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Applying the argument principle to count zeros of harmonic functions with poles

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Abstract. Several recent papers investigate the way in which the number of zeros of a complex-valued harmonic function depends on its coefficients by analyzing specific simple families. These families share two features: (1) the critical curve separating the sense-preserving and sense-reversing regions is a circle, and (2) the image of that circle is a well-understood parametric curve. In such cases, the harmonic analogue of the Argument Principle can be applied to count the zeros. In this paper, we illustrate the strengths and limitations of these techniques; we construct a new family of complex-valued harmonic functions with poles having some of these features. We obtain detailed zero-counting theorems for two subfamilies, and we illustrate how to obtain less detailed zero-counting theorems for the general family.

1. Introduction. We are interested in counting zeros of complex-valued harmonic polynomials and complex-valued harmonic functions with poles. At this point, it is well-known that the number of zeros of a harmonic polynomial is not simply a function of the degree but also depends on the coefficients. In the polynomial case, several researchers have explored this phenomenon, seeking sharp upper bounds on the number of zeros of a complex-valued harmonic polynomial. For example, if $f = h + \overline{g}$, with $\deg h = n$, $\deg g = m$, and $m \leq n$, Sheil-Small [SS92] conjectured that the maximum number of zeros of f is n^2. Wilmshurst [Wil98] proved this conjecture and showed that this bound is sharp. Quite recently, Sète and Zur [SZ21] showed that for every $n \leq k \leq n^2$, there exists a harmonic polynomial $f = h + \overline{g}$ with $\deg h = n$, $\deg g < n$, and with k zeros.

Additionally, a number of researchers have explored specific families of harmonic polynomials and harmonic functions with poles with the goal of

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describing very precisely how the number of zeros depends on the coefficients. In particular, Brilleslyper et al. [BBD+20] and Brooks et al. [BDH+22] consider

$$f_c(z) = z^n + c\overline{z}^k - 1, \quad c \in \mathbb{C} \setminus \{0\}, \tag{1.1}$$

and Brooks and Lee [Lee22, BL24] consider

$$f_c(z) = z^n + \frac{c}{\overline{z}^k} - 1, \quad c \in \mathbb{C} \setminus \{0\}. \tag{1.2}$$

Perhaps not surprisingly given the similarity between these functions, the method of analysis for both families is similar; for both families, the *critical curve* separating the sense-preserving and sense-reversing regions of the plane is a circle, and the *caustic* (the image of the critical curve under the function) is a known parametric curve—in the case of the family (1.1), the caustic is a hypocycloid, and for the family (1.2), it is an epicycloid. In both cases, it is then possible to use the harmonic analogue of the Argument Principle to count the zeros in the sense-reversing region by finding the winding number about the origin of the caustic.

In this paper, we illustrate both the power and limitations of this type of argument. We consider the family of complex-valued harmonic functions with poles

$$f_c(z) = z^n - z^{-k} - 1 + c\left(\frac{n}{n-m}\overline{z}^{n-m} - \frac{k}{k+m}\overline{z}^{-k-m}\right) \tag{1.3}$$

where

$$n, k, m \in \mathbb{Z} \setminus \{0\}, \qquad n-m,\, n+k,\, m+k \neq 0.$$

Generically, we assume that $c \in \mathbb{C} \setminus \{0\}$, but we will specialize to $c > 0$ in Theorems 4.4 and 5.3. By design, every member of the family has a critical curve that is a circle. Furthermore, there are several subfamilies for which the caustic is a hypocycloid or an epicycloid. For these subfamilies, the arguments in [BBD+20, BDH+22, BL24] can be modified to obtain similar theorems describing precisely how the number of zeros varies with the parameter c.

Specifically, in Theorems 4.4 and 5.3, we prove precise zero-counting theorems with $c > 0$ for the cases in which $m = -2k$ and $m = n - k$, respectively. Theorem 4.4 resembles Theorem 1.1 of [BBD+20] and Theorem 2 of [Lee22] in that the number of zeros is a monotone function of c. However, in Theorem 5.3, the number of zeros is not a monotone function of c.

We also illustrate the limitations of this kind of argument. For most choices of n, k, and m, the caustic is not a special, well-understood curve. Although the Argument Principle still applies, there is no obvious way to obtain the kind of detailed geometric information about the caustic that would be needed for a zero-counting argument. In such cases, Rouché's Theorem gives a powerful alternative for counting zeros. Along these lines, we use Rouché's Theorem to count the zeros for large and small $|c|$ for a broad subfamily. We prove:

THEOREM 6.1. *Let* f_c *be as in* (1.3) *with* $c = \rho e^{i\tau} \neq 0$, $n, k, m > 0$, *and* $n - m > 0$. *Then there exist positive constants* ρ_1 *and* ρ_N *such that*

(1) *if* $0 < \rho < \rho_1$, f_c *has* $n + 3k + m$ *zeros,*
(2) *if* $\rho > \rho_N$, f_c *has* $3n + k - m$ *zeros.*

Even for those subfamilies whose caustic is a hypocycloid or an epicycloid, making precise zero-counting arguments may be difficult. For example, when $m = n-k$, the caustic is a hypocycloid or an epicycloid for all $c \in \mathbb{C}\setminus\{0\}$, but the size and rotation of this curve depend on both the modulus and argument of c, allowing the function to gain and lose zeros in a way that is difficult to track as c varies. In Example 5.5, we show that taking $c \in \mathbb{C} \setminus \mathbb{R}$ can yield numbers of zeros that do not follow the pattern given in Theorem 5.3, suggesting that the theorem will not easily extend to non-real c.

The structure of the paper is as follows. In Section 2, we state many fundamental results about harmonic functions and prove a formula for the total number of zeros. In Section 3, we provide the necessary background on hypocycloids and epicycloids, and we prove a result about the general family (1.3). In Sections 4 and 5, we prove Theorems 4.4 and 5.3, which give detailed zero counts for the subfamilies where $m = -2k$ and where $m = n - k$. In Section 6, we illustrate the power of Rouché's Theorem when the caustic is not a hypocycloid or epicycloid by proving Theorem 6.1. We close with concluding remarks in Section 7.

2. Background.

Recall that a function $\phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if it is twice continuously differentiable and satisfies Laplace's equation $\phi_{xx} + \phi_{yy} = 0$. A function $f = u + iv : D \subseteq \mathbb{C} \to \mathbb{C}$ is a *complex-valued harmonic* function (or a *harmonic* function, for short) if both u and v are harmonic in D. It is well-known that if f is a complex-valued harmonic function on a simply connected domain $D \subseteq \mathbb{C}$, it can be written in the form $f = h + \overline{g}$, where h and g are analytic (see [Dur04]).

To determine when a complex-valued harmonic function $f = h + \overline{g}$ is sense-preserving or sense-reversing, we consider its Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. The mapping is sense-preserving when $J_f(z) > 0$ and sense-reversing when $J_f(z) < 0$. The points at which the Jacobian is zero form the *critical set.* There are two kinds of points in the critical set. The first are points at which $h'(z) = g'(z) = 0$. In all but trivial cases, these points form a discrete set, which we ignore. The second are points at which $|h'(z)| = |g'(z)|$ but both are non-zero. These are the points at which the *complex dilatation*

$$\omega(z) = \frac{g'(z)}{h'(z)}$$

has modulus 1. Observe that f is sense-preserving when $|\omega(z)| < 1$. For a more detailed discussion of the structure of the critical set see [Lyz92, SZ21].

DEFINITION 2.1. The *critical curve* is the set of $z \in \mathbb{C}$ for which $|\omega(z)| = 1$.

We note that the critical curve is a kind of level set. The geometry of such level sets and their generalizations are treated in [Bar02].

Because the image of a small circle about a zero of an analytic function has a positive winding number, whereas the image of a small circle about a zero of a co-analytic function has a negative winding number, it is reasonable to define the order of a zero of a harmonic function to be positive in the sense-preserving region and negative in the sense-reversing region. The following definition makes this precise.

DEFINITION 2.2 ([Dur04]). Let $f = h + \overline{g}$ be a complex-valued harmonic function and suppose $f(z_0) = 0$. Write

$$f(z) = a_0 + \sum_{j=r}^{\infty} a_j(z - z_0)^j + \overline{b_0 + \sum_{j=s}^{\infty} b_j(z - z_0)^j}.$$

If z_0 is in a sense-preserving region, we define the *order* of z_0 to be r and if z_0 is in a sense-reversing region, we define its order to be $-s$. If z_0 is in the critical set, we call it a *singular* zero.

