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The Riemann sphere of a C^*-algebra

by

Esteban Andruchow, Gustavo Corach, Lázaro Recht and Alejandro Varela

Dedicated to Professor Horacio Porta on the occasion of his 87th birthday

Abstract. Given the unital C^*-algebra $\mathcal{A}$, the unitary orbit of the projector $\tilde{p} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in the C^*-algebra $M_2(\mathcal{A})$ of 2×2 matrices with coefficients in $\mathcal{A}$ is called in this paper *the Riemann sphere $\mathcal{R}$ of $\mathcal{A}$.*

We show that $\mathcal{R}$ is a reductive homogeneous C^∞ manifold of the unitary group $\mathcal{U}_2(\mathcal{A}) \subset M_2(\mathcal{A})$ and carries the differential geometry deduced from this structure (including an invariant Finsler metric). Special attention is paid to the properties of geodesics and the exponential map. If the algebra $\mathcal{A}$ is represented in a Hilbert space H, in terms of local charts of $\mathcal{R}$, elements of the Riemann sphere may be identified with (graphs of) closed operators on H (bounded or unbounded).

In the first part of the paper, we develop several geometric aspects of $\mathcal{R}$ including a relation between the exponential map of the reductive connection and the *cross-ratio of subspaces of $H \times H$.*

In the last section we show some applications of the geometry of $\mathcal{R}$ to the geometry of operators on a Hilbert space. In particular, we define the notion of *bounded deformation of a closed operator* and give some relevant examples.

Contents

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1. Introduction. This paper is presented (in the spirit of Felix Klein's Erlanger Program) as a sort of "elliptic" counterpart of [4], where the authors develop aspects of the "hyperbolic" Poincaré half-space of a C^*-algebra. Given a unital C^*-algebra $\mathcal{A}$ we define the Riemann sphere $\mathcal{R}$ of $\mathcal{A}$ as follows. The unitary group $\mathcal{U}_2(\mathcal{A})$ of the C^*-algebra $M_2(\mathcal{A})$ of 2×2 matrices with coefficients in $\mathcal{A}$ operates on the space $\mathcal{P}$ of projections of $M_2(\mathcal{A})$ by inner automorphisms. We denote by $\mathcal{R}$ the $\mathcal{U}_2(\mathcal{A})$-orbit of the projection $\tilde{p}_0 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. If the algebra $\mathcal{A}$ is faithfully represented in the Hilbert space H and $M_2(\mathcal{A})$ is correspondingly represented in $H\oplus H$, then $\mathcal{R}$ consists of the orthogonal projections in $H\oplus H$ onto subspaces S of the form $S = \tilde{u}\,(H\oplus\{0\})$, $\tilde{u}\in\mathcal{U}_2(\mathcal{A})$. For example, if $\mathcal{A} = B(H)$ and $T : \mathcal{D}\to H$ (where $\mathcal{D}\subset H$ is dense) is a closed linear operator then the orthogonal projection $\tilde{p}$ onto the graph of T is in $\mathcal{R}$ (see Proposition 5.2).

As in the classical case where $\mathcal{A} = \mathbb{C}$, we define the "unitary sphere" $\mathcal{K}\subset\mathcal{A}^2$ as the unitary orbit $\mathcal{K} = \{\tilde{u}\mathbf{e}_1 : \tilde{u}\in\mathcal{U}_2(\mathcal{A}), \mathbf{e}_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$ and the Hopf fibration $\mathfrak{h} : \mathcal{K}\to\mathcal{R}$ by

$$\mathfrak{h}(\mathbf{x}) = \mathbf{x}\mathbf{x}^* = \begin{pmatrix} x_1x_1^* & x_1x_2^* \\ x_2x_1^* & x_2x_2^* \end{pmatrix}$$

where $\mathbf{x} = \left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right)$. It is a principal fibration with $\mathcal{U}\subset\mathcal{A}$ as its structure group ($\mathcal{U}$ = the unitary group of $\mathcal{A}$, acting on $\mathcal{K}$ by right multiplication).

In Section 3 we define the C^∞ structure of $\mathcal{R}$ by means of an appropriate atlas. The principal chart $\mathscr{C}_0 = (\mathcal{V}_0, \varphi_0, \mathcal{A})$ of this atlas gives a bijection φ_0 from $\mathcal{V}_0 = \{\tilde{p}\in\mathcal{R} : \|\tilde{p}-\tilde{p}_0\| < 1\}$ onto the algebra $\mathcal{A}$. These projections $\tilde{p}$ are of the form $\tilde{p} = \mathbf{x}\mathbf{x}^*$ where $\mathbf{x} = \left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right)\in\mathcal{K}$ is such that x_1 is invertible and the correspondence is given by $\tilde{p}\mapsto x_2x_1^{-1}$. As in projective geometry, we could call $\left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right)$ the "homogeneous coordinates" of $\tilde{p}$, and $a = x_2x_1^{-1}$ the "affine coordinate". For this reason we could call this chart the "projective chart".

Section 3.2.4 is concerned with the geometry of $\mathcal{R}$ as a reductive homogeneous space of the group $\mathcal{U}_2$. This differential-geometric structure contains an affine connection on $\mathcal{R}$ and its geodesics. It also contains an invariant Finsler metric which makes $\mathcal{R}$ into a metric space where geodesics are minimal curves (see [5, 6]).

The geometry of $\mathcal{R}$ as a reductive homogeneous space defines another atlas on $\mathcal{R}$ by means of the exponential map of the connection. In Section 3.2.5 we show that the principal chart of this atlas is of the form $(\mathcal{V}_0, \mathrm{Log}_{\tilde{p}_0}, \mathcal{W})$, where $\mathcal{V}_0$ is again $\{\tilde{p} \in \mathcal{R} : \|\tilde{p} - \tilde{p}_0\| < 1\}$, $\mathcal{W} = \{X \in (T\mathcal{R})_{\tilde{p}_0} : \|X\| < \pi/2\}$ and $\mathrm{Log}_{\tilde{p}_0}$ is the inverse of the exponential map at $\tilde{p}_0$. We call this chart the "geodesic chart" of $\mathcal{R}$.

The relation between the two principal charts has an interesting geometric meaning which we explain in Section 3.2.6. Loosely speaking, the homogeneous coordinates of $\tilde{p}$ produce the affine coordinates of $a = x_2 x_1^{-1}$, while the geodesic coordinates $X = \left(\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right) \in (T\mathcal{R})_{\tilde{p}_0}$ produce a kind of "polar coordinate" of $\tilde{p}$ given by an "angle" and a "phase" related to the pair $(\tilde{p}_0, \tilde{p})$.

We devote the final section (Section 5) to the study of examples and applications that we consider relevant. In Section 5.3, given an unbounded densely defined closed operator T on a Hilbert space H, we show that there exists a unique minimal geodesic on $\mathcal{R}$ joining $\tilde{p}_0$ to $P_{\mathrm{Gr}(T)}$. Notice that $P_{\mathrm{Gr}(T)}$ is in the boundary of $\mathcal{V}_0$. In particular, we analyze the case of the operator $-i\frac{d}{dx}$ on $L^2[0,1]$. We also study geodesics on $\mathcal{R}$ with conjugate points and compute the index of some of these geodesics related to Fredholm operators. In Section 5.4 we define a notion of (one-parameter) bounded deformation of closed operators as well as the notion of optimal deformation. The unique minimal geodesic joining $\tilde{p}_0$ to $P_{\mathrm{Gr}(T)}$, where T is a closed unbounded operator, is an optimal bounded deformation of the closed operator T. In Section 5.7 we exhibit types of C^*-algebras where $\mathcal{V}_0$ is dense in $\mathcal{R}$. We remark that $\mathcal{V}_0$ is not dense in $\mathcal{R}$ when $\mathcal{A} = B(H)$ and H is infinite-dimensional.

A second part of this paper will be devoted to the description and usage of a non-commutative Kähler structure on $\mathcal{R}$, which will be defined as an "elliptic" counterpart to the one defined in [4].

2. Preliminaries. We will denote by $\mathcal{A}$ a unital C^*-algebra, $\mathcal{G} \subset \mathcal{A}$ its group of invertible elements and $\mathcal{U}$ the unitary subgroup of $\mathcal{G}$. We say that $a \in \mathcal{A}$ is *anti-self-adjoint* if $a^* = -a$. The C^*-algebra of 2×2 matrices with entries in $\mathcal{A}$ will be denoted by $M_2(\mathcal{A})$, and the corresponding group of units and the unitary subgroup will be denoted by $\mathcal{G}_2$ and $\mathcal{U}_2$. Denote by $\mathcal{A}^2$ the right C^* $\mathcal{A}$-module

$$\mathcal{A}^2 = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathcal{A} \right\}$$

and also write $\mathcal{A}_t^2 = \{\hat{\mathbf{x}} = (x_1 \quad x_2) : x_1, x_2 \in \mathcal{A}\}$. We have maps

$$\mathcal{A}^2 \to \mathcal{A}_t^2, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \mathbf{x}^* = (x_1^* \quad x_2^*),$$

$$\mathcal{A}_t^2 \to \mathcal{A}^2, \qquad \hat{\mathbf{x}} = (x_1 \quad x_2) \mapsto \hat{\mathbf{x}}^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}.$$

Next we have products

$$\mathcal{A}^2 \times \mathcal{A}_t^2 \to M_2(\mathcal{A}), \qquad \mathbf{x}\hat{\mathbf{y}} = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix},$$

$$\mathcal{A}_t^2 \times \mathcal{A}^2 \to \mathcal{A}, \qquad \hat{\mathbf{x}}\mathbf{y} = x_1 y_1 + x_2 y_2.$$

Observe that the inner product in the C^* $\mathcal{A}$-module $\mathcal{A}^2$ is given by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^*\mathbf{y} = x_1^* y_1 + x_2^* y_2$.

The algebra $M_2(\mathcal{A})$ is identified with the C^*-algebra $\mathcal{L}_{\mathcal{A}}(\mathcal{A}^2)$ of $\mathcal{A}$-linear bounded adjointable operators [11], where we fix the standard basis of $\mathcal{A}^2$ given by $\{\mathbf{e}_1, \mathbf{e}_2\}$ with $\mathbf{e}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\mathbf{e}_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. With this identification, $T \in \mathcal{L}_{\mathcal{A}}(\mathcal{A}^2)$ is represented by the matrix $\tilde{t} = \left(\begin{smallmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{smallmatrix}\right)$:

$$Tx = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t_{11}x_1 + t_{12}x_2 \\ t_{21}x_1 + t_{22}x_2 \end{pmatrix}.$$

Then T^* is given by $\tilde{t}^* = \begin{pmatrix} t_{11}^* & t_{21}^* \\ t_{12}^* & t_{22}^* \end{pmatrix}$.

Note that $\mathcal{U}_2$ is the subgroup of $M_2(\mathcal{A})$ that preserves the quadratic form $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}^*\mathbf{y}$ when acting on the left by $\mathbf{x} \mapsto \tilde{u}\mathbf{x}$ for $\mathbf{x} \in \mathcal{A}^2$ and $\tilde{u} \in \mathcal{U}_2$.

Definition 2.1. A pair of vectors $\mathbf{x}, \mathbf{y} \in \mathcal{A}^2$ will be called a *unitary basis* of $\mathcal{A}^2$ if it is of the form $\mathbf{x} = \tilde{u}(\mathbf{e}_1)$ and $\mathbf{y} = \tilde{u}(\mathbf{e}_2)$ for $\tilde{u} \in \mathcal{U}_2$ where $\mathbf{e}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{e}_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

Notice that (if $\mathbf{x}$, $\mathbf{y}$ are a unitary basis of $\mathcal{A}^2$) we have the *Fourier identity* $\mathbf{z} = \mathbf{x}\langle \mathbf{x}, \mathbf{z} \rangle + \mathbf{y}\langle \mathbf{y}, \mathbf{z} \rangle$ for every $\mathbf{z} \in \mathcal{A}^2$.

Definition 2.2. A vector $\mathbf{x} \in \mathcal{A}^2$ is called a *unitary vector* if it is of the form $\mathbf{x} = \tilde{u}\mathbf{e}_1$ for some $\tilde{u} \in \mathcal{U}_2$.

Notice that every unitary vector is the first component of a unitary basis.

Now we come to the central topic of this paper. Recall that the unitary group $\mathcal{U}_2 = \mathcal{U}_2(\mathcal{A})$ operates on the space of all projections of the algebra $M_2(\mathcal{A})$ by the rule

$$L_{\tilde{u}}(\tilde{p}) = \tilde{u}\tilde{p}\tilde{u}^{-1} \quad \text{for } \tilde{u} \in \mathcal{U}_2.$$

The geometry related to this action is studied for example in [5].

Let $\tilde{p}_0$ be the projector $\tilde{p}_0 = \mathbf{e}_1\mathbf{e}_1^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

DEFINITION 2.3. The *Riemann sphere* $\mathcal{R}$ of the C^*-algebra $\mathcal{A}$ is the orbit

$$\mathcal{R} = \{L_{\tilde{u}}(\tilde{p}_0) : \tilde{u} \in \mathcal{U}_2\}.$$

A key role in the study of $\mathcal{R}$ is played by the space $\mathcal{K}$ defined as follows.

DEFINITION 2.4. We define the *unitary sphere* $\mathcal{K}$ in $\mathcal{A}^2$ as

$$\mathcal{K} = \{\mathbf{x} \in \mathcal{A}^2 : \exists \tilde{u} \in \mathcal{U}_2 \text{ such that } \tilde{u}\mathbf{e}_1 = \mathbf{x}\}. \tag{2.1}$$

DEFINITION 2.5. The *Hopf fibration* over $\mathcal{R}$ is the map

$$\mathfrak{h} : \mathcal{K} \to \mathcal{R}, \qquad \mathfrak{h}(\mathbf{x}) = \tilde{p}_{\mathbf{x}}, \tag{2.2}$$

where $\tilde{p}_{\mathbf{x}} = \mathbf{x}\mathbf{x}^*$. The unitary group $\mathcal{U}$ of $\mathcal{A}$ operates by right multiplication on $\mathcal{K}$ and is compatible with the projection $\mathfrak{h}$: $\mathfrak{h}(\mathbf{x}u) = \mathfrak{h}(\mathbf{x})$ for $\mathbf{x} \in \mathcal{K}$ and $u \in \mathcal{U}$.

PROPOSITION 2.6. *The Hopf fibration is equivariant under the action of $\mathcal{U}_2$. More explicitly,*

$$\mathfrak{h}(\tilde{u}\mathbf{x}) = \tilde{u}\mathfrak{h}(\mathbf{x})\tilde{u}^* \qquad \forall \tilde{u} \in \mathcal{U}_2,\ \mathbf{x} \in \mathcal{K}.$$

Proof. Indeed, $\mathfrak{h}(\tilde{u}\mathbf{x}) = \tilde{u}\mathbf{x}(\tilde{u}\mathbf{x})^* = \tilde{u}\mathbf{x}\mathbf{x}^*\tilde{u}^* = \tilde{u}\mathfrak{h}(\mathbf{x})\tilde{u}^*$. ∎

NOTATION 2.7. Given $\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \tilde{u}\mathbf{e}_1$ for $\tilde{u} \in \mathcal{U}_2$ ($\mathbf{x} \in \mathcal{K}$), we will denote by $[\mathbf{x}]$ the right $\mathcal{A}$-module generated by $\mathbf{x}$, that is,

$$[\mathbf{x}] = \mathbf{x}\mathcal{A} = \left\{\left(\begin{smallmatrix} x_1 a \\ x_2 a \end{smallmatrix}\right) \in \mathcal{A}^2 : a \in \mathcal{A}\right\}. \tag{2.3}$$

PROPOSITION 2.8. *If $\mathbf{x} \in \mathcal{K}$ and $[\mathbf{x}] = \mathbf{x}\mathcal{A}$ is the right $\mathcal{A}$-module generated by $\mathbf{x}$, then*

$$\operatorname{im} \tilde{p}_{\mathbf{x}} = [\mathbf{x}].$$

Proof. Since $\mathbf{x}^*\mathbf{x} = 1$ for each $\mathbf{x} \in \mathcal{K}$, we have $\tilde{p}_{\mathbf{x}}\mathbf{x} = (\mathbf{x}\mathbf{x}^*)\mathbf{x} = \mathbf{x}$ and then $\mathbf{x} \in \operatorname{im} \tilde{p}_{\mathbf{x}}$, which implies that $[\mathbf{x}] \subset \operatorname{im} \tilde{p}_{\mathbf{x}}$ (since $\operatorname{im} \tilde{p}_{\mathbf{x}}$ is an $\mathcal{A}$-submodule of $\mathcal{A}^2$).

The opposite inclusion is evident: if $\mathbf{z} = \tilde{p}_{\mathbf{x}}\mathbf{w}$ then $\mathbf{z} = \mathbf{x}\mathbf{x}^*\mathbf{w} \in [\mathbf{x}]$ and hence $\operatorname{im}(\tilde{p}_{\mathbf{x}}) \subset [\mathbf{x}]$. ∎

PROPOSITION 2.9. *Suppose that $\mathbf{x}, \mathbf{z} \in \mathcal{K}$. The following statements are equivalent:*

(1) $[\mathbf{x}] = [\mathbf{z}]$,
(2) $\tilde{p}_{\mathbf{x}} = \tilde{p}_{\mathbf{z}}$,
(3) $\exists \tilde{u} \in \mathcal{U}$ *such that* $\mathbf{z} = \mathbf{x}\tilde{u}$.

Proof. (1)⇔(2) is evident (see Proposition 2.8).

(3)⇒(2) is clear: if $u \in \mathcal{U}$ and $\mathbf{z} = \mathbf{x}u$, then $\tilde{p}_{\mathbf{z}} = \mathbf{x}u(\mathbf{x}u)^* = \mathbf{x}\mathbf{x}^* = \tilde{p}_{\mathbf{x}}$.

(2)⇒(3) If $\tilde{p}_{\mathbf{z}} = \tilde{p}_{\mathbf{x}}$, since $\mathbf{z} = \tilde{p}_{\mathbf{z}}\mathbf{z} = \tilde{p}_{\mathbf{x}}\mathbf{z} = \mathbf{x}\mathbf{x}^*\mathbf{z}$ it is enough to prove that $\mathbf{x}^*\mathbf{z} \in \mathcal{U}$, that is, $(\mathbf{x}^*\mathbf{z})(\mathbf{z}^*\mathbf{x}) = 1$ and $(\mathbf{z}^*\mathbf{x})(\mathbf{x}^*\mathbf{z}) = 1$. But $(\mathbf{x}^*\mathbf{z})(\mathbf{z}^*\mathbf{x}) = \mathbf{x}^*\tilde{p}_{\mathbf{z}}\mathbf{x} = \mathbf{x}^*\tilde{p}_{\mathbf{x}}\mathbf{x} = \mathbf{x}^*\mathbf{x} = 1$ and similarly $(\mathbf{z}^*\mathbf{x})(\mathbf{x}^*\mathbf{z}) = 1$. ∎

We mention two natural vector bundles associated to $\mathcal{R}$, the *tautological vector bundle* $\mathcal{T} \xrightarrow{\mathrm{pr}} \mathcal{R}$ (the *bundle of images*) and the *co-tautological vector bundle* $\mathcal{T}' \xrightarrow{\mathrm{pr}'} \mathcal{R}$ (the *bundle of kernels*), defined as follows:

$$\begin{aligned} \mathcal{T} &= \{(\tilde{p}, \mathbf{x}) : \tilde{p} \in \mathcal{R},\ \mathbf{x} \in \operatorname{im} \tilde{p}\} \quad \text{and} \quad \mathrm{pr}(\tilde{p}, \mathbf{x}) = \tilde{p}, \\ \mathcal{T}' &= \{(\tilde{p}, \mathbf{x}) : \tilde{p} \in \mathcal{R},\ \mathbf{x} \in \ker \tilde{p}\} \quad \text{and} \quad \mathrm{pr}'(\tilde{p}, \mathbf{x}) = \tilde{p}. \end{aligned} \tag{2.4}$$

Observe that the Hopf fibration is the "classical bundle of bases" of the tautological bundle (see Section 4.2 for more details). We will also show in Section 4.2 the relation between the co-tautological vector bundle $\mathcal{T}'$ and the tangent bundle $T\mathcal{R}$.

3. The smooth structure of $\mathcal{R}$

3.1. The C^∞ structure on $\mathcal{K}$. Let us start by constructing a C^∞ structure $\mathfrak{A}_{\mathcal{K}}$ on $\mathcal{K}$ (a C^∞ atlas $\mathfrak{A}_{\mathcal{K}}$). This is done by identifying open sets in $\mathcal{K}$ with appropriate C^∞ manifolds so that the transition functions are C^∞ too. Define an open neighborhood $\mathcal{K}_0$ of $\mathbf{e}_1$ by

$$\mathcal{K}_0 = \left\{\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathcal{K} : x_1 \in \mathcal{G}\right\} \tag{3.1}$$

and the map

$$\psi_0 : \mathcal{K}_0 \to \mathcal{A} \times \mathcal{U}, \qquad \psi_0(\mathbf{x}) = (x_2 x_1^{-1}, u),$$

where $x_1 = ru$ is the polar decomposition with r positive and u unitary.

We now write the inverse $\Psi_0 : \mathcal{A} \times \mathcal{U} \to \mathcal{K}$ of ψ_0,

$$\Psi_0(a, u) = \begin{pmatrix} (1 + a^*a)^{-1/2} u \\ a(1 + a^*a)^{-1/2} u \end{pmatrix}. \tag{3.2}$$

We will call the chart given by $\mathcal{C}_0 = (\mathcal{K}_0, \psi_0, \mathcal{A} \times \mathcal{U})$ the *principal chart* of the C^∞ atlas $\mathfrak{A}_{\mathcal{K}}$.

