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Unique ergodicity for non-invertible function systems on an interval

by

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Abstract. We study random dynamical systems of certain continuous functions on the unit interval. We use bounded variation to provide sufficient conditions for unique ergodicity of these systems. Several classes of examples are provided.

1. Introduction. Unique ergodicity of dynamical systems is a classical property which has been intensively studied in various settings for many years. Already in the 1950s, Oxtoby [O], Kakutani [Ka], and many others obtained deep results about uniquely ergodic dynamical systems.

Recently, there has been rapid progress in studying ergodicity for random dynamical systems. It turned out that combining probabilistic and dynamical systems tools allows us to obtain new information about the complex behaviour of stochastic dynamical systems.

Unique ergodicity of random homeomorphisms on intervals has been almost completely examined by Alsedà and Misiurewicz [AM], Czudek and Szarek [CS], Deroin, Kleptsyn, and Navas [DKN], Gharaei and Homburg [GH], Barański and Śpiewak [BS] and many others. In turn, unique ergodicity for random homeomorphisms on the circle was studied, e.g., by Malicet [M] and Navas [N]. However, it seems that the case of non-invertible random maps has not yet been completely analyzed. Some results were obtained for certain classes of non-invertible iterated function systems (see [BOS, HH^+, HK^+, Kl]).

In this note, we aim at formulating a new and quite general sufficient condition for unique ergodicity of non-invertible random maps. It uses the variation of random composition.

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Let μ be a Borel probability measure on the space $C([0,1],[0,1])$, denoted briefly by $C([0,1])$, of all continuous functions $g:[0,1]\to[0,1]$ equipped with the supremum norm $|\cdot|$. For brevity we will write gh for the composition $g\circ h$ of $g,h\in C([0,1])$.

Let $\Gamma\subset C([0,1])$ be a closed set such that $\mu(\Gamma)=1$. There is no loss of generality in assuming that $\Gamma=\operatorname{supp}\mu$. The pair (Γ,μ) will be called a *stochastic dynamical system on* $C([0,1])$.

Let $\mathbf{x}$ be a $[0,1]$-random variable, and let $\mathbf{g}$ be a Γ-random variable. Assume that $\mathbf{x}$ and $\mathbf{g}$ are independent, and that they have distributions ν and μ, respectively. Then we easily see that the distribution of $\mathbf{g}(\mathbf{x})$ is equal to the convolution $\mu*\nu$ of the measures μ and ν (see for instance [F]) defined by the formula

$$\int_{[0,1]} f(x)\,\mathrm{d}(\mu*\nu)(x)=\int_{\Gamma}\int_{[0,1]} f(g(x))\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(g)$$

for every bounded Borel measurable function $f:[0,1]\to\mathbb{R}$.

A Borel measure ν is said to be μ-*invariant* (or μ-*stationary*) if

$$\mu*\nu=\nu,$$

or equivalently

$$\nu(\cdot)=\int_{\Gamma} g\nu(\cdot)\,\mathrm{d}\mu(g),$$

where $g\nu(A)=\nu(g^{-1}(A))$ for any Borel set $A\subset[0,1]$.

A μ-invariant probability measure is said to be μ-*ergodic* if it cannot be written as a proper convex combination of two different μ-invariant probability measures. Obviously, if ν is a unique μ-invariant probability measure, then it is ergodic. Therefore, a unique μ-invariant probability measure is called *uniquely ergodic*. On the other hand, it is well known that any two different μ-ergodic measures are mutually singular (see [DPZ]).

The main purpose of the paper is to provide a condition on (Γ,μ) which implies unique ergodicity. An important component of this condition is the bounded variation of the random iteration, as follows:

$$\text{(BV)}\qquad \mu^{\otimes\mathbb{N}}\Big(\Big\{(g_1,g_2,\ldots)\in\Gamma^{\mathbb{N}}:\sup_{n\ge1}\bigvee\nolimits_0^1 g_1\cdots g_n<\infty\Big\}\Big)>0,$$

where $\bigvee_0^1 g$ denotes the variation of the function g on $[0,1]$ (see Section 2 for the definition), and $\mu^{\otimes\mathbb{N}}$ is the product measure on the product space $\Gamma^{\mathbb{N}}$.

Our main result is

MAIN THEOREM 1.1. *Let* (Γ,μ) *be a stochastic dynamical system on* $C([0,1])$ *such that property* (BV) *is satisfied. Then the following statements hold:*

(i) *If* ν_1, ν_2 *are* μ*-ergodic measures concentrated on* $(0,1)$ *and such that*

$$\{0,1\} \cap \operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2 \neq \emptyset,$$

then $\nu_1 = \nu_2$.

(ii) *If* (Γ, μ) *has no atomic* μ*-invariant probability measures with finite support in the open interval* $(0,1)$*, then any two distinct* μ*-ergodic measures* ν_1 *and* ν_2 *on* $(0,1)$ *have disjoint supports.*

Under additional assumptions, we can deduce unique ergodicity.

Corollary 1.2. *Let* (Γ, μ) *be a stochastic dynamical system on* $C([0,1])$ *such that property* (BV) *is satisfied. Assume that either*

($\mathfrak{U}_1$) *for every* $x \in (0,1)$ *there exists* $g \in \Gamma$ *such that* $g(x) < x$*, or*
($\mathfrak{U}_2$) *for every* $x \in (0,1)$ *there exists* $g \in \Gamma$ *such that* $g(x) > x$.

Then, on $(0,1)$*, there is at most one* μ*-invariant probability measure. If additionally* $\nu(\{0\}) = \nu(\{1\}) = 0$ *for every* μ*-invariant probability measure, then* (Γ, μ) *is uniquely ergodic.*

Corollary 1.3. *Let* (Γ, μ) *be a stochastic dynamical system on* $C([0,1])$ *such that property* (BV) *is satisfied. Assume additionally that both* ($\mathfrak{U}_1$) *and* ($\mathfrak{U}_2$) *hold. If* $g((0,1)) \subset (0,1)$ *for every* $g \in \Gamma$ *and there exist* $g_0, g_1 \in \Gamma$ *with* $g_0(0), g_1(1) \in (0,1)$*, then* (Γ, μ) *is uniquely ergodic.*

The outline of the paper is as follows. Section 2 contains some notation and definitions. In Section 3 we collect some measure-theoretic results needed for our proofs. In Section 4 we will use (BV) and prove Theorem 1.1, Corollaries 1.2 and 1.3. Finally, in Section 5 we provide different sufficient conditions for (BV) which are easy to check in many examples. Furthermore, in this section, we present a broad class of uniquely ergodic stochastic dynamical systems on $(0,1)$. It includes μ-injective random systems, already studied in [BOS, HH$^+$], but our class is much broader. It includes, for instance, the random maps constructed from tableaux (see Section 5.3). In particular, we provide a uniquely ergodic stochastic dynamical system with $\mu^{\otimes\mathbb{N}}(\{(g_1, g_2, \ldots) \in \Gamma^{\mathbb{N}} : \sup_{n\geq 1} \bigvee_0^1 g_1 \cdots g_n < \infty\})$ less than 1 but greater than 0.

2. Preliminaries.

We denote by $\mathcal{M}([0,1])$ the space of all finite signed Borel measures on $[0,1]$, and by $\mathcal{M}^+([0,1])$ the cone of positive finite Borel measures. Let $\mathcal{P}([0,1])$ denote the convex subset of all Borel probability measures on $[0,1]$. The space $\mathcal{P}([0,1])$ is equipped with the *Wasserstein distance* d_W,

$$d_W(\nu_1, \nu_2) = \sup \Big| \int_{[0,1]} f \, \mathrm{d}\nu_1 - \int_{[0,1]} f \, \mathrm{d}\nu_2 \Big| \quad \text{for any } \nu_1, \nu_2 \in \mathcal{P}([0,1]),$$

where the supremum is taken over all Lipschitz functions $f : [0,1] \to \mathbb{R}$ with Lipschitz constant less than or equal to 1 (called 1-Lipschitz functions).