The functions we consider have poles. The theory of poles for harmonic functions has been developed by Suffridge and Thompson [ST00]:

DEFINITION 2.3 ([ST00]). Let f be a harmonic function on a domain $D \subseteq \mathbb{C}$. Suppose that the local representation of f around a pole z_0 is

$$f(z) = \sum_{j=-\ell}^{\infty} a_j(z - z_0)^j + \overline{\sum_{j=-m}^{\infty} b_j(z - z_0)^j} + 2A \log|z - z_0|$$

for some constant A and where ℓ and m are finite.

(1) If $a_{-\ell} \neq 0$ for some $\ell > 0$ and $\ell > m$, or $\ell = m$ with $|a_{-\ell}| > |b_{-\ell}|$, then f is sense-preserving near z_0 and f has a pole at z_0 of order ℓ.
(2) If $b_{-m} \neq 0$ for some $m > 0$ and $\ell < m$, or $\ell = m$ with $|a_{-m}| < |b_{-m}|$, then f is sense-reversing near z_0 and f has a pole at z_0 of order $-m$.

Two very useful tools for counting zeros of a meromorphic function are the Argument Principle and Rouché's Theorem. Fortunately, there is a harmonic analogue to the Argument Principle, proved in [ST00]:

Theorem 2.4 (Argument Principle for harmonic functions with poles). *Let f be harmonic, except for a finite number of poles, in a simply connected domain $D \subseteq \mathbb{C}$. Let C be a simple closed curve contained in D not passing through a pole or a zero, and let Ω be the open bounded region created by C. Suppose that f has no singular zeros in D. Let $Z_{f,C}$ be the sum of the orders of the zeros of f in Ω, and let $P_{f,C}$ be the sum of the orders of the poles of f in Ω. Then*

$$\frac{1}{2\pi}\Delta_C \arg f(z) = Z_{f,C} - P_{f,C}.$$

The quantity $\frac{1}{2\pi}\Delta_C \arg f(z)$ is called the *winding number* of the curve $f(C)$ about the origin.

Just as in the meromorphic case, the harmonic analogue of the Argument Principle leads to a harmonic analogue of Rouché's Theorem.

Theorem 2.5 (Rouché's Theorem for harmonic functions with poles). *Suppose f and g satisfy the hypotheses for the Argument Principle for Harmonic Functions with Poles. If f and g are harmonic inside and on the simple closed contour C, if $|f(z)| > |g(z)|$ at each point on C, and if f and g have no poles on C and no singular zeros inside C, then $Z_{f,C} - P_{f,C} = Z_{f+g,C} - P_{f+g,C}$.*

Proof. This result is not new. It can be proved using an adaptation of the argument by Brown and Churchill [BC09] used to prove the meromorphic version of the result. ■

A combination of Theorems 2.4 and 2.5 allows us to count the total number of zeros.

Lemma 2.6. *Let f be harmonic, except for a finite number of poles, in $\mathbb{C}$. Suppose that the critical curve Γ of f is bounded and that f can be written as $f(z) = p(z) + q(z)$, where $|p(z)| > |q(z)|$ on all circles of sufficiently large radius. Denote the total number of zeros of f by T_f. Let C be any sufficiently large circle such that all of the zeros of f are contained in the interior of C and $|p(z)| > |q(z)|$ on C. Let $W_{f,\Gamma}$ denote the winding number of $f(\Gamma)$ about the origin. Then*

$$T_f = |2(W_{f,\Gamma} + P_{f,\Gamma}) - (Z_{p,C} - P_{p,C} + P_{f,C})|. \tag{2.1}$$

Proof. Because p dominates on C and $f = p + q$, Theorem 2.5 yields

$$Z_{f,C} = Z_{p,C} - P_{p,C} + P_{f,C}. \tag{2.2}$$

By hypothesis, all zeros of f are contained in C, so $Z_{f,C}$ is equal to the sum of the orders of all the zeros of f. Comparing the two cases depending on whether f is sense-preserving or sense-reversing inside Γ, we see that

$$T_f = |2Z_{f,\Gamma} - Z_{f,C}|. \tag{2.3}$$

By Theorem 2.4,

$$Z_{f,\Gamma} = W_{f,\Gamma} + P_{f,\Gamma}. \tag{2.4}$$

Substituting (2.2) and (2.4) into (2.3) gives the result. ■

In Sections 4 and 5, the following corollary will be more convenient.

Corollary 2.7. *Let f be harmonic, except for a finite number of poles, in $\mathbb{C}$. Suppose that the critical curve Γ of f is bounded and that f can be written as $f(z) = az^{\ell} + q(z)$ for some $\ell \in \mathbb{Z} \setminus \{0\}$ and $a \in \mathbb{C} \setminus \{0\}$, where $|z^{\ell}| > |q(z)|$ on all circles of sufficiently large radius. Suppose that all of the poles of f are contained in Γ. Then*

$$T_f = \big|2W_{f,\Gamma} + P_{f,\Gamma} - |\ell|\big|. \tag{2.5}$$

Similarly, if $f(z) = a\overline{z}^{\ell} + q(z)$, then

$$T_f = \big|2W_{f,\Gamma} + P_{f,\Gamma} + |\ell|\big|. \tag{2.6}$$

3. Geometry of the caustics.

3. Geometry of the caustics. As we indicated in the introduction, we constructed the family (1.3) so that the critical curve is always a circle and so that some subfamilies map that circle to a hypocycloid or an epicycloid. We thus review the definitions of these parametric curves.

Definition 3.1. A *hypocycloid* centered at the origin is the curve traced by a fixed point on a circle of radius b rolling inside a larger origin-centered circle of radius a. The curve is given by the parametric equations

$$x(\phi) = (a-b)\cos(\phi) + b\cos\left(\frac{a-b}{b}\phi\right),$$
$$y(\phi) = (a-b)\sin(\phi) - b\sin\left(\frac{a-b}{b}\phi\right).$$

If the ratio $\frac{a}{b}$ is rational and is written in lowest terms as $\frac{p}{q}$, then the hypocycloid has p cusps, and each arc connects cusps that are q away from each other in a counterclockwise direction. Such a hypocycloid is called a (p,q) *hypocycloid*, and the range of ϕ values to trace the entire hypocycloid is $0 \le \phi < 2\pi q$.

By combining the parametric equations in this definition, we obtain the complex form of the equation of a hypocycloid:

$$H(\phi) = (a-b)e^{i\phi} + be^{i\phi(\frac{b-a}{b})}. \tag{3.1}$$

Definition 3.2. An *epicycloid* centered at the origin is the curve traced out by a fixed point on a circle of radius b rolling on an origin-centered circle of radius a. The curve is given by the parametric equations

$$x(\phi) = (a+b)\cos(\phi) - b\cos\left(\frac{a+b}{b}\phi\right),$$

$$y(\phi) = (a+b)\sin(\phi) - b\sin\left(\frac{a+b}{b}\phi\right).$$

If the ratio $\frac{a}{b}$ is rational and is written in lowest terms as $\frac{p}{q}$, then the epicycloid has p cusps, and each arc connects cusps that are q away from each other in a counterclockwise direction. Such an epicycloid is a (p, q) *epicycloid*, and the range of ϕ values to trace the entire epicycloid is $0 \leq \phi < 2\pi q$.

We can also write the equation of an epicycloid in complex form:

$$E(\phi) = (a+b)e^{i\phi} - be^{i\phi(\frac{a+b}{b})}. \tag{3.2}$$

Note that both hypocycloids and epicycloids can be parameterized using a linear combination of two complex exponentials with different periods. In our later analysis, we will need to determine for various real a and b whether the expression $(a-b)e^{i\phi} + be^{i\phi(\frac{b-a}{b})}$ parameterizes a hypocycloid or an epicycloid and determine its type. The next lemma summarizes the cases.