For each $\tilde{u} \in \mathcal{U}_2$ we describe the chart $\mathcal{C}_{\tilde{u}} = (\tilde{u}\mathcal{K}_0, \psi_{\tilde{u}}, \mathcal{A} \times \mathcal{U})$ of the atlas $\mathfrak{A}_{\mathcal{K}}$ (by acting with $\tilde{u}$ on the principal chart $\mathcal{C}_0$) where $\psi_u = \psi_0 \circ \tilde{u}^{-1} : \tilde{u}\mathcal{K}_0 \to \mathcal{A} \times \mathcal{U}$. Clearly the atlas $\mathfrak{A}_{\mathcal{K}} = \{\mathcal{C}_{\tilde{u}} : \tilde{u} \in \mathcal{U}_2\}$ defines a C^∞ structure on $\mathcal{K}$.

3.2. The C^∞ structure of $\mathcal{R}$. Given a unital C^*-algebra $\mathcal{B}$, the space

$$\mathcal{P}_2 = \{p \in \mathcal{B} : p^2 = p = p^*\}$$

is a C^∞ Banach submanifold of $\mathcal{B}$. This is well known and details can be found for example in [16, 5].

Remark 3.1. We now recall that the unitary group $\mathcal{U}_{\mathcal{B}}$ of $\mathcal{B}$ operates on $\mathcal{P}$ by inner automorphisms $L_u(p) = upu^*$. This action divides P into orbits and each such orbit is a homogeneous space of the group $\mathcal{U}_{\mathcal{B}}$. Moreover, analyzing the infinitesimal aspect of this action we can provide each such orbit with a *reductive homogeneous structure*. Details can be found in [5].

This reductive homogeneous structure provides each orbit with an invariant affine connection and the associated geometry including geodesics, curvature, etc. Details can also be found in [5].

These ideas apply in our case to the C^*-algebra $M_2 = M_2(\mathcal{A})$, and the orbit of $\tilde{p}_0$ under the action of the unitary group $\mathcal{U}_2$, i.e. the Riemann sphere $\mathcal{R}$ of the algebra $\mathcal{A}$, and we will use them freely along this paper.

3.2.1. *The* C^∞ *atlas* $\mathfrak{A}_{\mathcal{R}}$ *of* $\mathcal{R}$. We describe a specific C^∞ atlas on $\mathcal{R}$. We start by the *principal chart* $\mathscr{C}_0 = (\mathcal{V}_0, \varphi_0, \mathcal{A})$ of this atlas where

$$\mathcal{V}_0 = \mathfrak{h}\,(\mathcal{K}_0) = \{\mathbf{x}\mathbf{x}^* : \mathbf{x} \in \mathcal{K},\, x_1 \text{ invertible}\} \subset \mathcal{R} \tag{3.3}$$

and

$$\varphi_0 : \mathcal{V}_0 \to \mathcal{A}, \quad \varphi_0(\tilde{p}) = x_2 x_1^{-1} \quad \text{if } \tilde{p} = \tilde{p}_{\mathbf{x}} \text{ for } \mathbf{x} \in \mathcal{K}_0. \tag{3.4}$$

Observe that if $\tilde{p} = \tilde{p}_{\mathbf{z}}$ for another $\mathbf{z} \in \mathcal{K}_0$, then $\mathbf{z} = \mathbf{x}u$ for $u \in \mathcal{U}$ and $z_2 z_1^{-1} = x_2 x_1^{-1}$, and hence φ_0 is well defined.

Let us verify that φ_0 is injective. If $\varphi_0(\tilde{p}) = \varphi_0(\tilde{q})$ with $\tilde{p} = \tilde{p}_{\mathbf{x}}$, $\tilde{q} = \tilde{p}_{\mathbf{y}}$ for some $\mathbf{x}, \mathbf{y} \in \mathcal{K}_0$ satisfying $x_2 x_1^{-1} = y_2 y_1^{-1}$, it follows that $\mathbf{y} = \left(\begin{smallmatrix} y_1 \\ x_2 x_1^{-1} y_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ x_2 x_1^{-1} \end{smallmatrix}\right) y_1$. In order to prove that $\tilde{p} = \tilde{q}$ it is enough to show that $[\mathbf{x}] = [\mathbf{y}]$, since $\operatorname{im} \tilde{p} = [\mathbf{x}]$ and $\operatorname{im} \tilde{q} = [\mathbf{y}]$. On the one hand, $\mathbf{x} = \left(\begin{smallmatrix} 1 \\ x_2 x_1^{-1} \end{smallmatrix}\right) x_1 = \mathbf{y} y_1^{-1} x_1$ and hence $\mathbf{x} \in [\mathbf{y}]$. Analogously, $\mathbf{y} = \mathbf{x} x_1^{-1} y_1 \in [\mathbf{x}]$. Therefore $[\mathbf{x}] = [\mathbf{y}]$ and $\tilde{p} = \tilde{q}$.

To prove the surjectivity of φ_0 take any $a \in \mathcal{A}$. We need to find an $\mathbf{x} \in \mathcal{K}_0$ such that $x_2 x_1^{-1} = a$, and hence $\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ a \end{smallmatrix}\right) x_1$ should hold. To satisfy the condition $\mathbf{x}^*\mathbf{x} = 1$ we must have $x_1^*(1 \quad a^*)\left(\begin{smallmatrix} 1 \\ a \end{smallmatrix}\right) x_1 = 1$. Then $x_1^*(1 + a^*a) x_1 = 1$, which implies that $1 + a^*a = (x_1 x_1^*)^{-1}$. Every solution of this equation is of the form $x_1 = (1 + a^*a)^{-1/2} u$ with $u \in \mathcal{U}$. Now $\mathbf{x} = \left(\begin{smallmatrix} 1 \\ a \end{smallmatrix}\right)(1 + a^*a)^{-1/2} u$ must satisfy $\mathbf{x} = \tilde{u}\mathbf{e}_1$ for some $\tilde{u} \in \mathcal{U}_2$ and this is the case for

$$\tilde{u} = \begin{pmatrix} (1 + a^*a)^{-1/2} u & -a^*(1 + aa^*)^{-1/2} v \\ a(1 + a^*a)^{-1/2} u & (1 + aa^*)^{-1/2} v \end{pmatrix} \quad \text{with } v \in \mathcal{U}. \tag{3.5}$$

We now construct a chart $\mathscr{C}_{\tilde{u}} = (\mathcal{V}_{\tilde{u}}, \varphi_{\tilde{u}}, \mathcal{A})$ for $\tilde{u} \in \mathcal{U}_2$ as follows. We let

$$\mathcal{V}_{\tilde{u}} = L_{\tilde{u}}(\mathcal{V}_0) \quad \text{and} \quad \varphi_{\tilde{u}} : \mathcal{V}_{\tilde{u}} \to \mathcal{A}, \quad \varphi_{\tilde{u}} = \varphi_0 \circ L_{\tilde{u}^{-1}}.$$

Given two charts $\mathscr{C}_{\tilde{u}}$ and $\mathscr{C}_{\tilde{v}}$ where $\tilde{u}, \tilde{v} \in \mathcal{U}_2$ and $\mathcal{V}_{\tilde{u}} \cap \mathcal{V}_{\tilde{v}} \neq \emptyset$, let us compute the coordinate change. Let $\mathbf{x} = \tilde{u}\mathbf{e}_1$ and $\mathbf{y} = \tilde{v}\mathbf{e}_1$. We have

$$(\varphi_{\tilde{u}} \circ \varphi_{\tilde{v}}^{-1})(a) = ((\tilde{u}\tilde{v})^* \left(\begin{smallmatrix} 1 \\ a \end{smallmatrix}\right))_2 \Big(\big((\tilde{u}\tilde{v})^* \left(\begin{smallmatrix} 1 \\ a \end{smallmatrix}\right)\big)_1\Big)^{-1} \quad \forall a \in \varphi_{\tilde{v}}\left(\mathcal{V}_{\tilde{u}} \cap \mathcal{V}_{\tilde{v}}\right).$$

If we write $(\tilde{u}\tilde{v})^* = \left(\begin{smallmatrix} c & d \\ e & f \end{smallmatrix}\right) \in \mathcal{U}_2$, we have

$$(\varphi_{\tilde{p}_{\mathbf{x}}} \circ \varphi_{\tilde{p}_{\mathbf{z}}}^{-1})(a) = (e + fa)(c + da)^{-1}.$$

Remark 3.2. Observe that the change of coordinates for two charts in the atlas $\mathfrak{A}_{\mathcal{R}}$ is given by a "Möbius" transformation. Consequently, this atlas defines on $\mathcal{R}$ a C^∞ structure. We shall pursue the study of this complex structure elsewhere.

Remark 3.3. Note that $\tilde{p} \in \mathcal{V}_0$ if and only if there is an element $\mathbf{x} = \binom{x_1}{x_2} \in \operatorname{im} \tilde{p}_0$ (not necessarily satisfying $\mathbf{x}^*\mathbf{x} = 1$) such that x_1 is invertible, we will call such an $\mathbf{x}$ a *regular element* in $\operatorname{im} \tilde{p}_0$. Also observe that both $\operatorname{im} \tilde{p}_0$ and $\ker \tilde{p}_0$ are right $\mathcal{A}$-modules and that every regular element in $\operatorname{im} \tilde{p}_0$ is a generator of this $\mathcal{A}$-module. Moreover, the correspondence $\mathcal{V}_0 \ni \tilde{p} \mapsto a \in \mathcal{A}$ is independent of the choice of the regular element $\mathbf{x} \in \operatorname{im} \tilde{p}$. Now choose a faithful representation of $\mathcal{A}$ in a Hilbert space H so that elements $a \in \mathcal{A}$ correspond to operators $a : H \to H$. Consequently, $M_2(\mathcal{A})$ is faithfully represented in $B(H \oplus H)$.

Theorem 3.4. *With the previous notations, the following statements are equivalent for $\tilde{p} \in \mathcal{R}$:*

(1) $\tilde{p} \in \mathcal{V}_0$ *(see* (3.3)*),*
(2) $\tilde{p}$ *is the projection* $P_{\mathrm{Gr}(a)} \in B(H \oplus H)$ *onto the graph of the operator* $a = \varphi_0(\tilde{p})$ *(see* (3.4)*),*
(3) $\|\tilde{p} - \tilde{p}_0\| < 1$,
(4) p_{11} *is invertible if* $\tilde{p} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$.

Proof. We will denote the elements of $\mathcal{A}$ with the same letters as their representations on $B(H)$. First we will prove that item (1) implies (2). Suppose that $\tilde{p} = \mathbf{x}\mathbf{x}^* = \begin{pmatrix} x_1x_1^* & x_1x_2^* \\ x_2x_1^* & x_2x_2^* \end{pmatrix}$ for $\mathbf{x} \in \mathcal{K}$ with x_1 invertible. Then $a = \varphi_0(\tilde{p}) = x_2x_1^{-1}$, $\tilde{p} \in B(H \oplus H)$ satisfies $\tilde{p}^* = \tilde{p} = \tilde{p}^2$ and hence it is an orthogonal projection in $B(H \oplus H)$. Moreover, using $x_1^*x_1 + x_2^*x_2 = 1$ and hence $x_2^*x_2 = 1 - x_1^*x_1$, we can write

$$\tilde{p}\binom{h}{x_2x_1^{-1}h} = \binom{x_1x_1^*h + x_1x_2^*x_2x_1^{-1}h}{x_2x_1^*h + x_2x_2^*x_2x_1^{-1}h} = \binom{x_1x_1^*h + x_1(1 - x_1^*x_1)x_1^{-1}h}{x_2x_1^*h + x_2(1 - x_1^*x_1)x_1^{-1}h} = \binom{h}{x_2x_1^{-1}h}.$$

This implies that $\mathrm{Gr}(a) \subset \operatorname{im} \tilde{p}$ for $a = x_2x_1^{-1}$. Finally,

$$\tilde{p}\binom{h}{k} = \binom{x_1x_1^*h + x_1x_2^*k}{x_2x_1^*h + x_2x_2^*k} = \binom{x_1x_1^*h + x_1x_2^*k}{x_2x_1^{-1}(x_1x_1^*h + x_1x_2^*k)},$$

which proves the inclusion $\operatorname{im} \tilde{p} \subset \mathrm{Gr}(a)$.

To prove that item (2) implies (3), observe first that $\binom{h}{k} = \binom{x}{ax} \oplus \binom{-a^*y}{y}$ for every $\binom{h}{k} \in H \oplus H$ with $\binom{x}{ax} \in \mathrm{Gr}(a)$ orthogonal to $\binom{-a^*y}{y} \in \mathrm{Gr}(a)^\perp$ (see Lemma 5.4). Then $\left\|\binom{h}{k}\right\| = 1$ implies that $\left\|\binom{x}{ax}\right\|^2 + \left\|\binom{-a^*y}{y}\right\|^2 = 1$ and hence

$$\|P_{\mathrm{Gr}(a)} - \tilde{p}_0\| = \sup_{\|\binom{h}{k}\|=1} \left\|\binom{x}{ax} - \binom{x - a^*y}{0}\right\| = \sup_{\|\binom{x}{ax}\|^2 + \|\binom{-a^*y}{y}\|^2 = 1} \left\|\binom{a^*y}{ax}\right\|.$$

We know that in general $\|\tilde{p}-\tilde{p}_0\| \leq 1$ (see [19, Corollary 2]). If it is equal to 1, there exist sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ with $\langle a^*y, a^*y\rangle + \langle ax, ax\rangle \to 1$. This would imply that $x_n \to 0$ and $y_n \to 0$ (since $\|x\|^2 + \|ax\|^2 + \|a^*y\|^2 + \|y^2\| = 1$), a contradiction. Then $\|P_{\mathrm{Gr}(a)} - \tilde{p}_0\| < 1$.

The proof that item (3) implies (4) follows if we consider the $1,1$ entry of $\tilde{p}-\tilde{p}_0$ and observe that $\|\tilde{p}-\tilde{p}_0\| < 1$. This necessarily implies that $(\tilde{p}-\tilde{p}_0)_{1,1} = p_{11} - 1$ has norm less than 1, and hence p_{11} is invertible.

Finally, if p_{11} is invertible and $\tilde{p} = \mathbf{x}\mathbf{x}^*$ for $\mathbf{x} \in \mathcal{K}$, it follows that $p_{11} = x_1x_1^*$ and therefore x_1 is invertible. Then $\tilde{p} \in \mathcal{V}_0$. ∎

Remark 3.5. Theorem 3.4 suggests the following considerations. Given an unbounded operator $T : D \to H$ with dense domain $D \subset H$ and closed graph we will show that the orthogonal projection $P_{\mathrm{Gr}(T)} : H \oplus H \to H \oplus H$ belongs to the Riemann sphere $\mathcal{R}(H)$ of the algebra $B(H)$. Since we have the obvious embedding $\mathcal{R} \subset \mathcal{R}(H)$ it is natural to ask about the relative position of $P_{\mathrm{Gr}(T)}$ with respect to $\mathcal{R}$. We will give some partial answers to this question in Section 5.2.

3.2.2. *The tangent map of the principal chart.* We describe here the tangent map $(T\varphi_0)_{\tilde{p}} : (T\mathcal{R})_{\tilde{p}} \to (T\mathcal{A})_{\varphi_0(p)}$ $(= \mathcal{A})$ of the principal chart φ_0 (see (3.3) and (3.4)). Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{K}_0 & & \\ {\scriptstyle \mathfrak{h}|_{\mathcal{K}_0}}\downarrow & \searrow^{\psi_0} & \\ \mathcal{V}_0 & \xrightarrow{\ \varphi_0\ } & \mathcal{A} \end{array}$$

where $\mathcal{K}_0 = \{\mathbf{x} \in \mathcal{K} : x_1 \text{ is invertible}\}$ (see 3.2.1), $\mathfrak{h}|_{\mathcal{K}_0}$ is the Hopf fibration over $\mathcal{V}_0$ and $\psi_0(\mathbf{x}) = x_2(x_1)^{-1}$. Now fix $\tilde{p} \in \mathcal{R}$ and $\mathbf{x} \in \mathcal{K}$ such that $\mathfrak{h}\mathbf{x} = \tilde{p}$. Then we have

$$(T\varphi_0)_{\tilde{p}}Y = (T\psi_0)_{\mathbf{x}}\,\kappa_{\mathbf{x}}(Y) \quad \text{for each } Y \in (T\mathcal{R})_{\tilde{p}}, \tag{3.6}$$

where κ is the structure morphism defined in 4.3. It is easy to check that (3.6) is independent of the choice of $\mathbf{x}$ with $\mathfrak{h}(\mathbf{x}) = \tilde{p}$. Explicitly,

$$(T\varphi_0)_{\tilde{p}}Y = (T\psi_0)_{\mathbf{x}}Y\mathbf{x} = (T\psi_0)_{\mathbf{x}}\binom{(Y\mathbf{x})_1}{(Y\mathbf{x})_2} = (Y\mathbf{x})_2x_1^{-1} - x_2x_1^{-1}(Y\mathbf{x})_1x_1^{-1},$$

where we write $Y\mathbf{x} = \binom{(Y\mathbf{x})_1}{(Y\mathbf{x})_2}$. The inverse map φ_0^{-1} of φ_0 is given by

$$\varphi_0^{-1}(a) = \mathbf{a}c^2\mathbf{a}^* = \tilde{p}$$

where $\mathbf{a} = \binom{1}{a} \in \mathcal{A}^2$ and $c = (1 + a^*a)^{-1/2} \in \mathcal{A}$ (note that $\mathbf{x} = \mathbf{a}c \in \mathcal{K}$). The tangent map $(T\varphi_0^{-1})_a : (T\mathcal{A})_a \to (T\mathcal{R})_{\tilde{p}}$ is given by

$$(T\varphi_0^{-1})\dot{a} = \dot{\mathbf{a}}c^2\mathbf{a}^* + \mathbf{a}c^2(\dot{\mathbf{a}})^* - \mathbf{a}b\mathbf{a}^*$$

where $\dot{a} \in (T\mathcal{A})_a$ $(= \mathcal{A})$, $\dot{\mathbf{a}} = \binom{0}{\dot{a}}$ and $b = c^{-2}(\dot{a}^*a + a^*\dot{a})c^{-2}$.

The Finsler structure of $\mathcal{R}$ may be translated to a Finsler structure on the manifold $\mathcal{A}$ assigning to each tangent vector $\dot{a} \in (T\mathcal{A})_a$ the norm $|||\dot{a}||| = \|\dot{\mathbf{a}}c^2\mathbf{a}^* + \mathbf{a}c^2(\dot{\mathbf{a}})^* - \mathbf{a}b\mathbf{a}^*\|$ (the standard operator norm of $M_2(\mathcal{A})$).

3.2.3. *Riemann sphere projectors in C^*-algebras.* In this section we give an intrinsic characterization of Riemann spheres of C^*-algebras. Let $\mathcal{M}$ be a unital C^*-algebra.

DEFINITION 3.6. A self-adjoint projector $p \in \mathcal{M}$ is called a *Riemann sphere projector* (rsp) if p is conjugate to $1-p$, i.e. there exists an invertible $g \in \mathcal{G}_{\mathcal{M}}$ such that $gpg^{-1} = 1-p$.

Note that if p is an rsp, there is a unitary element $u \in \mathcal{M}$ such that $upu^{-1} = 1-p$ (this can be shown by an easy argument involving the polar decomposition of g in the definition).

From now on we assume p is an rsp in $\mathcal{M}$, so $upu^{-1} = 1-p$ where u is unitary. Define the subalgebra $\mathcal{A}$ of $\mathcal{M}$ by

$$\mathcal{A} = p\mathcal{M}p.$$

Note that $\mathcal{A}$ is a C^*-algebra with unit p. Consider the map $J : \mathcal{M} \to M_2(\mathcal{A})$ with $J(a) = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ where $x = pap$, $y = paup$, $z = pu^{-1}ap$ and $t = pu^{-1}agu$, and inverse $J^{-1}\left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) = x + yu^{-1} + uz + utu^{-1}$.

PROPOSITION 3.7. *J is a C^*-algebra isomorphism and* $J(p) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

The unitary orbit of p in $\mathcal{M}$ is consequently isomorphic to the Riemann sphere of $M_2(\mathcal{A})$ and also p and $1-p$ are in the same connected component of the space of projectors of the algebra $\mathcal{M}$.

The contents of Section 3.2.3 follow the development in [15].

3.2.4. *The Riemann sphere $\mathcal{R}$ as a reductive homogeneous space.* As we have seen, the Riemann sphere $\mathcal{R}$ of the algebra $\mathcal{A}$ is a subspace $\mathcal{R} \subset \mathcal{P}_2(\mathcal{A})$. Also it is clear that the group $\mathcal{U}_2$ operates on $\mathcal{P}_2(\mathcal{A})$ by inner automorphisms. In fact, $\mathcal{R}$ is by definition an orbit of this action, which makes $\mathcal{R}$ a homogeneous space of the group $\mathcal{U}_2$. This situation is studied in [6, Section 5] for the general case. In particular, $\mathcal{R}$ is a reductive homogeneous space of the group $\mathcal{U}_2$ and consequently it carries an invariant connection, which we will call the *standard connection*, whose covariant derivative is given by

$$D_X Y = X \cdot Y + [Y, [X, \tilde{p}]]$$

where $X \in (T\mathcal{R})_{\tilde{p}}$, Y is a vector field tangent to $\mathcal{R}$ in a neighborhood of $\tilde{p}$ and where $X \cdot Y$ is the directional derivative of Y in the "ambient" algebra $M_2(\mathcal{A})$ ($X \cdot Y = \frac{d}{dt}\big|_{t=0} Y(\gamma(t))$ for a curve $\gamma(t)$ in $\mathcal{R}$, $\gamma(0) = \tilde{p}$, and $\dot{\gamma}(0) = X$).

