STUDIA MATHEMATICA

Online First version

The rational Gurarii space and its linear isometry group

by

Ondřej Kurka and Maciej Malicki

Abstract. We show that the classes of partial isometries in finite-dimensional polyhedral spaces and in finite-dimensional rational polyhedral spaces do not have the weak amalgamation property. This implies that the linear isometry group of the rational Gurarii space does not have a comeager conjugacy class. Our methods also demonstrate that the classes of finite-dimensional polyhedral spaces and of finite-dimensional rational polyhedral spaces fail to have the Hrushovski property.

1. Introduction. In this paper, we investigate amalgamation of partial isometries in finite-dimensional polyhedral spaces. We prove that the classes $\mathcal{P}_\mathbb{Q}$ and $\mathcal{P}_\mathbb{R}$ of finite-dimensional polyhedral spaces over $\mathbb{Q}$ and over $\mathbb{R}$, respectively, do not satisfy the Hrushovski property, and that the classes $(\mathcal{P}_\mathbb{Q})_1$ and $(\mathcal{P}_\mathbb{R})_1$ of partial isometries in the respective spaces do not have the weak amalgamation property. In particular, this implies that the linear isometry group $\mathrm{Aut}(\mathbf{G}_\mathbb{Q})$ of the Fraïssé limit $\mathbf{G}_\mathbb{Q}$ of $\mathcal{P}_\mathbb{Q}$, called the rational Gurarii space, does not admit a comeager conjugacy class.

Our investigation is motivated by a question posed by Sabok [10], namely whether the linear isometry group $\mathrm{LIso}(\mathbf{G})$ of the Gurarii space $\mathbf{G}$ has the automatic continuity property. Recall that a Polish (i.e., separable and completely metrizable) group has the automatic continuity property if all its homomorphisms into separable topological groups are continuous. This notion was first studied in the setting of C^*-algebras and Banach algebras. For example, Sakai [11], resolving a conjecture of Kaplansky, showed that all derivations on a C^*-algebra are norm-continuous. More recently, Tsankov [13] established the automatic continuity for the unitary group $U(\mathbf{H})$ of the separable Hilbert space $\mathbf{H}$, while Sabok [10] obtained the same for the isometry group $\mathrm{Iso}(\mathbf{U})$ of the Urysohn space $\mathbf{U}$.

2020 *Mathematics Subject Classification*: Primary 54H11; Secondary 46B20, 03E15.

Key words and phrases: rational Gurarii space, linear isometry group, conjugacy classes, weak amalgamation property.

Received 1 May 2025; revised 21 November 2025.

Published online 8 June 2026.

DOI: 10.4064/sm250501-5-12

He developed a general framework (see also [7] for a similar but simpler approach), where properties of automorphism groups of complete metric structures are studied via certain countable substructures, and their automorphism groups. His proof ultimately rests on a result of Solecki [12], who showed that the class $\mathcal{U}_0$ of finite metric spaces with rational distances has the Hrushovski property. Consequently, the isometry group $\mathrm{Aut}(\mathbf{U}_0)$ of the rational Urysohn space $\mathbf{U}_0$, the Fraïssé limit of $\mathcal{U}_0$, has the automatic continuity property.

As a matter of fact, an intermediate property, called ample generics, plays a key role there. Recall that a Polish group G has ample generics if it has a comeager diagonal n-conjugacy class for every $n \in \mathbb{N}$, i.e., there exists a comeager orbit under the action of G on G^n by diagonal conjugation. It was proved by Kechris and Rosendal [6] that a group with ample generics has the automatic continuity property. Let us mention that Tsankov's work on $U(\mathbf{H})$ also requires that certain countable counterpart of the separable Hilbert space, studied by Rosendal [9], has ample generics.

It is known that $\mathrm{LIso}(\mathbf{G})$ does not have ample generics. In fact, it does not even admit a comeager conjugacy class – indeed, [2] shows that no countably infinite group has a generic representation in $\mathrm{LIso}(\mathbf{G})$. Nevertheless, the question of whether this group has the automatic continuity property remains open. Turning to the rational Gurarii space $\mathbf{G}_{\mathbb{Q}}$, essentially nothing has been established in this direction so far. In this paper we prove that $\mathrm{Aut}(\mathbf{G}_{\mathbb{Q}})$ has no comeager conjugacy class, which indicates that Sabok's method may not extend to the Gurarii space. Still, it remains conceivable that a different notion of a rational Gurarii space could be better suited for such an approach.

2. Fraïssé classes. In model theory, a structure M is a set with relations, functions, and constants. The symbols denoting these objects, together with their arities, are referred to as the signature of M.

Let $\mathcal{K}$ be a class of structures in a fixed signature that is closed under isomorphism. Following [6], we say that $\mathcal{K}$ is *hereditary* if it is closed under taking substructures; it has *joint embedding property* (JEP) if any two $A, B \in \mathcal{K}$ can be embedded into a single $C \in \mathcal{K}$; it has *amalgamation property* (AP) if for every $A, B, C \in \mathcal{K}$ and embeddings $\alpha\colon A \to B$ and $\beta\colon A \to C$ there is $D \in \mathcal{K}$ and embeddings $\gamma\colon B \to D$, $\delta\colon C \to D$ such that $\gamma \circ \alpha = \delta \circ \beta$. Finally, $\mathcal{K}$ has *weak amalgamation property* (WAP) if for every $A \in \mathcal{K}$ there is $A' \in \mathcal{K}$ and an embedding $\phi\colon A \to A'$ such that for every $B, C \in \mathcal{K}$ and embeddings $\alpha\colon A' \to B$, $\beta\colon A' \to C$ there is $D \in \mathcal{K}$ and embeddings $\gamma\colon B \to D$, $\delta\colon C \to D$ such that $\gamma \circ \alpha \circ \phi = \delta \circ \beta \circ \phi$. Note that since $\mathcal{K}$ is closed under isomorphism, we can assume that in the definitions above, only extensions ϕ, α, β, instead of all embeddings, are considered.

Let $\mathcal{K}$ be a class of finitely generated structures in a fixed signature that is closed under isomorphism, and countable up to isomorphism. We say that $\mathcal{K}$ is a *Fraïssé class* if it is hereditary, has JEP and AP. By the classical theorem due to Fraïssé [3], if $\mathcal{K}$ is a Fraïssé class, then there exists a unique up to isomorphism countable structure M such that $\mathcal{K}$ is the class of finitely generated structures embeddable in M, and M has the extension property, i.e., for any $A, B \in \mathcal{K}$ with $A \subseteq B$, and embedding $\alpha : A \to M$, there is an embedding $\beta : B \to M$ that extends α. We call this M the *Fraïssé limit* of $\mathcal{K}$.

A class $\mathcal{K}$ of structures has the *n-Hrushovski property* if for any $A \in \mathcal{K}$, and any n-tuple $f_1, \dots, f_n$ of partial automorphisms of A, there exists $B \in \mathcal{K}$ such that $A \subseteq B$, and every f_i can be extended to an automorphism of B. It has the *Hrushovski property* if it has the n-Hrushovski property for every $n \in \mathbb{N}$.

For a class $\mathcal{K}$ of structures, let $\mathcal{K}_1$ be the class of all partial automorphisms of elements from $\mathcal{K}$, i.e., pairs $(A, f : B \to C)$ such that $A, B, C \in \mathcal{K}$, $B, C \subseteq A$, and f is an isomorphism between B and C. Then $\alpha : A \to A'$ is an embedding of $(A, f : B \to C)$ into $(A', f' : B' \to C')$ if it embeds A into A' and $\alpha \circ f \subseteq f' \circ \alpha$. In the same way, we define classes $\mathcal{K}_n$ of elements of the form $(A, f_1 : B_1 \to C_1, \dots, f_n : B_n \to C_n)$, where $A \in \mathcal{K}$ and f_i are partial automorphisms of A. JEP, AP and WAP can be defined for $\mathcal{K}_n$ analogously to $\mathcal{K}$.

Recall that a topological group is *Polish* if it is separable and completely metrizable. A Polish group G has a dense (non-meager, etc.) diagonal n-conjugacy class if there is a dense (non-meager, etc.) orbit in the action of G on G^n defined by $g.(g_1, \dots, g_n) = (gg_1g^{-1}, \dots, gg_ng^{-1})$, $g, g_1, \dots, g_n \in G$. It has *ample generics* if it has a comeager diagonal n-conjugacy class for every $n \in \mathbb{N}$.

It is well known that the automorphism group $\mathrm{Aut}(M)$ of a countable structure M, with the topology inherited from M^M, where M is regarded as a discrete space, is a Polish group. The following characterization was proved for Fraïssé limits of classes $\mathcal{K}$ of finite structures in [6], and generalized to classes $\mathcal{K}$ of finitely generated structures (and any countable structures M) in [8]:

THEOREM 2.1. *Let M be the Fraïssé limit of a Fraïssé class $\mathcal{K}$, and let $n \geq 1$. The group* $\mathrm{Aut}(M)$ *has a comeager diagonal n-conjugacy class iff $\mathcal{K}_n$ has JEP and WAP.*

3. Polyhedral spaces. We now recall the notion of a polyhedral space, considered here both in the rational and real settings. First, let us formulate the following fact that is well known (at least in the real setting).

In what follows, in a vector space over $\mathbb{Q}$, the notions of a convex hull, an absolutely convex hull and an extreme point are defined in the same way as in a real vector space, with the difference that rational numbers are used instead of real numbers.

FACT 3.1. *Let X be a finite-dimensional normed space over $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$. Then the following assertions are equivalent:*

(i) *The closed unit ball B_X of X is the absolutely convex hull of a finite number of points in X.*
(ii) *There are functionals $u_1^*, \ldots, u_n^*$ in the dual space $X^* = \mathcal{L}(X, \mathbb{F})$ such that the norm of X can be expressed by*

$$\|x\| = \max\{|u_i^*(x)| : 1 \leq i \leq n\}. \tag{$*$}$$

Proof. We consider this fact in the setting $\mathbb{F} = \mathbb{R}$ to be well known and understood, so we concentrate only on the setting $\mathbb{F} = \mathbb{Q}$. We can assume that $X = \mathbb{Q}^d$ for some d. By $e_1, \ldots, e_d$ we denote the canonical basis of $\mathbb{Q}^d$.