LEMMA 3.3. *Let $a, b \in \mathbb{R} \setminus \{0\}$ with $\frac{a}{b}$ rational and $a \neq b$. Consider the curve C parameterized by*

$$F(\phi) = (a-b)e^{i\phi} + be^{i\phi(\frac{b-a}{b})}. \tag{3.3}$$

(1) *If $a > b > 0$, then C is a hypocycloid of type (p, q) where $\frac{p}{q}$ is the unique way of writing $\frac{a}{b}$ as a ratio of natural numbers with* $\gcd(p, q) = 1$.
(2) *If $a > 0 > b$, C is an epicycloid whose type is determined by $\frac{a}{-b}$, as above.*
(3) *If $b > a > 0$, C is an epicycloid whose type is determined by $\frac{a}{b-a}$.*
(4) *If $0 > b > a$, C is a hypocycloid whose type is determined by $\frac{-a}{-b}$, rotated about the origin by π radians.*
(5) *If $b > 0 > a$, C is an epicycloid whose type is determined by $\frac{-a}{b}$, rotated about the origin by π radians.*
(6) *If $0 > a > b$, C is an epicycloid whose type is determined by $\frac{-a}{a-b}$, rotated about the origin by π radians.*

Proof. The last three cases follow from the first three by factoring a -1 out of the right-hand side of (3.3) and applying cases (1)–(3) with a and b replaced by $-a$ and $-b$. Thus it suffices to prove (1)–(3).

For (1), if $a > b > 0$, then $F(\phi)$ is the same as (3.1), and the ratio that determines the type of the hypocycloid is $\frac{a}{b}$. Likewise, for (2), if $a > 0 > b$, $F(\phi)$ is of the form (3.2) with b replaced by the positive number $-b$, so C is an epicycloid with type determined by $\frac{a}{-b}$.

For (3), if $b > a > 0$, then $b > b - a > 0$. Make the change of variable $\varphi = \frac{b-a}{b}\phi$ to obtain

$$\begin{aligned} F\left(\frac{b}{b-a}\varphi\right) &= (a-b)e^{i\varphi(\frac{b}{b-a})} + be^{i\varphi} \\ &= be^{i\varphi} - (b-a)e^{i\varphi(\frac{b}{b-a})}. \end{aligned}$$

This final expression follows the form of (3.2) and hence gives an epicycloid for which the radius of the fixed circle is $a = b - (b - a)$ and the radius of the rolling circle is $b - a$. Thus, the type of the epicycloid is determined by $\frac{a}{b-a}$. Note that $\frac{a}{b-a} = \left(\frac{b}{a} - 1\right)^{-1}$ is rational because $\frac{a}{b}$ is a rational number different from 1. ■

We now explore the geometric properties shared by all members of the family (1.3). To begin with, the family was constructed so that every member has a circular critical curve.

Proposition 3.4. *Let f_c be as in* (1.3). *The critical curve is the circle $\mathit{\Gamma}_c$ with equation*

$$|z| = R_c := |c|^{1/m}.$$

Proof. We compute the complex dilatation, ignoring the removable singularities. We find

$$\omega(z) = \frac{c(nz^{n-m-1} + kz^{-k-m-1})}{nz^{n-1} + kz^{-k-1}} = \frac{c}{z^m},$$

which has modulus 1 if and only if $|z| = |c|^{1/m}$. ■

Furthermore, we constructed the family so that the caustic is related to hypocycloids and epicycloids.

Proposition 3.5. *Let f_c be as in* (1.3) *with $c = \rho e^{i\tau} \neq 0$. Suppose that $m \neq -2k, 2n$. Let*

$$p_c(z) = z^n + \frac{cn}{n-m}\bar{z}^{n-m} \quad \textit{and} \quad q_c(z) = -z^{-k} - \frac{ck}{k+m}\bar{z}^{-k-m}, \tag{3.4}$$

so that

$$f_c(z) = p_c(z) + q_c(z) - 1.$$

Then

1. *the image $p_c(\mathit{\Gamma}_c)$ of the critical circle is a hypocycloid or epicycloid centered at $z = 0$, rotated by $\frac{n\tau}{2n-m}$, and scaled by $\frac{1}{n-m}\rho^{n/m}$,*
2. *the image $q_c(\mathit{\Gamma}_c)$ of the critical circle is a hypocycloid or epicycloid centered at $z = 0$, rotated by $\frac{k\tau}{2k+m}$, and scaled by $\frac{1}{k+m}\rho^{-k/m}$.*

The exact nature of the curves and their types depend on n, k, and m and can be determined using Lemma 3.3.

Proof. Parameterize the critical circle $\mathit{\Gamma}_c$ by $\theta \mapsto \rho^{1/m} e^{i\theta}$. Then $p_c(\mathit{\Gamma}_c)$ is parameterized by

$$\begin{aligned}\theta \mapsto \rho^{n/m} e^{in\theta} &+ \rho e^{i\tau} \frac{n}{n-m} \rho^{\frac{n-m}{m}} e^{-i(n-m)\theta} \\ &= \frac{1}{n-m} \rho^{n/m} [(n-m) e^{in\theta} + n e^{i(\tau + (m-n)\theta)}] \\ &= \frac{1}{n-m} \rho^{n/m} e^{i\frac{n\tau}{2n-m}} [(n-m) e^{in(\theta - \frac{\tau}{2n-m})} + n e^{i(m-n)(\theta - \frac{\tau}{2n-m})}].\end{aligned}$$

Let $\phi = n\big(\theta - \frac{\tau}{2n-m}\big)$. Then $p_c(\mathit{\Gamma}_c)$ is parameterized by

$$\phi \mapsto \frac{1}{n-m} \rho^{n/m} e^{i\frac{n\tau}{2n-m}} [(n-m) e^{i\phi} + n e^{i\phi(\frac{m-n}{n})}].$$

The expression in square brackets matches (3.3) with $a = 2n - m$ and $b = n$, so the claim about $p_c(\mathit{\Gamma}_c)$ follows from Lemma 3.3.

A similar calculation that we omit shows that $q_c(\mathit{\Gamma}_c)$ is parameterized by

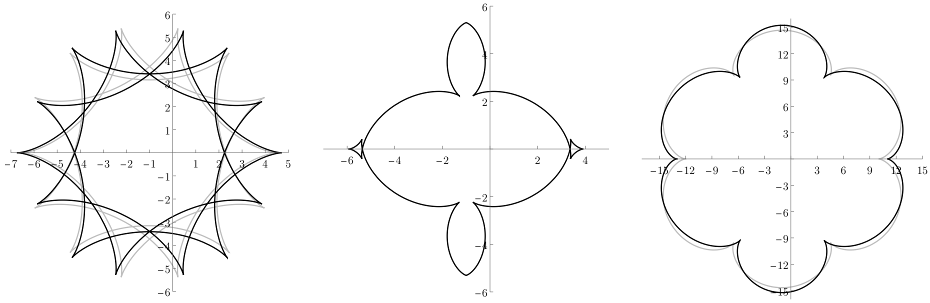
$$\phi \mapsto \frac{1}{k+m} \rho^{-k/m} e^{i\frac{k\tau}{2k+m}} [-(k+m) e^{i\phi} - k e^{i\phi(-\frac{k+m}{k})}]. \tag{3.5}$$

The change of variable from θ to ϕ requires that $m \neq -2k$. The expression in square brackets again matches the form in (3.3), except that now $a = -2k - m$ and $b = -k$. Thus the claim about $q_c(\mathit{\Gamma}_c)$ also holds. ∎

In general, adding two functions that individually parameterize a hypocycloid or epicycloid will not give a function that parameterizes another such curve, so this proposition does not imply that all members of our family map the critical circle to a known parametric curve. It does, however, give several important insights.

First, note that the scaling factors for both $p_c(\mathit{\Gamma}_c)$ and $q_c(\mathit{\Gamma}_c)$ involve powers of ρ. If, for example, n, k, and m are all positive, then for sufficiently large ρ, the image of the critical circle under f_c will be very nearly the same as its image under $p_c - 1$, whereas for sufficiently small ρ, the image of the critical circle under f_c will be very nearly the same as its image under $q_c - 1$ (see Figure 1). Analogous statements hold for other choices of n, k, and m if n and k have the same sign. In all these cases, we can obtain zero-counting theorems for small ρ and large ρ using our knowledge of p_c and q_c. Such a result is the subject of Section 6.