REMARK 3.8. Given a curve $\tilde{p}_t \in \mathcal{R}$ with $t \in [0,1]$, the differential equation $\frac{d}{dt}\tilde{g}_t = \left[\frac{d}{dt}\tilde{p}_t, \tilde{p}_t\right]\tilde{g}_t$ with initial condition $\tilde{g}_0 = 1$ has a solution $\tilde{g}_t \in \mathcal{U}_2$

and the action of $\tilde{g}_t$ on tangent vectors produces the parallel transport of the connection along the curve $\tilde{p}_t$ (see [6]).

Consequently, geodesics $\gamma(t)$ in $\mathcal{R}$ are defined by the condition

$$\frac{D}{dt}\dot{\gamma}(t) = 0$$

and they are explicitly given by

$$\gamma(t) = e^{t\tilde{X}}\tilde{p}e^{-t\tilde{X}}$$

where $X \in (T\mathcal{R})_{\tilde{p}}$ and $\tilde{X} = [X, \tilde{p}]$. The curve $\gamma(t)$ is the unique geodesic satisfying $\gamma(0) = \tilde{p}$ and $\dot{\gamma}(0) = X$. Therefore the exponential map is given by $\mathrm{Exp}_{\tilde{p}}(X) = \gamma(1)$ where $X \in (T\mathcal{R})_{\tilde{p}}$ and γ is the unique geodesic satisfying $\gamma(0) = \tilde{p}$ and $\dot{\gamma}(0) = X$. Observe that $\mathrm{Exp}_{\tilde{p}}(X)$ is defined for every $X \in (T\mathcal{R})_{\tilde{p}}$ and has the explicit form

$$\mathrm{Exp}_{\tilde{p}}(X) = e^{\tilde{X}}\tilde{p}e^{-\tilde{X}}.$$

REMARK 3.9. The exponential map $\mathrm{Exp}_{\tilde{p}_0} : (T\mathcal{R})_{\tilde{p}_0} \to \mathcal{R}$ is bijective from $V_0 = \{X \in (T\mathcal{R})_{\tilde{p}_0} : \|X\| < \pi/2\}$ to $U_0 = \{\tilde{p} \in \mathcal{R} : \|\tilde{p} - \tilde{p}_0\| < 1\}$, as we shall see later (see Theorem 3.16).

In what follows we denote by sinc the analytic function defined by $\mathrm{sinc}(x) = \sin(x)/x$, which is usually called the *cardinal sine.*

THEOREM 3.10. *If $\gamma : [0,1] \to \mathcal{R}$ is a geodesic with initial conditions $\gamma(0) = \tilde{p}_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\dot{\gamma}(0) = X = \left(\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right)$ then, considering the* $\mathrm{im}(\tilde{p}_0) \oplus \mathrm{im}(\tilde{p}_0)^\perp$ *decomposition,*

$$\begin{aligned}
\gamma(t) &= \begin{pmatrix} \cos^2|ta^*| & \cos|ta^*|(\mathrm{sinc}\,|ta^*|)ta \\ (\mathrm{sinc}\,|ta|)ta^*\cos|ta^*| & \sin^2|ta^*| \end{pmatrix} \\
&= \begin{pmatrix} \cos^2|ta^*| & \mathrm{sinc}(2|ta^*|)ta \\ \mathrm{sinc}(2|ta|)ta^* & \sin^2|ta^*| \end{pmatrix} \\
&= \begin{pmatrix} \cos|ta^*| \\ (\mathrm{sinc}\,|ta|)ta^* \end{pmatrix}\begin{pmatrix} \cos|ta^*| & ta\,\mathrm{sinc}\,|ta| \end{pmatrix} \\
&= (\cos^2|t\tilde{X}|)\tilde{p}_0 + (\sin^2|t\tilde{X}|)(1-\tilde{p}_0) + \mathrm{sinc}(2t|\tilde{X}|)\tilde{X}\tilde{\rho}_0
\end{aligned} \tag{3.7}$$

for $\tilde{X} = \left(\begin{smallmatrix} 0 & -a \\ a^* & 0 \end{smallmatrix}\right)$ and $\tilde{\rho}_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Proof. Recall that there is a unique geodesic such that

$$\dot{\gamma}(0) = X = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix},$$

which can be obtained by computing $\gamma(t) = e^{t\tilde{X}}\tilde{p}_0e^{-t\tilde{X}}$ with $\tilde{X} = \left(\begin{smallmatrix} 0 & -a \\ a^* & 0 \end{smallmatrix}\right)$ (see [15, 17]).

First, we will describe the unitary given by $e^{\tilde{X}}$ for $\tilde{X} = \left(\begin{smallmatrix} 0 & -a \\ a^* & 0 \end{smallmatrix}\right)$. If we separate the even and odd powers of the series we obtain

$$\tilde{X}^{2k} = (-1)^k \begin{pmatrix} |a^*|^{2k} & 0 \\ 0 & |a|^{2k} \end{pmatrix} \quad \text{for } k = 0, 1, \dots,$$

$$\tilde{X}^{2k+1} = (-1)^{k+1} \begin{pmatrix} 0 & |a^*|^{2k} a \\ -|a|^{2k} a^* & 0 \end{pmatrix} \quad \text{for } k = 0, 1, \dots,$$

and then

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \begin{pmatrix} |a^*|^{2k} & 0 \\ 0 & |a|^{2k} \end{pmatrix} = \begin{pmatrix} \cos |a^*| & 0 \\ 0 & \cos |a| \end{pmatrix},$$

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} \begin{pmatrix} 0 & -|a^*|^{2k} a \\ |a|^{2k} a^* & 0 \end{pmatrix} = \begin{pmatrix} \operatorname{sinc} |a^*| & 0 \\ 0 & \operatorname{sinc} |a| \end{pmatrix} \begin{pmatrix} 0 & -a \\ a^* & 0 \end{pmatrix}.$$

This implies that

$$e^{\tilde{X}} = \begin{pmatrix} \cos |a^*| & -(\operatorname{sinc} |a^*|) a \\ (\operatorname{sinc} |a|) a^* & \cos |a| \end{pmatrix} = \cos |\tilde{X}| + (\operatorname{sinc} |\tilde{X}|) \tilde{X} \tag{3.8}$$

since $|\tilde{X}| = \left(\begin{smallmatrix} |a^*| & 0 \\ 0 & |a| \end{smallmatrix}\right)$. And for $t \in \mathbb{R}_{\geq 0}$,

$$e^{t\tilde{X}} = \begin{pmatrix} \cos |ta^*| & -(\operatorname{sinc} |ta^*|) ta \\ (\operatorname{sinc} |ta|) ta^* & \cos |ta| \end{pmatrix} = \cos |t\tilde{X}| + \operatorname{sinc}(|t\tilde{X}|) \tilde{X}. \tag{3.9}$$

Then all the geodesics γ starting at $\gamma(0) = \tilde{p}_0$ are of the form

$$\gamma(t) = e^{t\tilde{X}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{-t\tilde{X}} = \begin{pmatrix} \cos^2 |ta^*| & \cos |ta^*| (\operatorname{sinc} |ta^*|) ta \\ (\operatorname{sinc} |ta|) ta^* \cos |ta^*| & \sin^2 |ta^*| \end{pmatrix}$$

$$= \begin{pmatrix} \cos |ta^*| \\ (\operatorname{sinc} |ta|) ta^* \end{pmatrix} \begin{pmatrix} \cos |ta^*| & ta \operatorname{sinc} |ta| \end{pmatrix}$$

where in the last equality we have used $(\operatorname{sinc} |ta^*|) ta = ta \operatorname{sinc} |ta|$.

In order to obtain the second equality in (3.7), we can observe that $ta^* \cos |ta^*| = (\cos |ta|) ta^*$ and that $\cos x \operatorname{sinc} x = \cos x \frac{\sin x}{x} = \frac{1}{2} \frac{\sin(2x)}{x} = \operatorname{sinc}(2x)$.

The last equality in (3.7) follows by direct computations. ∎

Remark 3.11. If the algebra $\mathcal{A}$ is faithfully represented in a Hilbert space H and $\left(\begin{smallmatrix} \xi \\ 0 \end{smallmatrix}\right) \in H \times \{0\} = \operatorname{im} \tilde{p}_0$, then $e^{t\tilde{X}} \left(\begin{smallmatrix} \xi \\ 0 \end{smallmatrix}\right) \in \operatorname{im} \gamma(t)$. Observe that $e^{t\tilde{X}} \left(\begin{smallmatrix} \xi \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos |ta^*| \xi \\ (\operatorname{sinc} |ta|) ta^* \xi \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos |ta^*| \xi \\ (\sin |ta|) v^* \xi \end{smallmatrix}\right)$ where $a = v|a|$ is the polar decomposition of a. For example, if $a \geq 0$, we have $e^{t\tilde{X}} \left(\begin{smallmatrix} \xi \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos(ta)\xi \\ \sin(ta)\xi \end{smallmatrix}\right)$, so that in the case where ξ is an eigenvector of a, $a\xi = \lambda\xi$, $e^{t\tilde{X}}\left(\begin{smallmatrix}\xi\\0\end{smallmatrix}\right)$ describes a circular movement in the two-dimensional plane generated by $\left(\begin{smallmatrix}\xi\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\\xi\end{smallmatrix}\right)$.

REMARK 3.12. Note that if we consider the algebra $\mathcal{A}$ represented in $B(H)$ we can also write the formula (3.7) as

$$\gamma(t) = \begin{pmatrix} \cos^2|ta^*| & \cos|ta^*|(\sin|ta^*|)u \\ (\sin|ta|)u^*\cos|ta^*| & (\sin|ta|)u^*(\sin|ta^*|)u \end{pmatrix}$$

where $a = u|a|$ is the polar decomposition of a (the partial isometry u might not belong to $\mathcal{A}$).

The space $\mathcal{R}$ also carries an invariant Finsler structure given by the C^*-algebra norm of $M_2(\mathcal{A})$. If $X \in (T\mathcal{R})_{\tilde{p}}$, X identifies with an element in $M_2(\mathcal{A})$ and has the corresponding norm. This Finsler structure on $\mathcal{R}$ allows us to define lengths of curves. In [17] it is shown that geodesics in $\mathcal{R}$ of length less than $\pi/2$ are minimal among curves joining given endpoints.

3.2.5. *The inverse of the exponential map in $\mathcal{R}$.* The standard connection of $\mathcal{R}$ defines the exponential map $\mathrm{Exp}_{\tilde{p}} : (T\mathcal{R})_{\tilde{p}} \to \mathcal{R}$ for $\tilde{p} \in \mathcal{R}$. In particular, the exponential map $\mathrm{Exp}_{\tilde{p}_0} : (T\mathcal{R})_{\tilde{p}_0} \to \mathcal{R}$ is given by

$$\mathrm{Exp}_{\tilde{p}_0}(X) = e^{\tilde{X}}\tilde{p}_0 e^{-\tilde{X}} \tag{3.10}$$

where $X \in (T\mathcal{R})_{\tilde{p}_0}$ and $\tilde{X} = [X, \tilde{p}_0]$ (explicitly, $X = \left(\begin{smallmatrix}0 & a\\a^* & 0\end{smallmatrix}\right)$ and $\tilde{X} = \left(\begin{smallmatrix}0 & -a\\a^* & 0\end{smallmatrix}\right)$ for $a \in \mathcal{A}$). It is well known that the exponential map is a diffeomorphism of a neighborhood $\mathcal{W}$ of $0 \in (T\mathcal{R})_{\tilde{p}_0}$ onto a neighborhood of $\tilde{p}_0$ in $\mathcal{R}$.

In Theorem 3.16 we will produce an explicit formula for the inverse map $\mathrm{Log}_{\tilde{p}_0}$ of the exponential map $\mathrm{Exp}_{\tilde{p}_0}$. The map $\mathrm{Log}_{\tilde{p}_0}$ will be defined on the open set $U_0 = \{\tilde{p} \in \mathcal{R} : \|[\tilde{p}_0, \tilde{p}]\| < 1/2\}$. The aforementioned formula involves the real analytic function

$$\mathrm{Asinc}(x) = \frac{\arcsin(x)}{x} \quad \text{for } x \in (-1, 1).$$

REMARK 3.13. For $\tilde{p} \in \mathcal{R}$, let $\tilde{\rho} = 2\tilde{p} - 1$ (the symmetry associated with $\tilde{p}$), and observe that $M_2 = M_2^0 \oplus M_2^1$ where $M_2^0 = \{\tilde{a} \in M_2 : \tilde{\rho}\tilde{a} = \tilde{a}\tilde{\rho}\}$ and $M_2^1 = \{\tilde{a} \in M_2 : \tilde{\rho}\tilde{a} = -\tilde{a}\tilde{\rho}\}$. Furthermore, the aforementioned decomposition defines on M_2 the structure of a $\mathbb{Z}_2$-graded algebra. In this context $(T\mathcal{R})_{\tilde{p}}$ may be identified with the self-adjoint part of M_2^1. In particular, the above formula for X reflects this fact.

At any $\tilde{p} \in \mathcal{R}$ the exponential $\mathrm{Exp}_{\tilde{p}}$ is given by $\mathrm{Exp}_{\tilde{p}}(X) = e^{\tilde{X}}\tilde{p}e^{-\tilde{X}}$ for X self-adjoint of degree 1 with respect to $\tilde{p}$ and $\tilde{X} = [X, \tilde{p}]$.

LEMMA 3.14. *Let $\tilde{p}_0 = \left(\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right)$ and $\tilde{p}$ be in $\mathcal{R}$ for $\tilde{p} = e^{\tilde{X}}\tilde{p}_0 e^{-\tilde{X}}$ for $\|\tilde{X}\| \leq \pi/2$ with $\tilde{X} = \left(\begin{smallmatrix}0 & -a\\a^* & 0\end{smallmatrix}\right)$ and $X = \left(\begin{smallmatrix}0 & -a\\a^* & 0\end{smallmatrix}\right) \in (T\mathcal{R})_{\tilde{p}_0}$. Then the following statements are equivalent:*

(1) $\|[\tilde p_0,\tilde p]\|<1/2$,
(2) $\|X\|=\|\tilde X\|<\pi/4$.

Proof. First note that $\tilde X$ anticommutes with $2\tilde p_0-1$ and then

$$
\begin{aligned}
(3.11)\qquad 1/2>\|[\tilde p_0,\tilde p]\| &= \|\tilde p_0 e^{\tilde X}\tilde p_0 e^{-\tilde X}-e^{\tilde X}\tilde p_0 e^{-\tilde X}\tilde p_0\|\\
&=\tfrac12\|2\tilde p_0 e^{\tilde X}\tilde p_0 e^{-\tilde X}-e^{\tilde X}\tilde p_0 e^{-\tilde X}+e^{\tilde X}\tilde p_0 e^{-\tilde X}-2e^{\tilde X}\tilde p_0 e^{-\tilde X}\tilde p_0)\|\\
&=\tfrac12\|(2\tilde p_0-1)e^{\tilde X}\tilde p_0 e^{-\tilde X}-e^{\tilde X}\tilde p_0 e^{-\tilde X}(2\tilde p_0-1)\|\\
&=\tfrac12\|e^{-\tilde X}\tilde p_0 e^{\tilde X}(2\tilde p_0-1)-e^{\tilde X}\tilde p_0 e^{-\tilde X}(2\tilde p_0-1)\|\\
&=\tfrac12\|e^{-\tilde X}\tilde p_0 e^{\tilde X}-e^{\tilde X}\tilde p_0 e^{-\tilde X}\|\\
&=\tfrac14\|2e^{-\tilde X}\tilde p_0 e^{\tilde X}-1-(2e^{\tilde X}\tilde p_0 e^{-\tilde X}-1)\|\\
&=\tfrac14\|e^{-\tilde X}(2\tilde p_0-1)e^{\tilde X}-e^{\tilde X}(2\tilde p_0-1)e^{-\tilde X})\|\\
&=\tfrac14\|e^{-2\tilde X}(2\tilde p_0-1)-e^{2\tilde X}(2\tilde p_0-1))\|=\tfrac14\|e^{-2\tilde X}-e^{2\tilde X}\|\\
&=\tfrac14\|1-e^{4\tilde X}\|.
\end{aligned}
$$

Therefore $\|e^{4\tilde X}-1\|<2$. This implies that $\|e^{4\tilde X}-1\|<2$ if and only if $\|[\tilde p_0,\tilde p]\|<1/2$. But since $\|e^{4\tilde X}-1\|=\sup_{i\theta\in\sigma(\tilde X)}|e^{4i\theta}-1|$, it follows that $\|[\tilde p_0,\tilde p]\|<1/2$ if and only if $|\theta|<\pi/4$ for $i\theta\in\sigma(\tilde X)$, which is equivalent to $\|\tilde X\|<\pi/4$. ∎

Lemma 3.15. *Let $\tilde p_0,\tilde p\in\mathcal R$ be such that $\|[\tilde p_0,\tilde p]\|<1/2$. Then $p_{11}\ge 1/2$ and $1/2\ge p_{22}$.*

Proof. The condition $\|[\tilde p_0,\tilde p]\|<1/2$ implies that $\|\tilde X\|<\pi/2$, and hence there exists a unique geodesic $\gamma(t)=e^{t\tilde X}\tilde p_0 e^{-t\tilde X}$, $t\in[0,1]$, between $\gamma(0)=\tilde p_0$ and $\gamma(1)=\tilde p$. This condition also implies that $\|\tilde p_0-\tilde p\|<1$, and hence p_{11} is positive definite and invertible.

Direct calculations show that $\|[\tilde p_0,\tilde p]\|=\|p_{12}\|=\|p_{21}\|<1/2$. Then if $\tilde p=\mathbf{x}\mathbf{x}^*$ for $\mathbf{x}=\left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right)\in\mathcal K$ where $x_1=(p_{11})^{1/2}$ is positive definite and invertible, we can write $\|x_1x_2^*x_2x_1\|=\|x_1|x_2|^2x_1\|<1/4$, since $p_{21}=x_2x_1$, $x_1^2+|x_2|^2=1$, and x_1 commutes with $|x_2|$. Moreover, we obtain

$$\big\||x_2|x_1\big\|=\big\|x_1^{1/2}|x_2|x_1^{1/2}\big\|<1/2,\quad\text{which implies}\quad |x_2|x_1\le 1/2$$

since $x_1^{1/2}|x_2|x_1^{1/2}\ge 0$.

We will now use the local cross-section $\sigma:\{\mathbf{x}\mathbf{x}^*:\mathbf{x}\in\mathcal K,\ x_1\in\mathcal G\}\to\mathcal K_0$ of $\mathfrak h$ (see Theorem 4.1.3 or (4.2)). Now define

$$\begin{pmatrix}\hat x_1(t)\\ \hat x_2(t)\end{pmatrix}=\sigma(\gamma(t)).$$

The entries satisfy $\hat{x}_1(0) = 1$, $\hat{x}_1(1) = x_1$ and $\hat{x}_2(1) = x_2$ since $\gamma(0) = \tilde{p}_0$ and $\gamma(1) = \tilde{p}$. Moreover, from the definition of σ it follows that $x_1(t) > 0$ for all $t \in [0,1]$. Now observe that by Lemma 3.14, since $\|t\tilde{X}\| < \pi/4$ for $t \in [0,1]$, we have $\|[\tilde{p}_0, \gamma(t)]\| < 1/2$ and therefore

$$\big\| x_1(t)|x_2(t)| \big\| < 1/2 \quad \text{for all } t \in [0,1]. \tag{3.12}$$

The function $g(s) = s\sqrt{1-s^2}$, $s \in [0,1]$, is positive in $(0,1)$, with $g(\sqrt{2}/2) = 1/2$ and $g(0) = g(1) = 0$.

Now, suppose that there exists $t_0 \in (0,1]$ such that $\|\hat{x}_1(t_0)\| < \sqrt{2}/2$. Then $g(\|\hat{x}_1(t_0)\|) < 1/2$. By the continuity of g, σ, and $\|\cdot\|$, and the fact that $g(\|\hat{x}_1(0)\|) = g(1) = 0$, there exists $\varepsilon \in (t_0, 1)$ such that $g(\|\hat{x}_1(\varepsilon)\|) = 1/2$. Then $\|x_1(\hat{\varepsilon})\| = \sqrt{2}/2$, and hence

$$\begin{aligned}\big\| \hat{x}_1(\varepsilon)|\hat{x}_2(\varepsilon)| \big\| &= \|\hat{x}_1(\varepsilon)(1-\hat{x}_1(\varepsilon)^2)^{1/2}\| = \|\hat{x}_1(\varepsilon)\|\sqrt{1-\|\hat{x}_1(\varepsilon)^2\|} \\ &= g(\|\hat{x}_1(\varepsilon)\|) = 1/2.\end{aligned}$$

This contradicts our hypothesis that $\|[\tilde{p}, \tilde{p}_0]\| < 1/2$. The issue arises because we had already established that $\big\| \hat{x}_1(t)|\hat{x}_2(t)| \big\| < 1/2$ for all $t \in [0,1]$ (see (3.12)) but we reached $1/2$, which is inconsistent with our assumption.

