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# Growth of masses of crystalline measures

by

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**Abstract.** Let $\mu$ be a measure on the Euclidean space $\mathbb{R}^d$ of unbounded total variation that is positive or translation bounded, and suppose its Fourier transform $\hat{\mu}$ in the sense of distributions is a pure point measure. We prove that the measure $\nu$ with the same support as $\hat{\mu}$ and masses equal to the squares of the masses of $\hat{\mu}$ is translation bounded. We also prove that if $\mu$ is as above and the restriction of its spectrum, i.e., of the support of $\hat{\mu}$, to each ball of fixed radius is a linearly independent set over $\mathbb{Z}$, then the measure $\hat{\mu}$ is also translation bounded. These results imply certain conditions for a crystalline measure to be a Fourier quasicrystal.

**1. Introduction.** A complex measure $\mu$ on the Euclidean space $\mathbb{R}^d$ with a locally finite support (that is, the support has finite intersection with any compact set) is called *crystalline* if $\mu$ is a tempered distribution and its Fourier transform $\hat{\mu}$ in the sense of distributions is also a measure with locally finite support. If both measures $|\mu|$ and $|\hat{\mu}|$ are tempered distributions, then $\mu$ is called a *Fourier quasicrystal*. Here and below we denote by $|\nu|(E)$ the variation of the complex measure $\nu$ on the set $E$.

Recently, Fourier quasicrystals and crystalline measures have been studied very actively. Many works are devoted to the investigations of their properties (see, for example, the survey papers [17, 12, 1]). In fact, Fourier quasicrystals are the form of Poisson formulas (see below); the latter were used in particular by D. Radchenko and M. Viazovska [20].

Special attention has been paid to Fourier quasicrystals of the form

$$\mu = \sum_{\lambda\in\Lambda} \delta_\lambda, \quad \lambda \in \mathbb{R}^d \tag{1.1}$$

($\delta_\lambda$, as usual, means the unit mass at $\lambda$). A. Olevskii and A. Ulanovskii [18, 19] for the case $d = 1$ established 1-1 correspondence between such mea-

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sures and sets of zeros of real-rooted exponential polynomials. W. Lawton and A. Tsikh [12] partially extended this result to an arbitrary dimension $d$. In [8] the result of Olevskii and Ulanovskii was generalized to Dirichlet series. F. Gonçalves [10] and F. Gonçalves and G. Vedana [11] investigated arbitrary measures $\mu$ on $\mathbb{R}$ with pure point measures $\hat{\mu}$ and obtained an analog of the result of Olevskii and Ulanovskii. Measures of the form (1.1) with $\lambda$ in a horizontal strip of finite width were studied in [9].

Returning to the definition of a Fourier quasicrystal, it is natural to ask whether we need additional conditions on the variations of the measures $\mu$ and $\hat{\mu}$ in order to prove that a crystalline measure is a Fourier quasicrystal. The answer is positive, as in [7] the second author constructed a crystalline measure which is not a Fourier quasicrystal.

In this paper, we investigate various additional conditions under which a crystalline measure becomes a Fourier quasicrystal. To this end, we investigate the properties of arbitrary tempered measures in $\mathbb{R}^d$ with a purely point Fourier transform $\hat{\mu}$. For $d = 1$, this is precisely the class of measures that was considered in [10, 11], but our study has no other common points with these papers.

To formulate our results, we recall some notions and properties related to the Fourier transform (see, e.g., [22]):

Denote by $\mathcal{D}$ the space of $C^\infty$-functions on $\mathbb{R}^d$ with compact support and by $\mathcal{S}$ the Schwartz space of $C^\infty$-functions $\varphi(t)$ on $\mathbb{R}^d$ with finite norms

$$\mathcal{N}_m(\varphi) = \sup_{\mathbb{R}^d} \{\max\{1, |t|^m\} \cdot \max_{\|k\| \le m} |D^k \varphi(t)|\}, \quad m = 0, 1, 2, \dots,$$

where

$$k = (k_1, \dots, k_d) \in (\mathbb{N} \cup \{0\})^d, \quad \|k\| = k_1 + \dots + k_d, \quad D^k = \partial_{x_1}^{k_1} \dots \partial_{x_d}^{k_d}.$$

The Fourier transform

$$\hat{\varphi}(y) = \int_{\mathbb{R}^d} \varphi(t) e^{-2\pi i \langle t, y\rangle}\, dt$$

is a continuous bijection of $\mathcal{S}$ onto $\mathcal{S}$. The elements of the space $\mathcal{S}'$ of all continuous linear functionals on $\mathcal{S}$ are called *tempered distributions*. For every $\Phi \in \mathcal{S}'$ its Fourier transform $\hat{\Phi}$ is defined by the equality

$$\hat{\Phi}(\varphi) = \Phi(\hat{\varphi}), \qquad \forall \varphi \in \mathcal{D}. \tag{1.2}$$

Since $\mathcal{D}$ is dense in $\mathcal{S}$, we find that $\hat{\Phi}$ belongs to $\mathcal{S}'$.