It is well known that the space $\mathcal{P}([0,1])$ with the Wasserstein distance d_W is a complete metric space and convergence in the Wasserstein metric is equivalent to weak convergence of measures.

Let $g : [0,1] \to [0,1]$, and let $[a,b] \subset [0,1]$. We shall denote by $\bigvee_a^b g$ the variation of g on $[a,b]$, i.e.

$$\bigvee_a^b g = \sup s_n(g),$$

where the supremum is taken over all partitions $a = x_0 < x_1 < \cdots < x_n = b$ and

$$s_n(g) = \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|.$$

All properties of the variation of a function used in our arguments can be found in [LM, Section 6.1] and in [Loj].

We denote by $|f|_L$ the Lipschitz constant of a Lipschitz function $f : [0,1] \to \mathbb{R}$.

Remark 2.1. From the definition of variation it follows that if f is a Lipschitz function, then

$$\bigvee_a^b fg \le |f|_L \bigvee_a^b g$$

for any $g : [0,1] \to [0,1]$.

We end this part of the paper with a simple observation on Lipschitz functions.

Remark 2.2. Let $g : I \to \mathbb{R}$ (I an interval in $\mathbb{R}$) be a continuous function non-differentiable on a countable set D' only. If the set D' has only finitely many accumulation points and $\sup_{x \in I \setminus D'} |g'(x)| < +\infty$, then we have $|g|_L = \sup_{x \in I \setminus D'} |g'(x)|$. On the other hand, we then also have $\sup_{x \in I \setminus D'} |g'(x)| = \operatorname{ess\,sup}_{x \in I} |g'(x)|$ and consequently $|g|_L = \operatorname{ess\,sup}_{x \in I} |g'(x)|$.

3. Some auxiliary measure-theoretic results. We start with two results – which may be of independent interest – that hold in any Polish space S. Analogously to the case of the interval $[0,1]$ we denote by $\mathcal{M}(S)$, $\mathcal{M}^+(S)$ and $\mathcal{P}(S)$ the families of all signed finite Borel measures, all finite Borel measures and all Borel probability measures on S, respectively.

Lemma 3.1. *Let S be a Polish space, and let $\nu_i \in \mathcal{M}(S)$, $i = 1, 2$, be such that $\nu_1 \le \nu_2$. The order interval*

$$[\nu_1, \nu_2] := \{\nu \in \mathcal{M}(S) : \nu_1 \le \nu \le \nu_2\}$$

is compact in every vector space topology on $\mathcal{M}(S)$ that coincides with the weak topology on $\mathcal{M}^+(S)$ and for which $\mathcal{M}^+(S)$ is closed in $\mathcal{M}(S)$.

Proof. One has $[\nu_1,\nu_2]=\nu_1+[0,\nu_2-\nu_1]$. Because the topology on $\mathcal{M}(S)$ makes addition continuous, it suffices to show that the order intervals $[0,\nu]$ are compact, for any $\nu\in\mathcal{M}^+(S)$. First, observe that

$$[0,\nu]=\{\hat\nu\in\mathcal{M}^+(S):\nu-\hat\nu\geq 0\}=\mathcal{M}^+(S)\cap(\nu-\mathcal{M}^+(S)).$$

$\mathcal{M}^+(S)$ is closed in the chosen vector space topology on $\mathcal{M}(S)$. Hence $\nu-\mathcal{M}^+(S)$ is closed, and therefore also $[0,\nu]$ is closed in $\mathcal{M}^+(S)$. Since ν is tight, for every $\varepsilon>0$ there exists K_ε compact such that $\nu(S\setminus K_\varepsilon)<\varepsilon$. Then also $\hat\nu(S\setminus K_\varepsilon)<\varepsilon$ for every $\hat\nu\in[0,\nu]$. So, $[0,\nu]$ is uniformly tight. Prokhorov's Theorem for positive measures (see e.g. [Bog, Theorem 8.6.2]) implies that $[0,\nu]$ is compact in the weak topology, hence in the vector space topology on $\mathcal{M}(S)$. ∎

The above lemma is used to obtain the following result.

Lemma 3.2. *Let S be a Polish space, and let $\nu_1,\nu_2\in\mathcal{M}^+(S)$ be given. Assume that there exists $\alpha>0$ such that for any $\epsilon>0$ one may find measures $\nu_1^\epsilon,\nu_2^\epsilon\in\mathcal{P}(S)$ with $d_W(\nu_1^\epsilon,\nu_2^\epsilon)<\epsilon$ satisfying*

$$\nu_i\geq\alpha\nu_i^\epsilon\quad\text{for } i=1,2.$$

Then the measures ν_1,ν_2 are not mutually singular.

Proof. Take two sequences of probability measures $(\nu_1^{1/n})_{n\geq1}$ and $(\nu_2^{1/n})_{n\geq1}$ satisfying $d_W(\nu_1^{1/n},\nu_2^{1/n})<1/n$ and $\nu_i\geq\alpha\nu_i^{1/n}$ for $i=1,2$. Since the sets

$$\{\nu\in\mathcal{M}^+(S):\alpha\nu\leq\nu_i\}\quad\text{for } i=1,2 \tag{3.1}$$

are weakly compact in $\mathcal{M}^+(S)$ according to Lemma 3.1, we may find an increasing sequence $(m_n)_{n\geq1}$ of integers such that $(\nu_1^{1/m_n})_{n\geq1}$ and $(\nu_2^{1/m_n})_{n\geq1}$ are weakly convergent. Moreover, as $d_W(\nu_1^{1/m_n},\nu_2^{1/m_n})<1/m_n$, they converge to the same measure, say $\hat\nu$. Obviously $\hat\nu\in\mathcal{P}(S)$ and, using again the fact that the sets defined by (3.1) are weakly compact, hence closed, we finally obtain

$$\nu_i\geq\alpha\hat\nu\quad\text{for } i=1,2,$$

which completes the proof. ∎

We continue with technical results that make use of the fact that the underlying space is $[0,1]$.

Lemma 3.3. *Let $\nu\in\mathcal{P}([0,1])$, and let $\delta\geq0$ and $0<c\leq1$ be such that $\nu([0,c))>\delta$. Then there exists $\bar c\in\operatorname{supp}\nu$ such that $0\leq\bar c<c$, $\nu([\bar c,c))>0$ and $\nu([0,\bar c])>\delta$.*

Proof. Since $\nu([0,c))>0$, we have $[0,c)\cap\operatorname{supp}\nu\neq\emptyset$. Define

$$c^*:=\sup([0,c)\cap\operatorname{supp}\nu).$$

Then $c^* \in \operatorname{supp}\nu$ and $0 \le c^* \le c$. If $c^* = c$, then there exists a strictly increasing sequence (c_n) in $[0,c) \cap \operatorname{supp}\nu$ such that $c_n \uparrow c^*$. Then

$$\delta < \nu([0,c^*)) = \nu\Big(\bigcup_{n=1}^{\infty} [0,c_n]\Big) = \lim_{n\to\infty} \nu([0,c_n]).$$

Pick n_0 such that $\nu\big([0,c_{n_0}]\big) > \delta$. Then $c_{n_0+1} \in (c_{n_0}, c^*) \cap \operatorname{supp}\nu$. So, one obtains $\nu((c_{n_0}, c^*)) > 0$. Then $\bar{c} := c_{n_0}$ has the desired properties.