(i)⇒(ii): Let the unit ball of $X = \mathbb{Q}^d$ be the absolutely convex hull of points $a_1, a_2, \ldots, a_m$ in $\mathbb{Q}^d$. Let B denote the absolutely convex hull in $\mathbb{R}^d$ of the same set of points. Since B contains εe_i for a sufficiently small $\varepsilon > 0$ (we can take $1/\|e_i\|_X$) for $1 \leq i \leq d$, the point $0 \in \mathbb{R}^d$ belongs to the interior of B. Hence B is the unit ball of a norm $|||\cdot|||$ on $\mathbb{R}^d$. Let us verify that $|||x||| = \|x\|_X$ for every $x \in \mathbb{Q}^d$, or equivalently $B_X = B \cap \mathbb{Q}^d$. We need to check that if a point $x \in \mathbb{Q}^d$ can be expressed as a convex combination of $\pm a_1, \ldots, \pm a_m$, then it can be expressed as a convex combination of $\pm a_1, \ldots, \pm a_m$ with rational coefficients. Note that x is a convex combination of at most $d + 1$ points $\pm a_k$ that form vertices of a simplex. Then the coefficients are uniquely determined, and since x and all a_k's belong to $\mathbb{Q}^d$, the coefficients must be rational.

We know from the real case that there are functionals $u_1^*, \ldots, u_n^*$ such that the norm of X can be expressed by $(*)$. However, we need to show that these functionals can be chosen to be $\mathbb{Q}$-valued on $\mathbb{Q}^d$, or equivalently $u_j^*(e_i) \in \mathbb{Q}$ for $1 \leq j \leq n, 1 \leq i \leq d$. For this purpose, let $u_1^*, \ldots, u_n^*$ be the extreme points of the dual unit ball of the space $(\mathbb{R}^d, |||\cdot|||)$. Equivalently, $u_1^*, \ldots, u_n^*$ are those functionals that are equal to 1 on some $(d-1)$-dimensional polytope contained in the unit sphere. Thus, each u_j^* is equal to 1 for d linearly independent points $\pm a_k$. These d points have rational coordinates and uniquely determine the numbers $u_j^*(e_i)$, so these numbers must be rational.

(ii)⇒(i): Let $u_1^*, \ldots, u_n^*$ in the dual space $X^* = \mathcal{L}(\mathbb{Q}^d, \mathbb{Q})$ be such that the norm of X can be expressed by $(*)$. Let us extend each u_j^* to $\mathbb{R}^d$ in the obvious way and equip $\mathbb{R}^d$ with $|||\cdot|||$ given by the same formula $(*)$. This is clearly a pseudonorm, and we check first that it is a norm. Given $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ different from 0, we need to verify that $|||x||| > 0$.

Assume the opposite, i.e., $|||x||| = 0$, and so $u_j^*(x) = 0$ for $1 \leq j \leq n$. Let U denote the matrix

$$U = (u_j^*(e_i))_{1\leq j\leq n,\, 1\leq i\leq d}.$$

Considering x as a one-column matrix, we obtain $Ux = 0$, and so U has rank less than d when considered as a matrix over $\mathbb{R}$. At the same time, U is a matrix consisting of rational numbers, and it has the same rank if considered over $\mathbb{Q}$. This is because the rank of U is equal to the size of the biggest square submatrix with non-zero determinant. Hence, there is $\widetilde{x} \in \mathbb{Q}^d$ different from 0 such that $U\widetilde{x} = 0$. But then $\|\widetilde{x}\|_X = 0$, which is not possible. Thus $|||\cdot|||$ is a norm indeed.

Let $a_1, \dots, a_m$ be the extreme points of the unit ball of $(\mathbb{R}^d, |||\cdot|||)$ (we know from the real case that there are only finitely many of them). Each a_k belongs to $\mathbb{Q}^d$, since $u_j^*(a_k) = \pm 1$ for d linearly independent functionals u_j^*, while these functionals are $\mathbb{Q}$-valued on $\mathbb{Q}^d$ and together with the values ± 1 uniquely determine a_k.

To show that the points $a_1, \dots, a_m$ witness that (i) holds, we need to check that if a point $x \in \mathbb{Q}^d$ can be expressed as a convex combination of $a_1, \dots, a_m$, then it can be expressed as a convex combination of $a_1, \dots, a_m$ with rational coefficients. This can be done in the same way as in the proof of the implication (i)⇒(ii). ∎

The spaces that satisfy the equivalent conditions (i) and (ii) above are called *polyhedral*. For a polyhedral space X, the extreme points of the unit ball B_X are called *vertices* of B_X. In other words, the set of vertices of B_X is the smallest set whose convex hull is B_X.

Let us note that if the dual space X^* is polyhedral and $u_1^*, \dots, u_n^*$ is the enumeration of vertices of B_{X^*}, then $(*)$ holds for each $x \in X$.

FACT 3.2.

(a) *If X is a polyhedral space, then its dual space $X^* = \mathcal{L}(X, \mathbb{F})$ is polyhedral.*
(b) *If Y is a subspace of a polyhedral space X, then both Y and the quotient X/Y are polyhedral.*
(c) *If X and Y are polyhedral spaces, then their ℓ_∞-sum and ℓ_1-sum, that is, the space $X \oplus Y$ with the norms $\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$ and $\|(x, y)\|_1 = \|x\| + \|y\|$, are polyhedral spaces.*
(d) *If $f : X \to X$ is a surjective linear isometry on a polyhedral space X, then there is $n \in \mathbb{N}$ such that f^n is the identity on X.*

Proof. (a) We know from the above comment that if X^* is polyhedral, then X is polyhedral. Thus, if X^{**} is polyhedral, then X^* is polyhedral. It remains to note that X^{**} is isometric to X.

(b) It is easy to see that Y is polyhedral. The dual of X/Y is isometric to the annihilator $Y^\perp = \{u^* \in X^* : (\forall y \in Y)(u^*(y) = 0)\}$, which, being a subspace of X^*, is polyhedral. It follows that X/Y is polyhedral.

(c) If the norm on X is given by $\|x\| = \max\{|u_i^*(x)| : 1 \le i \le n\}$ and the norm on Y is given by $\|y\| = \max\{|v_j^*(y)| : 1 \le j \le m\}$, then

$$\|(x,y)\|_\infty = \max(\{|u_i^*(x)| : 1 \le i \le n\} \cup \{|v_j^*(y)| : 1 \le j \le m\}),$$

so the ℓ_∞-sum of X and Y is polyhedral. Concerning the ℓ_1-sum of X and Y, we just point out that its dual is isometric to the ℓ_∞-sum of X^* and Y^*.

(d) Considering the restriction of f to the vertices of B_X, we obtain a permutation π. It is sufficient to note that π^n is the identity permutation for some n. ∎

4. Two classes of linear spaces. Let $\mathbb{F}$ be a field, and let $L_\mathbb{F}$ be the signature consisting of functional symbols $f_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n)$, $\alpha_i \in \mathbb{F}$, and a constant symbol $\mathbf{0}$. Let $N_\mathbb{F}$ be the signature obtained by expanding $L_\mathbb{F}$ by unary relational symbols $\|\cdot\|_\alpha$, $\alpha \in \mathbb{F}$. Every linear space V over $\mathbb{F}$ can be regarded as a structure in the signature $L_\mathbb{F}$ by defining functions $f_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n$, and $\mathbf{0}$ as the zero vector. Analogously, a linear space over $\mathbb{F}$ with a norm $\|\ \|$ taking values in $\mathbb{F}$ can be regarded as a structure in the signature $N_\mathbb{F}$ by putting $\|x\|_\alpha$ iff $\|x\| = \alpha$.

Let $\mathcal{L}_\mathbb{Q}$ (resp. $\mathcal{L}_\mathbb{R}$) be the class of finite-dimensional linear spaces over $\mathbb{Q}$ (resp. $\mathbb{R}$) (and linear injections as embeddings). Let $\mathcal{P}_\mathbb{Q}$ (resp. $\mathcal{P}_\mathbb{R}$) be the class of finite-dimensional linear spaces over $\mathbb{Q}$ (resp. $\mathbb{R}$), with polyhedral norms taking values in $\mathbb{Q}$ (resp. $\mathbb{R}$) (and linear isometric injections as embeddings). The following is well known and easy to verify:

Proposition 4.1. *$\mathcal{L}_\mathbb{Q}$, $\mathcal{L}_\mathbb{R}$, $\mathcal{P}_\mathbb{Q}$, and $\mathcal{P}_\mathbb{R}$ have JEP and WAP. In particular, because $\mathcal{L}_\mathbb{Q}$ and $\mathcal{P}_\mathbb{Q}$ are countable up to isomorphism, these classes are Fraïssé.*

The Fraïssé limit $\mathbf{L}_\mathbb{Q}$ of $\mathcal{L}_\mathbb{Q}$ is an infinite-dimensional countable linear space over $\mathbb{Q}$. The Fraïssé limit $\mathbf{G}_\mathbb{Q}$ of $\mathcal{P}_\mathbb{Q}$ is a normed linear space over $\mathbb{Q}$ that we will call the *rational Gurarii space*. Clearly, $\mathrm{Aut}(\mathbf{L}_\mathbb{Q})$ is the linear isomorphism group of $\mathbf{L}_\mathbb{Q}$, and $\mathrm{Aut}(\mathbf{G}_\mathbb{Q})$ is the linear isometry group of $\mathbf{G}_\mathbb{Q}$.

Recall that the *Gurarii space* $\mathbf{G}$ is the unique separable Banach space having the property that for any $\epsilon > 0$, finite-dimensional Banach spaces $E \subseteq F$, and isometric embedding $\phi : E \to \mathbf{G}$, there is a linear embedding $\Phi : F \to \mathbf{G}$ extending ϕ such that in addition $(1-\epsilon)\|x\| < \|\Phi(x)\| < (1+\epsilon)\|x\|$ for all $x \in F \setminus \{0\}$. It can also be characterized (see, e.g., [4, Theorem 2.7]) as the unique separable Banach space such that for any $\epsilon > 0$, finite-dimensional Banach spaces $E \subseteq F$, and isometric embedding $\phi : E \to \mathbf{G}$, there is an isometric embedding $\Phi : F \to \mathbf{G}$ such that $\|\Phi \restriction E - \phi\| < \epsilon$. Note that finite-dimensional rational Banach spaces, as defined in [5, Section 2.2], are exactly completions of spaces from $\mathcal{P}_\mathbb{Q}$. Moreover, $\mathbf{G}$ is the completion of $\mathbf{G}_\mathbb{Q}$. We sketch the proof of this folklore result for the sake of comprehensiveness.