Perhaps more interestingly, there are several choices of n, k, and m for which the two curves $p_c(\mathit{\Gamma}_c)$ and $q_c(\mathit{\Gamma}_c)$ combine to create a hypocycloid or epicycloid. This occurs, for example, when $m = -2k$ and when $m = n - k$. When $m = -2k$, the proof of Proposition 3.5 shows that $p_c(\mathit{\Gamma}_c)$ is a hypocycloid or epicycloid, but the proof fails for $q_c(\mathit{\Gamma}_c)$. In this case, $q_c(\mathit{\Gamma}_c)$ contains only the origin for $c > 0$, so the caustic is the hypocycloid or epicycloid given by $p_c(\mathit{\Gamma}_c)$ shifted to the left. When $m = n - k$, $p_c(\mathit{\Gamma}_c)$ and $q_c(\mathit{\Gamma}_c)$ are both



$c = 0.02$ $c = 0.5$ $c = 2$

Fig. 1. The caustic for $n = 7$, $k = 3$, and $m = 8$. When c is small, the caustic is very near the image under $q_c - 1$, which is shown as the gray curve. When c is an intermediary value, the caustic does not resemble a hypocycloid or epicycloid. When c is large, the caustic is very near to the image under $p_c - 1$, again shown in gray.

hypocycloids or epicycloids of the same type that add to create another hypocycloid or epicycloid. These subfamilies are the subject of Sections 4 and 5. We have also observed that $m = 2n$ gives rise to hypocycloids and epicycloids, but this case does not fundamentally differ from the subfamily addressed in Section 4 because $p_c(\mathit{\Gamma}_c) = \{0\}$ in this case.

For the remainder of the paper, we restrict to the case in which $n > -k$. The cases for $n < -k$ are closely related.

4. The first subfamily. In this section, we explore the subfamily where $q_c(\mathit{\Gamma}_c)$ reduces to $\{0\}$. As shown in the proof of Proposition 3.5, $p_c(\mathit{\Gamma}_c)$ is a hypocycloid or epicycloid, and we will exactly determine when each case occurs and the type of the hypocycloid or epicycloid. This explicit description of the caustic will allow us to determine the winding number of the caustic and therefore the total number of zeros of f_c.

Throughout this section, let $m = -2k$. Equation (1.3) simplifies to

$$f_c(z) = z^n - z^{-k} - 1 + c\left(\frac{n}{n+2k}\bar{z}^{n+2k} + \bar{z}^k\right). \tag{4.1}$$

Our first result describes the image under f_c of the critical circle. As stated in the previous section, we assume that $n > -k$.

Lemma 4.1. *Suppose $c > 0$, $n, k \in \mathbb{Z}$, n odd, $\gcd(n,k) = 1$, $n > -k$, $n \neq 0, -2k$, and $k \neq 0$.*

(1) *If $n > 0$ and $n > -2k$, then the image $f_c(\mathit{\Gamma}_c)$ of the critical circle is a $(2n+2k, n)$ hypocycloid centered at $z = -1$.*
(2) *If $n < 0$, then $f_c(\mathit{\Gamma}_c)$ is a $(2n+2k, -n)$ epicycloid centered at -1.*
(3) *If $n > 0$ and $n < -2k$, then $f_c(\mathit{\Gamma}_c)$ is a $(2n+2k, -n-2k)$ epicycloid centered at -1 that has been rotated around -1 by π radians.*

Remark 4.2. The image $f_c(\mathit{\Gamma}_c)$ is still a hypocycloid or an epicycloid if n is even or $\gcd(n,k) > 1$, but the fraction that determines the type of the curve is not in lowest terms, and the resulting curve has fewer cusps than the lemma states. It would be possible to treat these cases with similar arguments, but we do not do so in this paper.

Proof of Lemma 4.1. We show that $f_c(\mathit{\Gamma}_c) + 1$ is a hypocycloid or epicycloid centered at $z = 0$. We cannot directly apply Proposition 3.5 since $m = -2k$, but the algebra is similar to showing that $p_c(\mathit{\Gamma}_c)$ is a hypocycloid or epicycloid. However, $\tau = 0$ because c is real, so the change of variable is simpler.

Parameterize the circle $\mathit{\Gamma}_c$ by $\theta \mapsto R_c e^{i\theta}$ for $\theta \in [0, 2\pi)$ to obtain

$$\begin{aligned} f_c(R_c e^{i\theta}) + 1 &= R_c^n e^{in\theta} - R_c^{-k} e^{-ik\theta} + \frac{cn}{n+2k} R_c^{n+2k} e^{-i\theta(n+2k)} + cR_c^k e^{-ik\theta} \\ &= R_c^n e^{in\theta} + \frac{cn}{n+2k} R_c^{n+2k} e^{-i\theta(n+2k)} + e^{-ik\theta}(cR_c^k - R_c^{-k}). \end{aligned}$$

Recall that $R_c = c^{-\frac{1}{2k}}$. Then $cR_c^k - R_c^{-k} = c \cdot c^{-1/2} - c^{1/2} = 0$, so

$$f_c(R_c e^{i\theta}) + 1 = R_c^n e^{in\theta} + \frac{cn}{n+2k} R_c^{n+2k} e^{-i\theta(n+2k)}.$$

Let $\phi = n\theta$ and let

$$b = \frac{cn}{n+2k} R_c^{n+2k} = c^{-\frac{n}{2k}} \frac{n}{n+2k} \quad \text{and} \quad a = R_c^n + b = c^{-\frac{n}{2k}} \frac{2n+2k}{n+2k},$$

which satisfy

$$\frac{a-b}{b} = \frac{n+2k}{n}.$$

Making these substitutions yields the curve parameterized by

$$\phi \mapsto (a-b)e^{i\phi} + be^{i\phi(\frac{b-a}{b})}.$$

We now apply Lemma 3.3. Because $a - b = R_c^n > 0$, the possible sign combinations of a and b are $a > b > 0$, $a > 0 > b$, and $0 > a > b$.

For (1), note that if $n > 0$ and $n > -2k$, then $a > b > 0$. In this case, $f_c(\mathit{\Gamma}_c) + 1$ is a hypocycloid whose type is determined by the ratio $\frac{a}{b} = \frac{2n+2k}{n}$. Because n is odd and $\gcd(n,k) = 1$, this fraction is in lowest terms, so $f_c(\mathit{\Gamma}_c) + 1$ is a $(2n + 2k, n)$ hypocycloid.

For (2), if $n < 0$ (and $n > -k$), then $a > 0 > b$. By Lemma 3.3, $f_c(\mathit{\Gamma}_c) + 1$ is an epicycloid of type $(2n + 2k, -n)$. Because $n < 0$, the change of variables $\phi = n\theta$ above reverses the orientation of the curve, so $f_c(\mathit{\Gamma}_c) + 1$ is negatively-oriented.

Finally, for (3), if $n > 0$ and $n < -2k$, then $0 > a > b$. By Lemma 3.3, $f_c(\mathit{\Gamma}_c) + 1$ is an epicycloid with type determined by $\frac{-a}{a-b} = \frac{2n+2k}{-n-2k}$, rotated by π. Because this fraction is in lowest terms, the type of the epicycloid is $(2n + 2k, -n - 2k)$. ■

Remark 4.3. In light of the generalization of [BBD$^+$20] to the case of complex c in [BDH$^+$22], one might ask if $f_c(\Gamma_c)$ is also a hypocycloid or epicycloid for all complex $c \neq 0$. In the proof of Lemma 4.1, the four terms reduce to two terms because $cR_c^k - R_c^{-k} = 0$. When $c \in \mathbb{C} \setminus \mathbb{R}_+$, this equality no longer holds, so $f_c(\Gamma_c)$ is a more complicated curve and in general is not a hypocycloid or an epicycloid.

The explicit description of the caustic allows us to calculate the winding number of this curve. For ease of notation, let $W_c = W_{f_c,\Gamma_c}$ denote the winding number of $f_c(\Gamma_c)$ about the origin. We can count the total number of zeros of f_c by applying Corollary 2.7, which requires W_c, information about the leading term of f_c, and orders of poles of f_c. We note that the only pole of f_c is at the origin, which is contained in Γ_c. Since $n > -k$, the order of the pole at the origin is always k. Because the method is very similar to the method used to prove Theorem 1.1 of [BBD$^+$20] and Theorem 1.1 of [BDH$^+$22], we omit many of the details of the proof.