If $c^* < c$, then $\nu((c^*, c)) = 0$. Consequently, $\nu([0,c^*]) > \delta$. If $\nu(\{c^*\}) > 0$, then $\bar{c} := c_*$ satisfies the required properties. If $\nu(\{c^*\}) = 0$, then $c^* \in \operatorname{supp}\nu$ implies that for every $\epsilon > 0$, $\nu((c^* - \epsilon, c^* + \epsilon)) > 0$. As $\nu([c^*, c)) = 0$, for all $\epsilon > 0$ sufficiently small we have $\nu((c^* - \epsilon, c^*)) > 0$. Consequently, there exists a strictly increasing sequence (c_n) in $\operatorname{supp}\nu$ with $c_n \uparrow c^*$. A similar reasoning to the case $c^* = c$ above now yields $\bar{c} := c_{n_0}$ with the required properties. ∎

LEMMA 3.4. *Let $\nu \in \mathcal{P}([0,1])$, $0 < \delta \le 1$ and $0 \le c < 1$ be such that $\nu([0,c)) < \delta$ and $\nu(\{c\}) = 0$. Then there exists $\tilde{c} \in \operatorname{supp}\nu$ such that $c < \tilde{c} \le 1$, $\nu((c,\tilde{c}]) > 0$ and $\nu([0,\tilde{c})) < \delta$.*

Proof. The idea of the proof is similar to that of Lemma 3.3. First, since $\nu(\{c\}) = 0$,

$$\nu((c,1]) = 1 - \nu([0,c))) > 1 - \delta \ge 0.$$

Therefore, $(c,1] \cap \operatorname{supp}\nu \ne \emptyset$. Define

$$c_* := \inf((c,1] \cap \operatorname{supp}\nu).$$

Then $c_* \in \operatorname{supp}\nu$ and $c \le c_* \le 1$. If $c_* = c$, then there exists a strictly decreasing sequence (c_n) in $\operatorname{supp}\nu$ such that $c_n \downarrow c_*$. Because $\nu(\{c\}) = 0$, one obtains

$$\delta > \nu([0,c)) = \nu([0,c^*]) = \nu\Big(\bigcap_{n=1}^{\infty} [0,c_n)\Big) = \lim_{n\to\infty} \nu([0,c_n)).$$

Then one proceeds similarly to the proof of Lemma 3.3, also in the case $c_* > c$. ∎

In preparation for the proof of the main result, Theorem 1.1, we state a crucial property of 'ordering of mass' for two mutually singular probability measures. The proof uses elementary techniques, but requires a delicate case-by-case analysis.

PROPOSITION 3.5 (Mass ordering principle). *Let $\nu_1, \nu_2 \in \mathcal{P}([0,1])$ be mutually singular. Then there exist $a, b \in [0,1]$, $a < b$, such that $\nu_i([a,b]) > 0$ for $i = 1,2$ and*

$$\textit{either} \quad \nu_1([0,b)) < \nu_2([0,a]) \quad \textit{or} \quad \nu_2([0,b)) < \nu_1([0,a]).$$

Proof. Since ν_1 and ν_2 are mutually singular, $\nu_1 \neq \nu_2$. Hence, there exists $c \in (0,1)$ such that $\nu_1([0,c]) \neq \nu_2([0,c])$, say $\nu_1([0,c]) < \nu_2([0,c])$ (otherwise relabel). Since ν_1 and ν_2 are mutually singular, one must have $\nu_1(\{c\}) = 0$ or $\nu_2(\{c\}) = 0$ (or both).

CASE 1: $\nu_2(\{c\}) = 0$. Then $0 < \nu_2([0,c]) = \nu_2([0,c)])$. So, Lemma 3.3 yields $0 \leq \bar{c} < c$ such that $0 < \nu_2([\bar{c},c)) = \nu_2([\bar{c},c])$ and

$$\nu_1([0,c)) < \nu_1([0,c]) < \nu_2([0,\bar{c}]). \tag{3.2}$$

If $\nu_1([\bar{c},c]) > 0$, then we can take $a = \bar{c}$ and $b = c$. If $\nu_1([\bar{c},c]) = 0$, then in particular $\nu_1(\{c\}) = 0$. Inequalities (3.2) and Lemma 3.4 yield $c < \tilde{c} \leq 1$ such that $\nu_1((c,\tilde{c}]) > 0$ and $\nu_1([0,\tilde{c})) < \nu_2([0,\bar{c}])$. Then $a := \bar{c}$ and $b := \tilde{c}$ have the stated properties.

CASE 2: $\nu_1(\{c\}) = 0$ (and $\nu_2(\{c\}) > 0$). We have $\nu_1([0,c)) < \nu_2([0,c])$ and $\nu_1(\{c\}) = 0$ in this case. So, Lemma 3.4 yields $c < \tilde{c} \leq 1$ such that $\nu_1((c,\tilde{c}]) > 0$ and $\nu_1([0,\tilde{c})) < \nu_2\big([0,c]\big)$. Define $a := c$ and $b := \tilde{c}$. Then $\nu_2([a,b]) \geq \nu_2(\{c\}) > 0$ and a, b have the desired properties. ∎

LEMMA 3.6. *Let ν be a μ-invariant probability measure on $[0,1]$, and assume that $\Gamma = \operatorname{supp}\mu$. If $(\mathfrak{U}_1)$ holds, then $0 \in \operatorname{supp}\nu$. Analogously, $(\mathfrak{U}_2)$ implies that $1 \in \operatorname{supp}\nu$.*

Proof. Assume $(\mathfrak{U}_1)$: for every $x \in (0,1]$ there is some function $g_0 \in \Gamma$ such that $g_0(x) < x$. Since $\Gamma = \operatorname{supp}\mu$, for every $x \in (0,1]$ we obtain

$$\mu(\{g \in \Gamma : g(x) < x\}) > 0.$$

Take $z \in \operatorname{supp}\nu$ minimal. If $z = 0$, the proof of the first statement is complete. If $z > 0$, then by the above,

$$\mu(\{g \in \Gamma : g(z) < z\}) > 0.$$

Further, $\nu([0,z)) = 0$. On the other hand, by μ-invariance, we see that

$$\nu([0,z)) = \int_{\Gamma} \nu(g^{-1}([0,z)))\,d\mu(g) \geq \int_{\{g\in\Gamma\,:\,g(z)<z\}} \nu(g^{-1}([0,z)))\,d\mu(g) > 0,$$

because $\nu(g^{-1}([0,z))) > 0$ due to the fact that $g^{-1}([0,z))$ is an open neighbourhood of $z \in \operatorname{supp}\nu$ for every g with $g(z) < z$. Since this is impossible, $z = 0$.

The second statement is proven similarly. ∎

4. Proofs of Theorem 1.1 and Corollaries 1.2 and 1.3. To prove the main theorem, let us collect some lemmas.

LEMMA 4.1. *Let (Γ, μ) be a stochastic dynamical system on $C([0,1])$. Suppose that there exist two different measures $\nu_1, \nu_2 \in \mathcal{P}([0,1])$, concen-*

trated on $(0,1)$ *and satisfying*

$$\{0,1\} \cap \operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2 \neq \emptyset.$$

Then there exist sequences $(\tilde{z}_n)_{n\geq 1}$ *and* $(\hat{z}_n)_{n\geq 1}$ *on* $[0,1]$ *such that the open intervals* Z_n *with ends* $\tilde{z}_n$ *and* $\hat{z}_n$ *are pairwise disjoint, and* $\nu_1(Z_n) > 0$ *and* $\nu_2(Z_n) > 0$ *for* $n \geq 1$.