PROPOSITION 4.2. *The completion of the rational Gurarii space $\mathbf{G}_{\mathbb{Q}}$ is the Gurarii space $\mathbf{G}$.*

Proof. Let $\mathbf{H}$ be the completion of $\mathbf{G}_{\mathbb{Q}}$, and let $\| \ \|$ denote the norm on $\mathbf{G}$. We start with the following observation.

CLAIM. *Fix $\epsilon > 0$, a set $A = \{a_0, \ldots, a_k\}$ of unit vectors in $\mathbf{G}_{\mathbb{Q}}$, $f : A \to \mathbf{G}$ that gives rise to an isometry between* span A *(in $\mathbf{H}$) and* span $f[A]$, *and a unit vector $y \in \mathbf{G}$. There are unit vectors $y_0', \ldots, y_k', y' \in \mathbf{G}$ such that*

- $\|f(a_i) - y_i'\| < \epsilon$, $\|y - y'\| < \epsilon$,
- $a_i \mapsto y_i'$ *gives rise to an isometric isomorphism of* span A *and* span $\{y_0', \ldots, y_k'\}$,
- *the linear space over $\mathbb{Q}$ spanned by $\{y_0', \ldots, y_k', y'\}$ is in $\mathcal{P}_{\mathbb{Q}}$.*

To show the claim, note that Lemma 2.4 in [5] and its proof show that there exists a norm $\| \ \|_0'$ on $B_0' = \operatorname{span}\{f(a_0), \ldots, f(a_k), y\}$ that is ϵ-equivalent to $\| \ \|$ on B_0' (i.e., $(1-\epsilon)\|x\| < \|x\|_0' < (1+\epsilon)\|x\|$ for $x \in B_0'$, $x \neq \mathbf{0}$), equal to $\| \ \|$ on $\operatorname{span}\{f(a_0), \ldots, f(a_k)\}$, equal to 1 at y, and that turns B_0' into a rational space in the sense that the linear space over $\mathbb{Q}$ spanned by $\{f(a_0), \ldots, f(a_k), y\}$ equipped with $\| \ \|_0'$ is in $\mathcal{P}_{\mathbb{Q}}$. It is easy to observe (see, e.g., [1, Lemma 1.6]) that then there exists a seminorm $\| \ \|'$ on $B' = (B_0', \| \ \|) \oplus (B_0', \| \ \|_0')$ that extends $\| \ \|$ and $\| \ \|_0'$, and moreover $\|(f(a_i), \mathbf{0}) - (\mathbf{0}, f(a_i))\|' < \epsilon$, $\|(y, \mathbf{0}) - (\mathbf{0}, y)\|' < \epsilon$. By the above characterization of the Gurarii space, we can assume that actually $B'/\| \ \|'$ is a subspace of $\mathbf{G}$ such that $\|f(a_i) - (\mathbf{0}, f(a_i))\| < \epsilon$, $\|y - (\mathbf{0}, y)\| < \epsilon$. In other words, $y_i' = (\mathbf{0}, f(a_i))$, $y' = (\mathbf{0}, y)$ witness that the claim holds.

Now, to finish the proof of the proposition, we will construct an isometric injection $f : \mathbf{G}_{\mathbb{Q}} \to \mathbf{G}$ whose image is dense in $\mathbf{G}$, and $f(\mathbf{0}) = \mathbf{0}$. Let $X = \{x_i\}$ be all the unit vectors in $\mathbf{G}_{\mathbb{Q}}$, and let $\{y_i\}$ be a dense subset of $\mathbf{G} \setminus \{0\}$. Put $f_0(\mathbf{0}) = \mathbf{0}$, and suppose that injections $f_i : X_i \to \mathbf{G}$, $i \leq n$, have been defined so that, for $1 \leq i \leq n$,

- $X_i \subseteq X$ is finite and $X_{i-1} \subseteq X_i$,
- f_i gives rise to an isometry between $\operatorname{span} X_i$ and $\operatorname{span} f_i[X_i]$,
- $\|f_i(x) - f_{i-1}(x)\| < 2^{-i}$ for $x \in X_{i-1}$,
- $x_i \in \operatorname{span} X_{2i}$ if $2i \leq n$,
- $\|y - y_i\| < 2^{-i}$ for some $y \in \operatorname{span} f_{2i+1}[X_{2i+1}]$ if $2i+1 \leq n$.

For odd n, we use the above characterization of the Gurarii space to construct a required f_{n+1}. For even n, suppose that $y_{n/2} \notin \operatorname{span} f_n[X_n]$. We fix $y_0', \ldots, y_k'$, $y' \in \mathbf{G}$ as in the claim for a sufficiently small ϵ, $a_0, \ldots, a_k$ enumerating X_n, $f = f_n$ and $y = y_{n/2}/\|y_{n/2}\|$. Then, applying amalgamation in $\mathcal{P}_{\mathbb{Q}}$, we find $l \in \mathbb{N}$ such that f_{n+1} defined by $f_{n+1}(a_i) = y_i'$, $f_{n+1}(x_l) = y'$ witnesses that $\operatorname{span} X_n \cup \{x_l\}$ (in $\mathbf{H}$) is isometric to $\operatorname{span} f_{n+1}[X_n] \cup \{y'\}$. Finally, we define $f(x_n) = \lim_m f_m(x_n)$. ■

Now we point out that, similarly to $\mathrm{Aut}(\mathbf{H}_{\mathbb{Q}})$, where $\mathbf{H}_{\mathbb{Q}}$ is a countable counterpart of the separable Hilbert space studied in [9], $\mathrm{Aut}(\mathbf{L}_{\mathbb{Q}})$ has ample generics.

PROPOSITION 4.3. *The classes $\mathcal{L}_{\mathbb{Q}}$ and $\mathcal{L}_{\mathbb{R}}$ have the Hrushovski property. In particular,* $\mathrm{Aut}(\mathbf{L}_{\mathbb{Q}})$ *has ample generics.*

Proof. We show that $\mathcal{L}_{\mathbb{Q}}$ has the n-Hrushovski property, and that $(\mathcal{L}_{\mathbb{Q}})_n$ has WAP, for $n = 1$. The proofs for $n > 1$ and for $\mathcal{L}_{\mathbb{R}}$ are analogous. Fix $(X, f : E \to F) \in (\mathcal{L}_{\mathbb{Q}})_1$. Without loss of generality, we can assume that $E \cup F$ generates X. Fix a basis $x_0, \ldots, x_n$ in X such that, for some $j, k \leq n$, $E = \mathrm{span}\,\{x_0, \ldots, x_k\}$, and $F = \mathrm{span}\,\{x_j, \ldots, x_n\}$ (i.e., $E \cap F = \mathrm{span}\,\{x_j, \ldots, x_k\}$). Since the range of f is F, we have $n = j + k$. We extend f by putting $x_i = f(x_{n-i})$ for $i < j$.

It follows that there is a subfamily $\mathcal{F} \subseteq (\mathcal{L}_{\mathbb{Q}})_1$ that has AP, and is cofinal in $(\mathcal{L}_{\mathbb{Q}})_1$, i.e., for every $(X, f : E \to F) \in (\mathcal{L}_{\mathbb{Q}})_1$ there is $(X', f' : E' \to F') \in \mathcal{F}$ that embeds $(X, f : E \to F)$. In particular, $(\mathcal{L}_{\mathbb{Q}})_1$ has weak amalgamation. Indeed, for $(A, f : B \to C) \in (\mathcal{L}_{\mathbb{Q}})_1$, fix $(A_0, f_0 : A_0 \to A_0) \in (\mathcal{L}_{\mathbb{Q}})_1$ such that f_0 extends f. As A_0 is obviously invariant under every extension of f_0, for any $(A_1, f_1 : B_1 \to C_1), (A_2, f_2 : B_2 \to C_2) \in (\mathcal{L}_{\mathbb{Q}})_1$ such that f_1, f_2 extend f_0, we can assume that $A_1 \cap A_2 = A_0$. Write $A_1 = A_0 \oplus A_1'$, $f_1 = (f_0, f_1')$, $A_2 = A_0 \oplus A_2'$, $f_2 = (f_0, f_2')$. It is immediate that $(A_0 \oplus A_1' \oplus A_2', (f_0, f_1', f_2'))$ is the required amalgam. Note that JEP amounts to amalgamation over the trivial space, so $(\mathcal{L}_{\mathbb{Q}})_1$ has JEP as well. Using Theorem 2.1, we conclude that $\mathrm{Aut}(\mathbf{L}_{\mathbb{Q}})$ has a comeager conjugacy class.

In the same way, we can show that $(\mathcal{L}_{\mathbb{Q}})_n$ has JEP and WAP for every $n \in \mathbb{N}$, i.e., $\mathrm{Aut}(\mathbf{L}_{\mathbb{Q}})$ has ample generics. ∎

THEOREM 4.4. *The class $\mathcal{P}_{\mathbb{Q}}$ does not have the 1-Hrushovski property, and $(\mathcal{P}_{\mathbb{Q}})_1$ does not have WAP. In particular,* $\mathrm{Aut}(\mathbf{G}_{\mathbb{Q}})$ *does not have a comeager conjugacy class.*

Proof. The major part of the proof that $(\mathcal{P}_{\mathbb{Q}})_1$ does not have WAP will be accomplished in the next section, where we prove Lemma 5.1. According to this lemma, it is sufficient to find an example of $\mathcal{O} = (C, f : A \to B)$ for which there is no extension D of C that admits an extension $g : D \to D$ of f that is a surjective linear isometry. Such an example demonstrates at the same time that $\mathcal{P}_{\mathbb{Q}}$ does not have the 1-Hrushovski property. Although the following example is quite simple, perhaps it is not clear where the method comes from. A kind of clarification will be provided later in Proposition 4.5.