A positive measure $\nu$ on $\mathbb{R}^d$ is tempered if and only if

$$\log \nu(B(0, r)) = O(\log r), \qquad r \to \infty,$$

where, as usual, $B(x,r)$ denotes the ball with center $x$ and radius $r$ (see, e.g., [5, Lemma 1]). It is clear that for a pure point measure

$$\hat{\mu} = \sum_{\gamma\in\varGamma} b_\gamma \delta_\gamma \tag{1.3}$$

with a countable $\varGamma$, the measure $|\hat{\mu}|$ is tempered if and only if

$$\log \sum_{\gamma\in\varGamma,\,|\gamma|<r} |b_\gamma| = O(\log r), \qquad r\to\infty.$$

A measure $\mu$ in $\mathbb{R}^d$ is *translation bounded* if

$$\sup_{x\in\mathbb{R}^d} |\mu|(B(x,1)) < \infty$$

(see [14]). Clearly, every translation bounded measure $\mu$ satisfies the condition

$$|\mu|(B(x,r)) \le Cr^d, \qquad r\ge 1,$$

with some $C<\infty$. In particular, it follows that $|\mu|$ is tempered.

When $\mu$ is a tempered measure and $\hat{\mu}$ is defined as in (1.3), the equality (1.2) has the form

$$\sum_{\gamma\in\varGamma} b_\gamma \varphi(\gamma) = \int_{\mathbb{R}^d} \hat{\varphi}(x)\,\mu(dx), \qquad \forall \varphi\in\mathcal{D}.$$

For $\mu=\sum_{\lambda\in\varLambda} a_\lambda\delta_\lambda$ the above equality becomes

$$\sum_{\gamma\in\varGamma} b_\gamma \varphi(\gamma) = \sum_{\lambda\in\varLambda} a_\lambda \hat{\varphi}(\lambda), \qquad \forall \varphi\in\mathcal{D},$$

and is called the *generalized Poisson formula.* If $a_\lambda\ne 0$ for all $\lambda\in\varLambda$ and $b_\gamma\ne 0$ for all $g\in\varGamma$, we say that $\varLambda$ is the *support* of $\mu$ and $\varGamma$ is the *spectrum* of $\mu$.

With the above notations and properties, we have the following result:

THEOREM 1.1 ([6]). *Let $\mu$ and $\hat{\mu}$ be tempered measures on $\mathbb{R}^d$ and let $\mu$ have a polynomially discrete support, that is,*

$$|t-t'| \ge c\min\{1,\,|t|^{-h}\} \qquad \forall t,t'\in \operatorname{supp}\mu.$$

*Then $|\mu|$ is tempered. If, in addition, $\hat{\mu}$ has a polynomially discrete support, then $\mu$ is a Fourier quasicrystal.*

In this paper, we obtain the following results:

THEOREM 1.2. *Let $\mu$ be a positive or translation bounded measure on $\mathbb{R}^d$ and its Fourier transform $\hat{\mu}$ be a pure point measure* (1.3). *Then the measure*

$$\nu = \sum_{\gamma\in\varGamma} |b_\gamma|^2 \delta_\gamma \tag{1.4}$$

*is translation bounded. If $\Gamma$ is locally finite and the numbers $\#(\Gamma \cap B(0,r))$ grow polynomially as $r \to \infty$, then $|\hat{\mu}|$ is tempered. If, in addition, $\mu$ is a crystalline measure, then $\mu$ is a Fourier quasicrystal.*