Theorem 4.4. *Let f_c be as in (4.1), where $c > 0$, $n, k \in \mathbb{Z}$, n odd, $\gcd(n,k) = 1$, $n \neq 0, -k$, $k \neq 0$, and $n + 2k \neq 0$. Suppose also that $n > -k$.*

(1) *If $k > 0$, let $N = \frac{|n|+1}{2}$ and let*

$$Z_0 = \begin{cases} 3n + 3k & \text{if } n > 0, \\ n + 3k & \text{if } n < 0, \end{cases} \qquad \text{and} \qquad Z_N = \begin{cases} n + 3k & \text{if } n > 0, \\ 3n + 3k & \text{if } n < 0. \end{cases}$$

Then there are N critical c-values c_j with $0 < c_1 < \cdots < c_N$ such that

- (a) *if $0 < c < c_1$, f_c has Z_0 zeros,*
- (b) *if $c_j < c < c_{j+1}$ for some $1 \leq j \leq N-1$, f_c has $Z_0 - 4j$ zeros,*
- (c) *if $c > c_N$, f_c has Z_N zeros.*

(2) *If $k < 0$, let $N = \frac{|n+2k|+1}{2}$ and let*

$$Z_0 = n - k \quad \text{and} \quad Z_N = 3n + 3k.$$

Let $\delta = \operatorname{sgn}(n + 2k)$. Then there are N critical c-values c_j with $0 < c_1 < \cdots < c_N$ such that

- (a) *if $0 < c < c_1$, f_c has Z_0 zeros,*
- (b) *if $c_j < c < c_{j+1}$ for some $1 \leq j \leq N-1$, f_c has $Z_0 + (4j-1)\delta - 1$ zeros,*
- (c) *if $c > c_N$, f_c has Z_N zeros.*

In case (1), note that $Z_N = Z_0 - 2|n|$, so f_c always loses zeros as c grows. In case (2), f_c gains zeros when $\delta = 1$ and loses zeros when $\delta = -1$.

Proof. For (1), suppose that $k > 0$. Following Corollary 2.7, we must compute the winding number W_c, the order of the pole at the origin, and the dominant term. As commented above, we have $P_{f_c,\Gamma_c} = k$. Since $n > -k$

and $k > 0$, the dominant term is $\frac{n}{n+2k}\overline{z}^{n+2k}$, and $n + 2k > 0$. It remains to compute W_c. We consider the cases $n > 0$ and $n < 0$.

If $n > 0$, Lemma 4.1 shows that $f_c(\Gamma_c)$ is a $(2n + 2k, n)$ hypocycloid centered at $z = -1$. Since $n < \frac{2n+2k}{2}$, the hypocycloid is positively oriented. The radius of the hypocycloid is scaled by $c^{-\frac{n}{2k}}$, so the hypocycloid grows indefinitely as $c \to 0$ and approaches a single point as $c \to \infty$. Following a similar argument to [BBD$^+$20, Lemma 2.5], the hypocycloid has $N = \frac{n+1}{2}$ distinct intersections with the real axis to the right of -1. All of these intersections are double intersections except for the rightmost intersection. We find

$$W_c = \begin{cases} n & \text{if } 0 < c < c_1, \\ n - 2j & \text{if } c_j < c < c_{j+1} \text{ for some } 1 \leq j \leq N - 1, \\ 0 & \text{if } c > c_N. \end{cases}$$

Substituting these winding numbers, the order of the pole, and the dominant term into (2.6) yields the claimed total numbers of zeros.

If $n < 0$, $f_c(\Gamma_c)$ is a $(2n + 2k, -n)$ negatively oriented epicycloid centered at -1. Now, as c grows, the epicycloid grows as well. By adapting the work of [BL24, Lemma 9], one can show that the epicycloid has $N = \frac{-n+1}{2}$ distinct intersections with the real axis to the right of -1, and all of these intersections are double intersections except for the leftmost intersection. We find

$$W_c = \begin{cases} 0 & \text{if } 0 < c < c_1, \\ -2j & \text{if } c_j < c < c_{j+1} \text{ for some } 1 \leq j \leq N - 1, \\ n & \text{if } c > c_N. \end{cases}$$

As above, applying (2.6) gives the total number of zeros.

For (2), suppose that $k < 0$. Since $n > -k$, we also have $n > 0$. While the order of the pole is still k, the dominant term is now z^n. To apply Corollary 2.7, we once again compute the winding number W_c. We consider the cases $n > -2k$ and $n < -2k$.

If $n > -2k$, then Lemma 4.1 shows that $f_c(\Gamma_c)$ is a $(2n + 2k, n)$ hypocycloid centered at $z = -1$. However, now $n > \frac{2n+2k}{2}$, so the hypocycloid is negatively oriented. As $c \to \infty$, the hypocycloid grows indefinitely. Adapting Lemma 2.5 of [BBD$^+$20], one can prove that the hypocycloid has $N = \frac{n+2k+1}{2}$ distinct intersections with the real axis to the right of -1, and all intersections are double intersections except for the rightmost intersection. We find

$$W_c = \begin{cases} 0 & \text{if } 0 < c < c_1, \\ -2j + 1 & \text{if } c_j < c < c_{j+1} \text{ for some } 1 \leq j \leq N - 1, \\ -n - 2k & \text{if } c > c_N. \end{cases}$$

Inputting this data into (2.5) gives the claimed total number of zeros.

If $n < -2k$, $f_c(\Gamma_c)$ is a $(2n+2k, -n-2k)$ positively-oriented epicycloid centered at $z = -1$ that has been rotated by π radians. The epicycloid grows as c grows. By adapting Lemma 9 of [BL24], one can prove that the epicycloid intersects the real axis $\frac{-n-2k+1}{2}$ distinct times to the right of -1, and all intersections are double intersections except for the leftmost intersection. We find

$$W_c = \begin{cases} 0 & \text{if } 0 < c < c_1, \\ 2j & \text{if } c_j < c < c_{j+1} \text{ for some } 1 \le j \le N-1, \\ -n-2k & \text{if } c > c_N. \end{cases}$$

Substituting the winding number into (2.5) yields the total number of zeros. ∎

5. The second subfamily. Throughout this section, we let $m = n - k \neq 0$ and $k \neq 0$ so that

$$f_c(z) = z^n - z^{-k} - 1 + c\left(\frac{n}{k}\bar{z}^k - \frac{k}{n}\bar{z}^{-n}\right), \qquad c \in \mathbb{C} \setminus \{0\}. \tag{5.1}$$

We again assume $n > -k$ and $\gcd(n,k) = 1$.

This choice of m leads to another subfamily for which the caustic is a hypocycloid or an epicycloid, so as in the previous section, we can use the Argument Principle to count the zeros. However, it is much more difficult to prove that $f_c(\Gamma_c)$ is a hypocycloid or epicycloid in this case. When $m = -2k$ and $c > 0$, $q_c(\Gamma_c)$ contains only the origin, so the proof of Lemma 4.1 is straightforward. Now with $m = n - k$, neither $p_c(\Gamma_c)$ nor $q_c(\Gamma_c)$ is $\{0\}$. Reparameterizing $q_c(\Gamma_c)$ in (3.5) by $\varphi = -\frac{n}{k}\phi$ (and taking $m = n-k$) shows that $p_c(\Gamma_c)$ and $q_c(\Gamma_c)$ are parametric curves of the same type. While we do not expect hypocycloids or epicycloids of differing types to combine to create a known parametric curve, it is also not always the case that hypocycloids or epicycloids of the same type will combine to form another such curve. In the following lemma, we show that $f_c(\Gamma_c)$ is a hypocycloid or epicycloid, which is a surprising result in light of the previous discussion. The proof requires a change of variable, and the resulting parameterized curve is somewhat unwieldy.

LEMMA 5.1 (The Duck Lemma). *Let f_c be as in* (5.1) *with $c = \rho e^{i\tau} \in \mathbb{C} \setminus \{0\}$, $n > -k$, and $\gcd(n,k) = 1$. The image $f_c(\Gamma_c)$ of the critical curve is a hypocycloid or epicycloid centered at $z = -1$ and rotated by $\alpha\frac{n-k}{n+k}$, where*

$$\alpha = \arg\left(\frac{n}{k}\rho^{\frac{n}{n-k}} e^{i\tau\frac{n}{n-k}} - \rho^{-\frac{k}{n-k}} e^{i\tau\frac{k}{n-k}}\right).$$

The image $f_c(\Gamma_c)$ is formed by rolling a circle of radius A about a circle of *radius* $\frac{n+k}{n}A$*, where*

$$A = \left| \frac{n}{k}\rho^{\frac{n}{n-k}} e^{i\tau\frac{n}{n-k}} - \rho^{-\frac{k}{n-k}} e^{i\tau\frac{k}{n-k}} \right|. \tag{5.2}$$

In particular,

1. *if* $n, k > 0$*, the image is a hypocycloid of type* $(n+k, n)$*,*
2. *if* $n > 0, k < 0$*, the image is an epicycloid of type* $(n+k, -k)$*,*
3. *if* $n < 0, k > 0$*, the image is an epicycloid of type* $(n+k, -n)$*.*