Let $C = \mathbb{Q}^2$ with the norm $\|(x, y)\| = |x| + |y|$. Let $A = \{(t, 0) : t \in \mathbb{Q}\}$, $B = \{(t, t) : t \in \mathbb{Q}\}$ and $f(t, 0) = \left(\frac{1}{2}t, \frac{1}{2}t\right), t \in \mathbb{Q}$. Assume that suitable D and g exist. Since the norm of D is polyhedral, there is $n \in \mathbb{N}$ such that

$g^n = \mathrm{id}_D$. Let us consider $a_i = (2^{-i}, 0)$ for $0 \le i \le n-1$. Then

$$\begin{aligned}\|a_0 - f(a_{n-1})\| &= \left\|(1,0) - \left(\tfrac{1}{2}2^{-(n-1)}, \tfrac{1}{2}2^{-(n-1)}\right)\right\| = |1-2^{-n}| + 2^{-n} = 1,\\ \|a_{i+1} - f(a_i)\| &= \left\|(2^{-(i+1)}, 0) - \left(\tfrac{1}{2}2^{-i}, \tfrac{1}{2}2^{-i}\right)\right\|\\ &= |2^{-(i+1)} - 2^{-(i+1)}| + 2^{-(i+1)} = 2^{-(i+1)},\end{aligned}$$

and so

$$\begin{aligned}1 &= \|a_0 - f(a_{n-1})\| = \|g^n(a_0) - g(a_{n-1})\|\\ &= \left\|\sum_{i=0}^{n-2} g^{n-i}(a_i) - g^{n-(i+1)}(a_{i+1})\right\|\\ &\le \sum_{i=0}^{n-2} \|g^{n-i}(a_i) - g^{n-(i+1)}(a_{i+1})\| = \sum_{i=0}^{n-2} \|g(a_i) - a_{i+1}\|\\ &= \sum_{i=0}^{n-2} \|f(a_i) - a_{i+1}\| = \sum_{i=0}^{n-2} 2^{-(i+1)} = 1 - 2^{-(n-1)},\end{aligned}$$

a contradiction. ∎

In the next proposition, we provide more aspects of the results stated in Theorem 4.4. Here, we can consider both fields $\mathbb{F} = \mathbb{Q}$ and $\mathbb{F} = \mathbb{R}$.

PROPOSITION 4.5. *For every $\mathcal{O} = (C, f : A \to B) \in (\mathcal{P}_{\mathbb{F}})_1$, the following assertions are equivalent:*

(1) *There exist $\mathcal{O}_1 \in (\mathcal{P}_{\mathbb{F}})_1$ and an embedding $i : \mathcal{O} \to \mathcal{O}_1$ such that for any $\mathcal{O}_2, \mathcal{O}_3 \in (\mathcal{P}_{\mathbb{F}})_1$ and embeddings $j : \mathcal{O}_1 \to \mathcal{O}_2$, $k : \mathcal{O}_1 \to \mathcal{O}_3$, there are $\mathcal{O}_4 \in (\mathcal{P}_{\mathbb{F}})_1$ and embeddings $\widetilde{j} : \mathcal{O}_2 \to \mathcal{O}_4$, $\widetilde{k} : \mathcal{O}_3 \to \mathcal{O}_4$ such that*

$$\widetilde{j} \circ j \circ i = \widetilde{k} \circ k \circ i.$$

(2) *There exists $\mathcal{O}' \in (\mathcal{P}_{\mathbb{F}})_1$ of the form $\mathcal{O}' = (E, g : E \to E)$ that admits an embedding $i' : \mathcal{O} \to \mathcal{O}'$.*
(3) *There exists $n \in \mathbb{N}$ such that for any $a_0, \ldots, a_{n-1} \in A$, we have*

$$\|a_0 - f(a_{n-1})\| \le \sum_{i=0}^{n-2} \|a_{i+1} - f(a_i)\|.$$

Proof. (1)⇒(2): This is nothing but Lemma 5.1.

(2)⇒(1): We put $\mathcal{O}_1 = \mathcal{O}' = (E, g : E \to E)$ and $i = i' : C \to E$. Let $\mathcal{O}_2 = (C_2, f_2 : A_2 \to B_2)$, $\mathcal{O}_3 = (C_3, f_3 : A_3 \to B_3) \in (\mathcal{P}_{\mathbb{F}})_1$ and embeddings $j : \mathcal{O}_1 \to \mathcal{O}_2$, $k : \mathcal{O}_1 \to \mathcal{O}_3$ be given. Considering the norm

$\|(c_2, c_3)\| = \|c_2\| + \|c_3\|$ on $C_2 \oplus C_3$, we define

$$C_4 = (C_2 \oplus C_3)/N, \quad \text{where} \quad N = \{(j(e), -k(e)) : e \in E\},$$
$$\widetilde{j} : C_2 \to C_4, \quad \widetilde{j}(c_2) = (c_2, 0) + N, \quad c_2 \in C_2,$$
$$\widetilde{k} : C_3 \to C_4, \quad \widetilde{k}(c_3) = (0, c_3) + N, \quad c_3 \in C_3,$$
$$A_4 = \operatorname{span}(\widetilde{j}(A_2) \cup \widetilde{k}(A_3)), \quad B_4 = \operatorname{span}(\widetilde{j}(B_2) \cup \widetilde{k}(B_3)).$$

To show that

$$f_4(\widetilde{j}(a_2)) = \widetilde{j}(f_2(a_2)),\ a_2 \in A_2, \qquad f_4(\widetilde{k}(a_3)) = \widetilde{k}(f_3(a_3)),\ a_3 \in A_3,$$

well-defines a linear mapping f_4 on A_4, we need to check that

$$\widetilde{j}(a_2) = \widetilde{k}(a_3) \implies \widetilde{j}(f_2(a_2)) = \widetilde{k}(f_3(a_3)),$$

or equivalently $(a_2, -a_3) \in N \Rightarrow (f_2(a_2), -f_3(a_3)) \in N$ for $a_2 \in A_2, a_3 \in A_3$. For $(a_2, -a_3) \in N$, there is $e \in E$ such that $a_2 = j(e)$ and $a_3 = k(e)$, and so $(f_2(a_2), -f_3(a_3)) = (f_2(j(e)), -f_3(k(e))) = (j(g(e)), -k(g(e))) \in N$. Thus, f_4 is indeed well defined. Let us show that it is an isometry of A_4 onto B_4. Clearly, $f_4(A_4) \subseteq B_4$, and regarding the opposite inclusion $B_4 \subseteq f_4(A_4)$, for $b_4 \in B_4$ of the form $b_4 = \widetilde{j}(b_2) + \widetilde{k}(b_3)$, we can put $a_4 = \widetilde{j}(f_2^{-1}(b_2)) + \widetilde{k}(f_3^{-1}(b_3))$ and obtain $f_4(a_4) = b_4$. Further, let us pick $a_4 \in A_4$, so it is of the form

$$a_4 = \widetilde{j}(a_2) + \widetilde{k}(a_3) = (a_2, a_3) + N,$$

while

$$f_4(a_4) = \widetilde{j}(f_2(a_2)) + \widetilde{k}(f_3(a_3)) = (f_2(a_2), f_3(a_3)) + N.$$

Since g is a surjective linear isometry on E, it follows that

$$\begin{aligned}
\|f_4(a_4)\| &= \inf_{e \in E} \big(\|f_2(a_2) + j(e)\| + \|f_3(a_3) - k(e)\|\big) \\
&= \inf_{e \in E} \big(\|f_2(a_2) + j(g(e))\| + \|f_3(a_3) - k(g(e))\|\big) \\
&= \inf_{e \in E} \big(\|f_2(a_2) + f_2(j(e))\| + \|f_3(a_3) - f_3(k(e))\|\big) \\
&= \inf_{e \in E} \big(\|a_2 + j(e)\| + \|a_3 - k(e)\|\big) = \|a_4\|.
\end{aligned}$$

Now, let us check that the choice $\mathcal{O}_4 = (C_4, f_4 : A_4 \to B_4)$ works. For $c_2 \in C_2$, we have

$$\|\widetilde{j}(c_2)\| = \|(c_2, 0) + N\| = \inf_{e \in E} (\|c_2 + j(e)\| + \|k(e)\|) = \|c_2\|.$$

Analogously, $\|\widetilde{k}(c_3)\| = \|c_3\|$ for $c_3 \in C_3$, and it follows from the definition of f_4 that both $\widetilde{j}$ and $\widetilde{k}$ are embeddings of $\mathcal{O}_2$, resp. $\mathcal{O}_3$, into $\mathcal{O}_4$. Finally, for $e \in E$,

$$(\widetilde{j} \circ j)(e) = (j(e), 0) + N = (0, k(e)) + N = (\widetilde{k} \circ k)(e),$$

in particular, $\widetilde{j} \circ j \circ i = \widetilde{k} \circ k \circ i$.

(2)⇒(3): We can suppose that E is an extension of C and g is an extension of f. Since the norm of E is polyhedral, there is $n \in \mathbb{N}$ such that $g^n = \mathrm{id}_E$. Given $a_0, \dots, a_{n-1} \in A$, we denote $x_i = g^{n-i}(a_i)$, obtaining

$$\begin{aligned}\|a_0 - f(a_{n-1})\| &= \|g^n(a_0) - g(a_{n-1})\| \\ &= \|x_0 - x_{n-1}\| = \Big\|\sum_{i=0}^{n-2}(x_i - x_{i+1})\Big\| \\ &\le \sum_{i=0}^{n-2}\|x_i - x_{i+1}\| = \sum_{i=0}^{n-2}\|g^{n-i}(a_i) - g^{n-i-1}(a_{i+1})\| \\ &= \sum_{i=0}^{n-2}\|g(a_i) - a_{i+1}\| = \sum_{i=0}^{n-2}\|f(a_i) - a_{i+1}\|.\end{aligned}$$

(3)⇒(2): Let E_0 be the space C^n with the norm $\|(c_0, \dots, c_{n-1})\| = \sum_{i=0}^{n-1}\|c_i\|$, and with the isometry

$$g_0 : (c_0, \dots, c_{n-1}) \mapsto (c_1, \dots, c_{n-1}, c_0).$$

Let N be the subspace of E_0 generated by all vectors of the form

$$(0, \dots, 0, a, -f(a), 0, \dots, 0) \quad \text{and} \quad (-f(a), 0, \dots, 0, a),$$

where $a \in A$.