THEOREM 1.3. *Let $\mu$ be a positive or translation bounded measure on $\mathbb{R}^d$ and suppose its Fourier transform $\hat{\mu}$ is a pure point measure* (1.3)*. If there exists $\eta > 0$ such that the sets $\Gamma \cap B(x,\eta)$ are linearly independent over $\mathbb{Z}$ for all $x \in \mathbb{R}^d$, then $\hat{\mu}$ is translation bounded. In particular, if $\mu$ is a positive crystalline measure with spectrum linearly independent over $\mathbb{Z}$, then $\mu$ is a Fourier quasicrystal.*

Recall that the set $X \subset \mathbb{R}^d$ *is linearly independent over* $\mathbb{Z}$ if for any finite number of elements $x_1, \dots, x_n \in X$ and arbitrary $m_1, \dots, m_n \in \mathbb{Z}$ the equality $m_1x_1 + \dots + m_nx_n = 0$ implies $m_1 = \dots = m_n = 0$. Note that crystalline measures with supports linearly independent over $\mathbb{Z}$ were considered by Y. Meyer [15].

COROLLARY 1.4. *Let the measure $\mu$ from Theorem* 1.3 *have a uniformly discrete* ([1]) *support $\Lambda$ and let $\inf_{\lambda\in\Lambda} |\mu(\lambda)| > 0$. Then there exist a positive integer $N$, lattices $L_1, \dots, L_N$ in $\mathbb{R}^d$ of rank d* (*some of them may coincide*)*, points $\lambda_1, \dots, \lambda_N \in \Lambda$, and bounded sets $S_1, \dots, S_N \subset \mathbb{R}^d$ such that the support of $\mu$ is a finite union of shifts of these lattices and*

$$\mu = \sum_{j=1}^{N} \sum_{x\in L_j+\lambda_j} \Big[\sum_s \beta_{j,s} e^{2\pi i\langle x,\alpha_{j,s}\rangle}\Big]\delta_x \tag{1.5}$$

*with $\alpha_{j,s} \in S_j$, $\beta_{j,s} \in \mathbb{C}$, and $\sum_s |\beta_{j,s}| < \infty$ for all $j = 1, \dots, N$.*

The paper is organized as follows. In Section 2 we collect the definitions and properties of almost periodic functions. Three necessary preliminary results are given in Section 3. Section 4 is devoted to the proofs of Theorems 1.2 and 1.3.

**2. Almost periodic functions.** The proofs of our theorems are based on some results of the classical theory of almost periodic functions (see [2] for $d = 1$ and [14, 21] for arbitrary $d$).

DEFINITION 2.1. A continuous function $f$ on $\mathbb{R}^d$ is *almost periodic* if for any $\varepsilon > 0$ the set of $\varepsilon$-*almost periods* of $f$,

$$E_\varepsilon = \Big\{\tau \in \mathbb{R}^d : \sup_{x\in\mathbb{R}^d} |f(x+\tau) - f(x)| < \varepsilon\Big\},$$

is relatively dense, i.e., $E_\varepsilon \cap B(x,L) \neq \emptyset$ for all $x \in \mathbb{R}^d$ and some $L$ depending on $\varepsilon$.

([1]) A set $A$ is *uniformly discrete* if $\inf_{x,x'\in A,\, x\neq x'} |x - x'| > 0$.

Each almost periodic function in $\mathbb{R}^d$ is bounded, a finite sum or product of almost periodic functions is also almost periodic, and the uniform limit in $\mathbb{R}^d$ of almost periodic functions is almost periodic too. In particular, every absolutely convergent Dirichlet series

$$D(x) = \sum_{\omega\in\Omega} a_\omega e^{2\pi i\langle x,\omega\rangle}, \qquad \sum_{\omega\in\Omega} |a_\omega| < \infty, \tag{2.1}$$

with a countable $\Omega \subset \mathbb{R}^d$ is almost periodic. In this case

$$\hat{D} = \sum_{\omega\in\Omega} a_\omega \delta_\omega.$$

For every almost periodic $f$ and every $\omega \in \mathbb{R}^d$ one considers the Fourier coefficients $c_\omega(f) \in \mathbb{C}$ defined by

$$c_\omega(f) = \lim_{R\to\infty} \frac{1}{v_d R^d} \int_{B(x,R)} f(t) e^{-2\pi i\langle t,\omega\rangle}\, dt,$$

where $v_d$ is the volume of the unit ball in $\mathbb{R}^d$. The limit exists uniformly in $x \in \mathbb{R}^d$. The set $\{\omega : c_\omega(f) \neq 0\}$ is at most countable. It is easy to check that the uniform convergence of the series (2.1) implies $c_\omega(D) = a_\omega$ for all $\omega \in \mathbb{R}^d$. Then an analog of the Parseval identity holds:

$$\sum_{\omega\in\mathbb{R}^d} |c_\omega(f)|^2 = \lim_{R\to\infty} \frac{1}{v_d R^d} \int_{B(x,R)} |f(t)|^2\, dt. \tag{2.2}$$