Proof. Suppose $0 \in \operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2$; the case when $1 \in \operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2$ can be proven similarly. Let $\hat{z}_1 \in (0,1]$ be arbitrary. Since $0 \in \operatorname{supp} \nu_i$ and ν_i is concentrated on $(0,1)$, we obtain $\nu_i((0,\hat{z}_1)) > 0$ for $i = 1,2$. Now take a sequence $y_n \in (0,1)$ converging to 0. Then there exists an element y_k in this sequence such that $\nu_i((y_k,\hat{z}_1)) > 0$ for $i = 1,2$. Indeed, if not, i.e. $\nu_i((y_n, z_1)) = 0$ for all $n \in \mathbb{N}$, then by continuity of measures, we obtain $\nu_i((0,\hat{z}_1)) = \lim_{n\to\infty} \nu_i(\bigcup_{m=1}^n (y_m,\hat{z}_1)) = 0$, which violates our assumptions. Set $y_k := \tilde{z}_1$ and $Z_1 = (\tilde{z}_1,\hat{z}_1)$. Now, consider the open neighbourhood $[0,\tilde{z}_1)$ of 0. Again, $\nu_i((0,\tilde{z}_1)) > 0$ since ν_i is concentrated on $(0,1)$. Thus we can repeat the procedure: set $\tilde{z}_1 := \hat{z}_2$ and find $\tilde{z}_2$ such that $\nu_i((\tilde{z}_2,\hat{z}_2)) > 0$. Continue like this to find $(\tilde{z}_n)_{n\geq 1}$ and $(\hat{z}_n)_{n\geq 1}$. ■

As a consequence of Lemma 4.1 we get the following.

LEMMA 4.2. *Let* (Γ,μ) *be a stochastic dynamical system on* $C([0,1])$. *Suppose that there exist two different* μ*-invariant ergodic probability measures* ν_1, ν_2 *satisfying*

$$\operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2 \neq \emptyset.$$

If there is no atomic μ*-invariant measure with finite support in the open interval* $(0,1)$, *then there exist sequences* $(\tilde{z}_n)_{n\geq 1}$ *and* $(\hat{z}_n)_{n\geq 1}$ *in* $[0,1]$ *such that the intervals* Z_n *with ends* $\tilde{z}_n$ *and* $\hat{z}_n$ *are pairwise disjoint, and* $\nu_1(Z_n) > 0$ *and* $\nu_2(Z_n) > 0$ *for* $n \geq 1$.

Proof. Choose $z \in \operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2$. If $z \in \{0,1\}$, then Lemma 4.1 applies and the condition about atomic μ-invariant measures is not needed to draw the conclusion. Thus, consider the case $0 < z < 1$. Since ν_1 and ν_2 are μ-invariant, there exists a measurable set $\widetilde{\Gamma} \subseteq \Gamma$ (depending on ν_1, ν_2) with $\mu(\widetilde{\Gamma}) = 1$ such that $g(\operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2) \subseteq \operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2$ for all $g \in \widetilde{\Gamma}$. Therefore,

$$\Upsilon := \bigcup_{n=1}^{\infty} \widetilde{\Gamma}^n(z) \subseteq \operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2.$$

Moreover, $\#\Upsilon = \infty$ by the assumption that there is no atomic μ-invariant measure $\nu \in \mathcal{P}([0,1])$ with finite support intersecting the interval $(0,1)$. Indeed, if Υ were finite, then by compactness there would be a μ-invariant Borel probability measure on Υ, and its support would intersect $(0,1)$ non-trivially and be finite, which contradicts the assumptions. Obviously $\Upsilon \subset \operatorname{supp} \nu_i$ for

$i = 1, 2$. Let $(w_n)_{n\geq 1}$ be a sequence of points in Υ. Without loss of generality, we may assume that this sequence is monotonic. Let $r_n := \min\{|w_n - w_{n-1}|/3, |w_n - w_{n+1}|/3\}$ for $n \geq 2$ and $r_1 = |w_1 - w_2|/3$. Then, obviously, $B(w_i, r_i) \cap B(w_j, r_j) = \emptyset$ for $i \neq j$. Again, since $\nu_i(B(w_n, r_n)) > 0$ for $i = 1, 2$ and $n \in \mathbb{N}$, we may choose $\tilde{z}_n, \hat{z}_n \in B(w_n, r_n)$ such that the interval Z_n with ends $\tilde{z}_n, \hat{z}_n$ satisfies $\nu_1(Z_n) > 0$ and $\nu_2(Z_n) > 0$ for $n \in \mathbb{N}$. ∎

Proof of Theorem 1.1. (i) The proof will rely on Lemma 3.2. Supposing that, contrary to our claim, there exist two different μ-ergodic measures $\nu_1, \nu_2 \in \mathcal{P}([0,1])$ satisfying $\{0,1\} \cap \operatorname{supp} \nu_1 \cap \operatorname{supp} \nu_2 \neq \emptyset$, we obtain a contradiction. Namely, we shall prove that ν_1, ν_2 are not mutually singular, contradicting the well-known fact that they must be.

Fix an $\epsilon > 0$. From Proposition 3.5 it follows that there exist $a, b \in [0,1]$, $a < b$, such that, say,

$$\nu_1([0,b)) < \nu_2([0,a]) \quad \text{and} \quad \nu_1([a,b]), \nu_2([a,b]) > 0.$$

Set

$$\delta := \nu_1([a,b]) \wedge \nu_2([a,b]).$$

For $\omega = (g_1, g_2, \ldots) \in \Gamma^{\mathbb{N}}$, denote

$$V_n(\omega) := \textstyle\bigvee_0^1 g_1 \cdots g_n.$$

Then condition (BV) implies that there exists $K > 0$ such that

$$\beta := \mu^{\otimes \mathbb{N}}\Big(\Big\{\omega \in \Gamma^{\mathbb{N}} : \sup_{n\geq 1} V_n(\omega) < K\Big\}\Big) > 0.$$

Let $\tilde{\epsilon} = \beta\epsilon/2$. Choose $M \in \mathbb{N}$ such that $M > K/\tilde{\epsilon}$, and let $N \in \mathbb{N}$ be such that $N\gamma > M$, where

$$\gamma := (\nu_2([0,a]) - \nu_1([0,b)))/2 > 0.$$

By Lemma 4.1, we may choose pairwise disjoint intervals $Z_1, \ldots, Z_N$ such that $\nu_1(Z_j) > 0$ and $\nu_2(Z_j) > 0$ for $j = 1, \ldots, N$. Now, by the Individual Birkhoff Ergodic Theorem, we choose points $x_j, y_j \in Z_j$ such that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}_{[0,b)}(g_i \cdots g_1(x_j)) = \nu_1([0,b)) \qquad \mu^{\otimes \mathbb{N}}\text{-a.s.}$$

and

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}_{[0,a]}(g_i \cdots g_1(y_j)) = \nu_2([0,a]) \qquad \text{for } j \in \{1, \ldots, N\},\ \mu^{\otimes \mathbb{N}}\text{-a.s.}$$

These equations, in turn, give

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} (\mathbf{1}_{[0,a]}(g_i \cdots g_1(y_j)) - \mathbf{1}_{[0,b)}(g_i \cdots g_1(x_j))) = \nu_2([0,a]) - \nu_1([0,b))$$

for $j \in \{1, \dots, N\}$, $\mu^{\otimes\mathbb{N}}$-a.s. Consequently, there exists $n \in \mathbb{N}_{\geq N}$ such that

$$\frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{1}_{[0,a]}\left(g_i \cdots g_1(y_j)\right) - \mathbf{1}_{[0,b)}(g_i \cdots g_1(x_j))\right) > \gamma \tag{4.1}$$

for every $j \in \{1, \dots, N\}$ and every $\omega \in \hat{\Omega} \subset \Gamma^n$ with $\mu^{\otimes n}(\hat{\Omega}) > 1 - \beta/2$. Further, we have

$$\begin{aligned}\mu^{\otimes n}(\{\omega \in \Gamma^n : \textstyle\bigvee_0^1 g_n \cdots g_1 < K\}) &= \mu^{\otimes n}(\{\omega \in \Gamma^n : \textstyle\bigvee_0^1 g_1 \cdots g_n < K\}) \\ &\geq \mu^{\otimes\mathbb{N}}\Big(\Big\{\omega \in \Gamma^{\mathbb{N}} : \sup_{n\geq 1} V_n(\omega) < K\Big\}\Big) = \beta,\end{aligned}$$

and thus

$$\mu^{\otimes n}(\omega \in \hat{\Omega} : \textstyle\bigvee_0^1 g_n \cdots g_1 < K) \geq \beta/2.$$