Let E be the quotient E_0/N. Let us denote

$$\widetilde{c} = (c, 0, \dots, 0) + N, \quad c \in C.$$

Since $g_0(N) = N$, the mapping

$$g(x + N) = g_0(x) + N, \quad x + N \in E,$$

is well-defined, and moreover it is an isometry. For $a \in A$, we have

$$\begin{aligned}g(\widetilde{a}) &= g((a, 0, \dots, 0) + N) = g_0((a, 0, \dots, 0)) + N \\ &= (0, \dots, 0, a) + N = (f(a), 0, \dots, 0) + N = \widetilde{f(a)}.\end{aligned}$$

Hence, it remains to show that $c \mapsto \widetilde{c}$ is an isometry. For $c \in C$, we have

$$\|\widetilde{c}\| = \inf_{u \in N}\|(c, 0, \dots, 0) + u\|,$$

thus $\|\widetilde{c}\| \le \|c\|$ by the choice $u = 0$. On the other hand, a general $u \in N$ is of the form

$$u = (a_0 - f(a_{n-1}), a_1 - f(a_0), \dots, a_{n-1} - f(a_{n-2})),$$

for which, by the inequality from (3),

$$\begin{aligned}\|(c, 0, \dots, 0) + u\| &= \|c + a_0 - f(a_{n-1})\| + \sum_{i=0}^{n-2}\|a_{i+1} - f(a_i)\| \\ &\ge \|c + a_0 - f(a_{n-1})\| + \|a_0 - f(a_{n-1})\| \ge \|c\|,\end{aligned}$$

yielding $\|\widetilde{c}\| \geq \|c\|$. We obtain $\|\widetilde{c}\| = \|c\|$ for $c \in C$, finishing the proof of (2). ∎

To close this section, we show that the two-dimensional example from the proof of Theorem 4.4 actually disproves the 1-Hrushovski property also for the class of general finite-dimensional normed spaces.

Proposition 4.6. *Let X be a finite-dimensional normed linear space containing points e_1, e_2 such that $\|ae_1 + be_2\| = |a| + |b|$ for all $a, b \in \mathbb{F}$. Then there is no surjective linear isometry on X that maps e_1 to $\frac{1}{2}(e_1 + e_2)$.*

Proof. Suppose that T is such an isometry. Without loss of generality, we suppose that X is the linear span of the sequence $e_1, e_2, Te_1, Te_2, T^2e_1, T^2e_2, \ldots$, since T maps this span into itself. Let n be the smallest number such that $T^{n+1}e_2$ belongs to the linear span of $e_1, e_2, Te_2, T^2e_2, \ldots, T^ne_2$. We claim that $e_1, e_2, Te_2, T^2e_2, \ldots, T^ne_2$ is a basis of X. Since these vectors are linearly independent, we need only show that each T^me_1 and each T^me_2 belong to their linear span.

For suitable $\alpha, \beta_0, \beta_1, \ldots, \beta_n$, we have

$$T^{n+1}e_2 = \alpha e_1 + \beta_0 e_2 + \beta_1 Te_2 + \cdots + \beta_n T^n e_2.$$

It is easy to show by induction that

$$T^k e_1 = \frac{1}{2^k} e_1 + \frac{1}{2^k} e_2 + \frac{1}{2^{k-1}} Te_2 + \cdots + \frac{1}{2} T^{k-1} e_2,$$

so

$$\begin{aligned} T^{n+k+1}e_2 &= \alpha T^k e_1 + \beta_0 T^k e_2 + \beta_1 T^{k+1} e_2 + \cdots + \beta_n T^{n+k} e_2 \\ &= \alpha \left[\frac{1}{2^k} e_1 + \frac{1}{2^k} e_2 + \frac{1}{2^{k-1}} Te_2 + \cdots + \frac{1}{2} T^{k-1} e_2 \right] \\ &\quad + \beta_0 T^k e_2 + \beta_1 T^{k+1} e_2 + \cdots + \beta_n T^{n+k} e_2. \end{aligned}$$

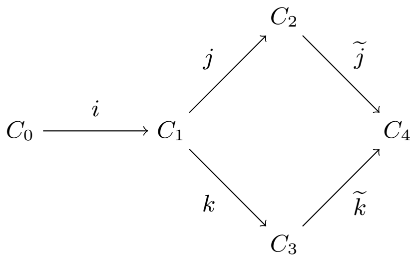
We first find by induction that each T^me_2 belongs to the linear span of $e_1, e_2, Te_2, T^2e_2, \ldots, T^ne_2$, and then the same holds for each T^me_1.

From the formula for T^ke_1, we see that $T^{n+1}e_1$ is a convex combination of the basic vectors $e_1, e_2, Te_2, T^2e_2, \ldots, T^ne_2$, in which each coefficient is positive. Since $T^{n+1}e_1$ has norm 1, as also do $e_1, e_2, Te_2, T^2e_2, \ldots, T^ne_2$, the whole convex hull of $e_1, e_2, Te_2, T^2e_2, \ldots, T^ne_2$ consists of vectors of norm 1. Hence, the norm is smooth in $T^{n+1}e_1$. The same is true for e_1, as there is obviously an isometry that maps e_1 to $T^{n+1}e_1$. But this is impossible, because the norm is not smooth in e_1 due to our assumptions. ∎

5. Key lemma. This section is devoted to the proof of the following lemma important for our proof that $(\mathcal{P}_{\mathbb{F}})_1$ does not have WAP (where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$).

Lemma 5.1. *Let $\mathcal{O}_0 = (C_0, f_0 : A_0 \to B_0) \in (\mathcal{P}_{\mathbb{F}})_1$ be such that there exist $\mathcal{O}_1 = (C_1, f_1 : A_1 \to B_1) \in (\mathcal{P}_{\mathbb{F}})_1$ and an embedding $i : \mathcal{O}_0 \to \mathcal{O}_1$ such that for any $\mathcal{O}_2 = (C_2, f_2 : A_2 \to B_2)$, $\mathcal{O}_3 = (C_3, f_3 : A_3 \to B_3) \in (\mathcal{P}_{\mathbb{F}})_1$ and embeddings $j : \mathcal{O}_1 \to \mathcal{O}_2$, $k : \mathcal{O}_1 \to \mathcal{O}_3$, there are $\mathcal{O}_4 = (C_4, f_4 : A_4 \to B_4) \in (\mathcal{P}_{\mathbb{F}})_1$ and embeddings $\widetilde{j} : \mathcal{O}_2 \to \mathcal{O}_4$, $\widetilde{k} : \mathcal{O}_3 \to \mathcal{O}_4$ such that $\widetilde{j} \circ j \circ i = \widetilde{k} \circ k \circ i$.*

Then there exists $\mathcal{O}' \in (\mathcal{P}_{\mathbb{F}})_1$ of the form $\mathcal{O}' = (E, f : E \to E)$ that admits an embedding $i' : \mathcal{O}_0 \to \mathcal{O}'$.



We start by introducing the following notation:

- $\mathfrak{D}$ denotes the space of all finitely supported $\alpha : \mathbb{Z} \to C_1$ with the norm $\|\alpha\|_{\mathfrak{D}} = \sum_{k\in\mathbb{Z}} \|\alpha(k)\|_{C_1}$,
- $f_{\mathfrak{D}}$ denotes the shift given by $[f_{\mathfrak{D}}(\alpha)](k) = \alpha(k-1)$,
- $\mathfrak{N} \subseteq \mathfrak{D}$ is defined as the subspace generated by all vectors of the form $(\dots, 0, 0, -f_1(a), a, 0, 0, \dots)$, where $a \in A_1$ and it can appear at any coordinate,
- $\mathfrak{C} = \mathfrak{D}/\mathfrak{N}$,
- $f_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}$ is given by $f_{\mathfrak{C}}(\alpha + \mathfrak{N}) = f_{\mathfrak{D}}(\alpha) + \mathfrak{N}$; this is well defined since $f_{\mathfrak{D}}$ maps $\mathfrak{N}$ into $\mathfrak{N}$, and moreover $f_{\mathfrak{C}}$ is an isometry,
- $j_{\mathfrak{C}} : C_1 \to \mathfrak{C}$ is given by $j_{\mathfrak{C}}(c) = (\dots, 0, c, 0, \dots) + \mathfrak{N}$, where c appears on the 0th coordinate; it follows from the next claim applied to $k = l = 0$ that $j_{\mathfrak{C}} : C_1 \to \mathfrak{C}$ is an isometry.

Claim 5.2. *Let $k \le l$ be integers and let $\mathfrak{D}_{[k,l]}$ and $\mathfrak{N}_{[k,l]}$ be the subspaces of $\mathfrak{D}$ and $\mathfrak{N}$ of all points supported by $\{k, k+1, \dots, l\}$. Then*

$$\|\alpha + \mathfrak{N}\|_{\mathfrak{C}} = \inf\{\|\alpha + \eta\|_{\mathfrak{D}} : \eta \in \mathfrak{N}_{[k,l]}\}, \quad \alpha \in \mathfrak{D}_{[k,l]}.$$

Consequently, the subspace $\{\alpha + \mathfrak{N} : \alpha \in \mathfrak{D}_{[k,l]}\}$ of $\mathfrak{C}$ is isometric to $\mathfrak{D}_{[k,l]}/\mathfrak{N}_{[k,l]}$, and thus it is polyhedral.

Proof. Let us fix $\alpha \in \mathfrak{D}_{[k,l]}$. The inequality "$\le$" is clear. To prove "$\ge$", given $\eta \in \mathfrak{N}$, we want to find $\eta' \in \mathfrak{N}_{[k,l]}$ such that $\|\alpha + \eta'\|_{\mathfrak{D}} \le \|\alpha + \eta\|_{\mathfrak{D}}$. We can write

$$\eta = (\dots, 0, -f_1(a_{p+1}), a_{p+1} - f_1(a_{p+2}), \dots, a_{q-1} - f_1(a_q), a_q, 0, \dots),$$

where $a_r - f_1(a_{r+1})$ appears at the rth coordinate, once η is supported by $\{p, p+1, \dots, q\}$. Now, if $p < k$, then replacing a_{p+1} by 0 does not increase

$\|\alpha+\eta\|_{\mathfrak{D}}$, because

$$\begin{aligned}\|-f_1(a_{p+1})\|_{C_1}+&\|\alpha(p+1)+a_{p+1}-f_1(a_{p+2})\|_{C_1}\\ &=\|a_{p+1}\|_{C_1}+\|\alpha(p+1)+a_{p+1}-f_1(a_{p+2})\|_{C_1}\\ &\geq\|\alpha(p+1)-f_1(a_{p+2})\|_{C_1}.\end{aligned}$$

Analogously, if $q>l$, then replacing a_q by 0 does not increase $\|\alpha+\eta\|_{\mathfrak{D}}$, as

$$\begin{aligned}\|\alpha(q-1)+a_{q-1}-&f_1(a_q)\|_{C_1}+\|a_q\|_{C_1}\\ &=\|\alpha(q-1)+a_{q-1}-f_1(a_q)\|_{C_1}+\|f_1(a_q)\|_{C_1}\\ &\geq\|\alpha(q-1)+a_{q-1}\|_{C_1}.\end{aligned}$$