Note that by a result of Y. Meyer [16, Theorem 3.8], if the Fourier transform $\hat{f}$ of an almost periodic function $f$ is a measure, then it has the form

$$\hat{f} = \sum_{\omega\in\Omega} c_\omega(f)\delta_\omega, \qquad \sum_{|\omega|<r} |c_\omega(f)| < \infty, \quad \forall r < \infty.$$

**3. Preliminary results.** We shall need the following multidimensional version of the Kronecker lemma (cf. [3, Chapter III]). We provide a proof for completeness.

PROPOSITION 3.1. *Let $x_1, \dots, x_N \in \mathbb{R}^d$ be linearly independent over $\mathbb{Z}$ and $\theta_1, \dots, \theta_N$ be arbitrary real numbers. Then for every $\varepsilon > 0$ there exist $t \in \mathbb{R}^d$ and $p_j \in \mathbb{Z}$, $j = 1, \dots, N$, such that*

$$|\langle x_j, t\rangle - \theta_j - p_j| < \varepsilon, \quad j = 1, \dots, N. \tag{3.1}$$

*Proof.* We set

$$f(t) = 1 + e^{2\pi i[\langle x_1,t\rangle - \theta_1]} + \dots + e^{2\pi i[\langle x_N,t\rangle - \theta_N]}.$$

It is enough to prove that $\sup_{t\in\mathbb{R}^d} |f(t)| = N + 1$. For $q \in \mathbb{N}$ we have

$$f^q(t) = \sum_s \alpha_s e^{2\pi i\langle\beta_s,t\rangle}, \tag{3.2}$$

where

$$\beta_s = m_1 x_1 + \cdots + m_N x_N, \tag{3.3}$$

with $m_j \in \mathbb{Z}$, $0 \le m_j \le q$, $j = 1, \ldots, N$, and $m_1 + \cdots + m_N \le q$. Therefore, the number of terms on the right-hand side of (3.2) is at most $(q+1)^N$.

Further, $f^q(t) = (1 + z_1 + \cdots + z_N)^q$, where $z_j = e^{2\pi i[\langle x_j, t\rangle - \theta_j]}$, and every term can be written as

$$C_{m_1,\ldots,m_N} e^{2\pi i[\langle m_1 x_1 + \cdots + m_N x_N, t\rangle - \phi_s]},$$

where

$$\phi_s = m_1\theta_1 + \cdots + m_N\theta_N,$$

and the (multi-binomial) coefficient $C_{m_1,\ldots,m_N}$ coincides with the coefficient of $z_1^{m_1} \cdots z_N^{m_N}$ in the expansion of $(1 + z_1 + \cdots + z_N)^q$, and is therefore positive.

It follows from the linear independence of $x_1, \ldots, x_N$ that the monomial corresponding to $\beta_s$ is uniquely determined. Hence the corresponding coefficient $\alpha_s$ has the form

$$\alpha_s = C_{m_1,\ldots,m_N} e^{-2\pi i\phi_s}. \tag{3.4}$$

Consequently, we have

$$\sum_s |\alpha_s| = \sum_{m_1,\ldots,m_N} C_{m_1,\ldots,m_N} = (N+1)^q.$$

On the other hand,

$$\alpha_s = \lim_{R\to\infty} \frac{1}{v_d R^d} \int_{B(x,R)} e^{-2\pi i\langle t,\beta_s\rangle} f^q(t)\,dt.$$

Therefore, if $\sup_{t\in\mathbb{R}^d} |f(t)| = r < N+1$, then $|\alpha_s| \le r^q$ and $\sum_s |\alpha_s| \le r^q(q+1)^N$. But this is impossible because $r^q(q+1)^N < (N+1)^q$ for large enough $q$. ∎

Remark 3.2. Without the linear independence assumption we may have two distinct representations for some $\beta_s$ in (3.3). Then the above argument is valid if a unique $\phi_s$ in (3.4) (up to an integer) corresponds to these two representations (collections of $m_j$). Therefore, we can replace the linear independence condition of $x_j$ with the following: "if $m_1x_1 + \cdots + m_Nx_N = 0$, then $m_1\theta_1 + \cdots + m_N\theta_N \in \mathbb{Z}$". It is easy to prove that the latter condition is also necessary for the existence of $t \in \mathbb{R}^d$ in (3.1) for all $\varepsilon > 0$. The one-dimensional version of the Kronecker lemma was stated in exactly this way in [13].