Set

$$\Omega := \{\omega \in \hat{\Omega} : \textstyle\bigvee_0^1 g_n \cdots g_1 < K\}.$$

From (4.1) it follows that for any $\omega \in \Omega$ and $j \in \{1, \dots, N\}$ the set

$$I(j, \omega) := \{1 \leq i \leq n : g_i \cdots g_1(y_j) \leq a < b \leq g_i \cdots g_1(x_j)\}$$

has n_j elements with $n_j \geq \gamma n$. Since n and N have been chosen such that $n \geq N$ and $N\gamma > M \geq 1$, we get $1 \leq M < n_j \leq n$. Let $i_1^{(j)}, \dots, i_{n_j}^{(j)}$ be an enumeration of $I(j, \omega)$. Concatenate these into a string of numbers

$$s := (i_1^{(1)}, \dots, i_{n_1}^{(1)},\ \dots\ , i_1^{(N)}, \dots, i_{n_N}^{(N)}).$$

Then s has length $\sum_{j=1}^{N} n_j \geq N \cdot n\gamma > nM$, while each entry of s has been chosen from n numbers. Therefore, s must contain at least M numbers which are equal. For a fixed j, all $i_k^{(j)}$ are distinct. Hence, for any $\omega \in \Omega$ there exist $1 \leq i < n$ and distinct $j_1(\omega), \dots j_M(\omega) \in \{1, \dots, N\}$ such that

$$g_i \cdots g_1(x_{j_l(\omega)}) \geq b \quad \text{and} \quad g_i \cdots g_1(y_{j_l(\omega)}) \leq a \qquad \text{for all } l = 1, \dots, M.$$

Consequently,

$$|g_n \cdots g_{i+1}([a, b])| < \tilde{\epsilon}.$$

If not, by the obvious property of variation, we would obtain

$$\begin{aligned}\textstyle\bigvee_0^1 g_n \cdots g_1 &\geq \sum_{l=1}^{M} \textstyle\bigvee_{x_{j_l(\omega)}}^{y_{j_l(\omega)}} g_n \cdots g_1 \\ &\geq M|g_n \cdots g_{i+1}([a, b])| \geq M\tilde{\epsilon} > K \qquad \text{for } \omega \in \Omega,\end{aligned}$$

contrary to the definition of Ω.

We are now in a position to define the measures ν_1^ϵ and ν_2^ϵ. For any $\omega = (g_1, \dots, g_n) \in \Omega$ we define $T(\omega) = \omega' = (g_n, \dots, g_l)$, where l is the largest integer strictly smaller than n such that

$$|g_n \cdots g_l([a, b])| < \tilde{\epsilon}.$$

Set

$$\Omega_i = T(\Omega) \cap \Gamma^i \quad \text{for } i = 1, \ldots, n.$$

We easily see that for $i \neq j$ the sets $\Omega_i \times \Gamma^{\mathbb{N}}$ and $\Omega_j \times \Gamma^{\mathbb{N}}$ are disjoint. Therefore, we have

$$\sum_{j=1}^{n} \mu^{\otimes j}(\Omega_j) = \sum_{j=1}^{n} \mu^{\otimes \mathbb{N}}(\Omega_j \times \Gamma^{\mathbb{N}}) \geq \mu^{\otimes n}(\Omega) \geq \beta/2.$$

Finally, define

$$\tilde{\nu}_i^{\epsilon}(A) := \sum_{j=1}^{n} \int_{\Omega_j} \frac{\nu_i(g^{-1}(A) \cap [a,b])}{\nu_i([a,b])} \, \mu^{\otimes i}(\mathrm{d}g) \ \text{ for any Borel set } A \text{ and } i = 1, 2,$$

and observe that since ν_1, ν_2 are μ-invariant we obtain

$$\begin{aligned}
\tilde{\nu}_i^{\epsilon}(A) &= \sum_{j=1}^{n} \int_{\Omega_i} \frac{\nu_i(g^{-1}(A) \cap [a,b])}{\nu_i([a,b])} \, \mu^{\otimes j}(\mathrm{d}g) \\
&\leq \sum_{j=1}^{n} \int_{\Omega_j} \frac{\nu_i(g^{-1}(A))}{\nu_i([a,b])} \, \mu^{\otimes j}(\mathrm{d}g) \\
&= \sum_{j=1}^{n} \int_{\Omega_j \times \Gamma^{n-j}} \frac{\nu_i(g^{-1}(A))}{\nu_i([a,b])} \, \mu^{\otimes n}(\mathrm{d}g) \\
&= \int_{\bigcup_{j=1}^{n} \Omega_j \times \Gamma^{n-j}} \frac{\nu_i(g^{-1}(A))}{\nu_i([a,b])} \, \mu^{\otimes n}(\mathrm{d}g) \leq \int_{\Gamma^n} \frac{g_* \nu_i(A)}{\nu_i([a,b])} \, \mu^{\otimes n}(\mathrm{d}g) \\
&= \frac{\nu_i(A)}{\nu_i([a,b])} \quad \text{for any Borel set } A \text{ and } i = 1, 2.
\end{aligned} \tag{4.2}$$

We see that

$$\begin{aligned}
\tilde{\nu}_i^{\epsilon}([0,1]) &:= \sum_{j=1}^{n} \int_{\Omega_j} \frac{\nu_i(g^{-1}([0,1]) \cap [a,b])}{\nu_i([a,b])} \, \mu^{\otimes j}(\mathrm{d}g) \\
&= \sum_{j=1}^{n} \int_{\Omega_j} \frac{\nu_i([a,b])}{\nu_i([a,b])} \, \mu^{\otimes j}(\mathrm{d}g) \\
&= \sum_{j=1}^{n} \mu^{\otimes j}(\Omega_j) \geq \beta/2 \quad \text{for } i = 1, 2.
\end{aligned} \tag{4.3}$$

Note that $\tilde{\nu}_1^{\epsilon}([0,1]) = \tilde{\nu}_2^{\epsilon}([0,1])$. Moreover, for any 1-Lipschitz function f we have

$$(4.4)\quad \Big|\int_{[0,1]} f(x)\,\tilde\nu_1^\epsilon(\mathrm{d}x) - \int_{[0,1]} f(x)\,\tilde\nu_2^\epsilon(\mathrm{d}x)\Big|$$
$$\le \sum_{i=1}^n \int_{\Omega_i} \Big| \int_{[a,b]} f(g(x))\,\frac{\nu_1(\mathrm{d}x)}{\nu_1([a,b])} - \int_{[a,b]} f(g(x))\frac{\nu_2(\mathrm{d}x)}{\nu_2([a,b])}\Big|\,\mu^{\otimes i}(\mathrm{d}g)$$
$$\le \sum_{i=1}^n \int_{\Omega_i} \Big(\int_{[a,b]}\int_{[a,b]} |f(g(x)) - f(g(y))|\,\frac{\nu_1(\mathrm{d}x)}{\nu_1([a,b])}\,\frac{\nu_2(\mathrm{d}y)}{\nu_2([a,b])}\Big)\,\mu^{\otimes i}(\mathrm{d}g)$$
$$\le \sum_{i=1}^n \int_{\Omega_i} \Big(\int_{[a,b]}\int_{[a,b]} |g(x) - g(y)|\,\frac{\nu_1(\mathrm{d}x)}{\nu_1([a,b])}\,\frac{\nu_2(\mathrm{d}y)}{\nu_2([a,b])}\Big)\,\mu^{\otimes i}(\mathrm{d}g) \le \tilde\epsilon.$$

Finally, define

$$\nu_i^\epsilon(\cdot) = \frac{\tilde\nu_i^\epsilon(\cdot)}{\tilde\nu_i^\epsilon([0,1])} \quad \text{for } i = 1,2$$

and observe that by (4.2) and (4.3) we have

$$\nu_i \ge \alpha\nu_i^\epsilon \quad \text{for } i = 1,2$$

with $\alpha = \beta\delta/2$ and

$$d_W(\nu_1^\epsilon, \nu_2^\epsilon) < 2\tilde\epsilon/\beta = \epsilon,$$

by (4.3) and (4.4). The application of Lemma 3.2 completes the proof of (i).

