}$.

The remainder of this proof now works as indicated in Section 5 below. ■

5. A glimpse at the other inclusion of the Calderón formula. In order to fully justify why separability is only used for one direction of the proof of Theorem 1.1, we give a short overview of the proof of the inclusion $[X_0, X_1]_\theta \subseteq X_0^{1-\theta}X_1^{\theta}$ in [15].

A motivation for the Banach case can be found in [19, Chap. 4], where it is argued that admissible functions F are determined by their boundary values through the Poisson formula

$$\begin{aligned}F(z) &= \int_{\mathbb{R}} F(\mathrm{i}t)P_0(z, \mathrm{i}t)\,\mathrm{d}t + \int_{\mathbb{R}} F(1+\mathrm{i}t)P_1(z, 1+\mathrm{i}t)\,\mathrm{d}t\\ &=: (1 - \mathrm{Re}(z))x_0 + \mathrm{Re}(z)x_1,\end{aligned} \tag{6}$$

where P_0 and P_1 denote the components of the Poisson kernel for $\overline{\mathbb{S}}$, while x_0 and x_1 define elements in X_0 and X_1 respectively. The above equality can be turned into a product estimate of the form $|F(\theta)| \leq x_0^{1-\theta}x_1^{\theta}$ a.e., showing

$[X_0, X_1]_\theta \subseteq X_0^{1-\theta} X_1^\theta$ with continuous inclusion due to the validity of the triangle inequality for X_j-valued integrals.

As a preliminary result in the quasi-Banach case, Kalton showed in [13, Thm. 4.4] and [11, Thm. 2.2] that for a quasi-Banach function space X, A-convexity is equivalent to so-called *p-convexity* for some $p \in (0,1]$, which means that there is $C \geq 1$ depending only on p such that for all $g_0, \dots, g_n \in X$ we have

$$\Big\| \Big(\sum_{k=0}^{n} |g_k|^p \Big)^{1/p} \Big\|_X \leq C \Big(\sum_{k=0}^{n} \|g_k\|_X^p \Big)^{1/p}.$$

In [11, Thm. 2.2], it is also shown that X is p-convex if and only if X is q-convex for all $q \in (0, p]$.

Straightforward estimates then show that the space $X_0 + X_1$ is p-convex [15, Thm. 3.4]. In particular, we may assume that X_0, X_1 and $X_0 + X_1$ are p-convex for the same range of parameters $q \in (0, p]$.

First, in [15, Thm. 3.4] Kalton and Mitrea used the fact that every p-convex quasi-Banach function space X admits the *p-convexification* $X^p := \{f \in \mathrm{L}^0(\Omega) \mid |f|^{1/p} \in X\}$, which is endowed with $\|f\|_{X^p} := \||f|^{1/p}\|_X^p$ and after renorming satisfies a triangle inequality [18, Prop. 3.23], so it is a Banach function space. The idea of the proof is then to mimic equation (6) for $|F|^q$, which – and this is a remarkable observation – can be turned back into an estimate for $|F|$ via a limiting process as $q \to 0$.

Indeed, if $f = F(\theta) \in [X_0, X_1]_\theta$ for some $F \in \mathcal{F}$, then $|F|^q$ is Banach-valued for all $q \in (0, p]$ but not analytic in general. By using scalar-valued theory, it is shown in [15, Thm. 3.4] that $|F|^q$ satisfies the sub-mean value formula

$$|F(z)(\,\cdot\,)|^q \leq \fint_0^{2\pi} |F(z + r\mathrm{e}^{\mathrm{i}t})(\,\cdot\,)|^q \,\mathrm{d}t = \Big[\fint_0^{2\pi} |F(z + r\mathrm{e}^{\mathrm{i}t})|^q \,\mathrm{d}t \Big](\,\cdot\,)$$

whenever $B_r(z)$ is compactly contained in $\mathbb{S}$. We stress that the above is an a.e. estimate on Ω and that the right-hand side is a vector-valued Riemann integral. Interchanging the inner variable and the integral was a non-trivial consequence of analyticity of F; compare with [18, Lem. 7.5].

Applying a positive functional φ (of which there are plenty, because the dual space of $(X_0 + X_1)^q$ contains a saturated Banach function space by virtue of the weak order unit; see [25, Ch. 71, Thm. 4(a)] and [20, Prop. 2.5]) yields a scalar-valued subharmonic function $\varphi(|F(\,\cdot\,)|^q)$. By using the Poisson formula for subharmonic functions on $\overline{\mathbb{S}}$, it is shown in [15, Thm. 3.4] that

$$\varphi(|F(\theta)|^q) \leq \int_{\mathbb{R}} P_0(\theta, \mathrm{i}t)\, \varphi(|F(\mathrm{i}t)|^q) \,\mathrm{d}t + \int_{\mathbb{R}} P_1(\theta, 1 + \mathrm{i}t)\, \varphi(|F(1 + \mathrm{i}t)|^q) \,\mathrm{d}t.$$