We also need two results on the connection between measures and their Fourier transforms.

PROPOSITION 3.3 (cf. [8, Theorem 1]). *If $\mu$ is a nonnegative tempered measure on $\mathbb{R}^d$ and $\hat{\mu}$ is a complex measure, then $\mu$ is translation bounded.*

*Proof.* For a $C^\infty$-function $\psi(t) \not\equiv 0$ such that $\operatorname{supp}\psi \subset B(0,1)$ we consider

$$\varphi(y) = \int_{\mathbb{R}^d} \overline{\psi(t-y)}\psi(t)\,dt.$$

Then we obtain

$$\operatorname{supp}\varphi \subset B(0,2), \quad \hat{\varphi}(x) = |\hat{\psi}(x)|^2 \geq 0, \quad \hat{\varphi}(x) \not\equiv 0,$$

and therefore there exist $\eta > 0$, $r > 0$, and $x_0 \in \mathbb{R}^d$ such that $\hat{\varphi}(x) > \eta$ for all $x \in B(x_0, r)$. Then for any $t \in \mathbb{R}^d$ we consecutively have

$$\begin{aligned}
\mu(B(t,r)) &\leq \eta^{-1} \int_{B(t,r)} \hat{\varphi}(x - t + x_0)\,\mu(dx) \\
&\leq \eta^{-1} \int_{\mathbb{R}^d} \hat{\varphi}(x - t + x_0)\,\mu(dx) \\
&= \eta^{-1} \int_{\mathbb{R}^d} \varphi(y) e^{2\pi i \langle t - x_0, y\rangle}\,\hat{\mu}(dy) \\
&\leq \eta^{-1} \max_{\mathbb{R}^d} |\varphi(y)|\,|\hat{\mu}|(B(0,2)).
\end{aligned}$$

Since $|\hat{\mu}|$ is a measure, we conclude that $|\hat{\mu}|(B(0,2))$ is finite and the measure $\mu$ is translation bounded. ■

PROPOSITION 3.4. *Let $\mu$ be a translation bounded measure on $\mathbb{R}^d$ and $\varphi \in \mathcal{S}$. Then the function $\mu \star \varphi$ is bounded on $\mathbb{R}^d$ by a constant that depends neither on shifts of $\mu$ or $\varphi$ nor on their multiplying by $e^{iat}$, $a \in \mathbb{R}$.*

*Proof.* For $E \subset \mathbb{R}^d$, $h \in \mathbb{R}^d$, set

$$\varphi_h(t) = \varphi(t+h), \qquad \mu_h(E) = \mu(E+h), \qquad \nu(E) = \mu(-E).$$

Then for any $h_1, h_2, x \in \mathbb{R}^d$ we get

$$(\mu_{h_2} \star \varphi_{h_1})(x) = \int_{\mathbb{R}^d} \varphi(x - t + h_1)\,\mu_{h_2}(dt) = \int_{\mathbb{R}^d} \varphi(t)\,\nu_{-x-h_1-h_2}(dt).$$

Since the measure $\nu$ is also translation bounded, we obtain

$$|\nu_{-x-h_1-h_2}|(B(0,r)) \leq C_1(\max\{1, r\})^d,$$

where $C_1$ does not depend on $h_1, h_2, x$. Since $\varphi \in \mathcal{S}$, we have

$$|\varphi(t)| \leq C_2(\max\{1,\, |t|\})^{-d-1}.$$

It follows that

$$|(\mu_{h_2} \star \varphi_{h_1})(x)| \le C_2\Big[|\nu_{-x-h_1-h_2}|(B(0,1)) + \int\limits_{|t|>1} |s|^{-d-1}\, |\nu_{-x-h_1-h_2}|(ds)\Big]$$

$$= C_2(d+1)\int\limits_1^\infty \frac{|\nu_{-x-h_1-h_2}|(B(0,s))}{s^{d+2}}\, ds \le (d+1)C_1C_2.$$

Clearly, the constants $C_1, C_2$ do not depend on any additional multiplier $e^{iat}$, $a \in \mathbb{R}$. ∎