(ii) As in case (i), we argue by contradiction. Supposing that contrary to our claim, there exist two distinct μ-ergodic measures $\nu_1, \nu_2 \in \mathcal{P}([0,1])$ satisfying $\operatorname{supp}\nu_1 \cap \operatorname{supp}\nu_2 \ne \emptyset$, Lemma 4.2 instead of Lemma 4.1 and the reasoning as in the proof of (i) show that ν_1, ν_2 are not mutually singular, contradicting their ergodicity. ∎

Proof of Corollary 1.2. Since probability measures are convex combinations of ergodic ones, it suffices to show that there is at most one μ-ergodic measure. Let ν_1, ν_2 be two μ-ergodic measures on $(0,1)$. By Lemma 3.6, we find that $0 \in \operatorname{supp}\nu_1 \cap \operatorname{supp}\nu_2$ or $1 \in \operatorname{supp}\nu_1 \cap \operatorname{supp}\nu_2$. Thus, by Theorem 1.1(i), we see that $\nu_1 = \nu_2$. If moreover every μ-invariant probability measure gives zero mass to the sets $\{0\}$ and $\{1\}$, then there exists exactly one μ-invariant probability measure on $[0,1]$. Indeed, by compactness, there exists a μ-invariant probability measure ν on $[0,1]$ and since $\nu(\{0\}) = \nu(\{1\}) = 0$, this measure is concentrated on $(0,1)$. By the above, there can be at most one μ-invariant probability measure on $(0,1)$. This proves unique ergodicity. ∎

Proof of Corollary 1.3. Due to Corollary 1.2, it is enough to show that $\nu(\{0\}) = \nu(\{1\}) = 0$. Assume, contrary to our claim, that $\nu(\{0,1\}) > 0$. Since ν is μ-invariant and $g((0,1)) \subset (0,1)$ for every $g \in \Gamma$ by assumption, we see that $g^{-1}(\{0,1\})$ equals either $\emptyset$, $\{0\}$, $\{1\}$ or $\{0,1\}$, and consequently

$$\begin{aligned}\nu(\{0,1\}) &= \int_{\Gamma} \nu(g^{-1}(\{0,1\}))\,\mu(\mathrm{d}g) \\ &= \int_{\{g\in\Gamma\,:\,g^{-1}(\{0,1\})=\{0,1\}\}} \nu(\{0,1\})\,\mu(\mathrm{d}g) \\ &\quad + \int_{\{g\in\Gamma\,:\,g^{-1}(\{0,1\})=\{1\}\}} \nu(\{1\})\,\mu(\mathrm{d}g) \\ &\quad + \int_{\{g\in\Gamma\,:\,g^{-1}(\{0,1\})=\{0\}\}} \nu(\{0\})\,\mu(\mathrm{d}g) \\ &\le \int_{\{g\in\Gamma\,:\,g(\{0,1\})\subset\{0,1\}\}} \nu(\{0,1\})\,\mu(\mathrm{d}g) \\ &< \int_{\Gamma} \nu(\{0,1\})\,\mu\mathrm{d}(g) = \nu(\{0,1\}),\end{aligned}$$

where the strict inequality is due to the fact that $\nu(\{0,1\}) > 0$ and the complement of the set $\{g \in \Gamma : g(\{0,1\}) \subset \{0,1\}\}$ is non-empty and has positive μ-measure, because it contains g_0 and g_1, by assumption and $\Gamma = \operatorname{supp}\mu$. This contradiction completes the proof. ∎

5. Various sufficient conditions and examples. In this section, we demonstrate that condition (BV) holds in various examples. First, we show a construction that allows us to give a large family of examples consisting of Γ generated by finitely many piecewise monotonic continuous maps. The construction is such that condition (BV) is satisfied. By introducing maps that satisfy assumptions (i) and (ii) in Corollary 1.2, the existence of a unique μ-invariant probability measure is then guaranteed by this corollary. Second, we construct an example generated by two maps on the unit interval, with one being highly oscillating in regions. It has unbounded variation. Nevertheless, condition (BV) is satisfied due to the particular structure of the other map, which compensates for this behaviour.

Two families of examples can be immediately provided (see e.g. [AM, CS]):

5.1. Monotonic maps. If all $g \in \Gamma$ are monotonic, then $\bigvee_0^1 g_1 \cdots g_n \le 1$ for every $g_1, \ldots, g_n \in \Gamma$ and condition (BV) holds. If we additionally assume that both $(\mathfrak{U}_1)$ and $(\mathfrak{U}_2)$ are satisfied, then, according to Corollary 1.2, (Γ, μ) has at most one μ-invariant measure, which is concentrated on $(0,1)$. Moreover, if there exist $g_0, g_1 \in \Gamma$ with $g_0(0) \in (0,1)$ and $g_1(1) \in (0,1)$, then (Γ, μ) is uniquely ergodic, by Corollary 1.3.

5.2. μ-injective maps. In [BOS] the authors introduced the so-called μ-injective maps. Recall that the stochastic dynamical system (Γ, μ) on $C([0,1])$ is said to be *μ-injective* if for every $x \in [0,1]$,

$$\int_{\Gamma} \# g^{-1}(x)\, \mu(\mathrm{d}g) \leq 1.$$

Lemma 6.6 in [HH+] shows that every μ-injective system (Γ, μ) on the space $C([0,1])$ satisfies condition (BV). If, additionally, the hypotheses of Corollary 1.2 or 1.3 hold, then the system has at most one μ-invariant probability measure on $(0,1)$ or is uniquely ergodic.

5.3. Systems constructed by tableaux. An important source of examples of random dynamical systems (Γ, μ) with unique ergodicity is provided by the following result.

PROPOSITION 5.1. *Let $\mathcal{I} = \{I_1, \ldots, I_m\}$ be a finite covering of $[0,1]$ by closed intervals such that for every $g \in \Gamma$ and for every $I \in \mathcal{I}$,*

(1) *g is monotonic on I,*
(2) *there exists $J \in \mathcal{I}$ such that $g(I) \subset J$.*

Then for any probability measure μ on $C([0,1])$ with $\operatorname{supp} \mu = \Gamma$, *condition* (BV) *is satisfied.*

Proof. Let $\omega = (g_1, g_2, \ldots) \in \Gamma^{\mathbb{N}}$, and let $\mu \in \mathcal{P}$ be given. For any $g \in \Gamma$, according to the monotonicity of g on any $I \in \mathcal{I}$, one has

$$\bigvee\nolimits_0^1 g \leq \sum_{I \in \mathcal{I}} \bigvee\nolimits_I g \leq \sum_{I \in \mathcal{I}} |g(b_I) - g(a_I)| \leq |\mathcal{I}|,$$

where $I = [a_I, b_I]$ and $|\mathcal{I}|$ indicates the number of intervals in $\mathcal{I}$. So all $g \in \Gamma$ are of bounded variation. The assumptions satisfied by any $g \in \Gamma$ imply further the existence of a map $\phi_g : \mathcal{I} \to \mathcal{I}$ such that $g(I) \subset \phi_g(I)$. Thus, for given ω and any $k \in \mathbb{N}_{\geq 2}$, define $\phi_\omega^k := \phi_{g_2} \circ \cdots \circ \phi_{g_k}$. Then, because each g is monotonic on each interval from $\mathcal{I}$, for any $n \in \mathbb{N}_{\geq 2}$,

$$\begin{aligned} V_n(\omega) &= \bigvee\nolimits_0^1 g_1 \cdots g_n \leq \sum_{I \in \mathcal{I}} \bigvee\nolimits_I g_1 \cdots g_n \leq \sum_{I \in \mathcal{I}} \bigvee\nolimits_{\phi_{g_n}(I)} g_1 \cdots g_{n-1} \\ &\leq \sum_{I \in \mathcal{I}} \bigvee\nolimits_{\phi_\omega^n(I)} g_1 \leq |\mathcal{I}| \cdot \max_{I \in \mathcal{I}} \bigvee\nolimits_I g_1 \leq |\mathcal{I}| \bigvee\nolimits_0^1 g_1 \leq |\mathcal{I}|^2. \end{aligned}$$