Therefore, we can shrink the support of η until it is a subset of $\{k,k+1,\dots,l\}$, without increasing $\|\alpha+\eta\|_{\mathfrak{D}}$. ∎

Claim 5.3. *$(f_{\mathfrak{C}}\circ j_{\mathfrak{C}})(a)=(j_{\mathfrak{C}}\circ f_1)(a)$ for each $a\in A_1$.*

Proof. We have

$$\begin{aligned}(f_{\mathfrak{C}}\circ j_{\mathfrak{C}})(a)&=f_{\mathfrak{C}}((\dots,0,a,0,\dots)+\mathfrak{N})\\ &=f_{\mathfrak{D}}((\dots,0,a,0,\dots))+\mathfrak{N}=(\dots,0,0,a,\dots)+\mathfrak{N}\\ &=(\dots,0,f_1(a),0,\dots)+\mathfrak{N}=j_{\mathfrak{C}}(f_1(a))=(j_{\mathfrak{C}}\circ f_1)(a)\end{aligned}$$

for each $a\in A_1$. ∎

Let us consider a sequence $\{u_k^*\}_{k\in\mathbb{Z}}$ in the dual unit ball $B_{C_1^*}$ of C_1 such that $u_k^*(f_1(a))=u_{k+1}^*(a)$ for all $k\in\mathbb{Z}$ and $a\in A_1$. Then the sequence has the property that $\sum_{k\in\mathbb{Z}}u_k^*(\eta(k))=0$ for each $\eta\in\mathfrak{N}$. It follows that the functional

$$\mathfrak{u}^*(\alpha+\mathfrak{N})=\sum_{k\in\mathbb{Z}}u_k^*(\alpha(k)),\quad \alpha\in\mathfrak{D},$$

is well-defined, and moreover it belongs to $B_{\mathfrak{C}^*}$. Indeed, for $\alpha\in\mathfrak{D}$, we have $|\mathfrak{u}^*(\alpha+\mathfrak{N})|=|\sum_{k\in\mathbb{Z}}u_k^*(\alpha(k))|\leq\sum_{k\in\mathbb{Z}}\|u_k^*\|_{C_1^*}\|\alpha(k)\|_{C_1}\leq\sum_{k\in\mathbb{Z}}\|\alpha(k)\|_{C_1}$ $=\|\alpha\|_{\mathfrak{D}}$, as well as $|\mathfrak{u}^*(\alpha+\mathfrak{N})|\leq\|\alpha+\eta\|_{\mathfrak{D}}$ for $\eta\in\mathfrak{N}$, hence $|\mathfrak{u}^*(\alpha+\mathfrak{N})|\leq$ $\inf_{\eta\in\mathfrak{N}}\|\alpha+\eta\|_{\mathfrak{D}}=\|\alpha+\mathfrak{N}\|_{\mathfrak{C}}$, which gives $\|\mathfrak{u}^*\|_{\mathfrak{C}^*}\leq 1$.

Let $\psi_1,\dots,\psi_l$ be an enumeration of the vertices of $B_{C_1^*}$. For $1\leq n\leq l$, we denote $\psi_{n,0}=\psi_n$ and choose $\psi_{n,-1},\psi_{n,1}\in B_{C_1^*}$ such that $\psi_{n,-1}(f_1(a))=$ $\psi_{n,0}(a)$ and $\psi_{n,0}(f_1(a))=\psi_{n,1}(a)$ for every $a\in A_1$. There are numbers $s_{n,m}\in\mathbb{F}$ and $t_{n,m}\in\mathbb{F}$ such that $\sum_{m=1}^l|s_{n,m}|\leq 1$ and $\sum_{m=1}^l|t_{n,m}|\leq 1$ for $1\leq n\leq l$ and

$$\psi_{n,1}=\sum_{m=1}^l s_{n,m}\psi_m,\quad \psi_{n,-1}=\sum_{m=1}^l t_{n,m}\psi_m,\quad 1\leq n\leq l.$$

Recursively we define, for $j \geq 1$,

$$\psi_{n,j+1} = \sum_{m=1}^{l} s_{n,m}\psi_{m,j}, \quad \psi_{n,-j-1} = \sum_{m=1}^{l} t_{n,m}\psi_{m,-j}, \quad 1 \leq n \leq l.$$

By induction, $\|\psi_{n,\pm j}\|_{C_1^*} \leq 1$ and $\psi_{n,j}(f_1(a)) = \psi_{n,j+1}(a)$, resp. $\psi_{n,-j-1}(f_1(a)) = \psi_{n,-j}(a)$ for every $a \in A_1$. Therefore, for $1 \leq n \leq l$,

$$\mathfrak{u}_n^*(\alpha + \mathfrak{N}) = \sum_{k\in\mathbb{Z}} \psi_{n,k}(\alpha(k)), \quad \alpha \in \mathfrak{D},$$

defines a functional with $\|\mathfrak{u}_n^*\|_{\mathfrak{C}^*} \leq 1$.

CLAIM 5.4. *For $c \in i(C_0)$, we have*

$$\|x\|_{\mathfrak{C}} = \sup\{|\mathfrak{u}_n^*(f_{\mathfrak{C}}^k(x))| : 1 \leq n \leq l,\, k \in \mathbb{Z}\}, \quad x \in \operatorname{span}\{f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(c)) : k \in \mathbb{Z}\}.$$

Proof. We define

$$|||x|||_{\mathfrak{C}} = \sup\big(\{\tfrac{1}{2}\|x\|_{\mathfrak{C}}\} \cup \{|\mathfrak{u}_n^*(f_{\mathfrak{C}}^k(x))| : 1 \leq n \leq l,\, k \in \mathbb{Z}\}\big), \quad x \in \mathfrak{C},$$

which is an equivalent norm on $\mathfrak{C}$ satisfying $\frac{1}{2}\|x\|_{\mathfrak{C}} \leq |||x|||_{\mathfrak{C}} \leq \|x\|_{\mathfrak{C}}$, and such that $f_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}$ is an isometry also with respect to $|||\cdot|||_{\mathfrak{C}}$. To prove the claim, we show that the norms $\|\cdot\|_{\mathfrak{C}}$ and $|||\cdot|||_{\mathfrak{C}}$ coincide on the linear span of the points $f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(c))$, $k \in \mathbb{Z}$.

Notice first that $\|j_{\mathfrak{C}}(e)\|_{\mathfrak{C}} = |||j_{\mathfrak{C}}(e)|||_{\mathfrak{C}}$ for every $e \in C_1$. Indeed, for some n with $1 \leq n \leq l$, we have $\psi_n(e) = \|e\|_{C_1}$, so $\|j_{\mathfrak{C}}(e)\|_{\mathfrak{C}} \geq |||j_{\mathfrak{C}}(e)|||_{\mathfrak{C}} \geq \mathfrak{u}_n^*(j_{\mathfrak{C}}(e)) = \mathfrak{u}_n^*((\dots,0,e,0,\dots)+\mathfrak{N}) = \psi_{n,0}(e) = \psi_n(e) = \|e\|_{C_1} = \|j_{\mathfrak{C}}(e)\|_{\mathfrak{C}}$.

It is enough to prove that $\|w\|_{\mathfrak{C}} = |||w|||_{\mathfrak{C}}$ for w of the form

$$w = \sum_{k=0}^{\nu} w_k f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(c)).$$

We can assume that $\nu \geq 1$. Let us put

$$C_2 = C_3 = \operatorname{span} \bigcup_{k=0}^{\nu} f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(C_1)).$$

We equip C_2 with the restriction of the norm $\|\cdot\|_{\mathfrak{C}}$ and C_3 with the restriction of the norm $|||\cdot|||_{\mathfrak{C}}$. Then C_2 is polyhedral by Claim 5.2, and to see that C_3 is polyhedral, we prove that

$$\|x\|_{C_3} = \sup\big(\{\tfrac{1}{2}\|x\|_{\mathfrak{C}}\} \cup \{|\mathfrak{u}_n^*(f_{\mathfrak{C}}^k(x))| : 1 \leq n \leq l, -\nu \leq k \leq 0\}\big), \quad x \in C_3.$$

Given $x = \sum_{r=0}^{\nu} f_{\mathfrak{C}}^r(j_{\mathfrak{C}}(c_r)) \in C_3$, it is sufficient to show that the function

$$k \mapsto \max\{|\mathfrak{u}_n^*(f_{\mathfrak{C}}^k(x))| : 1 \leq n \leq l\}$$

is non-increasing for $k \geq 0$ and non-decreasing for $k \leq -\nu$. Let us note that $\mathfrak{u}_n^*(f_{\mathfrak{C}}^k(x)) = \mathfrak{u}_n^*(\sum_{r=0}^{\nu} f_{\mathfrak{C}}^{k+r}(j_{\mathfrak{C}}(c_r))) = \sum_{r=0}^{\nu} \psi_{n,k+r}(c_r)$ for each $k \in \mathbb{Z}$. In the case $k \geq 0$, we have

$$\begin{aligned}\mathfrak{u}_n^*(f_{\mathfrak{C}}^{k+1}(x)) &= \sum_{r=0}^{\nu} \psi_{n,k+1+r}(c_r) = \sum_{r=0}^{\nu}\sum_{m=1}^{l} s_{n,m}\psi_{m,k+r}(c_r) \\ &= \sum_{m=1}^{l} s_{n,m} \sum_{r=0}^{\nu} \psi_{m,k+r}(c_r) = \sum_{m=1}^{l} s_{n,m}\mathfrak{u}_m^*(f_{\mathfrak{C}}^{k}(x)),\end{aligned}$$

and consequently

$$|\mathfrak{u}_n^*(f_{\mathfrak{C}}^{k+1}(x))| \leq \sum_{m=1}^{l} |s_{n,m}|\,|\mathfrak{u}_m^*(f_{\mathfrak{C}}^{k}(x))| \leq \max\{|\mathfrak{u}_m^*(f_{\mathfrak{C}}^{k}(x))| : 1 \leq m \leq l\}.$$

Similarly, for $k \leq -\nu$, we have

$$\begin{aligned}\mathfrak{u}_n^*(f_{\mathfrak{C}}^{k-1}(x)) &= \sum_{r=0}^{\nu} \psi_{n,k-1+r}(c_r) = \sum_{r=0}^{\nu}\sum_{m=1}^{l} t_{n,m}\psi_{m,k+r}(c_r) \\ &= \sum_{m=1}^{l} t_{n,m} \sum_{r=0}^{\nu} \psi_{m,k+r}(c_r) = \sum_{m=1}^{l} t_{n,m}\mathfrak{u}_m^*(f_{\mathfrak{C}}^{k}(x)),\end{aligned}$$

and consequently

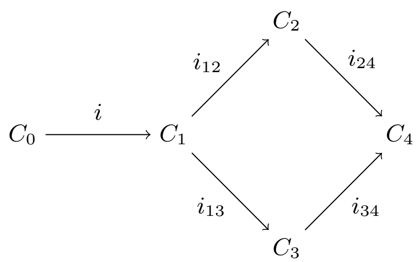
$$|\mathfrak{u}_n^*(f_{\mathfrak{C}}^{k-1}(x))| \leq \sum_{m=1}^{l} |t_{n,m}|\,|\mathfrak{u}_m^*(f_{\mathfrak{C}}^{k}(x))| \leq \max\{|\mathfrak{u}_m^*(f_{\mathfrak{C}}^{k}(x))| : 1 \leq m \leq l\}.$$

Hence, the formula for $\|\cdot\|_{C_3}$ holds, and C_3 is polyhedral indeed.