## 4. Proofs of Theorems 1.2 and 1.3

*Proof of Theorem 1.2.* Let $\varphi \in \mathcal{S}$ be a nonnegative even function with compact support such that $\varphi(y) \equiv 1$ for $|y| < 1$. We have

$$\mu \star \hat\varphi(t) = \int\limits_{\mathbb{R}^d} [\widehat{\varphi(-y)e^{2\pi i\langle t,y\rangle}}](x)\, \mu(dx) = \int\limits_{\mathbb{R}^d} \varphi(-y)e^{2\pi i\langle t,y\rangle}\, \hat\mu(dy)$$

$$= \sum_{\gamma\in \operatorname{supp}\varphi} b_\gamma \varphi(\gamma) e^{2\pi i\langle t,\gamma\rangle}.$$

Since $\hat\mu$ is a measure, we see that $\sum_{\gamma\in\operatorname{supp}\varphi} |b_\gamma|$ is finite and the last sum is absolutely convergent. Therefore, $\mu \star \hat\varphi$ is an almost periodic function. We also see from the above equality that

$$\widehat{\mu \star \hat\varphi} = \sum_{\gamma\in\operatorname{supp}\varphi} \varphi(\gamma) b_\gamma \delta_\gamma.$$

By Meyer's result (see the end of Section 2), the Fourier coefficients of the function $\mu \star \hat\varphi$ are

$$c_\gamma(\mu \star \hat\varphi) = b_\gamma \varphi(\gamma).$$

Using Parseval's equality (2.2), we find that the variation of the measure $\nu$ from (1.4) can be estimated as follows:

$$|\nu|(B(0,1)) = \sum_{|\gamma|<1} |b_\gamma|^2$$

$$\le \sum_{\gamma\in\mathbb{R}^d} |\varphi(\gamma) b_\gamma|^2 = \sum_{\gamma\in\mathbb{R}^d} |c_\gamma(\mu \star \hat\varphi)|^2$$

$$= \lim_{R\to\infty} \frac{1}{v_d R^d} \int\limits_{B(0,R)} |(\mu \star \hat\varphi)(x)|^2\, dx.$$

It follows from Proposition 3.3 that under the conditions of the theorem, the measure $\mu$ is translation bounded. Hence by Proposition 3.4, the last integral is bounded by a constant $C$. If we replace $\varphi(y)$ with $\varphi(y - y_0)$ (here $y_0 \in \mathbb{R}^d$ can be arbitrary), then $\hat\varphi$ changes by a factor of $e^{-2\pi i\langle x,y_0\rangle}$ and by

Proposition 3.4, the constant $C$ remains the same. We conclude that

$$\nu(B(y_0,1)) \le C. \tag{4.1}$$

Consequently, the measure $\nu$ is translation bounded, and $\nu(B(0,r)) \le C' r^d$ with some constant $C'$.

Finally, if

$$\#\{\gamma \in \Gamma : |\gamma| < r\} = O(r^\rho), \qquad r \to \infty,$$

with some $\rho < \infty$, then by the Cauchy–Bunyakovskii inequality

$$\sum_{|\gamma|<r} |b_\gamma| \le \Big[\sum_{|\gamma|<r} |b_\gamma|^2\Big]^{1/2} \cdot [\#\{\gamma \in \Gamma : |\gamma| < r\}]^{1/2} = O(r^{(d+\rho)/2})$$

as $r$ tends to infinity. Therefore, $|\hat{\mu}| \in \mathcal{S}'$. If $\mu$ is a crystalline measure, we conclude that it is a Fourier quasicrystal. ∎

REMARK 4.1. By (4.1), the masses $b_\gamma$ are uniformly bounded as well. Hence we can replace the exponent 2 in (1.4) with any $q > 2$. We do not know whether (1.4) is valid for $q < 2$ under the conditions of Theorem 1.2.