Thus, $\sup_n V_n(\omega) < \infty$ for any ω. ∎

REMARK 5.2. The assumptions of Proposition 5.1 are in particular fulfilled under the following condition, which is often easy to verify and thus provides a large class of examples which satisfy (BV):

Assume there exists an essential partition $\mathcal{I} = \{I_1, \ldots, I_m\}$ of $[0,1]$ into intervals such that

(1) every $g \in \Gamma$ is monotonic on each $I \in \mathcal{I}$,
(2) crossing the graph of a $g \in \Gamma$ from $I_j \times I_k$ to $I_{j'} \times I_{k+1}$ or $I_{j'} \times I_{k-1}$ (i.e. changing a partition interval in vertical direction) can only happen at a corner point of $I_j \times I_k$.

Then, for any probability measure μ on $C([0,1])$ with $\operatorname{supp}\mu = \Gamma$, the assumptions of Proposition 5.1 are satisfied.

By Corollary 1.2, Proposition 5.1 above implies the following statement about the number of μ-invariant probability measures on $(0,1)$.

Corollary 5.3. *Let the assumptions of Proposition* 5.1 *be satisfied. If moreover condition* $(\mathfrak{U}_1)$ *or* $(\mathfrak{U}_2)$ *holds, then the stochastic dynamical system* (Γ, μ) *has at most one* μ*-invariant probability measure on* $(0,1)$.

Let us now provide a concrete example that falls into the above class. Let us consider the discrete system $\Gamma = \{g_1, g_2, g_3\}$ defined by the drawing below.

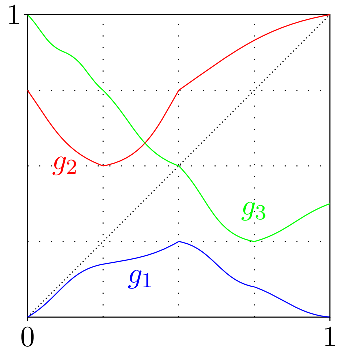

Fig. 1. A family of functions satisfying Proposition 5.1

Let μ be any probability measure on Γ with $\mu(\{g_i\}) \neq 0$ for all $i = 1,2,3$. The dotted lines in the drawing indicate a partition, as in the condition of Remark 5.2. Thus, the system satisfies (BV). Moreover, conditions $(\mathfrak{U}_1)$ and $(\mathfrak{U}_2)$ are satisfied (g_1 is below the diagonal and g_2 is above the diagonal). Finally, $g_i((0,1)) \subset (0,1)$, $g_2(0) \in (0,1)$ and $g_3(1) \in (0,1)$. Thus, by Corollary 1.3, the system (Γ, μ) is uniquely ergodic. Note that if we assume that $\mu(\{g_i\}) > 1/2$ for some $i \in \{1,2,3\}$, then the stochastic dynamical system (Γ, μ) will not be μ-injective.

5.4. Compensation of unbounded variation. Consider a sequence of closed intervals $I_n := [a_n, b_n]$ with non-empty interior, contained in $[0,1]$, disjoint and such that $b_{n+1} < a_n$, while $a_n \downarrow 0$. Put $a_0 := 1$, so $b_1 < 1$. Let $J_n := [b_n, a_{n-1}]$. Thus, the intervals I_n, J_m are essentially pairwise disjoint and

$$[0,1] = \{0\} \cup \bigcup_{n=1}^{\infty} (I_n \ \cup \ J_n).$$

We shall define continuous maps $f_i : [0,1] \to [0,1]$, $i = 1,2$, piecewise linear, such that $f_1(x) < x$ for all $0 < x \le 1$ and $f_2(x) > x$ for all $0 \le x < 1$. Moreover, the construction will be so that $f_1(I_n) = I_{n+1}$ and $f_1(J_n) = J_{n+1}$, while $\bigvee_{a_n}^{b_n} f_1 \uparrow \infty$ as $n \to \infty$. The map f_2 will be monotonic on $[0,1]$.

Let (m_n) be a strictly increasing sequence satisfying

$$(2m_n - 1)|I_{n+1}| \to \infty \quad \text{as } n \to \infty. \tag{5.1}$$

Put $\Delta_n := |I_n|/(2m_n - 1)$ for $n \ge 1$. We define f_1 on I_n to be the piecewise linear continuous function that is linear on each subinterval $[a_n + k\Delta_n, a_n + (k+1)\Delta_n]$, $k = 0, 1, \dots, 2m_n - 1$, and

$$f_1(a_n + k\Delta_n) := \begin{cases} a_{n+1} & \text{for } k \text{ even, } 0 \le k \le 2m_n - 2, \\ b_{n+1} & \text{for } k \text{ odd, } 1 \le k \le 2m_n - 1, \end{cases} \tag{5.2}$$

for $n \ge 1$ and $f_1(a_0) := a_1$. Thus, f_1 on I_n has m_n maximal values b_{n+1} and m_n minimal values a_{n+1} and 'oscillates linearly' between these. Note that $f_1(a_n) = a_{n+1}$ and $f_1(b_n) = b_{n+1}$.

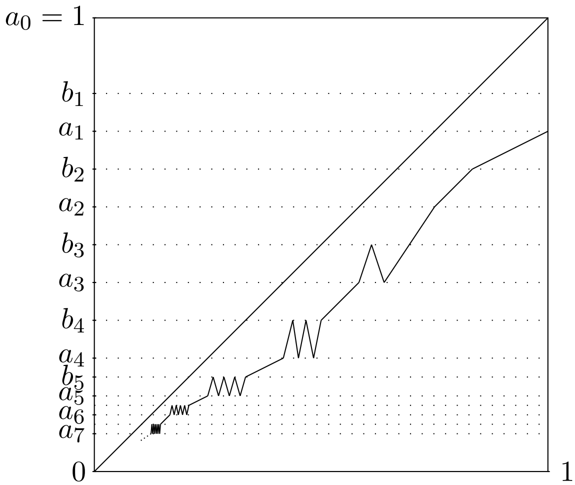

Fig. 2. f_1 with $m_n = n$

Moreover,

$$\operatorname*{ess\,sup}_{x \in I_n} |f_1'(x)| = \frac{b_{n+1} - a_{n+1}}{\Delta_n} = \frac{|I_{n+1}|}{|I_n|}(2m_n - 1). \tag{5.3}$$

Further, $f_1(0) := 0$ and on J_n the function f_1 is linear, increasing and such that $f_1(b_n)=b_{n+1}$ and $f_1(a_{n-1})=a_n$. In particular, $f_1(1)=f_1(a_0)=a_1<1$. Then we have

$$\operatorname*{ess\,sup}_{x\in J_n}|f_1'(x)| = \frac{|J_{n+1}|}{|J_n|}. \tag{5.4}$$

Note that by construction $f_1(x) < x$ for all $0 < x \le 1$. Since

$$\textstyle\bigvee_0^1 f_1 \ge \bigvee_{a_n}^{b_n} f_1 = (2m_n - 1)|I_{n+1}| \to \infty \quad \text{as } n \to \infty,$$

by (5.1), we obtain $\bigvee_0^1 f_1 = \infty$.

We now define f_2 on I_n and J_n to be piecewise linear with slope $0 < S_n^I < 1$ and $0 < S_n^J < 1$ respectively – to be chosen later – and $f_2(1) = 1$. Thus, f_2 is increasing on $(0, 1]$ and $f_2(x) > x$ for $0 < x < 1$. Hence, $\lim_{x\downarrow 0} f_2(x)$ exists. Extend f_2 continuously to 0. Then necessarily $f_2(0) > 0$.