Next, let

$$A_2 = A_3 = \operatorname{span}\bigcup_{k=0}^{\nu-1} f_{\mathfrak{C}}^{k}(j_{\mathfrak{C}}(C_1)), \quad B_2 = B_3 = \operatorname{span}\bigcup_{k=1}^{\nu} f_{\mathfrak{C}}^{k}(j_{\mathfrak{C}}(C_1)),$$

where we consider the restriction of $\|\cdot\|_{\mathfrak{C}}$ on A_2, B_2 and the restriction of $|\!|\!|\cdot|\!|\!|_{\mathfrak{C}}$ on A_3, B_3. If we define $i_{12} = i_{13} = j_{\mathfrak{C}}$ and $f_2 = f_3 = f_{\mathfrak{C}}|_{A_2}$, then $i_{12} : C_1 \to C_2, i_{13} : C_1 \to C_3, f_2 : A_2 \to B_2$ and $f_3 : A_3 \to B_3$ are isometries. By Claim 5.3, $(f_2 \circ i_{12})(a) = (i_{12} \circ f_1)(a)$ (equivalently $(f_3 \circ i_{13})(a) = (i_{13} \circ f_1)(a)$) for each $a \in A_1$, and so i_{12} and i_{13} are embeddings from $\mathcal{O}_1$ into $\mathcal{O}_2 = (C_2, f_2 : A_2 \to B_2)$ and $\mathcal{O}_3 = (C_3, f_3 : A_3 \to B_3)$.



Now, there are $\mathcal{O}_4 = (C_4, f_4 : A_4 \to B_4) \in (\mathcal{P}_{\mathbb{F}})_1$ and embeddings $i_{24} : \mathcal{O}_2 \to \mathcal{O}_4$, $i_{34} : \mathcal{O}_3 \to \mathcal{O}_4$ such that $i_{24} \circ i_{12} \circ i = i_{34} \circ i_{13} \circ i$. Note that $(i_{24} \circ i_{12})(c) = (i_{34} \circ i_{13})(c)$, that is, $i_{24}(j_{\mathfrak{C}}(c)) = i_{34}(j_{\mathfrak{C}}(c))$. By induction, we can prove for $0 \le k \le \nu$ that

$$i_{24}(f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(c))) = i_{34}(f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(c))),$$

since for $0 \le k \le \nu - 1$ we can write

$$\begin{aligned} i_{24}(f_{\mathfrak{C}}^{k+1}(j_{\mathfrak{C}}(c))) &= (i_{24} \circ f_2)(f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(c))) = (f_4 \circ i_{24})(f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(c))) \\ &= (f_4 \circ i_{34})(f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(c))) = (i_{34} \circ f_3)(f_{\mathfrak{C}}^k(j_{\mathfrak{C}}(c))) \\ &= i_{34}(f_{\mathfrak{C}}^{k+1}(j_{\mathfrak{C}}(c))). \end{aligned}$$

Finally, it follows that $i_{24}(w) = i_{34}(w)$ and $\|w\|_{C_2} = \|i_{24}(w)\|_{C_4} = \|i_{34}(w)\|_{C_4} = \|w\|_{C_3}$, that is, $\|w\|_{\mathfrak{C}} = |||w|||_{\mathfrak{C}}$. ■

Before the next claim, let us make one more remark on the above introduced functionals $\psi_{n,k}$ and numbers $s_{n,m}$ and $t_{n,m}$. Let S be the matrix $(s_{n,m})_{n,m=1}^l$ and T be the matrix $(t_{n,m})_{n,m=1}^l$, and let $s_{n,m}^{(j)}$ and $t_{n,m}^{(j)}$ be the entries of the matrices S^j and T^j. Then it is straightforward to show by induction on $j \ge 0$ that

$$\psi_{n,j} = \sum_{m=1}^{l} s_{n,m}^{(j)} \psi_m, \quad \psi_{n,-j} = \sum_{m=1}^{l} t_{n,m}^{(j)} \psi_m, \quad 1 \le n \le l,$$

and

$$\sum_{m=1}^{l} |s_{n,m}^{(j)}| \le 1, \quad \sum_{m=1}^{l} |t_{n,m}^{(j)}| \le 1, \quad 1 \le n \le l.$$

CLAIM 5.5. *For $c \in i(C_0)$, $\nu \in \mathbb{N}$ and $\varepsilon > 0$, there is $\eta \in \mathfrak{N}$ such that $\|c + \eta(0)\|_{C_1} + \sum_{0<|k|\le\nu} \|\eta(k)\|_{C_1} < \varepsilon$.*

Proof. We can assume that $\|c\|_{C_1} = 1$. Since there is a cluster point of the sequence $\{(S^j, T^j)\}_{j=0}^{\infty}$, we can choose $\tau > \sigma \ge 0$ such that $\tau \ge 3\sigma$, $\tau \ge \sigma + 2\nu + 1$ and

$$|s_{n,m}^{(\tau)} - s_{n,m}^{(\sigma)}| \le \varepsilon/l, \quad |t_{n,m}^{(\tau)} - t_{n,m}^{(\sigma)}| \le \varepsilon/l, \quad 1 \le n, m \le l.$$

Then, for $j \ge 0$, using $S^{\tau+j} - S^{\sigma+j} = S^j(S^\tau - S^\sigma)$, we obtain $|s_{n,m}^{(\tau+j)} - s_{n,m}^{(\sigma+j)}| = |\sum_{o=1}^{l} s_{n,o}^{(j)}(s_{o,m}^{(\tau)} - s_{o,m}^{(\sigma)})| \le (\varepsilon/l) \sum_{o=1}^{l} |s_{n,o}^{(j)}|$. From this and from the same for T, we get

$$|s_{n,m}^{(\tau+j)} - s_{n,m}^{(\sigma+j)}| \le \varepsilon/l, \quad |t_{n,m}^{(\tau+j)} - t_{n,m}^{(\sigma+j)}| \le \varepsilon/l, \quad 1 \le n, m \le l.$$

Let us choose $R \in \mathbb{N}$ with $R \ge 2/\varepsilon$, and let

$$x = \sum_{p=0}^{2R-1} (-1)^p f_{\mathfrak{C}}^{p(\tau-\sigma)}(j_{\mathfrak{C}}(c)).$$

That is, $x = \sum_{r=0}^{R-1} [f_{\mathfrak{C}}^{2r(\tau-\sigma)}(j_{\mathfrak{C}}(c)) - f_{\mathfrak{C}}^{(2r+1)(\tau-\sigma)}(j_{\mathfrak{C}}(c))]$, so for $k \in \mathbb{Z}$ and $1 \leq n \leq l$,

$$\mathfrak{u}_n^*(f_{\mathfrak{C}}^k(x)) = \sum_{r=0}^{R-1} \big[\psi_{n,2r(\tau-\sigma)+k}(c) - \psi_{n,(2r+1)(\tau-\sigma)+k}(c)\big].$$

Let us consider three cases for r:

- If $2r(\tau-\sigma)+k \geq \sigma$, then

$$\begin{aligned}\psi_{n,2r(\tau-\sigma)+k}(c)-&\psi_{n,(2r+1)(\tau-\sigma)+k}(c)\\ &= \sum_{m=1}^{l} s_{n,m}^{(2r(\tau-\sigma)+k)}\psi_m(c) - \sum_{m=1}^{l} s_{n,m}^{((2r+1)(\tau-\sigma)+k)}\psi_m(c)\\ &= \sum_{m=1}^{l} (s_{n,m}^{(2r(\tau-\sigma)+k)} - s_{n,m}^{((2r+1)(\tau-\sigma)+k)})\psi_m(c),\end{aligned}$$

and so

$$|\psi_{n,2r(\tau-\sigma)+k}(c) - \psi_{n,(2r+1)(\tau-\sigma)+k}(c)| \leq \sum_{m=1}^{l} (\varepsilon/l)|\psi_m(c)| \leq l(\varepsilon/l)\|c\|_{C_1} = \varepsilon.$$

- If $(2r+1)(\tau-\sigma)+k \leq -\sigma$, then

$$\begin{aligned}\psi_{n,2r(\tau-\sigma)+k}(c)-&\psi_{n,(2r+1)(\tau-\sigma)+k}(c)\\ &= \sum_{m=1}^{l} t_{n,m}^{(-[2r(\tau-\sigma)+k])}\psi_m(c) - \sum_{m=1}^{l} t_{n,m}^{(-[(2r+1)(\tau-\sigma)+k])}\psi_m(c)\\ &= \sum_{m=1}^{l} (t_{n,m}^{(-[2r(\tau-\sigma)+k])} - t_{n,m}^{(-[(2r+1)(\tau-\sigma)+k])})\psi_m(c),\end{aligned}$$

and so

$$|\psi_{n,2r(\tau-\sigma)+k}(c) - \psi_{n,(2r+1)(\tau-\sigma)+k}(c)| \leq \sum_{m=1}^{l} (\varepsilon/l)|\psi_m(c)| \leq l(\varepsilon/l)\|c\|_{C_1} = \varepsilon.$$

- In the remaining case $2r(\tau-\sigma)+k < \sigma$ and $(2r+1)(\tau-\sigma)+k > -\sigma$, equivalently $\frac{-\sigma-k}{2(\tau-\sigma)} - \frac{1}{2} < r < \frac{\sigma-k}{2(\tau-\sigma)}$, we use simply

$$|\psi_{n,2r(\tau-\sigma)+k}(c) - \psi_{n,(2r+1)(\tau-\sigma)+k}(c)| \leq 2\|c\|_{C_1} = 2.$$

This may happen for at most one r, since $\frac{\sigma-k}{2(\tau-\sigma)} \leq 1 + \frac{-\sigma-k}{2(\tau-\sigma)} - \frac{1}{2}$, as $\tau \geq 3\sigma$.