Thus, we conclude that $\|\hat{x}_1(t)\| > \sqrt{2}/2$ for all $t \in [0,1]$, and

$$\begin{aligned}\|\hat{x}_1(t)\|^2 = \|\hat{x}_1(t)^2\| &= \big\| 1 - |\hat{x}_2(t)|^2 \big\| \\ &= 1 - \big\| |\hat{x}_2(t)|^2 \big\| > 1/2.\end{aligned}$$

This implies that $\|\hat{x}_2(t)\|^2 < 1/2$, which gives $|\hat{x}_2(t)| \leq \sqrt{2}/2$. This implies that $\sqrt{1-\hat{x}_1(t)^2} \leq \sqrt{2}/2$ and therefore $1 - \hat{x}_1(t)^2 \leq 1/2$. Then $1/2 \leq \hat{x}_1(t)^2$ for all $t \in [0,1]$. Hence we have $1/2 \leq \hat{x}_1(1)^2 = x_1^2 = \tilde{p}_{11}$.

In order to prove that $1/2 \geq \tilde{p}_{22}$ observe that since $x_1^2 \geq 1/2$, it follows that $1 - |x_2|^2 \geq 1/2$ and $1/2 \geq |x_2|^2$. Thus we obtain $1/2 \geq \big\| |x_2| \big\|^2 = \|x_2\| = \|x_2^*\|$ and finally $1/2 \geq |x_2^*|^2 = p_{22}$. ∎

Theorem 3.16. *The exponential map* $\mathrm{Exp}_{\tilde{p}_0}$ (*see* (3.10)) *is a diffeomorphism* $\mathrm{Exp}_{\tilde{p}_0} : \{X \in (T\mathcal{R})_{\tilde{p}_0} : \|X\| < \pi/2\} \to \{\tilde{p} \in \mathcal{R} : \|\tilde{p} - \tilde{p}_0\| < 1\}$. *Moreover, if* $U_0 = \{\tilde{p} \in \mathcal{R} : \|[\tilde{p}_0, \tilde{p}]\| < 1/2\}$ *and* $V_0 = \{X \in (T\mathcal{R})_{\tilde{p}_0} : \|X\| < \pi/4\}$, *there exists an inverse map* $\mathrm{Log}_{\tilde{p}_0} : U_0 \to V_0$ *that is a diffeomorphism given by*

$$\mathrm{Log}_{\tilde{p}_0}(\tilde{p}) = \tilde{\rho}_0 \,\mathrm{Asinc}(2|[\tilde{p}_0, \tilde{p}]|)[\tilde{p}_0, \tilde{p}] \tag{3.13}$$

where $\tilde{\rho}_0 = 2\tilde{p}_0 - 1$.

We call the triple $(U_0, V_0, \mathrm{Log}_{\tilde{p}_0})$ the *geodesic chart* at $\tilde{p}_0$.

Proof. Similar computations to those made in (3.11) lead to the equivalence between the properties $\|\tilde{p} - \tilde{p}_0\| < 1$ and $\|\tilde{X}\| < \pi/2$ for $\tilde{p} = e^{\tilde{X}}\tilde{p}_0 e^{-\tilde{X}}$. Therefore, $\mathrm{Exp}_{\tilde{p}_0} : \{X \in (T\mathcal{R})_{\tilde{p}_0} : \|X\| < \pi/2\} \to \{\tilde{p} \in \mathcal{R} : \|\tilde{p} - \tilde{p}_0\| < 1\}$ is onto, and since $\|\tilde{p}_0 - \tilde{p}\| < 1$ implies that there is a unique geodesic between

$\tilde{p}_0$ and $\tilde{p}$, of the form $\mathrm{Exp}_{\tilde{p}_0}(t\tilde{X})$, the map $\mathrm{Exp}_{\tilde{p}_0}$ is also injective (see for example [1, Lemma 2.6]). Then $\mathrm{Exp}_{\tilde{p}_0} : \{X \in (T\mathcal{R})_{\tilde{p}_0} : \|X\| < \pi/2\} \to \{\tilde{p} \in \mathcal{R} : \|\tilde{p} - \tilde{p}_0\| < 1\}$ is a diffeomorphism (see [17]).

Using again Lemma 3.14 and the fact we mentioned above, that $\|\tilde{p} - \tilde{p}_0\| < 1$ is equivalent to $\|\tilde{X}\| < \pi/2$, it can be proved that $U_0 \subset \{\tilde{p} \in \mathcal{R} : \|\tilde{p} - \tilde{p}_0\| < 1\}$. Now we will prove that $\mathrm{Log}_{\tilde{p}_0} : U_0 \to V_0$ is the inverse of $\mathrm{Exp}_{\tilde{p}_0} : V_0 \to U_0$.

To prove that the formula (3.13) of the inverse holds, we will use the following expression from (3.7):

$$\mathrm{Exp}_{\tilde{p}_0}(X) = (\cos^2|\tilde{X}|)\tilde{p}_0 + (\sin^2|\tilde{X}|)(1 - \tilde{p}_0) + \mathrm{sinc}(2|\tilde{X}|)\tilde{X}\tilde{\rho}_0 \tag{3.14}$$

for an anti-self-adjoint codiagonal element $\tilde{X}$ such that $X\tilde{p}_0 - \tilde{p}_0 X = \tilde{X}$. Put $X = \tilde{\rho}_0 \,\mathrm{Asinc}(2|[\tilde{p}_0, \tilde{p}]|)[\tilde{p}_0, \tilde{p}]$. Then $\mathrm{Exp}_{\tilde{p}_0}(X) = e^{\tilde{X}}\tilde{p}_0 e^{-\tilde{X}}$ for

$$\tilde{X} = -\,\mathrm{Asinc}(2|[\tilde{p}_0, \tilde{p}]|)[\tilde{p}_0, \tilde{p}] \quad \text{with} \quad |\tilde{X}| = \tfrac{1}{2}\arcsin(2|[\tilde{p}_0, \tilde{p}]|). \tag{3.15}$$

Note here that the condition $\|[\tilde{p}_0, \tilde{p}]\| = \||[\tilde{p}_0, \tilde{p}]|\| < 1/2$ implies that arcsin and Asinc are defined and C^∞ at $2|[\tilde{p}_0, \tilde{p}]|$. Using the fact that

$$\cos^2\big(\tfrac{1}{2}\arcsin(2x)\big) = \tfrac{1}{2}(1 + \sqrt{1 - 4x^2})$$

and that $\tilde{p} = \mathbf{x}\mathbf{x}^* = \begin{pmatrix} x_1x_1^* & x_1x_2^* \\ x_2x_1^* & x_2x_2^* \end{pmatrix}$ for $\mathbf{x} \in \mathcal{V}_0$ (see Theorem 3.4), we will prove first that $\frac{1}{2}(1 + \sqrt{1 - 4|[\tilde{p}_0, \tilde{p}]|^2})_{11} = p_{11} = x_1x_1^* = x_1^2$ (where we can assume that x_1 can be taken invertible and positive). Note that, since $x_1^2 + |x_2|^2 = 1$ and x_1 commutes with $|x_2|$, we have

$$\begin{aligned} &\big(2\cos^2\big(\tfrac{1}{2}\arcsin(2|[\tilde{p}_0, \tilde{p}]|)_{11}\big) - 1\big)^2 \\ &\quad = 1 - 4(|[\tilde{p}_0, \tilde{p}]|^2)_{11} = 1 - 4x_1^2|x_2|^2 = 1 - 4x_1^2(1 - x_1^2) = (2x_1^2 - 1)^2. \end{aligned} \tag{3.16}$$

We can use Lemma 3.15 here to obtain $2x_1^2 - 1 \geq 0$, since $x_1^2 = p_{11} \geq 1/2$. This implies that $(1 - 4x_1^2(1 - x_1^2))^{1/2} = 2x_1^2 - 1$ and then $\frac{1}{2}(1 + \sqrt{1 - 4|[\tilde{p}_0, \tilde{p}]|^2})_{11} = x_1^2 = p_{11}$, which is the equality $\mathrm{Exp}_{\tilde{p}_0}(X)_{11} = \tilde{p}_{11}$.

For the $\tilde{p}_{22}$ entry we can reason similarly, but using instead the fact that in this case $\sin^2\big(\frac{1}{2}\arcsin(2x)\big) = \frac{1}{2}(1 - \sqrt{1 - 4x^2})$, and obtain

$$\begin{aligned} &\big(1 - 2\sin^2\big(\tfrac{1}{2}\arcsin(2|[\tilde{p}_0, \tilde{p}]|_{22})\big)\big)^2 \\ &\qquad = 1 - 4(|[\tilde{p}_0, \tilde{p}]|^2)_{22} = 1 - 4|p_{12}|^2 \\ &\qquad = 1 - 4(x_1x_2^*)^*x_1x_2^* = 1 - 4x_2x_1^2x_2^* \\ &\qquad = 1 - 4x_2(1 - x_2^*x_2)x_2^* = 1 - 4(|x_2^*|^2 - |x_2^*|^4) = (1 - 2|x_2^*|^2)^2. \end{aligned}$$

Then, since $|x_2^*|^2 = p_{22} \leq 1/2$ (see Lemma 3.15), it follows that $1 - 2|x_2^*|^2 \geq 0$, and hence we obtain $1 - 2\sin^2\big(\frac{1}{2}\arcsin(2|[\tilde{p}_0, \tilde{p}]|_{22})\big) = 1 - 2|x_2^*|^2$. Therefore

$$\mathrm{Exp}_{\tilde{p}_0}(X)_{22} = \sin^2\big(\tfrac{1}{2}\arcsin(2|[\tilde{p}_0, \tilde{p}]|_{22})\big) = |x_2^*|^2 = p_{22}.$$

Considering the last term of (3.14), the codiagonal of $\mathrm{Exp}_{\tilde{p}_0}(X)$, observe that if $[\tilde{p}_0,\tilde{p}] = |[\tilde{p}_0,\tilde{p}]|\tilde{\nu}$ is the polar decomposition of $[\tilde{p}_0,\tilde{p}]$, we can write (see (3.15))

$$\begin{aligned}\operatorname{sinc}(2|\tilde{X}|)\tilde{X}\tilde{\rho}_0 &= -\tfrac{1}{2}\sin(2|\tilde{X}|)\tilde{\nu}\tilde{\rho} = -\tfrac{1}{2}\sin(\arcsin(2|[\tilde{p}_0,\tilde{p}]|))\tilde{\nu}\tilde{\rho}_0\\ &= -|[\tilde{p}_0,\tilde{p}]|\tilde{\nu}\tilde{\rho}_0 = -[\tilde{p}_0,\tilde{p}]\tilde{\rho}_0.\end{aligned}$$

Then, since $[\tilde{p}_0,\tilde{p}] = \left(\begin{smallmatrix} 0 & p_{12}\\ -p_{21} & 0\end{smallmatrix}\right)$, the codiagonal of $\tilde{p}$ coincides with the product $-[\tilde{p}_0,\tilde{p}]\rho_0$.

Therefore we have proved that $\mathrm{Exp}_{\tilde{p}_0}(-\tilde{\rho}_0 \operatorname{Asinc}(2|[\tilde{p}_0,\tilde{p}]|)\,[\tilde{p}_0,\tilde{p}]) = \tilde{p}$ for $\tilde{p}\in U_0$. ∎

Remark 3.17. The formula (3.13) does not give the inverse of $\mathrm{Exp}_{\tilde{p}_0}$ in the domain $\{\tilde{p}\in\mathcal{R} : \|\tilde{p}-\tilde{p}_0\| < 1\} \supset \{\tilde{p}\in\mathcal{R} : \|[\tilde{p}_0,\tilde{p}]\| < 1/2\}$. An example where $1/2 < \|\tilde{p}-\tilde{p}_0\| < 1$ and $\mathrm{Exp}_{\tilde{p}_0}(\tilde{\rho}_0 \operatorname{Asinc}(2|[\tilde{p}_0,\tilde{p}]|)[\tilde{p}_0,\tilde{p}])(\tilde{p}) \neq \tilde{p}$ is

$$\tilde{p} = \begin{pmatrix} \cos^2(\pi/3) & \sin(\pi/3)\cos(\pi/3)\\ \sin(\pi/3)\cos(\pi/3) & \sin^2(\pi/3)\end{pmatrix} = \begin{pmatrix} 1/4 & \sqrt{3}/4\\ \sqrt{3}/4 & 3/4\end{pmatrix} \in M_2(\mathbb{C}).$$

Remark 3.18. In general, if $\mathrm{Exp}_{\tilde{p}_0}(X) = \tilde{p}$ and $\|X\| < \pi/2$, we will say that $X \in (T\mathcal{R})_{\tilde{p}_0}$ is the *geodesic coordinate* of $\tilde{p}$. In this way we have geodesic coordinates in $(T\mathcal{R})_{\tilde{p}_0}$ for points $\tilde{p}\in\mathcal{R}$ such that $\|[\tilde{p}_0,\tilde{p}]\| < 1/2$.

Remark 3.19. Consider a representation of the algebra $\mathcal{A}$ into a Hilbert space H and the corresponding representation of $M_2(\mathcal{A})$ in $H\oplus H$. Next, write $[\tilde{p}_0,\tilde{p}] = |[\tilde{p}_0,\tilde{p}]|\tilde{u}$, the polar decomposition of $[\tilde{p}_0,\tilde{p}]$, where $\tilde{u}$ is the partial isometry, and observe that $\tilde{u}$ commutes with $|[\tilde{p}_0,\tilde{p}]|$ since $[\tilde{p}_0,\tilde{p}]$ is anti-self-adjoint. Then we can write

$$\mathrm{Log}_{\tilde{p}_0}(\tilde{p}) = \tfrac{1}{2}\arcsin(2|[\tilde{p}_0,\tilde{p}]|)\tilde{\rho}_0\tilde{u},$$

the polar decomposition of $\mathrm{Log}_{\tilde{p}_0}(\tilde{p})$. In this formula we may interpret the positive part $\frac{1}{2}\arcsin(2|[\tilde{p}_0,\tilde{p}]|)$ as a kind of "unoriented" angle between $\tilde{p}_0$ and $\tilde{p}$ and the partial isometry $\tilde{\rho}_0\tilde{u}$ as a *partial imaginary unit* in the sense that $(\tilde{\rho}_0\tilde{u})^2 = -\tilde{q}$ where $\tilde{q}$ is the projection $\tilde{u}^*\tilde{u}$.

3.2.6. *Geometric interpretation of the logarithm.* We start with an example.

Let $\mathcal{A} = \mathbb{C}$ so $M_2 = M(2,\mathbb{C})$ is the C^*-algebra of 2×2 complex matrices. The $\mathcal{U}_2$ orbit of $\mathbf{e}_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is the unit sphere $\mathcal{K} = S^3$ of $\mathbb{C}^2$. Also note that the Riemann sphere $\mathcal{R}$ is the original Riemann sphere which is here represented by the orbit of the projector $\tilde{p}_0 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ under the action of $\mathcal{U}_2$ (of course $\mathcal{R}$ is diffeomorphic to the projective line $\mathbb{P}^1(\mathbb{C})$ "=" S^2). Finally, the Hopf fibration is the original Hopf fibration given by $\mathfrak{h} : \mathcal{K}\to\mathcal{R}$,

$$\mathbf{z}\mapsto \mathfrak{h}(\mathbf{z}) = \tilde{p}_z, \quad \text{where} \quad \mathbf{z} = \begin{pmatrix} z_1\\ z_2\end{pmatrix}, \quad \tilde{p}_z = \mathbf{z}\mathbf{z}^* = \begin{pmatrix} |z_1|^2 & z_1\overline{z_2}\\ \overline{z_1}z_2 & |z_2|^2\end{pmatrix}.$$

Let $\tilde{p} \in \mathcal{R}$ be such that $\| [\tilde{p}_0, \tilde{p}] \| < 1/2$ and write $X = \mathrm{Log}_{\tilde{p}_0}(\tilde{p})$. According to the explicit formula for $\mathrm{Log}_{\tilde{p}_0}$ we may write

$$X = \frac{\arcsin(2|[\tilde{p}_0, \tilde{p}]|)}{2} \tilde{\rho}_0 \tilde{u},$$

where $[\tilde{p}_0, \tilde{p}] = |[\tilde{p}_0, \tilde{p}]|\tilde{u}$ ($\tilde{u}$ a partial isometry) is the polar decomposition of $[\tilde{p}_0, \tilde{p}]$. Notice that $|[\tilde{p}_0, \tilde{p}]|$ commutes with $[\tilde{p}_0, \tilde{p}]$. Observe that $|[\tilde{p}_0, \tilde{p}]| = |z_1|\,|z_2|$ (a scalar in $M(2, \mathbb{C})$). Now, since $|z_1|^2 + |z_2|^2 = 1$, there is a unique angle $0 \le \varphi \le \pi/2$ such that $|z_2| = \sin(\varphi)$ and $|z_1| = \cos(\varphi)$. Therefore, the positive part of X is exactly φ, so

$$X = \varphi \tilde{\rho}_0 \tilde{u}$$

is the polar decomposition of X. Finally, the positive part of the logarithm of $\tilde{p}$ is the Finsler distance $\mathrm{dist}(\tilde{p}_0, \tilde{p})$ in the Riemann sphere.

Next we produce a geometric interpretation of $[\tilde{p}_0, \tilde{p}]$ $(= |[\tilde{p}_0, \tilde{p}]|\tilde{u})$. Consider Figure 1, where we schematically represent $\tilde{p}_0$, $\tilde{p}$ and the (complex) lines $l_1 = \ker(\tilde{p})$, $l_2 = \mathrm{im}(\tilde{p})$, $l_3 = \mathrm{im}(\tilde{p}_0)$ and $l_4 = \ker(\tilde{p}_0)$. The correspondence $l_3 \ni x \mapsto y \in l_3$ defines a linear map $y = \alpha x$ from l_3 to l_3. The number α is the classical cross ratio of the ordered four points l_1, l_2, l_3, l_4 in the complex projective line. In our case, $[\tilde{p}_0, \tilde{p}]$ is

$$[\tilde{p}_0, \tilde{p}] = \begin{pmatrix} 0 & z_1 \overline{z_2} \\ -\overline{z_1} z_2 & 0 \end{pmatrix}$$

and therefore $[\tilde{p}_0, \tilde{p}]$ maps l_3 into l_4 and l_4 into l_3. We only describe the map $l_3 \to l_4$ (the other one is similar). The correspondence $x \to w$ in the picture has matrix $\beta = -\overline{z_1} z_2$ which determines $[\tilde{p}_0, \tilde{p}]$. We call this geometric construction the *complementary cross ratio.*

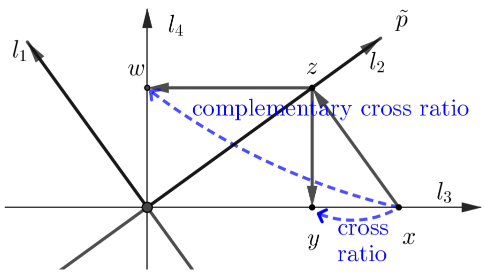


Fig. 1. Cross ratio and complementary cross ratio

With this example in mind we now turn to the general case. Observe first that the inverse (see [17]) of the diffeomorphism $\mathrm{Exp}_{\tilde{p}_0} : \{\tilde{p} \in \mathcal{R} : \{X \in (T\mathcal{R})_{\tilde{p}_0} : \|X\| < \pi/2\} \to \|\tilde{p} - \tilde{p}_0\| < 1\}$ defined in (3.10) allows us to determine the angle between $\tilde{p}$ and $\tilde{p}_0$ using the polar decomposition of the corresponding $X \in (T\mathcal{R})_{\tilde{p}_0}$ in some representation of $\mathcal{A}$. In what follows, we consider the formula of $\operatorname{Log}_{\tilde{p}_0}$ from (3.13) to obtain an expression of this angle. Let $\tilde{p} \in \mathcal{R}$ be such that $\|[\tilde{p}_0, \tilde{p}]\| < 1/2$. Then $\operatorname{Log}_{\tilde{p}_0} \tilde{p} = \operatorname{Asinc}(2|[\tilde{p}_0, \tilde{p}]|)(\tilde{\rho}_0[\tilde{p}_0, \tilde{p}])$ (see Theorem 3.16).

We can represent the algebra $\mathcal{A}$, and correspondingly $M_2(\mathcal{A})$, faithfully in the Hilbert space H (resp. $H \times H$) and refer the polar decompositions to this representation. Write the right polar decomposition of the bracket $[\tilde{p}_0, \tilde{p}]$ as $[\tilde{p}_0, \tilde{p}] = |[\tilde{p}_0, \tilde{p}]|\tilde{v}$. Note that $[\tilde{p}_0, \tilde{p}] = (-\tilde{v}^*)|[\tilde{p}_0, \tilde{p}]|$ is the left polar decomposition.

We claim that the polar decomposition of $\operatorname{Log}_{\tilde{p}_0} \tilde{p}$ is $\operatorname{Log}_{\tilde{p}_0} \tilde{p} = |\operatorname{Log}_{\tilde{p}_0} \tilde{p}|\tilde{u}$ where $|\operatorname{Log}_{\tilde{p}_0} \tilde{p}| = \frac{\arcsin 2|[\tilde{p}_0,\tilde{p}]|}{2}$ and where the partial isometry $\tilde{u}$ is $\tilde{u} = -\tilde{\rho}_0\tilde{v}^*$.