Since f_2 has only countably many points where it is not differentiable and 0 is their only accumulation point, Remark 2.2 gives

$$|f_2|_L = \operatorname*{ess\,sup}_{x\in[0,1]}|f_2'(x)| = \max\Big(\sup_{n\ge1} S_n^I, \sup_{n\ge1} S_n^J\Big) \le 1, \tag{5.5}$$

where $|f_2|_L$ denotes the Lipschitz constant of f_2. In particular, $\bigvee_0^1 f_2 \le 1$.

For any $m \in \mathbb{N}$ one has

$$|f_2 f_1^m|_L = \operatorname*{ess\,sup}_{x\in[0,1]} |f_2'(f_1^m(x)) \cdot f_1'(f_1^{m-1}(x)) \cdots f_1'(x)|, \tag{5.6}$$

because $f_2 f_1^m$ satisfies the hypotheses of Remark 2.2.

We shall estimate the essential supremum in (5.6) by considering the essential suprema on I_n and J_n separately. Note that $f_1(I_n) = I_{n+1}$ for $n \ge 1$. Thus,

$$\begin{aligned} E_n^{I,m} &:= \operatorname*{ess\,sup}_{x\in I_n} |f_2'(f_1^m(x)) \cdot f_1'(f_1^{m-1}(x)) \cdots f_1'(x)| \\ &\le \operatorname*{ess\,sup}_{x\in I_{n+m}} |f_2'(x)| \cdot \prod_{k=0}^{m-1} \operatorname*{ess\,sup}_{y\in I_{n+k}} |f_1'(y)| \\ &= S_{n+m}^I \cdot \prod_{k=0}^{m-1} \frac{|I_{n+k+1}|}{|I_{n+k}|}(2m_{n+k} - 1) = S_{n+m}^I \frac{|I_{n+m}|}{|I_n|} \prod_{k=0}^{m-1} (2m_{n+k} - 1). \end{aligned}$$

Similarly,

$$\begin{aligned} E_n^{J,m} &:= \operatorname*{ess\,sup}_{x\in J_n} |f_2'(f_1^m(x)) \cdot f_1'(f_1^{m-1}(x)) \cdots f_1'(x)| \\ &\le \operatorname*{ess\,sup}_{x\in J_{n+m}} |f_2'(x)| \cdot \prod_{k=0}^{m-1} \sup_{y\in J_{n+k}} |f_1'(y)| = S_{n+m}^J \cdot \prod_{k=0}^{m-1} \frac{|J_{n+k+1}|}{|J_{n+k}|} \\ &= S_{n+m}^J \frac{|J_{n+m}|}{|J_n|}. \end{aligned}$$

We may choose the intervals so that $|J_{n+1}|/|J_n| \le \theta < 1$ for all n. One has $|I_{n+1}|/|I_n| < 1$ by construction. So, if we take

$$S_\ell^I \le \prod_{k=1}^{\ell-1} (2m_k - 1)^{-1}, \tag{5.7}$$

then $E_n^{I,m} \le 1$ for all $n, m \in \mathbb{N}$. Moreover, since we are taking $S_\ell^J < 1$ for all ℓ to ensure that $f_2(x) > x$, we find that $E_n^{J,m} < \theta^m < 1$ for all $n, m \in \mathbb{N}$ too. Thus, we obtain the following.

LEMMA 5.4. *With the choices of* $[a_n, b_n]$ *and* S_n^I *and* S_n^J *as described above, the functions* f_1 *and* f_2 *are continuous on* $[0,1]$ *and such that*

(i) f_1 *has unbounded variation on* $[0,1]$,
(ii) $f_1(x) < x$ *on* $(0,1]$,
(iii) $f_2(x) > x$ *on* $[0,1)$,
(iv) *for any* $m \in \mathbb{N}$, $f_2 f_1^m$ *is Lipschitz on* $[0,1]$ *and* $|f_2 f_1^m|_L \le 1$.

Property (iv), combined with the non-expansiveness of f_2 (see (5.5)), yields the following result.

COROLLARY 5.5. *For any infinite sequence* $\omega = (f_2, f_{i_1}, f_{i_2}, \dots)$ *of* f_i $(i = 1,2)$ *that starts with* f_2, *one has, for all* $n \in \mathbb{N}$,

$$V_n(\omega) := \bigvee_0^1 f_2 f_{i_1} \cdots f_{i_{n-1}} \le 1$$

for $i_j \in \{1,2\}$.

Proof. We use induction on n. For $n = 1$, the result holds (see (5.5)). For $n = 2$, if $i_1 = 2$ then $\bigvee_0^1 f_2 f_2 \le 1$, since f_2, and also $f_2 f_2$, is a monotonic function. If $i_1 = 1$, then the result is simply Lemma 5.4(iv). Suppose the estimate has been proven for all n up to $n = N$ for all $\omega \in \{f_2\} \times \{f_1, f_2\}^{\mathbb{N}}$. Let $\omega = (f_2, f_{i_1}, \dots) \in \{f_2\} \times \{f_1, f_2\}^{\mathbb{N}}$. If the $i_1, \dots, i_{N+1}$ are all 1 or all 2, then the result follows from Lemma 5.4(iv) or non-expansiveness of f_2, respectively. So, let n_0 be the smallest integer such that $i_{n_0} = 1$. Let m be the largest non-negative integer such that $i_n = 1$ for all $n_0 \le n \le n_0 + m$.

Let $T : \{f_1, f_2\}^{\mathbb{N}} \to \{f_1, f_2\}^{\mathbb{N}}$ denote the shift $T(f_{i_1}, f_{i_2}, \dots) := (f_{i_2}, \dots)$. If $n_0 = 1$, then $T^{m+2}\omega$ starts with f_2. Thus (see Remark 2.1),

$$V_{N+1}(\omega) \le |f_2 f_1^{m+1}|_L \cdot V_{N-m-1}(T^{m+2}\omega) \le 1,$$

according to Lemma 5.4(iv) and the induction hypothesis. If $n_0 > 1$, then analogously

$$V_{N+1}(\omega) \le |f_2^{n_0-1}|_L \cdot V_{N-n_0+2}(T^{n_0-1}\omega) \le |f_2|_L^{n_0-1} \cdot V_{N-n_0+2}(T^{n_0-1}\omega).$$

Now, $\omega' := T^{n_0-1}\omega$ is an infinite sequence that starts with f_2. Hence, $V_{N-n_0+2}(\omega') \le 1$ by the induction hypothesis. The estimate follows. ∎

Note that the system (Γ, μ) for $\Gamma = \{f_1, f_2\}$ and $\mu(f_1) = p$, $\mu(f_2) = 1-p$ for some $p \in (0,1)$ satisfies condition (BV). In fact, we have

$$\mu^{\otimes \mathbb{N}}\Big(\Big\{(g_1, g_2, \ldots) \in \Gamma^{\mathbb{N}} : \sup_{n\geq 1} \bigvee\nolimits_0^1 g_1 \cdots g_n < \infty\Big\}\Big) = 1 - p < 1.$$

Indeed, from the above construction it follows that for $(g_1, \ldots, g_n) \in \Gamma^n$ such that $g_1 = f_1$ we have $\bigvee_0^1 g_1 \ldots g_n = \infty$. Finally, since $0, 1 \in \operatorname{supp} \nu$ for every μ-invariant measure ν, and since $f_i((0,1)) \subset (0,1)$, $f_1(1) \in (0,1)$ and $f_2(0) \in (0,1)$, Corollary 1.3 implies the following.

Proposition 5.6. *The system (Γ, μ), where $\Gamma = \{f_1, f_2\}$ and $\mu(f_1) = p$, $\mu(f_2) = 1 - p$ for $p \in (0,1)$, admits a unique μ-ergodic measure on* $(0,1)$.

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