Altogether,

$$\begin{aligned}|\mathfrak{u}_n^*(f_{\mathfrak{C}}^k(x))| &\leq \sum_{r=0}^{R-1} |\psi_{n,2r(\tau-\sigma)+k}(c) - \psi_{n,(2r+1)(\tau-\sigma)+k}(c)|\\ &\leq 2 + (R-1)\varepsilon,\end{aligned}$$

and using this for all $1 \le n \le l$ and $k \in \mathbb{Z}$, we deduce from Claim 5.4 that

$$\|x\|_{\mathfrak{E}} \le 2 + (R-1)\varepsilon \le R\varepsilon + (R-1)\varepsilon < 2R\varepsilon.$$

We have $x = \alpha + \mathfrak{N}$, where $\alpha \in \mathfrak{D}$ has $(-1)^p c$ on the coordinate $p(\tau - \sigma)$ for $0 \le p \le 2R-1$, and 0 elsewhere. For some $\eta \in \mathfrak{N}$, we have $\|\alpha + \eta\|_{\mathfrak{D}} < 2R\varepsilon$. Since $\tau \ge \sigma + 2\nu + 1$, we can write

$$\sum_{p=0}^{2R-1} \Big[\|(-1)^p c + \eta(p(\tau-\sigma))\|_{C_1} + \sum_{0<|k|\le\nu} \|\eta(p(\tau-\sigma)+k)\|_{C_1} \Big] \le \sum_{k\in\mathbb{Z}} \|\alpha(k) + \eta(k)\|_{C_1} = \|\alpha + \eta\|_{\mathfrak{D}} < 2R\varepsilon,$$

so we must have $\|(-1)^p c + \eta(p(\tau-\sigma))\|_{C_1} + \sum_{0<|k|\le\nu} \|\eta(p(\tau-\sigma)+k)\|_{C_1} < \varepsilon$ for some $0 \le p \le 2R-1$, from which we easily deduce the claim. ■

Let us define recursively

$$E_0 = F_0 = C_1, \quad E_{\nu+1} = f_1^{-1}(B_1 \cap E_\nu), \quad F_{\nu+1} = f_1(A_1 \cap F_\nu),$$

and let

$$E = \bigcap_{\nu=0}^{\infty} E_\nu.$$

Then E_ν is the subspace of C_1 of all points on which f_1 can be applied ν times and F_ν is the subspace of C_1 of all points on which f_1^{-1} can be applied ν times, so $F_\nu = f_1^\nu(E_\nu)$.

We have $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$, hence there is ν_0 such that $E_\nu = E$ for each $\nu \ge \nu_0$. It follows that $f_1|_E$ is a surjective linear isometry on E. Also, $F_\nu = f_1^\nu(E_\nu) = f_1^\nu(E) = E$ for each $\nu \ge \nu_0$.

Claim 5.6. *For every $\nu \in \mathbb{N} \cup \{0\}$, there is $\Gamma_\nu > 0$ such that for each $\varepsilon > 0$ and each sequence $a_0, a_1, \dots, a_\nu$ in A_1 with $\|f_1(a_{n-1}) - a_n\|_{C_1} < \varepsilon$ for $1 \le n \le \nu$, the distance from a_0 to $E_{\nu+1}$ is less than $\Gamma_\nu \varepsilon$.*

Similarly, for every $\nu \in \mathbb{N} \cup \{0\}$, there is $\Delta_\nu > 0$ such that for each $\varepsilon > 0$ and each sequence $b_0, b_1, \dots, b_\nu$ in B_1 with $\|f_1^{-1}(b_{n-1}) - b_n\|_{C_1} < \varepsilon$ for $1 \le n \le \nu$, the distance from b_0 to $F_{\nu+1}$ is less than $\Delta_\nu \varepsilon$.

Proof. We prove only the first part. For $\nu = 0$, this is clear, as $a_0 \in A_1 = E_{\nu+1}$. Concerning the induction step, let the statement hold for $\nu - 1$ with some $\Gamma_{\nu-1}$. There is $\varrho > 0$ such that for every $b \in B_1$ we have $\operatorname{dist}(b, B_1 \cap E_\nu) \le \varrho \operatorname{dist}(b, E_\nu)$; indeed, let δ be the distance from the unit sphere of $B_1/(B_1 \cap E_\nu)$ to $E_\nu/(B_1 \cap E_\nu)$ in $C_1/(B_1 \cap E_\nu)$; then

$$\delta \operatorname{dist}(b, B_1 \cap E_\nu) = \delta \|b + (B_1 \cap E_\nu)\| \le \operatorname{dist}(b + (B_1 \cap E_\nu), E_\nu/(B_1 \cap E_\nu)) = \operatorname{dist}(b, E_\nu),$$

so we can take $\varrho = 1/\delta$. By the induction hypothesis, $\mathrm{dist}(a_1, E_\nu) < \Gamma_{\nu-1}\varepsilon$, and so

$$\begin{aligned}\mathrm{dist}(a_0, E_{\nu+1}) &= \mathrm{dist}(f_1(a_0), B_1 \cap E_\nu) \leq \varrho\,\mathrm{dist}(f_1(a_0), E_\nu) \\ &\leq \varrho(\|f_1(a_0) - a_1\| + \mathrm{dist}(a_1, E_\nu)) < \varrho(1 + \Gamma_{\nu-1})\varepsilon.\end{aligned}$$

Hence, we can put $\Gamma_\nu = \varrho(1 + \Gamma_{\nu-1})$. ∎

Claim 5.7. *$i(C_0) \subseteq E$.*

Proof. Let $c \in i(C_0)$, let $\varepsilon > 0$ be arbitrary, and let ν be large enough to satisfy $E_{\nu+1} = F_{\nu+1} = E$. By Claim 5.5, there is $\eta \in \mathfrak{N}$ such that $\|c + \eta(0)\|_{C_1} + \sum_{0<|k|\leq\nu} \|\eta(k)\|_{C_1} < \varepsilon$. We can write

$$\eta = \sum_{k\in\mathbb{Z}} (\ldots, 0, 0, -f_1(a_k), a_k, 0, 0, \ldots),$$

where $a_k \in A_1$ appears on the kth coordinate and only finitely many a_k's are non-zero. We have $\varepsilon > \|c+\eta(0)\|_{C_1} = \|c+a_0-f_1(a_1)\|_{C_1}$ and $\varepsilon > \|\eta(k)\|_{C_1} = \|a_k - f_1(a_{k+1})\|_{C_1}$ for $0 < |k| \leq \nu$. Applying Claim 5.6 on the sequences $a_0, a_{-1}, \ldots, a_{-\nu}$ and $f_1(a_1), f_1(a_2), \ldots, f_1(a_{\nu+1})$, we obtain $\mathrm{dist}(a_0, E_{\nu+1}) < \Gamma_\nu\varepsilon$ and $\mathrm{dist}(f_1(a_1), F_{\nu+1}) < \Delta_\nu\varepsilon$. It follows that

$$\begin{aligned}\mathrm{dist}(c, E) &\leq \|c + a_0 - f_1(a_1)\|_{C_1} + \mathrm{dist}(a_0, E) + \mathrm{dist}(f_1(a_1), E) \\ &< (1 + \Gamma_\nu + \Delta_\nu)\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, we arrive at $c \in E$. ∎

To complete the proof of Lemma 5.1, we put $f = f_1|_E$ and $i' = i$. Then f is a surjective linear isometry on E, and Claim 5.7 guarantees that i' is an embedding of $\mathcal{O}_0 = (C_0, f_0 : A_0 \to B_0)$ into $(E, f : E \to E)$.

Acknowledgements. The authors are grateful to the referee for numerous suggestions that helped to improve this paper.

Funding. The first named author was supported by the Czech Science Foundation, project no. GAČR 22-07833K, and by the Academy of Sciences of the Czech Republic (RVO 67985840). The second author was partially supported by the National Science Centre, Poland under the Weave-UNISONO call in the Weave programme [grant no 2021/03/Y/ST1/00072].

References

[1] I. Ben Yaacov and W. Henson, *Generic orbits and type isolation in the Gurarij space*, Fund. Math. 237 (2017), 47–82.

[2] M. Doucha and M. Malicki, *Generic representations of countable groups*, Trans. Amer. Math. Soc. 372 (2019), 8249–8277.

[3] R. Fraïssé, *Theory of Relations*, Stud. Logic Found. Math. 145, North-Holland, Amsterdam 1986.

[4] J. Garbulińska and W. Kubiś, *Remarks on Gurarii space*, Extracta Math. 26 (2011), 235–269.
[5] J. Garbulińska and W. Kubiś, *A universal operator on the Gurarii space*, J. Operator Theory 73 (2015), 143–158.
[6] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. London Math. Soc. 94 (2007), 302–350.
[7] M. Malicki, *Consequences of the existence of ample generics and automorphism groups of homogeneous metric structures*, J. Symbolic Logic 81 (2016), 876–886.
[8] M. Malicki, *Remarks on weak amalgamation and large conjugacy classes in non-archimedean groups*, Arch. Math. Logic 61 (2021), 685–704.
[9] C. Rosendal, *A topological version of the Bergman property*, Forum Math. 21 (2009), 299–332.
[10] M. Sabok, *Automatic continuity for isometry groups*, J. Inst. Math. Jussieu 18 (2019), 561–590.
[11] S. Sakai, *On a conjecture of Kaplansky*, Tôhoku Math. J. 12 (1960), 31–33.
[12] S. Solecki, *Extending partial isometries*, Israel J. Math. 150 (2005), 315–331.
[13] T. Tsankov, *Automatic continuity for the unitary group*, Proc. Amer. Math. Soc. 141 (2013) 3673–3680.

Ondřej Kurka
Institute of Mathematics
Czech Academy of Sciences
115 67 Praha 1, Czech Republic
E-mail: kurka.ondrej@seznam.cz

Maciej Malicki
Faculty of Mathematics, Informatics and Mechanics
University of Warsaw
02-097 Warszawa, Poland
E-mail: mmalicki@mimuw.edu.pl