In order to explore the positive part of $\operatorname{Log}_{\tilde{p}_0} \tilde{p}$ we first describe $|[\tilde{p}_0, \tilde{p}]|$ as follows. Take $\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathcal{K}$ such that $\mathfrak{h}(\mathbf{x}) = \mathbf{x}\mathbf{x}^* = \left(\begin{smallmatrix} x_1x_1^* & x_1x_2^* \\ x_2x_1^* & x_2x_2^* \end{smallmatrix}\right) = \tilde{p}$ and x_1 is positive invertible. Such a choice is unique. Recall that the equality $|x_1|^2 + |x_2|^2 = 1$ implies that $|x_1| = x_1$ commutes with $|x_2|$.

Then if $x_2 = w|x_2|$ is the polar decomposition of x_2 we have $|[\tilde{p}_0, \tilde{p}]|^2 = \left(\begin{smallmatrix} x_1x_2^*x_2x_1 & 0 \\ 0 & x_2x_1^2x_2^* \end{smallmatrix}\right)$, so

$$|[\tilde{p}_0, \tilde{p}]| = \begin{pmatrix} x_1|x_2| & 0 \\ 0 & wx_1|x_2|w^* \end{pmatrix}.$$

So we have the following expression for $|\operatorname{Log}_{\tilde{p}_0} \tilde{p}|$:

$$|\operatorname{Log}_{\tilde{p}_0} \tilde{p}| = \begin{pmatrix} \frac{\arcsin(2x_1|x_2|)}{2} & 0 \\ 0 & \frac{\arcsin(2wx_1|x_2|w^*)}{2} \end{pmatrix}.$$

Now write $x_1 = \cos\varphi$ for a unique positive element $\varphi \in \mathcal{A}$ ($0 \le \varphi \le \pi/4$, see Lemma 3.15), and therefore $|x_2| = \sin\varphi$. So we have proved the following result.

Theorem 3.20. *Let $\tilde{p} \in \mathcal{R}$ such that $\|[\tilde{p}_0, \tilde{p}]\| < 1/2$. Then there exists a unique element $\varphi \in \mathcal{A}$ $(0 \le \varphi \le \pi/4)$ such that*

$$|\operatorname{Log}_{\tilde{p}_0} \tilde{p}| = \begin{pmatrix} \varphi & 0 \\ 0 & w\varphi w^* \end{pmatrix}$$

where $\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ is the element in $\mathcal{K}$ that projects onto $\tilde{p}$ with x_1 positive and invertible, and w is the partial isometry of the polar decomposition $x_2 = w|x_2|$.

We call the positive operator $\varphi \in \mathcal{A}$ the angle *between $\tilde{p}_0$ and $\tilde{p}$.*

Remark 3.21. Since $\operatorname{Log}_{\tilde{p}_0} \tilde{p}$ directs the geodesic in $\mathcal{R}$ from $\tilde{p}_0$ to $\tilde{p}$ in $\mathcal{R}$, it follows that its norm is the Finsler distance from $\tilde{p}_0$ to $\tilde{p}$, and therefore this distance is $\|\varphi\|$.

Remark 3.22. The geometric interpretation of the commutator $[\tilde{p}_0, \tilde{p}]$ is given by the constructions of "projection and section" illustrated in Figure 1 in exactly the same way. The correspondence $x \mapsto y$ in the picture is the classical cross ratio as defined by Zelikin [21]. In our case $[\tilde{p}_0, \tilde{p}]$ is given geometrically by the correspondence $x \mapsto w$ from l_3 to l_4 (and similarly $l_4 \to l_3$). We call this correspondence the *complementary cross ratio.*

4. The Hopf fibration. In this section we will describe more properties of the Hopf fibration defined in Definition 2.5. Recall that the total space is the sphere $\mathcal{K}$ in $\mathcal{A}^2$, the base is the Riemann sphere $\mathcal{R}$ of $\mathcal{A}$, the group is the unitary group $\mathcal{U}$ of the algebra $\mathcal{A}$ and the projection is

$$\mathfrak{h} : \mathcal{K} \to \mathcal{R} \quad \text{given by} \quad \mathfrak{h}(\mathbf{x}) = \mathbf{x}\mathbf{x}^* = \tilde{p}_{\mathbf{x}}.$$

Note that $\mathcal{U}$ operates on the right in $\mathcal{K}$ by $\mathbf{x}u = \left(\begin{smallmatrix} x_1 u \\ x_2 u \end{smallmatrix}\right)$ for $\mathbf{x} \in \mathcal{K}$ and $u \in \mathcal{U}$.

Theorem 4.1. *The Hopf fibration* $\mathfrak{h}$ *is a* C^∞ *principal bundle with structure group* $\mathcal{U}$.

Proof. We have to prove that

1. $\mathfrak{h}$ is a C^∞ map onto $\mathcal{R}$,
2. the fibers of $\mathfrak{h}$ are the $\mathcal{U}$-orbits of the action,
3. the map $\mathfrak{h}$ has C^∞ local cross sections.

1. We use the atlas defined on $\mathcal{K}$ (see Section 3.1). It suffices to prove this only for the case of the local identification associated with $\mathcal{K}_0$ (because local identifications are obtained by just acting with $\mathcal{U}_2$ on "basic identification" associated to $\mathcal{K}_0$). In this identification the map $\mathfrak{h}$ sends $(a, u) \mapsto \tilde{p}$, where $a \in \mathcal{A}$, $u \in \mathcal{U}$, $\tilde{p} = \mathbf{x}\mathbf{x}^*$ and $\mathbf{x} = \left(\begin{smallmatrix} (1+a^*a)^{-1/2}u \\ a(1+a^*a)^{-1/2}u \end{smallmatrix}\right)$, which is obviously smooth.

The map is clearly surjective since given $\tilde{p} \in \mathcal{R}$, we have $\tilde{p} = \tilde{u}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\tilde{u}^*$ and therefore $\tilde{p} = \tilde{u}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\tilde{u}^* = \tilde{u}\mathbf{e}_1\mathbf{e}_1^*\tilde{u}^*$ so that $\tilde{p} = \mathfrak{h}(\tilde{u}\mathbf{e}_1)$.

2. Here we will show that every fiber $\mathfrak{h}^{-1}(\tilde{p}_{\mathbf{x}})$ can be identified with the unitary group $\mathcal{U}$ of $\mathcal{A}$ and that every projector $\tilde{p}_{\mathbf{x}} \in \mathcal{R}$ is the image under $\mathfrak{h}$ of one of such fibers.

Let us consider first the fiber $\{\tilde{u}\mathbf{e}_1 \in \mathcal{K} : \tilde{u} \in \mathcal{U}_2 \text{ and } \tilde{u}\tilde{p}_0\tilde{u}^* = \tilde{p}_0\}$ over $\tilde{p}_0$. Here the equation $\tilde{u}\tilde{p}_0\tilde{u}^* = \tilde{p}_0$ for $\tilde{u} = \left(\begin{smallmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{smallmatrix}\right)$ is equivalent to $\left(\begin{smallmatrix} u_{1,1} \\ u_{2,1} \end{smallmatrix}\right)(u_{1,1}^* \quad u_{2,1}^*) = \tilde{p}_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This equation implies that $u_{1,1} \in \mathcal{U}$ and that $v_{2,1} = 0$. Moreover, using the fact that $\tilde{u} \in \mathcal{U}$ we can conclude that $u_{1,2} = 0$ and $u_{2,2} \in \mathcal{U}$ also. Hence, in this case the fiber is

$$\mathfrak{h}^{-1}(\tilde{p}_0) = \left\{ \begin{pmatrix} u_{1,1} & 0 \\ 0 & u_{2,2} \end{pmatrix} \mathbf{e}_1 : u_{1,1}, u_{2,2} \in \mathcal{U} \right\} = \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} : u \in \mathcal{U} \right\}, \tag{4.1}$$

which can be identified with $\mathcal{U}$.

Now consider the general case of $\tilde{p}_{\mathbf{x}} \in \mathcal{R}$ where $\mathbf{x} = \tilde{v}\mathbf{e}_1$ with $\tilde{v} \in \mathcal{U}_2$ and suppose that $\mathfrak{h}(\tilde{w}\mathbf{e}_1) = \tilde{p}_{\mathbf{x}}$ with $\tilde{w} \in \mathcal{U}_2$ is any other element of the fiber. Then we have $\tilde{w}\mathbf{e}_1\mathbf{e}_1^*\tilde{w}^* = \tilde{p}_{\mathbf{x}} = \tilde{v}\mathbf{e}_1\mathbf{e}_1^*\tilde{v}^*$ for $\tilde{w} \in \mathcal{U}_2$. Now, using $\tilde{p}_0 = \mathbf{e}_1\mathbf{e}_1^* = \tilde{v}^*\tilde{w}\mathbf{e}_1\mathbf{e}_1^*\tilde{w}^*\tilde{v} = \tilde{v}^*\tilde{w}\tilde{p}_0\tilde{w}^*\tilde{v}$ and $\tilde{v}^*\tilde{w} \in \mathcal{U}_2$, the description (4.1) proves that $\tilde{v}^*\tilde{w} = \left(\begin{smallmatrix} u_{1,1} & 0 \\ 0 & u_{2,2}\end{smallmatrix}\right)$ for $u_{1,1}, u_{2,2} \in \mathcal{U}$. Then

$$\tilde{w}\mathbf{e}_1 = \tilde{v}\begin{pmatrix} u_{1,1} & 0 \\ 0 & u_{2,2}\end{pmatrix}\mathbf{e}_1 = \begin{pmatrix} v_{1,1}u_{1,1} \\ v_{1,2}u_{1,1}\end{pmatrix} = \begin{pmatrix} v_{1,1} \\ v_{1,2}\end{pmatrix}u_{1,1} \quad \text{with } u_{1,1} \in \mathcal{U},$$

which proves that for every $\mathbf{x} \in \mathcal{K}$ the fiber of $\tilde{p}_{\mathbf{x}}$ can be identified with $\mathcal{U}$.

3. Consider the subset of $\mathcal{R}$ given by $V_0 = \mathfrak{h}(\mathcal{K}_0) = \{\mathbf{x}\mathbf{x}^* : \mathbf{x} \in \mathcal{K}$ and $x_1 \in \mathcal{G}\}$ where $\mathcal{K}_0$ is the domain of the map ψ_0 defined in (3.1), which is also the range of its inverse $\Psi_0 : \mathcal{A} \otimes \mathcal{U} \to \mathcal{K}_0$ (see (3.2)). Then V_0 is an open neighborhood of $\tilde{p}_0$ where we can define a section σ by

$$\begin{aligned} &\sigma : V_0 \to \mathcal{K}_0, \\ &\sigma(\tilde{p}_{\mathbf{x}}) = \Psi_0(a, 1) = \begin{pmatrix} (1 + a^*a)^{-1/2} \\ a(1 + a^*a)^{-1/2}\end{pmatrix} = \begin{pmatrix} 1 \\ a\end{pmatrix}(1 + a^*a)^{-1/2} \end{aligned} \tag{4.2}$$

for $\mathbf{x} = \Psi_0(a, u)$ with $u \in \mathcal{U}$ (or equivalently for $\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ such that $x_2x_1^{-1} = a$).

Let us see first that σ is well defined. Suppose that $\tilde{p}_{\mathbf{x}} = \mathbf{x}\mathbf{x}^* = \mathbf{z}\mathbf{z}^* = \tilde{p}_{\mathbf{z}}$. Then $\mathbf{x} = \mathbf{z}u$ for $u \in \mathcal{U}$ (see Proposition 2.9). Hence, if $x_1 = rv$ for $r > 0$ and $v \in \mathcal{U}$, then $z_1 = rvu$ for $v \in \mathcal{U}$ and $z_2 = x_2u$. Therefore $z_2z_1^{-1} = x_2uu^*x_1^{-1} = x_2x_1^{-1}$, which implies that $\sigma(\tilde{p}_{\mathbf{x}}) = \sigma(\tilde{p}_{\mathbf{z}})$.

Moreover, if we compose σ with the map $\psi_0 : \mathcal{K}_0 \to \mathcal{A} \times \mathcal{U}$ we obtain $\psi_0(\sigma(\tilde{p}_{\mathbf{x}})) = \psi_0(\Psi_0(x_2x_1^{-1}, 1)) = (x_2x_1^{-1}, 1)$, which is clearly C^∞ since $(\tilde{p}_{\mathbf{x}})_{2,1}\,(\tilde{p}_{\mathbf{x}})_{1,1}^{-1} = x_2x_1^*(x_1x_1^*)^{-1} = x_2x_1^*(x_1^*)^{-1}x_1^{-1} = x_2x_1^{-1}$ is an analytic function of two of the entries of $\tilde{p}_{\mathbf{x}} \in M_2(\mathcal{A})$. ∎

DEFINITION 4.2. Given $\tilde{p} \in \mathcal{R}$ we will say that $(x_1, x_2) \in \mathcal{A} \times \mathcal{A}$ is a *pair of homogeneous coordinates* for $\tilde{p}$ if $\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \operatorname{im}\tilde{p}$ and there exists an invertible element $\lambda \in \mathcal{A}$ such that $\mathbf{x}\lambda \in \mathcal{K}$.

Observe that every $\tilde{p} \in \mathcal{R}$ has a pair of homogeneous coordinates. Also note that if (x_1, x_2) and (x_1', x_2') are pairs of homogeneous coordinates of $\tilde{p}$ then there exists an invertible element λ in $\mathcal{A}$ such that $x_1' = x_1\lambda$ and $x_2' = x_2\lambda$.

REMARK 4.3. The open set $\mathcal{V}_0$ defined in (3.3) consists of all $\tilde{p}$ that have homogeneous coordinates (x_1, x_2) with x_1 invertible.

4.1. Relation between geodesic and homogeneous coordinates in $\mathcal{R}$. We now give an explicit expression for the relation between homogeneous coordinates and geodesic coordinates of an element $\tilde{p} \in U_0$.

Theorem 4.4. *Let* $\tilde{p} \in U_0$. *Consider the diagram*

$$\begin{array}{ccccc} & & \mathcal{K}_0 \subset \mathcal{K} & & \\ & & \big\downarrow \mathfrak{h} & \overset{\psi_0}{\searrow} & \\ V_0 \subset (T\mathcal{R})_{\tilde{p}_0} & \xrightarrow{\operatorname{Exp}_{\tilde{p}_0}} & U_0 \subset \mathcal{R} & \xrightarrow{\varphi_0} & \mathcal{A} \\ \Cup & & & & \Cup \\ X = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} & & \longmapsto & & v \tan |a| \end{array}$$

where $\operatorname{Log}_{\tilde{p}_0}(\tilde{p}) = X$ *and* v *comes from the polar decomposition* $a = v|a|$. *Then*

$$\varphi_0(\tilde{p}) = v \tan |a|.$$

Proof. Suppose that $\operatorname{Exp}_{\tilde{p}_0}(X) = \tilde{p}$ with $X = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}$ and $\|X\| = \|a\| = \|a^*\| < \pi/4$. We know from (3.7), considering that in Theorem 3.10 we have used $X = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$, that $\tilde{p} = \begin{pmatrix} \cos |a| \\ a \operatorname{sinc} |a| \end{pmatrix} (\cos |a| \quad (\operatorname{sinc} |a|) a^*)$ is a possible expression of $\tilde{p}$ in terms of a. Now consider $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos |a| \\ a(\operatorname{sinc} |a|) \end{pmatrix} = \begin{pmatrix} \cos |a| \\ v(\sin |a|) \end{pmatrix}$ where v is the isometry of the polar decomposition $a = v|a|$. Then $x_1 = \cos |a|$ is invertible since $\|a\| < \pi/4$ and $\mathbf{x}^*\mathbf{x} = (\cos |a|)^2 + |a|^2 (\operatorname{sinc} |a|)^2 = (\cos |a|)^2 + (\sin |a|)^2 = 1$. We can also find a unitary $\tilde{u} \in \mathcal{U}_2$ as in (3.5) such that $\mathbf{x} = \tilde{u}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and then $\mathbf{x} \in \mathcal{K}$. Therefore, $x_2 x_1^{-1} = v \sin |a| (\cos |a|)^{-1} = v \tan |a|$ and hence we obtain the formula $\varphi_0(\tilde{p}) = v \tan |a|$. ■

Remark 4.5. Consider the classical picture in Figure 2, where we have schematically represented homogeneous coordinates (x_1, x_2) for $\tilde{p}$. The "affine" coordinate $\varphi_0(\tilde{p})$ is $x_2 x_1^{-1}$, while the element $v \tan |a|$ is related to the geodesic coordinate of $\tilde{p}$, which is X. This suggests naming $|a|$ the *unoriented angle* between $\tilde{p}$ and $\tilde{p}_0$, and the partial isometry v becomes a "phase" related to the pair $(\tilde{p}_0, \tilde{p})$.

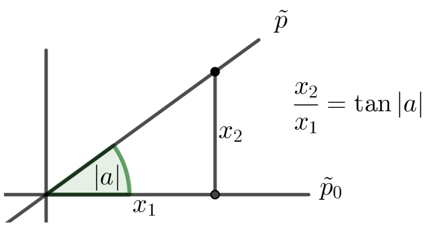


Fig. 2. Unoriented angle

Note that the angle $|a|$ coincides with the one denoted by φ in Theorem 3.20.

4.2. The canonical connection on the Hopf fibration. We will define a C^∞ *horizontal distribution* $H_{\mathbf{x}}$ of subspaces of the tangent spaces

$(T\mathcal{K})_{\mathbf{x}}$ for $\mathbf{x} \in \mathcal{K}$. This distribution turns out to be invariant under the right action of $\mathcal{U}$ on $\mathcal{K}$ and consequently defines a connection on the principal bundle $\mathcal{K} \to \mathcal{R}$. We will call this connection the *canonical connection* on the Hopf fibration.

Observe that given $\mathbf{x} \in \mathcal{K}$ the tangent space $(T\mathcal{K})_{\mathbf{x}}$ is described as follows:

$$(T\mathcal{K})_{\mathbf{x}} = \{\xi \in \mathcal{A}^2 : \langle \mathbf{x}, \xi \rangle \text{ is anti-self-adjoint}\}.$$

Observe that every $\mathbf{x} \in \mathcal{K}$ satisfies $\langle \mathbf{x}^*, \mathbf{x} \rangle = 1$.

At each point $\mathbf{x} \in \mathcal{K}$ we have the vertical tangent space $V_{\mathbf{x}} \subset (T\mathcal{K})_{\mathbf{x}}$ defined by

$$V_{\mathbf{x}} = \ker (T\mathfrak{h})_{\mathbf{x}},$$

where $T\mathfrak{h}$ is the tangent map. Clearly $V_{\mathbf{x}}$ is the image of the Lie algebra of $\mathcal{U}$ under the derivative at $u = 1$ of the action $u \mapsto \mathbf{x}u$. Therefore

$$V_{\mathbf{x}} = \{\mathbf{x}a : a \in \mathcal{A} \text{ anti-self-adjoint}\}.$$

Next we define the *horizontal space* $H_{\mathbf{x}} \subset (T\mathcal{K})_{\mathbf{x}}$ *at* $\mathbf{x} \in \mathcal{K}$ by

$$H_{\mathbf{x}} = \ker(\tilde{p}_{\mathbf{x}}),$$

where the vectors in $H_{\mathbf{x}}$ are considered as tangent vectors to $\mathcal{A}^2$ at $\mathbf{x}$ (note that if $\xi \in \ker(\tilde{p}_{\mathbf{x}})$ then $\langle \mathbf{x}, \xi \rangle = 0$).

Observe that $(T\mathcal{K})_{\mathbf{x}} = V_{\mathbf{x}} \oplus H_{\mathbf{x}}$. It is also clear that the map $TR_u : (T\mathcal{K})_{\mathbf{x}} \to (T\mathcal{K})_{\mathbf{x}u}$ (where R_u is the right multiplication and TR_u is the tangent map of R_u) satisfies

$$(TR_u)_{\mathbf{x}}(H_{\mathbf{x}}) = H_{\mathbf{x}u}.$$

This completes the statement at the beginning of this subsection about the definition of the canonical connection on the Hopf fibration.

REMARK 4.6. Clearly, the (left) action of $\mathcal{U}_2$ on $\mathcal{K}$ preserves the decomposition $(T\mathcal{K})_{\mathbf{x}} = V_{\mathbf{x}} \oplus H_{\mathbf{x}}$.

We finish this section by describing the tangent map $(T\mathfrak{h})_{\mathbf{x}} : (T\mathcal{K})_{\mathbf{x}} \to (T\mathcal{R})_{\tilde{p}_{\mathbf{x}}}$ for $\mathbf{x} \in \mathcal{K}$. Given $\xi \in (T\mathcal{K})_{\mathbf{x}}$ we clearly have $T\mathfrak{h}(\xi) = X = \xi\mathbf{x}^* + \mathbf{x}\xi^*$. Also note the identity $X\tilde{p}_{\mathbf{x}} = (1 - \tilde{p}_{\mathbf{x}})X$.