*Proof of Theorem 1.3.* It follows from Proposition 3.3 that under the assumptions of the theorem, the measure $\mu$ is translation bounded. Suppose for a contradiction that $\hat{\mu}$ is not translation bounded. Then the masses of $|\hat{\mu}|$ in balls of radius $\eta/2$ are unbounded, where $\eta > 0$ is the radius such that $\Gamma \cap B(x,\eta)$ is linearly independent over $\mathbb{Z}$, for any $x \in \mathbb{R}^d$. This means that there are points $y_n \in \mathbb{R}^d$, $n = 1, 2, \dots$, such that

$$\sum_{\gamma \in \Gamma \cap B(y_n,\eta/2)} |b_\gamma| > 5n.$$

If the set $\Gamma \cap B(y_n,\eta)$ is finite for some $n \in \mathbb{N}$, then we set $A_n = \Gamma \cap B(y_n,\eta)$. Otherwise, taking into account that $\hat{\mu}$ is a measure and that $|\hat{\mu}|(B(y_n,\eta)) < \infty$, we choose $A_n$ to be a finite subset of $\Gamma \cap B(y_n,\eta)$ such that

$$\sum_{\gamma \in \Gamma \cap B(y_n,\eta)\setminus A_n} |b_\gamma| < n. \tag{4.2}$$

Then

$$\sum_{\gamma \in A_n \cap B(y_n,\eta/2)} |b_\gamma| > 4n.$$

Applying Proposition 3.1, we can find $x_n \in \mathbb{R}^d$ and $m_\gamma \in \mathbb{Z}$ such that for all $\gamma \in A_n$,

$$\left| \langle x_n, \gamma\rangle + \frac{\arg b_\gamma}{2\pi} - m_\gamma \right| < \frac{1}{6}.$$

Therefore, for all $\gamma \in A_n$ we have

$$\Re e^{2\pi i \langle x_n,\gamma\rangle} b_\gamma > |b_\gamma|/2 > 0,$$

and

$$\Re \sum_{\gamma\in A_n\cap B(y_n,\eta/2)} e^{2\pi i\langle x_n,\gamma\rangle} b_\gamma > 2n. \tag{4.3}$$

Let $\varphi\in\mathcal{D}$ be such that $0\le\varphi(y)\le 1$, $\operatorname{supp}\varphi\subset B(0,\eta)$, and $\varphi(y)\equiv 1$ for $|y|<\eta/2$. It follows from (4.2) and (4.3) that

$$\begin{aligned}
&\Big|\int e^{2\pi i\langle x_n,y\rangle}\varphi(y-y_n)\,\hat\mu(dy)\Big| \ge \Big|\sum_{\gamma\in A_n}\varphi(\gamma-y_n)e^{2\pi i\langle x_n,\gamma\rangle}b_\gamma\Big| - \sum_{\gamma\in\Gamma\cap B(y_n,\eta)\setminus A_n}|b_\gamma|\\
&\ge \Re\sum_{\gamma\in A_n\cap B(y_n,\eta/2)} e^{2\pi i\langle x_n,\gamma\rangle}b_\gamma + \Re\sum_{\gamma\in A_n\setminus B(y_n,\eta/2)}\varphi(\gamma-y_n)e^{2\pi i\langle x_n,\gamma\rangle}b_\gamma - n \ge n.
\end{aligned}\tag{4.4}$$

On the other hand,

$$\begin{aligned}
\int e^{2\pi i\langle x_n,y\rangle}\varphi(y-y_n)\,\hat\mu(dy) &= \int e^{-2\pi i\langle y_n,x-x_n\rangle}\hat\varphi(x-x_n)\,\mu(dx)\\
&= \int e^{-2\pi i\langle y_n,x\rangle}\hat\varphi(x)\,\mu_{x_n}(dx).
\end{aligned}$$

By Proposition 3.4, the modulus of the integral on the right-hand side is bounded by a constant that does not depend on $x_n$ or $y_n$. This contradicts (4.4). Therefore, the measure $\hat\mu$ is translation bounded. If under the asumptions of the theorem the measures $\mu$ and $\hat\mu$ have locally finite supports, that is, $\mu$ is a crystalline measure, then $\mu$ is a Fourier quasicrystal. ∎

*Proof of Corollary.* It follows from Theorem 1.3 that $|\hat\mu|(B(0,r)) = O(r^d)$ as $r\to\infty$. Therefore, all the asumptions of [4, Theorem 3] are satisfied and we obtain the representation (1.5). ∎

QUESTION. Let $\mu$ be a crystalline measure, with $|\mu|$ a tempered measure. Is $|\hat\mu|$ a tempered measure? And if $\mu$ is moreover translation bounded? (2)

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(2) *Added in proof.* Negative answers to both questions have been obtained in [J. Mazáč, C. Richard, and N. Strungaru, *On almost periodicity in crystalline measures*, arXiv:2605.23884v1 (2026)].

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