Proof. We work with $f_c(\Gamma_c) + 1$. Parameterize Γ_c by $\theta \mapsto R_c e^{i\theta}$, $\theta \in [0, 2\pi)$. Writing $c = \rho e^{i\tau}$ and recalling that $R_c = \rho^{\frac{1}{m}} = \rho^{\frac{1}{n-k}}$, we find

$$\begin{aligned} f_c(R_c e^{i\theta}) + 1 &= R_c^n e^{in\theta} - R_c^{-k} e^{-ik\theta} + \frac{cn}{k} R_c^k e^{-ik\theta} - \frac{ck}{n} R_c^{-n} e^{in\theta} \\ &= \rho^{\frac{n}{n-k}} e^{in\theta} - \rho^{-\frac{k}{n-k}} e^{-ik\theta} + \frac{n}{k}\rho^{\frac{n}{n-k}} e^{i\tau} e^{-ik\theta} - \frac{k}{n}\rho^{-\frac{k}{n-k}} e^{i\tau} e^{in\theta}. \end{aligned}$$

Let $\phi = n\theta + \tau\frac{n}{n-k}$, so that $f_c(\Gamma_c) + 1$ is parameterized by

$$\begin{aligned} \phi \mapsto{}& \rho^{\frac{n}{n-k}} e^{i(\phi - \tau\frac{n}{n-k})} - \rho^{-\frac{k}{n-k}} e^{-i\frac{k}{n}(\phi - \tau\frac{n}{n-k})} \\ &+ \frac{n}{k}\rho^{\frac{n}{n-k}} e^{i\tau} e^{-i\frac{k}{n}(\phi - \tau\frac{n}{n-k})} - \frac{k}{n}\rho^{-\frac{k}{n-k}} e^{i\tau} e^{i(\phi - \tau\frac{n}{n-k})} \\ ={}& \left(\rho^{\frac{n}{n-k}} e^{-i\tau\frac{n}{n-k}} - \frac{k}{n}\rho^{-\frac{k}{n-k}} e^{-i\tau\frac{k}{n-k}} \right) e^{i\phi} \\ &+ \left(\frac{n}{k}\rho^{\frac{n}{n-k}} e^{i\tau\frac{n}{n-k}} - \rho^{-\frac{k}{n-k}} e^{i\tau\frac{k}{n-k}} \right) e^{-i\frac{k}{n}\phi} \\ ={}& \frac{k}{n}\left(\frac{n}{k}\rho^{\frac{n}{n-k}} e^{-i\tau\frac{n}{n-k}} - \rho^{-\frac{k}{n-k}} e^{-i\tau\frac{k}{n-k}} \right) e^{i\phi} \\ &+ \left(\frac{n}{k}\rho^{\frac{n}{n-k}} e^{i\tau\frac{n}{n-k}} - \rho^{-\frac{k}{n-k}} e^{i\tau\frac{k}{n-k}} \right) e^{-i\frac{k}{n}\phi}. \end{aligned}$$

The coefficients in the last expression are complex conjugates. In particular, if we write

$$Ae^{i\alpha} = \frac{n}{k}\rho^{\frac{n}{n-k}} e^{i\tau\frac{n}{n-k}} - \rho^{-\frac{k}{n-k}} e^{i\tau\frac{k}{n-k}},$$

where $A > 0$ and $\alpha \in \mathbb{R}$, we see that $f_c(\Gamma_c) + 1$ is parameterized by

$$\phi \mapsto \frac{k}{n} A e^{-i\alpha} e^{i\phi} + A e^{i\alpha} e^{-i\frac{k}{n}\phi}.$$

Finally, let $\psi = \phi - \alpha\frac{2n}{n+k}$ to obtain the new parameterization

$$\begin{aligned}\psi \mapsto & \frac{k}{n}Ae^{-i\alpha}e^{i(\psi+\alpha\frac{2n}{n+k})} + Ae^{i\alpha}e^{-i\frac{k}{n}(\psi+\alpha\frac{2n}{n+k})} \\ &= \frac{k}{n}Ae^{i\alpha\frac{n-k}{n+k}}e^{i\psi} + Ae^{i\alpha\frac{n-k}{n+k}}e^{-i\frac{k}{n}\psi} \\ &= \frac{A}{n}e^{i\alpha\frac{n-k}{n+k}}(ke^{i\psi} + ne^{-i\frac{k}{n}\psi}).\end{aligned}$$

Thus $f_c(\Gamma_c)+1$ is the curve parameterized by $\psi \mapsto ke^{i\psi}+ne^{-i\frac{k}{n}\psi}$ dilated by a factor of $\frac{A}{n}$ and rotated by $\alpha\frac{n-k}{n+k}$. Comparing to the form of the hypocycloid or epicycloid given in Lemma 3.3, we see that $a - b = k$ and $b = n$, or, equivalently, $a = n + k$ and $b = n$.

We now consider the three cases depending on the signs of n and k. Because $n > -k$, it follows that $a = n + k > 0$ in all cases.

(1) If $n, k > 0$, then $n + k > n > 0$. Thus $a > b > 0$, so by Lemma 3.3, $f_c(\Gamma_c)$ is a hypocycloid of type $(n + k, n)$.

(2) If $n > 0$ and $k < 0$, then $n > n + k > 0$. Thus $b > a > 0$, so by Lemma 3.3, $f_c(\Gamma_c)$ is an epicycloid of type $(n + k, -k)$.

(3) If $n < 0$ and $k > 0$, then $a > 0 > b$, so $f_c(\Gamma_c)$ is an epicycloid of type $(n + k, -n)$.

Thus, $f_c(\Gamma_c) + 1$ is a hypocycloid or epicycloid formed by rolling a circle of radius A about a circle of radius $\frac{n+k}{n}A$. ■

One distinctive feature of this subfamily is that $f_c(\Gamma_c)$ shrinks to a point for an appropriate $c > 0$ and $n, k > 0$:

COROLLARY 5.2. *Let* $c_* = \left(\frac{k}{n}\right)^{\frac{n-k}{n+k}}$ *with* $n, k > 0$. *Then* $f_{c_*}(\Gamma_{c_*})$ *is the single point* $z = -1$.

Proof. A simple calculation shows that if we take $\rho = c_*$ and $\tau = 0$ in (5.2), then $A = 0$. Because $f_c(\Gamma_c)$ is formed by rolling a circle of radius A about a circle of radius $\frac{n+k}{k}A$ centered at $z = -1$, when $A = 0$, the curve reduces to the single point $z = -1$. ■

As in Section 4, the explicit description of the caustic allows us to count the total number of zeros of f_c by way of Corollary 2.7. We count the zeros of f_c for any $c \in \mathbb{R}_{>0}$; afterwards, we discuss the behavior for $c \in \mathbb{R}$ and $c \in \mathbb{C}$.

THEOREM 5.3. *Let* f_c *be as in* (5.1), *with* $c \in \mathbb{R}_{>0}$, *and let* $c_* = \left(\frac{k}{n}\right)^{\frac{n-k}{n+k}}$. *Suppose* $n > -k$ *and* $\gcd(n, k) = 1$.

(1) *Suppose* $n > k > -k$. *Let* $N = \left\lfloor\frac{k+1}{2}\right\rfloor$ *and* $M = \left\lfloor\frac{k}{2}\right\rfloor + 1$. *There are* $N + M$ *critical values* $0 < c_1 < \cdots < c_N < c_* < c_{N+1} < \cdots < c_{N+M}$.

  (a) *If* $c < c_1$ *or* $c > c_{N+M}$, *then* f_c *has* $2n + 2k$ *zeros.*

  (b) *If* $c_j < c < c_{j+1}$ *for* $1 \le j \le N - 1$, *then* f_c *has* $2n + 2k - 4j$ *zeros if* n *is odd and* $2n + 2k - 4j + 2$ *zeros if* n *is even.*

   (c) *If* $c_N < c < c_{N+1}$*, then* f_c *has* $2n$ *zeros.*
   (d) *If* $c_{N+j} < c < c_{N+j+1}$ *for* $1 \le j \le M-1$*, then* f_c *has* $2n+4j-2$ *zeros.*

   The numbers of zeros for the case $k > n > 0 > -k$ *are similar, but with the roles of* n *and* k *and the orientation of* $f_c(\Gamma_c)$ *reversed.*

(2) *Suppose* $n > -k > k$*. Let* $P = \left\lfloor \frac{-k+1}{2} \right\rfloor$*. There are* P *critical values* $0 < c_1 < \cdots < c_P$*.*

   (a) *If* $c < c_1$*, then* f_c *has* $2n$ *zeros.*
   (b) *If* $c_j < c < c_{j+1}$ *for* $1 \le j \le P-1$*, then* f_c *has* $2n-4j$ *zeros if* n *is odd and* $2n-4j+2$ *zeros if* n *is even.*
   (c) *If* $c > c_P$*, then* f_c *has* $2n+2k$ *zeros.*

   The numbers of zeros for the case $k > 0 > n > -k$ *are similar, but with the roles of* n *and* k *and the orientation of* $f_c(\Gamma_c)$ *reversed.*

Proof. In each case, the sense-reversing region is inside the critical curve, so the pole at the origin is of order $-n$. The dominant term in both cases is z^n. As before, we denote the winding number of the $f_c(\Gamma_c)$ about the origin as W_c.