4.3. The structure morphism $\kappa : \mathfrak{R} \to T\mathcal{K}$. Define $\mathfrak{R} \to \mathcal{K}$ as the induced vector bundle

$$\begin{array}{ccc} \mathfrak{h}^*(T\mathcal{R}) & \longrightarrow & T\mathcal{R} \\ \downarrow & & \downarrow \\ \mathcal{K} & \xrightarrow{\ \mathfrak{h}\ } & \mathcal{R} \end{array}$$

where we write $\mathfrak{R}$ for $\mathfrak{h}^*(T\mathcal{R})$ as a bundle over $\mathcal{K}$. With this notation we define the *structure morphism* κ as a vector bundle morphism

$$\begin{array}{ccc} \mathfrak{R} & \xrightarrow{\ \kappa\ } & T\mathcal{K} \\ & \searrow & \downarrow \\ & & \mathcal{K} \end{array}$$

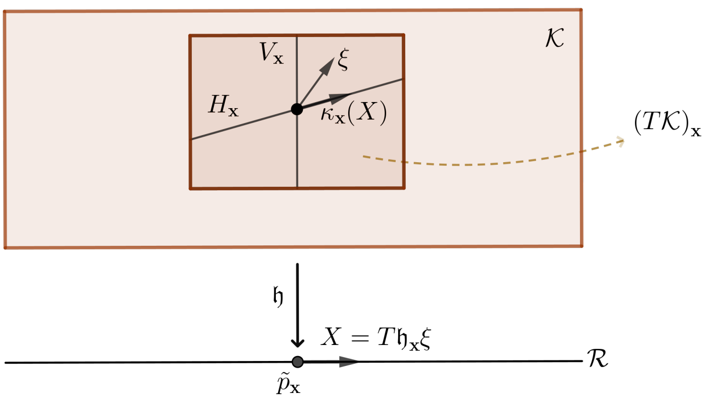
where $\kappa_{\mathbf{x}}(X) = X\mathbf{x}$ for each $\mathbf{x} \in \mathcal{K}$, $X \in (T\mathcal{R})_{\tilde{p}_{\mathbf{x}}}$ (notice that $X \in M_2(\mathcal{A})$, $\mathbf{x} \in \mathcal{A}^2$ and therefore $X\mathbf{x} \in \mathcal{A}^2$). Observe that $\kappa_{\mathbf{x}}(X) \in \ker(\tilde{p}_{\mathbf{x}}) = H_{\mathbf{x}}$. Also notice that $(T\mathfrak{h})_{\mathbf{x}}(\kappa_{\mathbf{x}}X) = X$ because of the identity $X\mathbf{x}\mathbf{x}^* + \mathbf{x}\mathbf{x}^*X = X$ (observe that X is self-adjoint).

We remark that the morphism κ has the following equivariance property:

$$\kappa_{\mathbf{x}u}(X) = (\kappa_{\mathbf{x}}(X))u$$

for $u \in \mathcal{U}$. This equivariance gives a way of constructing the tangent bundle $T\mathcal{R}$ out of the principal bundle $\mathcal{K} \to \mathcal{R}$ and the co-tautological bundle $\mathcal{T}'$ (see (2.4)).

The following schematic picture illustrates our constructions:



where the inner rectangle represents the tangent space $(T\mathcal{K})_{\mathbf{x}}$.

4.4. The Finsler metric on $\mathcal{R}$ and the structure form κ. Recall that $\mathcal{A}^2$ is a Hilbert C^*-module over $\mathcal{A}$ (acting on the right) in the usual way defining $\langle \mathbf{x}, \mathbf{y} \rangle = x_1^* y_1 + x_2^* y_2$. Then we have the following result:

THEOREM 4.7. *Let $X \in (T\mathcal{R})_{\tilde{p}}$, $\mathbf{x} \in \mathcal{K}$, $\mathfrak{h}(\mathbf{x}) = \tilde{p}$. Then*

$$\|X\| = \|\kappa_{\mathbf{x}}(X)\| = \|X\mathbf{x}\|.$$

Here $\|X\|$ is the Finsler norm in $\mathcal{R}$ of the tangent vector X (i.e. the usual norm of the self-adjoint matrix $X \in M_2(\mathcal{A})$), whereas $\|\kappa_{\mathbf{x}}(X)\|$ stands for the norm of $\kappa_{\mathbf{x}}(X)$ as an element of the C^* $\mathcal{A}$-module $\mathcal{A}^2$.

Proof. Suppose first that $\tilde{p} = \tilde{p}_0$. In this case, since $X \in (T\mathcal{R})_{\tilde{p}_0}$, we have $X = \left(\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right)$ for $a \in \mathcal{A}$. Then

$$\|X\|_{M_2(\mathcal{A})} = \|a\| = \left\|X\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right\|_{\mathcal{A}^2} = \left\|X\left(\begin{smallmatrix} 1u \\ 0 \end{smallmatrix}\right)\right\|_{\mathcal{A}^2} = \|X\mathbf{x}\|_{\mathcal{A}^2},$$

where $u \in \mathcal{U}$, $\mathbf{x} \in \mathcal{K}$ with $\mathfrak{h}(\mathbf{x}) = \mathbf{x}\mathbf{x}^* = \tilde{p}_0$, and $\| \, \|_{\mathcal{A}^2}$ is the norm of the Hilbert C^*-module $\mathcal{A}^2$.

The general case follows using the fact that given $\tilde{p} \in \mathcal{R}$, there is $\mathbf{z} \in \mathcal{K}$ such that $\tilde{p} = \tilde{p}_{\mathbf{z}} = \mathbf{z}\mathbf{z}^* = \tilde{u}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ with $\tilde{u} \in \mathcal{U}$. And every element of $(T\mathcal{R})_{\tilde{p}}$ is of the form $\tilde{u}X\tilde{u}^*$ for $X \in (T\mathcal{R})_{\tilde{p}_0}$. Hence $\|\tilde{u}X\tilde{u}^*\|_{M_2(\mathcal{A})} = \|X\|_{M_2(\mathcal{A})}$. And for $\mathbf{y}$ such that $\mathfrak{h}(y) = \tilde{p}$ we have $\mathbf{y} = \mathbf{z}v$ with $v \in \mathcal{U}$. Then

$$\begin{aligned} \|\,(\tilde{u}X\tilde{u}^*)\,\mathbf{y}\|_{\mathcal{A}^2} &= \|\,(\tilde{u}X\tilde{u}^*)\,\mathbf{z}v\|_{\mathcal{A}^2} = \left\|(\tilde{u}X\tilde{u}^*)\tilde{u}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)v\right\|_{\mathcal{A}^2} = \left\|\tilde{u}X\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)v\right\|_{\mathcal{A}^2} \\ &= \left\|v^*\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)X^*\tilde{u}^*\tilde{u}X\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)v\right\|^{1/2} = \left\|\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)X^*X\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right\|^{1/2} \\ &= \|X\|_{M_2(\mathcal{A})}, \end{aligned}$$

where the last equality was proved in the case of $\tilde{p} = \tilde{p}_0$. ∎

5. Examples

5.1. The finite-dimensional case. Consider $\mathcal{A} = M_n(\mathbb{C})$. The case where $n = 1$ is the classical Riemann sphere $\mathcal{R}$ and the classical Hopf fibration $\mathcal{K} \to \mathcal{R}$ over the Riemann sphere (see 2.5). In this case $\mathcal{R}$ is the one-dimensional complex projective line $\mathbb{P}(\mathbb{C})$ (diffeomorphic to S^2) and $\mathcal{K}$ is the unit sphere in $\mathbb{C}^2$, which is homeomorphic to S^3.

The case $n > 1$ involves the non-commutative C^*-algebra $M_n(\mathbb{C})$ of operators on $H = \mathbb{C}^n$. Here $M_2(\mathcal{A})$ is naturally identified with $M_{2n}(\mathbb{C})$ operating on $H \oplus H$, which is naturally identified with $\mathbb{C}^{2n}$. Also $\tilde{p}_0$ is the orthogonal projection in $\mathbb{C}^{2n}$ onto $\mathbb{C}^n \subset \mathbb{C}^{2n}$ as the subspace defined by $z_{n+1} = z_{n+2} = \cdots = z_{2n} = 0$. Therefore, the orbit $\mathcal{R}$ of $\tilde{p}_0$ under the action of $\mathcal{U}_2 \subset M_2(\mathcal{A})$ can be identified with the classical Grassmann manifold $\mathrm{Grass}_{n,2n}(\mathbb{C})$ of all n-dimensional subspaces of $\mathbb{C}^{2n}$.

We now describe the sphere $\mathcal{K}$ in $\mathcal{A}^2$ corresponding to the present situation. We have

$$\mathcal{K} = \left\{\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) : \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n : \mathbf{x} \text{ is an isometry}\right\}.$$

Observe that $\mathbf{x}^* = (x_1^* \quad x_2^*) : \mathbb{C}^{2n} \to \mathbb{C}^n$ and $\mathbf{x}\mathbf{x}^*$ is an orthogonal projection in $\mathbb{C}^{2n}$ so that $\mathfrak{h} : \mathcal{K} \to \mathcal{R}$ is given by the usual formula. The space $\mathcal{K}$ may be identified with the usual Stiefel manifold $\mathrm{St}_{n,2n}$ of orthogonal n-frames in $\mathbb{C}^{2n}$ and $\mathfrak{h}$ is therefore identified with the usual projection $\mathrm{St}_{n,2n} \to \mathrm{Grass}_{n,2n}$.

In this context the open set $\mathcal{V}_0$, the domain of the principal chart, consists of all orthogonal projections $\tilde{p} \in \mathcal{R}$ such that $\operatorname{im} \tilde{p}$ is the graph of a linear map $a : \mathbb{C}^n \to \mathbb{C}^n$ and $\varphi_0(\tilde{p}) = a$.

REMARK 5.1. Notice that $\mathcal{V}_0$ is dense in $\mathcal{R}$. In the standard CW-decomposition of $\mathcal{R}$, $\mathcal{V}_0$ is the *top cell* and has (real) dimension $4n^2$. See for example [14] for the real case. The complex case is similar.

5.2. Bounded and unbounded operators. In this subsection we present the closed operators on a Hilbert space H as elements of the Riemann sphere of the algebra $\mathcal{A} = B(H)$.

For a densely defined closed operator $T : \mathrm{Dom}(T) \to H$, it can be proved that its orthogonal projection $P_{\mathrm{Gr}(T)}$ onto the graph of T belongs to the Riemann sphere of $\mathcal{A} = L(H)$. These statements are formalized in the following result where we also provide formulas for these orthogonal projections.

PROPOSITION 5.2. *Let H be a Hilbert space and $T : \mathrm{Dom}(T) \to H$ a densely defined operator with closed graph. Then the orthogonal projection $\tilde{p}_T = P_{\mathrm{Gr}(T)}$ onto the graph $\mathrm{Gr}(T) = \{(h, T(h)) : h \in \mathrm{Dom}(T)\} \subset H \times H$ belongs to the unitary orbit of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ which is the Riemann sphere $\mathcal{R}$ of the algebra $B(H)$. Moreover, $P_{\mathrm{Gr}(T)}$ can be written as*

$$
\begin{aligned}
P_{\mathrm{Gr}(T)} &= \begin{pmatrix} 1 \\ T \end{pmatrix} (1 + T^*T)^{-1} (1 \quad T^*) \\
&= \begin{pmatrix} 1 & T^* \\ T & TT^* \end{pmatrix} \begin{pmatrix} 1 + T^*T & 0 \\ 0 & 1 + TT^* \end{pmatrix}^{-1} \\
&= \begin{pmatrix} (1 + T^*T)^{-1} & (1 + T^*T)^{-1} T^* \\ T(1 + T^*T)^{-1} & T(1 + T^*T)^{-1} T^* \end{pmatrix}.
\end{aligned} \tag{5.1}
$$

All the entries of the last matrix are bounded operators.

Proof. Consider the operator $\tilde{T}$ defined by

$$\tilde{T} = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}. \tag{5.2}$$

Since T is a closed and densely defined operator on a Hilbert space, T^* is also closed and densely defined (see [20, Theorem 5.3]) and $T^{**} = T$ [18, Theorem 1.8]. Moreover, $\tilde{T}^* = -\tilde{T}$ and $\mathrm{Dom}(\tilde{T}) = \mathrm{Dom}(\tilde{T}^*)$.

Now we will consider the norms. Given $\left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right) \in \mathrm{Dom}(\tilde{T})$ we have

$$\tilde{T} \left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right) = -\tilde{T}^* \left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right)$$

and therefore they have the same norm. This proves that $\tilde{T}$ is a normal operator in $H \times H$ (see [20, Section 5.6]). This implies that also $1 + \tilde{T}$ is normal and therefore invertible with a bounded inverse. This follows from $\tilde{T}^* = -\tilde{T}$ and the functional calculus of the self-adjoint operator $i\tilde{T}$.

Now consider the polar decomposition (see [20, Theorem 7.20])

$$1+\tilde{T} = US$$

where U is a unitary operator since $1+\tilde{T}$ is invertible.

Now let us analyze the operator S. Using the same reference cited above, S can be written as

$$S = |1+\tilde{T}| = ((1+\tilde{T})^*(1+\tilde{T}))^{1/2} = ((1+\tilde{T}^*)(1+\tilde{T}))^{1/2} = ((1-\tilde{T})(1+\tilde{T}))^{1/2} = (1-\tilde{T}^2)^{1/2}. \tag{5.3}$$

Then, using (5.2), a direct computation shows that $S^2 = |1+\tilde{T}|^2 = 1 + \begin{pmatrix} T^*T & 0 \\ 0 & TT^* \end{pmatrix} = \begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+TT^* \end{pmatrix}$ and hence

$$S = |1+\tilde{T}| = \begin{pmatrix} (1+T^*T)^{1/2} & 0 \\ 0 & (1+TT^*)^{1/2} \end{pmatrix}.$$

Thus S is invertible with bounded inverse (see [18, Proposition 3.18]) and we can write

$$U = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \begin{pmatrix} (1+T^*T)^{-1/2} & 0 \\ 0 & (1+TT^*)^{-1/2} \end{pmatrix}.$$

The first column of U is $\begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1/2} = \begin{pmatrix} (1+T^*T)^{-1/2} \\ T(1+T^*T)^{-1/2} \end{pmatrix}$ with an invertible first coordinate and with second coordinate $Z_T = T(1+T^*T)^{-1/2}$, which is usually called the *bounded transform* of T.

Then it follows that $\mathcal{A} = (1+T^*T)^{-1/2}\mathcal{A}$ and hence

$$\left[\begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1/2}\right] = \left[\begin{pmatrix} 1 \\ T \end{pmatrix}\right] = \begin{pmatrix} 1 \\ T \end{pmatrix}\mathcal{A}$$

can be identified with $\mathrm{Gr}(T)$.

The corresponding orthogonal projection $\tilde{p}_T = P_{\mathrm{Gr}(t)}$ onto the graph $\mathrm{Gr}(T)$ belongs to $\mathcal{R}$ and can be written as

$$\begin{aligned} \tilde{p}_T &= \begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1/2}(1+T^*T)^{-1/2}(1 \quad T^*) \\ &= \begin{pmatrix} (1+T^*T)^{-1} & (1+T^*T)^{-1}T^* \\ T(1+T^*T)^{-1} & T(1+T^*T)^{-1}T^* \end{pmatrix} \\ &= \begin{pmatrix} 1 & T^* \\ T & TT^* \end{pmatrix}\begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+TT^* \end{pmatrix}^{-1}. \ \blacksquare \end{aligned}$$

The following facts will be useful to establish the existence of minimal geodesics between graphs of operators.

Definition 5.3. The *inverse graph* (see [10]) of a densely defined operator T on $\mathrm{Dom}(T) \subset H$ is given by

$$\mathrm{invGr}(T) = \{(Tx, x) : x \in \mathrm{Dom}(T)\}. \tag{5.4}$$

Lemma 5.4. *If $T : \mathrm{Dom}(T) \to H$ is a densely defined closed operator on $\mathrm{Dom}(T) \subset H$, then*

$$\begin{aligned}\mathrm{Gr}(T)^{\perp} &= \{(-T^*x, x) : x \in \mathrm{Dom}(T^*)\}\\ &= \mathrm{invGr}(-T^*).\end{aligned}$$

Proof. We can use the unitary operator $V : H \oplus H \to H \oplus H$ defined by $V(x, y) = (-y, x)$ to write $\mathrm{Gr}(T^*) = V(\mathrm{Gr}(T)^{\perp})$ (see [18, Lemma 1.10]). Then, since $V^2 = -I$, we can write $\mathrm{Gr}(T)^{\perp} = -V^2(\mathrm{Gr}(T^{\perp})) = -V(\mathrm{Gr}(T^*))$ and therefore

$$\begin{aligned}\mathrm{Gr}(T)^{\perp} &= -V(\{(x, T^*x) : x \in \mathrm{Dom}(T^*)\})\\ &= -\{(-T^*x, x) : x \in \mathrm{Dom}(T^*)\}\\ &= \{(-T^*x, x) : x \in \mathrm{Dom}(T^*)\}. \ \blacksquare\end{aligned} \tag{5.5}$$

Proposition 5.5. *Let S, T be bounded operators acting in H.*

(1) *There exists a (minimal) geodesic of $\mathcal{R}$ joining $P_{\mathrm{Gr}(S)}$ and $P_{\mathrm{Gr}(T)}$ if and only if $\dim\ker(1 + T^*S) = \dim\ker(1 + S^*T)$. The minimal geodesic is unique if and only if these subspaces are trivial.*
(2) *If $S^* = S$ and $T^* = T$, the global unitary isomorphism Ω of $H \times H$ given by*

$$\Omega(h_1, h_2) = (h_2, -h_1)$$

maps pairs (h, Sh) with h in $\ker(1 + TS)$ to pairs (g, Tg) with g in $\ker(1 + ST)$. In particular, there always exists a minimal geodesic of $\mathcal{R}$ joining $P_{\mathrm{Gr}(S)}$ and $P_{\mathrm{Gr}(T)}$.

Proof. Note that $\mathrm{Gr}(T)^{\perp} = \{(-T^*g, g) : g \in H\}$. Hence, a pair $(h, Sh) \in \mathrm{Gr}(S)$ belongs to $\mathrm{Gr}(T)^{\perp}$ if and only if there exists $g \in H$ such that $h = -T^*g$ and $Sh = g$. Therefore, $h = -T^*g = -T^*Sh$, i.e., $h \in \ker(1+T^*S)$. Conversely, if $h \in \ker(1 + T^*S)$, then $(h, Sh) = (-T^*Sh, Sh) \in \mathrm{Gr}(T)^{\perp}$. Then

$$\mathrm{Gr}(S) \cap \mathrm{Gr}(T)^{\perp} = \{(h, Sh) : h \in \ker(1 + T^*S)\},$$

and $\dim \mathrm{Gr}(S) \cap \mathrm{Gr}(T)^{\perp} = \dim\ker(1 + T^*S)$. Similarly, $\mathrm{Gr}(T) \cap \mathrm{Gr}(S)^{\perp} = \{(g, Tg) : g \in \ker(1 + S^*T)\}$ with the same dimension as $\ker(1 + S^*T)$. The proof follows by recalling that given two subspaces V and W, a necessary and sufficient condition for the existence of a minimal geodesic joining the orthogonal projections P_V and P_W is the equality of the dimensions of $V \cap W^{\perp}$ and $V^{\perp} \cap W$; and that the minimal geodesic is unique if and only if these intersections are trivial (see [1, Theorem 4.5]).

Suppose now that S and T are self-adjoint. Note that $h \in \ker(1 + TS)$ means that $\Omega(h, Sh) = (Sh, -h) = (Sh, TSh)$ belongs to $\mathrm{Gr}(T)$, with $Sh \in$

$\ker(1+ST)$: $STSh = S(TSh) = S(-h) = -Sh$. That is, Ω maps $\mathrm{Gr}(S)\cap \mathrm{Gr}(T)^{\perp}$ into $\mathrm{Gr}(T)\cap\mathrm{Gr}(S)^{\perp}$. Similarly, Ω maps $\mathrm{Gr}(T)\cap\mathrm{Gr}(S)^{\perp}$ into $\mathrm{Gr}(S)\cap \mathrm{Gr}(T)^{\perp}$. Note that $\Omega^2 = -1$. ∎

If S or T are non-self-adjoint, there may not exist geodesics joining their graphs, as in the following example:

EXAMPLE 5.6. Consider the (unilateral) shift operator $\mathbf{S}$ in ℓ^2 defined by $\mathbf{S}(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Put $S = -2\mathbf{S}$ and $T = 1$. Then $\ker(1+T^*S) = \ker\big(1-2\mathbf{S}\big) = \ker\left(\frac{1}{2}-\mathbf{S}\right) = \{0\}$ (the shift has no eigenvalues). On the other hand, $\ker(1+S^*T) = \ker(1-2\mathbf{S}^*) = \ker\big(\frac{1}{2}-\mathbf{S}^*\big)$, which has dimension 1. Therefore $\mathrm{Gr}(1) = \{(x,x) : x\in\ell^2\}$ and $\mathrm{Gr}(-2\mathbf{S}) = \{(y, -2\mathbf{S}y) : y\in\ell^2\}$ cannot be joined by a geodesic of $\mathcal{R}$.

5.3. The unique minimal geodesic from $\tilde{p}_0$ to the graph of a closed operator. Let us describe explicitly the minimal geodesic γ of $\mathcal{R}$ with $\gamma(0) = \tilde{p}_0 = P_{\mathrm{Gr}(0)}$ and $P_{\mathrm{Gr}(T)}$, for $T : H \supset \mathrm{Dom}(T) \to H$ a closed operator. Recall from (5.1) the formula of the projection $P_{\mathrm{Gr}(T)}$:

$$P_{\mathrm{Gr}(T)} = \begin{pmatrix} (1+T^*T)^{-1} & (1+T^*T)^{-1}T^* \\ T(1+T^*T)^{-1} & T(1+T^*T)^{-1}T^* \end{pmatrix}.$$

Let $T = V|T|$ be the polar decomposition of T, where $|T|$ is a (possibly unbounded) non-negative self-adjoint operator, and $V : \overline{\mathrm{im}(T^*)} \to \overline{\mathrm{im}(T)}$ is a partial isometry.