Pulling φ out of the integrals by continuity and using positivity yields

$$\begin{aligned}|F(\theta)|^q &\le \int_{\mathbb{R}} P_0(\theta, \mathrm{i}t)|F(\mathrm{i}t)|^q \,\mathrm{d}t + \int_{\mathbb{R}} P_1(\theta, 1+\mathrm{i}t)|F(1+\mathrm{i}t)|^q \,\mathrm{d}t \\ &=: (1-\theta) f_0(q)^q + \theta f_1(q)^q,\end{aligned}$$

where the functions $f_j(q)^q$ are defined as X_j^q-valued Riemann integrals. Interchanging inner variables and integrals is again a tedious task (see [18, proof of Thm. 7.1, Ch. 7.3] for details), but it works and Jensen's inequality yields $|F(\theta)|^q \le (1-\theta) f_0(p)^q + \theta f_1(p)^q$ almost everywhere. By letting $q \to 0$, it is argued in [15, Thm. 3.4] that $|F(\theta)| \le f_0(p)^{1-\theta} f_1(p)^\theta$ almost everywhere. Finally, the triangle inequality for X_j^p-valued integrals (after renorming) yields

$$\begin{aligned}\|f_j(p)\|_{X_j} &= \left(\left\|\frac{1}{1-\theta}\int_{\mathbb{R}} P_j(\theta, j+\mathrm{i}t)|F(j+\mathrm{i}t)|^p \,\mathrm{d}t\right\|_{(X_j)^p}\right)^{1/p} \\ &\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} P_j(\theta, j+\mathrm{i}t)\big\||F(j+\mathrm{i}t)|^p\big\|_{(X_j)^p} \,\mathrm{d}t\right)^{1/p} \\ &\le \|F\|_{\mathcal{F}}.\end{aligned}$$

Acknowledgements. This article is based on the master's thesis of the second author [18], which contains further background and details on the topic. We wish to thank Emiel Lorist for guiding us through the literature on Banach function spaces and Marius Mitrea for a helpful exchange about Wolff reiteration.

References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Pure Appl. Math. 119, Academic Press, Orlando, FL, 1985.

[2] A. Amenta and P. Auscher, *Elliptic Boundary Value Problems with Fractional Regularity Data*, CRM Monogr. Ser. 37, Amer. Math. Soc., Providence, RI, 2018.

[3] P. Auscher and M. Egert, *Boundary Value Problems and Hardy Spaces for Elliptic Systems with Block Structure*, Progr. Math. 346, Birkhäuser, 2023.

[4] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), 113–190.

[5] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution. II*, Adv. Math. 24 (1977), 101–171.

[6] R. R. Coifman, Y. Meyer, and E. M. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal. 62 (1985), 304–335.

[7] M. Cwikel, M. Milman, and Y. Sagher, *Complex interpolation of some quasi-Banach spaces*, J. Funct. Anal. 65 (1986), 339–347.

[8] M. Cwikel, P. G. Nilsson, and G. Schechtman, *Interpolation of weighted Banach lattices. A characterization of relatively decomposable Banach lattices*, Mem. Amer. Math. Soc. 165 (2003), no. 737, vi+127 pp.

[9] M. E. Gomez and M. Milman, *Complex interpolation of H^p spaces on product domains*, Ann. Mat. Pura Appl. (4) 155 (1989), 103–115.

[10] Y. Huang, *Weighted tent spaces with Whitney averages: factorization, interpolation and duality*, Math. Z. 282 (2016), 913–933.

[11] N. J. Kalton, *Convexity conditions for non-locally convex lattices*, Glasgow Math. J. 25 (1984), 141–152.

[12] N. J. Kalton, *Analytic functions in non-locally convex spaces and applications*, Studia Math. 83 (1986), 275–303.

[13] N. J. Kalton, *Plurisubharmonic functions on quasi-Banach spaces*, Studia Math. 84 (1986), 297–324.

[14] N. J. Kalton, S. Mayboroda, and M. Mitrea, *Interpolation of Hardy–Sobolev–Besov–Triebel–Lizorkin spaces and applications to problems in partial differential equations*, in: Interpolation Theory and Applications, Contemp. Math. 445, Amer. Math. Soc., Providence, RI, 2007, 121–177.

[15] N. J. Kalton and M. Mitrea, *Stability results on interpolation scales of quasi-Banach spaces and applications*, Trans. Amer. Math. Soc. 350 (1998), 3903–3922.

[16] N. J. Kalton, N. T. Peck, and J. W. Roberts, *An F-Space Sampler*, London Math. Soc. Lecture Note Ser. 89, Cambridge Univ. Press, Cambridge, 1984.

[17] P. Kolwicz, K. Leśnik, and L. Maligranda, *Pointwise products of some Banach function spaces and factorization*, J. Funct. Anal. 266 (2014), 616–659.

[18] B. W. Kosmala, *Products of quasi-Banach lattices*, Master's thesis, Technische Universität Darmstadt, 2023.

[19] S. G. Kreĭn, Y. I. Petunīn, and E. M. Semënov, *Interpolation of Linear Operators*, Transl. Math. Monogr. 54, Amer. Math. Soc., 1982.

[20] E. Lorist and Z. Nieraeth, *Banach function spaces done right*, Indag. Math. (N.S.) 35 (2024), 247–268.

[21] H. Triebel, *Theory of Function Spaces. II*, Monogr. Math. 84, Birkhäuser, Basel, 1992.

[22] T. H. Wolff, *A note on interpolation spaces*, in: Harmonic Analysis (Minneapolis, MN, 1981), Lecture Notes in Math. 908, Springer, Berlin, 1982, 199–204.

[23] W. Yuan, *Complex interpolation and Calderón product of quasi-Banach spaces*, Frontiers Math. China 11 (2016), 753 – 762.

[24] W. Yuan, W. Sickel, and D. Yang, *Interpolation of Morrey–Campanato and related smoothness spaces*, Sci. China Math. 58 (2015), 1835–1908.

[25] A. C. Zaanen, *Integration*, North-Holland, Amsterdam, and Interscience Publishers John Wiley & Sons, New York, 1967.

Moritz Egert, Benjamin W. Kosmala
Department of Mathematics
Technical University of Darmstadt
64289 Darmstadt, Germany
E-mail: egert@mathematik.tu-darmstadt.de
kosmala@mathematik.tu-darmstadt.de