(1) We treat the case $n > k > -k$. (The case $k > n > 0 > -k$ is similar, except the roles of n and k are reversed and $f_c(\Gamma_c)$ is positively-oriented.) Because $n, k > 0$, by Lemma 5.1, the image of the critical curve is an $(n+k, n)$ hypocycloid. Using an argument similar to that in [BBD+20, Lemma 2.5], one can count the intersections of this hypocycloid with the ray $\mathbb{R}_{>-1}$, along with their multiplicities.

For $0 < c < c_*$ small enough, all intersections of the hypocycloid that are to the right of the center are also right of the origin. As c increases from 0 to c_*, the hypocycloid shrinks down to the point $z = -1$ and has $N = \left\lfloor \frac{k+1}{2} \right\rfloor$ distinct intersections with the ray $\mathbb{R}_{>-1}$. For $c > c_*$, as c increases, the hypocycloid grows indefinitely and has $M = \left\lfloor \frac{k}{2} \right\rfloor + 1$ distinct intersections with $\mathbb{R}_{>-1}$. Thus there are $N + M$ critical values $0 < c_1 < \cdots < c_N < c_* < c_{N+1} < \cdots < c_{N+M}$ at which the winding number changes. We find

$$W_c = \begin{cases} -k & \text{if } c < c_1 \text{ or if } c > c_{N+M}, \\ -k+2j & \text{if } c_j < c < c_{j+1} \text{ for } 1 \le j \le N-1 \text{ and } n \text{ odd}, \\ -k+2j-1 & \text{if } c_j < c < c_{j+1} \text{ for } 1 \le j \le N-1 \text{ and } n \text{ even}, \\ 0 & \text{if } c_N < c < c_{N+1}, \\ -2j+1 & \text{if } c_{N+j} < c < c_{N+j+1} \text{ for } 1 \le j \le M-1. \end{cases}$$

Substituting these values into (2.5) yields the claimed total numbers of zeros.

(2) We treat the case $n > -k > k$. (The case $k > 0 > n > -k$ is similar, except the roles of n and k are reversed and $f_c(\mathit{\Gamma}_c)$ is negatively-oriented.) By Lemma 5.1, the image of the critical curve is an $(n+k, -k)$ epicycloid. Using an argument similar to that in [BL24, Lemma 9], one can count the intersections of the epicycloid with the ray $\mathbb{R}_{>-1}$ and determine their multiplicities.

For $c > 0$, the epicycloid has $P = \lfloor \frac{-k+1}{2} \rfloor$ intersections with $\mathbb{R}_{>-1}$. As c approaches 0, the epicycloid shrinks to be entirely to the left of the imaginary axis, and as c approaches infinity, the epicycloid grows indefinitely. Thus there are P critical values $0 < c_1 < \cdots < c_P$ at which the winding number changes. We find that

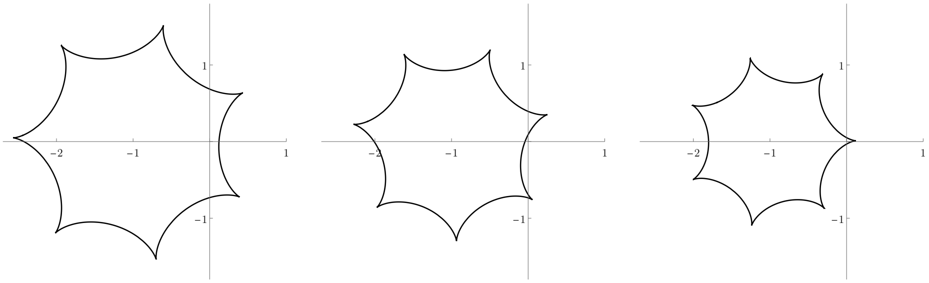
$$W_c = \begin{cases} 0 & \text{if } c < c_1, \\ 2j & \text{if } c_j < c < c_{j+1} \text{ for } 1 \le j \le P-1 \text{ and } n \text{ odd}, \\ 2j-1 & \text{if } c_j < c < c_{j+1} \text{ for } 1 \le j \le P-1 \text{ and } n \text{ even}, \\ -k & \text{if } c > c_P. \end{cases}$$

Substituting these values into (2.5) yields the claimed total numbers of zeros. ■

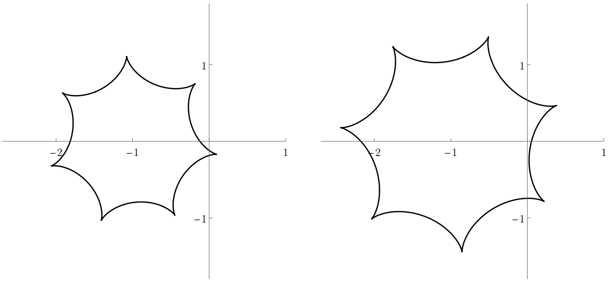
Remark 5.4. One can count the zeros of f_c for $c < 0$ using similar methods. We do not state the result here because the behavior in the epicycloid case is more complicated; if we allow $c < 0$, the epicycloids shrink down to a point as c increases and then rotate and grow, similar to the behavior we see above with the hypocycloids. However, no fundamentally new behaviors are exhibited.

Example 5.5. In Lemma 5.1, we showed that both the size and rotation of $f_c(\mathit{\Gamma}_c)$ depend on ρ and τ (where $c = \rho e^{i\tau}$). These dual dependences are notably different from Lemma 3.4 of [BDH$^+$22], where (for fixed n and k) the size depended only on ρ and the rotation depended only on τ. Because now the size and rotation of $f_c(\mathit{\Gamma}_c)$ are much more complex, it is impractical to state an analogue of Theorem 5.3 for non-real c.

We give an example to illustrate this impracticality. In Theorem 5.3, the number of zeros changes monotonically as $c > 0$ increases until c reaches some intermediary value c_*, at which point the number of zeros changes monotonically in the opposite direction. This pattern does not hold generally for complex c. Consider f_c as in (5.1) with $n = 6$, $k = 1$, and $\tau = \frac{\pi}{4}$. Figure 2 demonstrates that the winding number of $f_c(\mathit{\Gamma}_c)$ alternates between -1 and 0 as ρ increases. For the ρ values given in the subfigures, the numbers of zeros of f_c are respectively 14, 12, 14, 12, 14, which breaks the pattern of Theorem 5.3.



(a) $c = 0.1e^{i\frac{\pi}{4}}$ (b) $c = 0.15e^{i\frac{\pi}{4}}$ (c) $c = 0.215e^{i\frac{\pi}{4}}$



(d) $c = 0.25e^{i\frac{\pi}{4}}$ (e) $c = 0.36e^{i\frac{\pi}{4}}$

Fig. 2. The caustic for $n = 6$ and $k = 1$. The winding number for each plot alternates between -1 and 0.

6. Counting zeros for the general family. For most members of the family (1.3), the caustic is not a known parametric curve, so it is not possible to use the Argument Principle to obtain the sorts of detailed zero-counting theorems obtained for the two subfamilies discussed above. However, it is still often possible to count zeros using Rouché's Theorem. Although many such theorems are possible for the general family under different hypotheses on n, k, and m, we illustrate the idea by proving just one such theorem:

THEOREM 6.1. *Let f_c be as in* (1.3) *with $c = \rho e^{i\tau} \neq 0$, $n, k, m > 0$, and $n - m > 0$. Then there exist positive constants ρ_1 and ρ_N such that*

(1) *if $0 < \rho < \rho_1$, f_c has $n + 3k + m$ zeros,*
(2) *if $\rho > \rho_N$, f_c has $3n + k - m$ zeros.*

The idea of the proof is to use the decomposition from Proposition 3.5 to write

$$f_c(z) = p_c(z) + q_c(z) - 1$$

where, as in (3.4),

$$p_c(z) = z^n + \frac{cn}{n-m}\overline{z}^{n-m} \quad \text{and} \quad q_c(z) = -z^{-k} - \frac{ck}{k+m}\overline{z}^{-k-m}.$$

Recall from Proposition 3.5 that $p_c(\mathit{\Gamma}_c)$ is scaled by a factor of $\rho^{n/m}$ and

$q_c(\mathit{\Gamma}_c)$ is scaled by a factor of $\rho^{-k/m}$. Thus when ρ is large and z is on the critical circle, p_c dominates, whereas when ρ is small and z is on the critical circle, q_c dominates. We can therefore use Rouché's Theorem to compare $Z_{f_c,\mathit{\Gamma}_c} - P_{f_c,\mathit{\Gamma}_c}$ to the corresponding sums for p_c or q_c. The proof of the theorem requires the next lemma.