THEOREM 5.7. *With the current notations, we have*

$$\gamma(t) = e^{itZ}\tilde{p}_0 e^{-itZ} \text{ for } Z = \begin{pmatrix} 0 & i\arctan(|T|)V^* \\ -iV\arctan(|T|) & 0 \end{pmatrix}. \tag{5.6}$$

Proof. To verify (5.6), let us compute the even and odd powers of itZ. Note that

$$(iZ)^2 = \begin{pmatrix} -\arctan(|T|)V^*V\arctan(|T|) & 0 \\ 0 & -V(\arctan(|T|))^2V^* \end{pmatrix}.$$

Since V is a partial isometry with initial space $\overline{\mathrm{im}(|T|)}$ and final space $\overline{\mathrm{im}(T)}$, and $\arctan(T)$ is a continuous function with $\arctan(0) = 0$, it follows that $V^*V = P_{\overline{\mathrm{im}(|T|)}}$, and thus $V^*V\arctan(|T|) = \arctan(|T|)V^*V = \arctan(|T|)$. Therefore, we have

$$(iZ)^2 = \begin{pmatrix} -\left(\arctan(|T|)\right)^2 & 0 \\ 0 & -V\left(\arctan(|T|)\right)^2V^* \end{pmatrix}.$$

Similarly,

$$(iZ)^{2k} = (-1)^k \begin{pmatrix} -\left(\arctan(|T|)\right)^{2k} & 0 \\ 0 & -V\left(\arctan(|T|)\right)^{2k} V^* \end{pmatrix}.$$

The odd powers of iZ: $(iZ)^3$ equals

$$\begin{pmatrix} -(\arctan(|T|))^2 & 0 \\ 0 & -v(\arctan(|T|))^2 V^* \end{pmatrix} \begin{pmatrix} 0 & -\arctan(|T|)V^* \\ V\arctan(|T|) & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \arctan(|T|)V^*V\left(\arctan(|T|)\right)^2 V^* \\ -V\left(\arctan(|T|)\right)^3 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \left(\arctan(|T|)\right)^3 V^* \\ -V\left(\arctan(|T|)\right)^3 & 0 \end{pmatrix}.$$

Similarly,

$$(iZ)^{2k+1} = (-1)^k \begin{pmatrix} 0 & -\left(\arctan(|T|)\right)^{2k+1} V^* \\ V\left(\arctan(|T|)\right)^{2k+1} & 0 \end{pmatrix}.$$

Therefore,

$$e^{iZ} = \begin{pmatrix} \cos(\arctan(|T|)) & -\sin(\arctan(|T|)) \\ \sin(\arctan(|T|)) & \cos(\arctan(|T|)) \end{pmatrix}.$$

Notice the function identities $\cos(\arctan(t)) = \frac{1}{\sqrt{1+t^2}}$ and $\sin(\arctan(t)) = \frac{t}{\sqrt{1+t^2}}$. Then (since $T = V|T|$ and $T^* = |T|V^*$), e^{iZ} equals

$$\begin{pmatrix} (1+|T|^2)^{-1/2} & -(1+|T|^2)^{-1/2}|T|V^* \\ V|T|(1+|T|^2)^{-1/2} & V(1+|T|^2)^{-1/2}V^* \end{pmatrix}$$
$$= \begin{pmatrix} (1+|T|^2)^{-1/2} & -(1+|T|^2)^{-1/2}T^* \\ T(1+|T|^2)^{-1/2} & V(1+|T|^2)^{-1/2}V^* \end{pmatrix}.$$

Then, after straightforward computations, $e^{iZ}\tilde{p}_0 e^{-iZ}$ equals

$$\begin{pmatrix} (1+|T|^2)^{-1/2} & -(1+|T|^2)^{-1/2}T^* \\ T(1+|T|^2)^{-1/2} & V(1+|T|^2)^{-1/2}V^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} (1+|T|^2)^{-1/2} & (1+|T|^2)^{-1/2}T^* \\ -T(1+|T|^2)^{-1/2} & T(1+|T|^2)^{-1/2}V^* \end{pmatrix}$$
$$= \begin{pmatrix} (1+|T|^2)^{-1/2} & (1+|T|^2)^{-1/2}T^* \\ T(1+|T|^2)^{-1/2} & T(1+|T|^2)^{-1/2}T^* \end{pmatrix} = P_{\mathrm{Gr}(T)},$$

as claimed. ∎

Note that if T is bounded, then $\|Z\| = \|\arctan(|T|)\| = \arctan(\|T\|) < \pi/2$, while if T is unbounded, $\|Z\| = \|\arctan(|T|)\| = \pi/2$.

5.4. Bounded deformations of closed operators. In this section we consider operators T on a Hilbert space H.

Definition 5.8. A *bounded deformation* of a closed operator T on H is a family $\{T_t\}_{t\in[0,\alpha)}$, with $\alpha > 0$, of bounded operators T_t such that

- $t \mapsto T_t$ is continuous in the norm topology,
- $\lim_{t\to\alpha^-} \tilde{p}_t = \tilde{p}_T$ where $\tilde{p}_t$, $\tilde{p}_T$ are in the Riemann sphere $\mathcal{R}$ of the algebra $B(H)$, $\tilde{p}_t$ is the orthogonal projection on $\mathrm{Gr}(T_t)$, $\tilde{p}_T$ is the orthogonal projection on $\mathrm{Gr}(T)$ (cf. Proposition 5.2) and the limit is taken in the Finsler metric of the Riemann sphere $\mathcal{R}$.

In particular, if the bounded deformation $\{T_t\}_{t\in[0,\alpha)}$ of the operator T satisfies the condition

$$\mathrm{dist}(\tilde{p}_{t_0}, \tilde{p}_\alpha) = \text{length } \tilde{p}_t|_{t_0}^{\alpha} \quad \text{for every } t_0 \in [0,\alpha)$$

we will call it an *optimal bounded deformation*. Here $\mathrm{dist}(\tilde{p}_{t_0}, \tilde{p}_\alpha)$ stands for the Finsler distance in $\mathcal{R}$, and length $\tilde{p}_t|_{t_0}^{\alpha}$ means the Finsler length of the curve, where we write $\tilde{p}_\alpha$ for $\tilde{p}_T$.

In Theorem 5.10 and Corollary 5.13 we construct a specific optimal bounded deformation of any closed operator T on H.

Remark 5.9. Observe that for an operator T,

$$P_{\mathrm{invGr}(T)} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) P_{\mathrm{Gr}(T)} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),$$

which implies that $P_{\mathrm{invGr}(T)} \in \mathcal{R}$.

Theorem 5.10. *Let H be a Hilbert space,* $\mathrm{Grass}(H \oplus H)$ *the Riemann sphere of the C^*-algebra $B(H)$,* $\mathrm{Gr}(0) = H \oplus \{0\}$ *the graph of the null operator and* $\mathrm{Gr}(T)$ *the graph of a densely defined closed operator T with domain* $\mathrm{Dom}(T)$.

The unique minimal geodesic $\gamma : [0,1] \to \mathrm{Grass}(H \oplus H)$ such that $\gamma(0) = P_{\mathrm{Gr}(0)}$, $\gamma(1) = P_{\mathrm{Gr}(T)}$ and $\dot{\gamma}(0) = \left(\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right)$ with $\|a\| \le \pi/2$, consists of orthogonal projections onto the graphs

$$\gamma(t) = P_{\mathrm{Gr}(A(t))},$$

with

$$A(t) = ta^*(\mathrm{sinc}\,|ta^*|)(\cos|ta^*|)^{-1} = v \tan|ta^*| \in B(H)$$

for $t \in [0,1)$ and v the partial isometry of the polar decomposition $a^* = v|a^*|$.

Proof. Note that $\mathrm{ran}(P_{\mathrm{Gr}(0)}) = \mathrm{Gr}(0) = H \oplus \{0\}$, $\ker(P_{\mathrm{Gr}(0)}) = \mathrm{Gr}(0)^\perp$, $\mathrm{ran}(P_{\mathrm{Gr}(T)}) = \mathrm{Gr}(T) = \{(x, Tx) : x \in \mathrm{Dom}(T)\}$ and $\ker(P_{\mathrm{Gr}(T)}) = \mathrm{Gr}(T)^\perp$. Observe that $\mathrm{Gr}(0)^\perp \cap \mathrm{Gr}(T) = (\{0\} \oplus H) \cap \{(x, Tx) : x \in \mathrm{Dom}(T)\} = \{(0,0)\}$ for any T. Then we only need to prove that $\mathrm{Gr}(0) \cap \mathrm{Gr}(T)^\perp = \{(0,0)\}$ and use [1, Theorem 4.5]. With this objective, using Lemma 5.4, we express $\mathrm{Gr}(T)^{\perp} = \{(-T^*x, x) : x \in \mathrm{Dom}(T^*)\}$ and then obtain $\mathrm{Gr}(0) \cap \mathrm{Gr}(T)^{\perp} = (H \oplus \{0\}) \cap \{(-T^*x, x) : x \in \mathrm{Dom}(T^*)\} = \{(0,0)\}$, which proves uniqueness.

Let $\gamma : [0,1] \to \mathrm{Grass}(H \oplus H)$ be the unique geodesic that joins $P_{\mathrm{Gr}(0)} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ to $P_{\mathrm{Gr}(T)}$ such that $\dot\gamma(0) = \left(\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right)$ (see [1, Proposition 2.9]). This γ is of the form (see (3.7) and Theorem 3.10)

$$\gamma(t) = \begin{pmatrix} \cos|ta^*| \\ (\operatorname{sinc}|ta|)ta^* \end{pmatrix} (\cos|ta^*| \quad ta \operatorname{sinc}|ta|)$$

for $t \in [0,1]$. Moreover, the geodesics satisfy $\|\left(\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right)\| = \|a\| \le \pi/2$ (see [1, Proposition 3.1, Theorem 3.2]). For all $t \in [0,1)$, the vectors $\mathbf{x}(t) = \left(\begin{smallmatrix} \cos|ta^*| \\ (\operatorname{sinc}|ta|)ta^* \end{smallmatrix}\right)$ are in $\mathcal{K}_0$ because $\tilde v(t) = \left(\begin{smallmatrix} \cos|ta^*| & -\operatorname{sinc}|ta^*|ta \\ \operatorname{sinc}|ta|ta^* & \cos|ta| \end{smallmatrix}\right) \in \mathcal{U}_2$ and $\tilde v(t)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \mathbf{x}(t)$, and satisfy $\mathbf{x}(t)\mathbf{x}(t)^* \in \mathcal{V}_0$ (see (3.3)) since $\cos|ta^*|$ is invertible if $\|a\| \le \pi/2$.

Then, since $\gamma(t) \in \mathcal{V}_0$ for all $t \in [0,1)$, applying Theorem 3.4 to each projection $\gamma(t)$, we obtain

$$\gamma(t) = P_{\mathrm{Gr}(A(t))},$$

where

$$\begin{aligned} A(t) = \varphi_0 \begin{pmatrix} \cos|ta^*| \\ (\operatorname{sinc}|ta|)ta^* \end{pmatrix} &= (\operatorname{sinc}|ta|)ta^*(\cos|ta^*|)^{-1} \\ &= ta^*(\operatorname{sinc}|ta^*|)(\cos|ta^*|)^{-1} = v|ta^*|(\operatorname{sinc}|ta^*|)(\cos|ta^*|)^{-1} \\ &= v\tan|ta^*| \end{aligned}$$

for $t \in [0,1)$ and v is the partial isometry in the polar decomposition $a^* = v|a^*|$. ∎

Remark 5.11. The orthogonal projection onto the graph of any densely defined closed unbounded operator T is in the boundary of the domain of the image of the chart φ_0 (see (3.4)) when we identify the operators $a \in \mathcal{A}$ with the orthogonal projections $P_{\mathrm{Gr}(a)}$ onto their graphs (see Theorem 3.4).

Corollary 5.12. *For any unbounded operator T there is a unique bounded deformation $\{T_t\}_{t\in[0,1)}$ (see Definition 5.8) such that*

(1) $T_0 = 0$,
(2) $t \mapsto \tilde p_t$, $t \in [0,1]$, *is a geodesic in* $\mathcal{R}$,
(3) $\tilde p_1 = P_{\mathrm{Gr}(T)}$.

Proof. This follows from the properties of the unique minimal geodesic $\gamma(t) = P_{\mathrm{Gr}(A(t))}$, where $A(t) = ta^*(\operatorname{sinc}|ta^*|)(\cos|ta^*|)^{-1} = v\tan|ta^*| \in B(H)$ with $t \in [0,1]$ from Theorem 5.10. ∎

COROLLARY 5.13. *The deformation* $t \mapsto \{T_t\}_{t\in[0,1]} = \gamma(t) = P_{\mathrm{Gr}(A(t))}$ *with* $A(t) = ta^*(\operatorname{sinc}|ta^*|)(\cos|ta^*|)^{-1} = v\tan|ta^*| \in B(H)$ *is an optimal bounded deformation* (*see the comments after Definition* 5.8), *that is,*

$$\mathrm{length}|_{t_0}^{1}(\tilde{p}_{t_0}, \tilde{p}_1) = \mathrm{dist}(\tilde{p}_{t_0}, \tilde{p}_1)$$

for $t_0 \in [0,1)$, *where* length *and* dist (*distance in the Riemann sphere* $\mathcal{R}$) *are defined for the Finsler metric on* $\mathcal{R}$ (*see Section* 4.4).

5.5. The differential operator. We study here the particular case of an unbounded operator. The conclusions are stated in Theorem 5.15.

EXAMPLE 5.14 (A geodesic between $P_{\mathrm{Gr}(0)}$ and the orthogonal projection onto the graph of the differential operator $-i\frac{d}{dx}$). Consider the operator

$$-i\frac{d}{dx} : \mathcal{D} \to L^2[0,1] \tag{5.7}$$

given by $f \mapsto -if'$ for $f : [0,1] \to \mathbb{C}$ with domain

$$\mathcal{D} = \{f \in L^2[0,1] : f \text{ is absolutely continuous},\\ f' \in L^2[0,1] \text{ and } f(0) = f(1)\}. \tag{5.8}$$

This is a known densely defined closed self-adjoint unbounded operator on the Hilbert space $L^2[0,1]$ (see [18, Example 1.4]). Denote by

$$\Gamma = \mathrm{Gr}\left(-i\frac{d}{dx}\right) = \{(f, -if') : f \in \mathcal{D}\}$$

the graph of $-i\frac{d}{dx}$, which is closed in $L^2[0,1] \times L^2[0,1]$, and the orthogonal projection $P_\Gamma : L^2[0,1] \times L^2[0,1] \to L^2[0,1] \times L^2[0,1]$ onto Γ.

Using Lemma 5.4 and the fact that $-i\frac{d}{dx}$ is self-adjoint we have

$$\Gamma^\perp = \left\{\left(-\left(-i\frac{d}{dx}\right)^* g, g\right) : g \in \mathcal{D}\right\} = \left\{(ig', g) : g \in \mathcal{D}\right\}. \tag{5.9}$$

Theorem 5.10 establishes that there exists a unique geodesic joining $P_{\mathrm{Gr}(0)}$ with P_Γ. It can also be seen that

$$\begin{aligned}\mathcal{H}_{11} &:= \mathrm{im}(P_{\mathrm{Gr}(0)}) \cap \mathrm{im}(P_\Gamma) = [1] \times \{0\},\\ \mathcal{H}_{00} &:= \ker(P_{\mathrm{Gr}(0)}) \cap \ker(P_\Gamma) = \{0\} \times [1],\end{aligned} \tag{5.10}$$

where $[1] = \{f \in L^2[0,1] : f = \lambda 1,\ \lambda \in \mathbb{C}\}$. As in the general case, the matrix block decomposition of $P_{\mathrm{Gr}(0)}$ and P_Γ in $\mathcal{H}_{11} \oplus \mathcal{H}_{00}$ is

$$P_{\mathrm{Gr}(0)}|_{\mathcal{H}_{11}\oplus\mathcal{H}_{00}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_\Gamma|_{\mathcal{H}_{11}\oplus\mathcal{H}_{00}}. \tag{5.11}$$

In order to obtain a geodesic between $P_{\mathrm{Gr}(0)}$ and P_Γ, we need to study $P_\Gamma|_{\mathcal{H}_0}$, the orthogonal projection onto Γ restricted to

$$\mathcal{H}_0 := (\mathcal{H}_{11} \oplus \mathcal{H}_{00})^\perp = [1]^\perp \times [1]^\perp, \tag{5.12}$$

where $[1]^\perp = \{h \in L^2[0,1] : \int_0^1 h(x)\, dx = 0\}$. It is clear that $\mathcal{H}_{11}$, $\mathcal{H}_{00}$ and $\mathcal{H}_0$ reduce $P_{\mathrm{Gr}(0)}$ and P_Γ.

To describe the geodesic that connects $P_{\mathrm{Gr}(0)}$ to P_Γ, we need to calculate a more specific expression of P_Γ restricted to $\mathcal{H}_0 = [1]^\perp \times [1]^\perp$. Using the Fourier basis $\{\xi_n\}_{n\in\mathbb{Z}}$ of $L^2[0,1]$ given by $\xi_n(x) = e^{i2\pi nx}$, we find that $\big\{\frac{1}{\sqrt{1+(2\pi n)^2}}(\xi_n, 2\pi n\xi_n)\big\}_{n\in\mathbb{Z}\setminus\{0\}}$ is an orthonormal basis of $[1]^\perp \times [1]^\perp$. Then, using blocks in the basis $\{\xi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ of $[1]^\perp$ we deduce that for $(h,k) \in [1]^\perp \times [1]^\perp$,

$$P_\Gamma|_{[1]^\perp\times[1]^\perp} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} D_1 h & D_2 k \\ D_2 h & D_3 k \end{pmatrix} \tag{5.13}$$

for the following diagonal operators in the $\{\xi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ basis:

$$\begin{aligned} D_1 &= \operatorname{diag}\big(\{1/(1+(2\pi n)^2)\}_{n\in\mathbb{Z}\setminus\{0\}}\big),\\ D_2 &= \operatorname{diag}\big(\{2\pi n/(1+(2\pi n)^2)\}_{n\in\mathbb{Z}\setminus\{0\}}\big),\\ D_3 &= \operatorname{diag}\big(\{(2\pi n)^2/(1+(2\pi n)^2)\}_{n\in\mathbb{Z}\setminus\{0\}}\big). \end{aligned} \tag{5.14}$$

Note that D_1 and D_2 are positive semidefinite compact operators and D_3 is positive definite (invertible) bounded in $[1]^\perp$.

Following ideas from [1, 9] and splitting the basis $\{\xi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ as $\{\xi_n\}_{n<0} \cup \{\xi_n\}_{n>0}$, we can construct the self-adjoint operator $Z_0 : [1]^\perp \times [1]^\perp \to [1]^\perp \times [1]^\perp$ given by

(5.15)
$$Z_0 = i \begin{pmatrix} 0 & 0 & \operatorname{diag}_{n<0}\{-a_n\} & 0 \\ 0 & 0 & 0 & \operatorname{diag}_{n>0}\{a_n\} \\ \operatorname{diag}_{n<0}\{a_n\} & 0 & 0 & 0 \\ 0 & \operatorname{diag}_{n>0}\{-a_n\} & 0 & 0 \end{pmatrix}$$

for

$$a_n = \cos^{-1}\left(\frac{1}{\sqrt{4\pi^2 n^2 + 1}}\right) = \begin{cases} \tan^{-1}(2\pi n) & \text{if } n > 0, \\ -\tan^{-1}(2\pi n) & \text{if } n < 0. \end{cases} \tag{5.16}$$

Observe that $0 < \cos^{-1}\big(\frac{1}{\sqrt{4\pi^2+1}}\big) < a_n = a_{-n} < \pi/2$ for every $n \in \mathbb{N}$ and $\lim_{n\to\pm\infty} a_n = \pi/2$.

Then the unitary matrix e^{iZ_0} satisfies $e^{iZ_0} P_{\mathrm{Gr}(0)}|_{[1]^\perp\times[1]^\perp} e^{-iZ_0} = \big(\begin{smallmatrix} D_1 & D_2 \\ D_2 & D_3 \end{smallmatrix}\big)$ (see (5.13), (5.14)).

Using the representation as in (5.15), since for all $n \in \mathbb{Z}$ we have $\lim_{n\to\pm\infty} \cos^{-1}\big(\frac{1}{\sqrt{4\pi^2n^2+1}}\big) = \pi/2$ and $0 < \cos^{-1}\big(\frac{1}{\sqrt{4\pi^2n^2+1}}\big) < \pi/2$, we deduce that $\|Z_0\| = \pi/2$. Then we can apply the results from [17] or [1, Theorem 5.3]: the curve $\delta : [-1,1] \to \mathcal{P}([1]^\perp \oplus [1]^\perp)$ given by

$$\delta(t) = e^{itZ_0} P_{\mathrm{Gr}(0)}|_{[1]^\perp\oplus[1]^\perp} e^{-itZ_0} \quad \text{for } t \in [-1,1] \tag{5.17}$$

is minimal along its path considering the Finsler metric defined by the operator norm in $[1]^\perp \times [1]^\perp$. In particular, using our previous computations, this minimal geodesic joins $\delta(0) = P_{\mathrm{Gr}(0)}|_{[1]^\perp\times[1]^\perp}$ to $\delta(1) = P_\Gamma|_{[1]^\perp\oplus[1]^\perp}$. Moreover, this implies that, after restricting to $[1]^\perp\times[1]^\perp$, we have $\mathrm{dist}(P_{\mathrm{Gr}(0)}, P_\Gamma) = \pi/2$ and $\|P_{\mathrm{Gr}(0)} - P_\Gamma\| = 1$.