LEMMA 6.2. *Let p_c and q_c be as in* (3.4) *with $c = \rho e^{i\tau} \neq 0$, $n, k, m > 0$, and $n - m > 0$. Then*

(1) $Z_{p_c,\mathit{\Gamma}_c} - P_{p_c,\mathit{\Gamma}_c} = m - n$,
(2) $Z_{q_c,\mathit{\Gamma}_c} - P_{q_c,\mathit{\Gamma}_c} = -k$.

Proof. We first calculate the complex dilatation for p_c and q_c. Observe that

$$\omega_{p_c}(z) = \omega_{q_c}(z) = \frac{c}{z^m},$$

which has modulus 1 if and only if $|z| = \rho^{1/m}$. In other words, the circle $\mathit{\Gamma}_c$ separates the sense-preserving and sense-reversing regions for all three functions f_c, p_c, and q_c, with the sense-reversing region inside $\mathit{\Gamma}_c$ because $m > 0$.

For (1), note that p_c has a zero at the origin of order $-(n-m)$. Furthermore, if $z = re^{i\theta} \neq 0$ is any other zero of p_c,

$$r^n e^{in\theta} = -\frac{cn}{n-m} r^{n-m} e^{i(m-n)\theta}.$$

Taking the modulus of both sides and solving for r gives

$$r = \rho^{1/m}\left(\frac{n}{n-m}\right)^{1/m} > \rho^{1/m}.$$

Thus all other zeros of p_c are outside $\mathit{\Gamma}_c$. Because p_c has no poles, (1) follows.

For (2), q_c has a pole at the origin of order $-k-m$. Furthermore, if $z = re^{i\theta} \neq 0$ is a zero of q_c, then

$$r^{-k} e^{-ik\theta} = -\frac{ck}{k+m} r^{-k-m} e^{i(k+m)\theta} \tag{6.1}$$

and hence

$$r = \rho^{1/m}\left(\frac{k}{k+m}\right)^{1/m} < \rho^{1/m}.$$

Thus all zeros lie on a single circle in the sense-reversing region. Because (6.1) is satisfied by $2k + m$ values of θ and all zeros of q_c are simple,

$$Z_{q_c,\mathit{\Gamma}_c} - P_{q_c,\mathit{\Gamma}_c} = -(2k+m) - (-k-m) = -k,$$

as claimed. ∎

To bridge the gap between the behavior inside $\mathit{\Gamma}_c$ and the total number of zeros in the entire plane, we require the sum of the orders of all of the zeros.

LEMMA 6.3. *Let $n, k, m > 0$. If f_c has no singular zeros, then the sum of the orders of zeros of f_c in $\mathbb{C}$ is $n - k - m$.*

Proof. The proof uses Rouché's Theorem applied to a circle C_R centered at the origin of large radius. We have

$$Z_{f_c,C_R} = Z_{d,C_R} - P_{d,C_R} + P_{f_c,C_R},$$

where d denotes the dominant term of f_c. By our conditions on n, k, and m, we see that the dominant term of f_c is z^n, so $Z_{d,C_R} - P_{d,C_R} = n$. The order of the pole of f_c at the origin is $-k - m$. Thus, $Z_{f_c,C_R} = n - k - m$. ■

Proof of Theorem 6.1. In order to determine the total number of zeros, we need Z_{f_c,Γ_c}, the sum of the orders of the zeros in the sense-reversing region. We determine this number for sufficiently small and sufficiently large ρ using Rouché's Theorem.

For (1), we expect q_c to dominate on the critical circle for small ρ. By the calculations in Proposition 3.5, for $z \in \Gamma_c$ and $\rho \le 1$,

$$|p_c(z) - 1| \le |p_c(z)| + 1 \le \frac{2n - m}{n - m}\rho^{n/m} + 1 \le \frac{2n - m}{n - m} + 1.$$

On the other hand, for $z \in \Gamma_c$,

$$|q_c(z)| = \frac{1}{k + m}\rho^{-k/m}|(k + m)e^{i\phi} + ke^{i\phi(-\frac{k}{k+m})}| \ge \frac{m}{k + m}\rho^{-k/m}.$$

Therefore $|p_c(z) - 1| < |q_c(z)|$ on Γ_c if

$$\rho < \rho_1 := \left(\frac{m}{k + m} \cdot \frac{n - m}{3n - 2m}\right)^{m/k},$$

which is less than 1. Thus for $\rho < \rho_1$,

$$Z_{f_c,\Gamma_c} - P_{f_c,\Gamma_c} = Z_{q_c,\Gamma_c} - P_{q_c,\Gamma_c} = -k,$$

which, because f_c has a pole of order $-k - m$ at the origin, yields

$$Z_{f_c,\Gamma_c} = -2k - m.$$

Furthermore, because the sum of the orders of the zeros of f_c in the plane is $n - k - m$, the sum S of the orders of the zeros outside Γ_c satisfies

$$S - 2k - m = n - k - m.$$

That is, $S = n + k$. Because f_c has $2k + m$ zeros in the sense-reversing region and $n + k$ in the sense-preserving region, it has a total of $n + 3k + m$ zeros.

For (2), we expect p_c to be dominant on Γ_c for ρ large. For $z \in \Gamma_c$ and $\rho \ge 1$,

$$|q_c(z) - 1| \le |q_c(z)| + 1 \le \frac{2k + m}{k + m}\rho^{-k/m} + 1 \le \frac{2k + m}{k + m} + 1.$$

On the other hand, for $z \in \mathit{\Gamma}_c$,

$$|p_c(z)| \geq \frac{m}{n-m}\rho^{n/m}.$$

Thus $|q_c(z) - 1| < |p_c(z)|$ on $\mathit{\Gamma}_c$ if

$$\rho > \rho_N := \max\left\{1, \left(\frac{3k+2m}{k+m} \cdot \frac{n-m}{n}\right)^{m/n}\right\}.$$

Thus for $\rho > \rho_N$,

$$Z_{f_c,\mathit{\Gamma}_c} - P_{f_c,\mathit{\Gamma}_c} = Z_{p_c,\mathit{\Gamma}_c} - P_{p_c,\mathit{\Gamma}_c} = m - n,$$

which, because f_c has a pole of order $-k-m$ at the origin, yields

$$Z_{f_c,\mathit{\Gamma}_c} = -n - k.$$

Furthermore, because the sum of the orders of the zeros of f_c in the plane is $n-k-m$, the sum S of the orders of the zeros outside $\mathit{\Gamma}_c$ satisfies

$$S - n - k = n - k - m.$$

That is, $S = 2n - m$. Because f_c has $n+k$ zeros in the sense-reversing region and $2n-m$ in the sense-preserving region, it has a total of $3n+k-m$ zeros. ■

7. Conclusion.

The family of harmonic functions considered in this paper is the natural five-term analogue of the families of trinomials studied in [BBD+20, BDH+22, BL24], yet we can only prove complete analogues of the main zero-counting theorems for a few subfamilies, and then only for real c. For other members of the family, we proved a result that applies only to large and small $|c|$. The difficulty of analyzing this family—even when $f_c(\mathit{\Gamma}_c)$ is a known parametric curve—indicates that the geometric methods used by [BBD+20, BDH+22, BL24] cannot easily be applied to more general families.

In light of these difficulties, we are in need of new methods for counting the zeros of complex-valued harmonic functions. Currently, the main alternative to these geometric arguments is a Rouché's Theorem argument. Our work in Section 6 shows that there is a way to refine these arguments for functions that decompose as a sum of harmonic functions whose zeros are already understood. One possible direction for future work would involve constructing new families by taking combinations of well-understood families to determine whether or how properties of the constituent pieces are reflected in properties of the whole.

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