Now considering (5.10)–(5.12), (5.15), (5.17) and the decomposition $(\mathcal{H}_{11} \oplus \mathcal{H}_{00}) \oplus \mathcal{H}_0 = L^2([0,1]) \times L^2([0,1])$, we can describe the minimal geodesic $\gamma : [-1,1] \to \mathcal{P}(L^2[0,1] \times L^2[0,1])$:

(5.18)

$$\gamma(t) = \begin{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & 0 \\ 0 & \delta(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{itZ_0} \end{pmatrix} \begin{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & 0 \\ 0 & P_{\mathrm{Gr}(0)}|_{[1]^\perp\oplus[1]^\perp} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-itZ_0} \end{pmatrix}$$

with $\gamma(0) = P_{\mathrm{Gr}(0)}$ and $\gamma(1) = P_\Gamma$.

Now observe that the unitary e^{itZ_0} (see (5.16)) in its 2×2 block decomposition, restricted to $\{\xi_n\}_{n\in\mathbb{Z}\setminus\{0\}} \times \{\xi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, can be expressed as $e^{itZ_0} = \left(\begin{smallmatrix} A(t) & -B(t) \\ B(t) & A(t) \end{smallmatrix}\right)$, where the diagonal operators $A(t)$ and $B(t)$ are self-adjoint and invertible for $0 < t < 1$.

It can moreover be shown that for $0 \le t < 1$ the image of the projection $\delta(t)$ is also the graph of the self-adjoint bounded operator in $[1]^\perp \subset L^2[0,1]$ given by

$$B(t)A(t)^{-1} = \operatorname*{diag}_{n\in\mathbb{Z}\setminus\{0\}} \{\tan(t\tan^{-1}(2\pi n))\} \quad \text{for } 0 < t < 1. \tag{5.19}$$

Here $-\tan\left(\frac{t\pi}{2}\right) < \tan(t\tan^{-1}(2\pi x)) < \tan\left(\frac{t\pi}{2}\right)$ for all $x\in\mathbb{R}$ and $\|B(t)A(t)^{-1}\| = \tan\left(\frac{t\pi}{2}\right)$. Therefore,

$$\lim_{t\to1} \|B(t)A(t)^{-1}\| = \lim_{t\to1} \tan(t\pi/2) = +\infty. \tag{5.20}$$

Now, considering elements of the whole space, $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in L^2[0,1] \times L^2[0,1] = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \oplus \mathcal{H}_0 = [1] \times \{0\} \oplus \{0\} \times [1] \oplus [1]^\perp \times [1]^\perp$, we can write

$$\begin{aligned} \gamma(t)\begin{pmatrix} f \\ g \end{pmatrix} &= \begin{pmatrix} \int_0^1 f \\ 0 \end{pmatrix} + \begin{pmatrix} A(t)^2 & A(t)B(t) \\ B(t)A(t) & B(t)^2 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} \\ &= \begin{pmatrix} \hat{A}_d(t)^2 & \hat{A}(t)\hat{B}(t) \\ \hat{B}(t)\hat{A}(t) & \hat{B}(t)^2 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} \end{aligned} \tag{5.21}$$

where we denote by $\hat{A}(t)$ and $\hat{B}(t)$ the corresponding operators extended to $L^2[0,1]$ such that $\hat{A}(t)(1) = \hat{B}(t)(1) = 0$, $\hat{A}_d(t) = \hat{A}(t) + \mathrm{diag}\{d_j\}_{j\in\mathbb{Z}}$ with $d_j = 0$ for $j \ne 0$ and $d_0 = 1$. We also use $\hat{A}_d(t)\hat{B}(t) = \hat{B}(t)\hat{A}_d(t)$ and $\hat{A}(t)^2 + \mathrm{diag}\{d\} = \hat{A}_d(t)^2$. Also note that $\hat{A}_d(t) : L^2[0,1] \to L^2[0,1]$ is an invertible operator for $0 \le t < 1$.

Hence, with the notation used in Section 3.2, the $x_1 = \hat{A}_d(t)$ coordinate of $\mathbf{x} = \binom{\hat{A}_d(t)}{\hat{B}(t)}$ is invertible for $-1 < t < 1$, which implies that all the elements $\gamma(t) \in \mathcal{A}^2 = B(L^2[0,1])^2$ belong to the chart defined in (3.3) and (3.4).

We know from Theorem 5.10 that the $\gamma(t)$ are projections onto the graph of an operator for every t. In this case it can be proved that, in terms of the Fourier basis,

$$\gamma(t) = P_{\mathrm{Gr}(\hat{B}(t)\hat{A}_d(t)^{-1})}. \tag{5.22}$$

Therefore, since $\gamma(1) = P_\Gamma$, the entire geodesic $\gamma : [0,1] \to \mathcal{P}(L^2[0,1] \times L^2[0,1])$ consists of self-adjoint orthogonal projections onto graphs of operators.

Now denote by $\hat{D}_i$ the diagonal operators such that $\hat{D}_i(\xi_0) = 0$ and $\hat{D}_i(\xi_n) = D_i(\xi_n)$ for $n \neq 0$, where D_i are the ones obtained in (5.14) for $i = 1,2,3$. Then we have $\gamma(1) = P_\Gamma = \begin{pmatrix} \hat{D}_1+\mathrm{diag}\{d\} & \hat{D}_2 \\ \hat{D}_2 & \hat{D}_3 \end{pmatrix}$, expressed in terms of the basis $\{\xi_n\}_{n\in\mathbb{Z}} \times \{\xi_n\}_{n\in\mathbb{Z}}$. Since $\gamma(1)_{1,1} = \hat{D}_1 + \mathrm{diag}\{d\}$ is a compact operator, it is not invertible and hence is not in the domain of the chart defined in (3.3) of Section 3.2. Moreover, since $\gamma(t)_{1,1}$ is invertible for $0 \leq t < 1$ and $\gamma(0) = P_{\mathrm{Gr}(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we conclude that $\gamma(1) = P_\Gamma$ lies in the boundary of the principal chart $\mathscr{C}_0$ defined in (3.3), a fact that was proved in general in Theorem 5.10 and Remark 5.11.

We may summarize the above considerations as follows.

Theorem 5.15. *The unique geodesic $\gamma : [0,1] \to \mathcal{P}(L^2[0,1] \oplus L^2[0,1])$, defined in* (5.21), *that joins the orthogonal projection $\gamma(0) = P_{\mathrm{Gr}(0)} = P_{L^2[0,1]\oplus\{0\}}$ onto the graph $\mathrm{Gr}(0)$ of the zero operator to the orthogonal projection $\gamma(1) = P_\Gamma$ onto the graph Γ of the self-adjoint* (*unbounded, densely defined and closed*) *differentiation operator $-i\,\frac{d}{dx}$* (*see* (5.7)) *has the following properties:*

(1) *For every $t \in (0,1)$, $\gamma(t)$ is the orthogonal projection onto the graph $G_{T(t)}$ of the diagonal self-adjoint bounded operator $T(t) = \hat{B}(t)\hat{A}_d(t)^{-1} : L^2[0,1] \to L^2[0,1]$* (*see* (5.22)) *that can be written as*

$$T(t) = \begin{pmatrix} \mathrm{diag}_{n\in\mathbb{Z}_{<0}}\{\tan(t\tan^{-1}(2\pi n))\} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{diag}_{p\in\mathbb{Z}_{>0}}\{\tan(t\tan^{-1}(2\pi p))\} \end{pmatrix}$$

in terms of blocks determined by the subspaces generated by the respective subsets of the Fourier basis ($\xi_n(x) = e^{i2\pi nx}$, $n \in \mathbb{Z}$) *corresponding to $\{\xi_j\}_{j\in\mathbb{Z}_{<0}}$, $\{\xi_0\}$ and $\{\xi_j\}_{j\in\mathbb{Z}_{>0}}$ of $L^2[0,1]$.*

(2) *For $0 < t < 1$ the operator norm of $T(t)$ is $\|T(t)\| = \tan(t\pi/2)$ and hence*

$$\lim_{t\to 0}\|T(t)\| = 0 \quad \text{and} \quad \lim_{t\to 1}\|T(t)\| = +\infty$$

(*see* (5.20) *and the properties of* $B(t)A(t)^{-1}$).

(3) *Every projection $\gamma(t)$ with $0 \le t < 1$, as an element of $M_2(\mathcal{A})$ with $\mathcal{A} = B(L^2[0,1])$, belongs to the chart defined by* (3.3) *and* (3.4).

(4) *$\gamma(1) = P_\Gamma$ does not belong to the principal chart $\mathscr{C}_0$ defined in* (3.3) *and* (3.4)*; nevertheless, P_Γ lies on the boundary of this chart.*

(5) *$\{T(t)\}_{t\in[0,1)}$ is an optimal bounded deformation of $-i\frac{d}{dx}$ (see Definition* 5.8).

5.6. Conjugate parameter values. There are infinitely many geodesics joining two orthogonal projections $\tilde{p}_0$ and $\tilde{p}$ in the Grassmann manifold Grass(H) if and only if $\dim(\mathrm{im}(\tilde{p}_0)\cap\ker(\tilde{p})) = \dim(\ker(\tilde{p}_0)\cap\mathrm{im}(\tilde{p})) \neq 0$ (see [1]). Recall that $(T\mathcal{R})_{\tilde{p}_0}$ consists of matrices of the form $X = \left(\begin{smallmatrix} 0 & a \\ a^* & 0\end{smallmatrix}\right)$ for $a \in B(H)$. Also, the unique geodesic δ that satisfies the initial conditions $\delta(0) = \tilde{p}_0$ and $\dot{\delta}(0) = X = \left(\begin{smallmatrix} 0 & a \\ a^* & 0\end{smallmatrix}\right)$ is (see [1, Proposition 2.9])

$$\delta(t) = e^{t\tilde{X}}Pe^{-t\tilde{X}} \qquad \text{for } \tilde{X} = \left(\begin{smallmatrix} 0 & -a \\ a^* & 0\end{smallmatrix}\right).$$

THEOREM 5.16. *Let $P_{\mathrm{Gr}(0)}$ and Q be orthogonal projections that satisfy $\dim(\mathrm{im}(P_{\mathrm{Gr}(0)})\cap\ker(Q)) = \dim(\ker(P_{\mathrm{Gr}(0)})\cap\mathrm{im}(Q)) \neq 0$. Using the decomposition of $H\times H$ given by $H\times H = \mathcal{H}_{11}\oplus\mathcal{H}_{00}\oplus\mathcal{H}'\oplus\mathcal{H}_0$, with*

$$\mathcal{H}' = (\mathrm{im}(P_{\mathrm{Gr}(0)})\cap\ker(Q))\oplus(\ker(P_{\mathrm{Gr}(0)})\cap\mathrm{im}(Q)),$$

the geodesics $\gamma : [0,1]\to\mathcal{R}$ joining $P_{\mathrm{Gr}(0)}$ to Q and with length$(\gamma) \le \pi/2$ *are of the form*

$$\gamma_u(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\begin{smallmatrix} \cos^2(t\pi/2) & \cos(t\pi/2)(\sin(t\pi/2))u \\ (\sin(t\pi/2))(\cos(t\pi/2))u^* & \sin^2(t\pi/2)\end{smallmatrix}\right) & 0 \\ 0 & 0 & 0 & \delta_0(t)\end{pmatrix} \tag{5.23}$$

where

- *u is any isometric isomorphism between the subspaces* $\mathrm{im}(P_{\mathrm{Gr}(0)})\cap\ker(Q)$ *and* $\ker(P_{\mathrm{Gr}(0)})\cap\mathrm{im}(Q)$,
- $\dot{\gamma}|_{\mathcal{H}'}(0) = X = \left(\begin{smallmatrix} 0 & \frac{\pi}{2}u \\ \frac{\pi}{2}u^* & 0\end{smallmatrix}\right)$,
- *δ_0 is the unique geodesic between the reductions of $P_{\mathrm{Gr}(0)}$ and Q to $\mathcal{H}_0$,*
- *γ_u has minimal length $\pi/2$.*

Proof. The multiplicity of these geodesics only appears in $\mathcal{H}'$, which reduces $P_{\mathrm{Gr}(0)}$ and Q to the expressions $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right)$, respectively (see [1, Section 3]). In what follows we will focus on the geodesics restricted to $\mathcal{H}'$.

Observe that the tangent space at $P_{\mathrm{Gr}(0)}|_{\mathcal{H}'}$ is also formed by codiagonals $X = \left(\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right)$ but with $a \in B(\mathcal{H}_{01}, \mathcal{H}_{10})$, and the geodesics starting at $P_{\mathrm{Gr}(0)}$ are described as $e^{t\tilde{X}} P_{\mathrm{Gr}(0)} e^{-t\tilde{X}}$ for $\tilde{X} = \left(\begin{smallmatrix} 0 & -a \\ a^* & 0 \end{smallmatrix}\right)$.

Similarly to the computation in (3.8), we obtain

$$e^{\tilde{X}} = \begin{pmatrix} \cos|a^*| & -(\operatorname{sinc}|a^*|)a \\ (\operatorname{sinc}|a|)a^* & \cos|a| \end{pmatrix} = \cos|\tilde{X}| + (\operatorname{sinc}|\tilde{X}|)\tilde{X}, \tag{5.24}$$

because $|\tilde{X}| = \left(\begin{smallmatrix} |a^*| & 0 \\ 0 & |a| \end{smallmatrix}\right)$. Then using the fact that $e^{\tilde{X}}$ is a unitary operator and must satisfy $e^{\tilde{X}}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)e^{-\tilde{X}} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, it can be proved that $\cos|a^*| = \cos|a| = 0$ and hence $(\sin|a^*|)^2 = (\sin|a|)^2 = 1$. Thus, we obtain $|a| = \sum_{j=1}^m (n_j\pi + \pi/2)p_j$ with $n_j \in \mathbb{N} \cup \{0\}$ for p_j spectral projections of $|a|$ that satisfy $\sum_{j=1}^n p_j = 1$. Considering that $\tilde{X}$ must satisfy $\|\tilde{X}\| = \|a\| = \|a^*\| \leq \pi/2$ (length$(\gamma) \leq \pi/2$), we have $|a| = \frac{\pi}{2}$ and then $a = \frac{\pi}{2}u$, with a unitary isomorphism $u : \mathcal{H}_{01} \to \mathcal{H}_{10}$. Hence $a^* = \frac{\pi}{2}u^*$.

Therefore all the possible $\tilde{X}$ are of the form $\tilde{X} = \left(\begin{smallmatrix} 0 & -\frac{\pi}{2}u \\ \frac{\pi}{2}u^* & 0 \end{smallmatrix}\right)$, and all the unitaries $e^{t\tilde{X}}$ are (see (5.24))

$$e^{t\tilde{X}} = \begin{pmatrix} \cos|t\frac{\pi}{2}| & -(\operatorname{sinc}|t\frac{\pi}{2}|)t\frac{\pi}{2}u \\ (\operatorname{sinc}|t\frac{\pi}{2}|)t\frac{\pi}{2}u^* & \cos|t\frac{\pi}{2}| \end{pmatrix} = \begin{pmatrix} \cos(t\frac{\pi}{2}) & -\sin(t\frac{\pi}{2})u \\ \sin(t\frac{\pi}{2})u^* & \cos(t\frac{\pi}{2}) \end{pmatrix}.$$

Then considering the decomposition of $H \oplus H = \mathcal{H}_{00} \oplus \mathcal{H}_{11} \oplus \mathcal{H}' \oplus \mathcal{H}_0$ with $\mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01}$ (see [1, Section 3]) all the geodesics between the projections $P_{\mathrm{Gr}(0)}$ and Q can be parameterized more explicitly as

$$\gamma(t) = \begin{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & 0 & 0 \\ 0 & e^{t\tilde{X}}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)e^{-t\tilde{X}} & 0 \\ 0 & 0 & \delta_0(t) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\begin{smallmatrix} \cos^2(t\frac{\pi}{2}) & \cos(t\frac{\pi}{2})\sin(t\frac{\pi}{2})u \\ \sin(t\frac{\pi}{2})u^*\cos(t\frac{\pi}{2}) & \sin^2(t\frac{\pi}{2}) \end{smallmatrix}\right) & 0 \\ 0 & 0 & 0 & \delta_0(t) \end{pmatrix},$$

which is the expression for the geodesics γ_u of (5.23).

The minimality condition of the geodesics γ when $t \in [0,1]$ and $\|a\| \leq \frac{\pi}{2}$ follows from [1, Theorem 5.3 and Corollary 5.5]. ■

Recall that a classical Jacobi field is a field on a fixed geodesic γ that can be obtained by differentiating a family of perturbations of γ by geodesics that start and end at the same points as γ.

DEFINITION 5.17. Given a geodesic $\gamma(t)$, $t \in [0,1]$, a parameter value $t_0 \in [0,1]$ is called a *conjugate of* 0 *along* γ if there exists a non-trivial Jacobi field that vanishes at 0 and at t_0. In this case, the *index of* t_0 is the dimension of the space of Jacobi fields that vanish at 0 and at t_0. The parameter t_0 is called *conjugate* if this index is greater than zero.

The following is an example of conjugate values in $\mathcal{R}$ (see 5.9) involving Fredholm operators.

THEOREM 5.18. *Let T be a Fredholm operator of index zero and $n = \dim(\ker(T)) = \dim(\mathrm{im}(T)^\perp) > 0$. Then 1 is a conjugate parameter of* 0 *for the geodesic defined in* (5.23) *for $u = 1$, with $t \in [0,1]$, connecting the orthogonal projection onto the graph of the null operator $P_{\mathrm{Gr}(0)}$ and the orthogonal projection $P_{\mathrm{invGr}(T)}$ onto the inverse graph.*

Moreover, the index of this conjugate parameter has dimension n^2.

Proof. From Lemma 5.4 deduce $\mathrm{Gr}(T)^\perp = \{(-T^*x, x) : x \in \mathrm{Dom}(T^*)\} = \mathrm{invGr}(-T^*)$. And since $\mathrm{im}(P_{\mathrm{Gr}(0)}) = H \oplus \{0\}$ and $\ker(P_{\mathrm{Gr}(0)}) = \{0\} \times H$, we obtain

$$\begin{aligned} \mathcal{H}_{10} &= \mathrm{im}(P_{\mathrm{Gr}(0)}) \cap \ker(P_{\mathrm{invGr}(T)}) \\ &= \ker(T^*) \oplus \{0\} = \mathrm{im}(T)^\perp \oplus \{0\}, \\ \mathcal{H}_{01} &= \ker(P_{\mathrm{Gr}(0)}) \cap \mathrm{im}(P_{\mathrm{invGr}(T)}) = \{0\} \oplus \ker(T). \end{aligned} \tag{5.25}$$

Therefore, the condition $\dim(\ker(T)) = \dim(\mathrm{im}(T)^\perp) = n > 0$ implies that there exist infinitely many geodesics joining $P_{\mathrm{Gr}(0)}$ to $P_{\mathrm{invGr}(T)}$ (see [1]).

Then, we can use Theorem 5.16 and the expression for the geodesics γ_u from (5.23). We will differentiate curves of geodesics using the parameter s that describes unitaries $u(s)$ with fixed t. Observe that the only part that changes is in the $\mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01}$ space given by

$$\gamma_u|_{\mathcal{H}'}(t) = \begin{pmatrix} \cos^2(t\pi/2) & -\cos(t\pi/2)(\sin(t\pi/2))u \\ (\sin(t\pi/2))(\cos(t\pi/2))u^* & -\sin^2(t\pi/2) \end{pmatrix} \quad \text{for } t \geq 0.$$

We will construct a Jacobi field along the fixed geodesic γ_1. In this case we can consider the Jacobi field obtained after differentiating the geodesics perturbed by unitary curves $u(s)$ that depend on the parameter s close to $s = 0$ with $u(0) = 1$. Then, differentiating $\gamma_u|_{\mathcal{H}'}(t)$ with respect to s, we have

$$\begin{aligned} \mathcal{J}_{\dot u}(t) &= \frac{\partial}{\partial s}\bigg|_{s=0} (\gamma_{u(s)}|_{\mathcal{H}'}(t)) \\ &= \begin{pmatrix} 0 & -\cos(t\pi/2)(\sin(t\pi/2))\dot u(s) \\ (\sin(t\pi/2))(\cos(t\pi/2))\dot u^*(s) & 0 \end{pmatrix} \end{aligned}$$

for $t \in [0,1]$. Note that the derivatives $\dot{u}(s)$ belong to the space of anti-self-adjoint operators (elements of the Lie algebra of the unitary group) that has real dimension n^2 since $u : \mathcal{H}_{01} \to \mathcal{H}_{10}$ (each of dimension n). ∎

5.7. Density (and non-density) of the geodesic neighborhoods. Let us briefly examine examples of algebras where $\{\tilde{q} \in P_2(\mathcal{A}) : \|\tilde{q} - \tilde{p}_0\| < 1\}$ is dense in the orbit of $\tilde{p}_0$, and examples where it is not. The first example includes the case of finite matrices.

Example 5.19. Let $\mathcal{A}$ be a finite von Neumann factor, with (unique) normal, faithful and normalized trace τ. Then $M_2(\mathcal{A})$ is also a finite von Neumann factor with trace $\mathbf{Tr}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \frac{1}{2}\tau(a+d)$. In [2] it was shown that any pair of projections in a finite factor, in the same connected component (i.e., in the same unitary orbit, or equivalently, with equal trace), can be joined with a minimal geodesic. Pick as usual $\tilde{p}_0 \in M_2(\mathcal{A})$, $\tilde{p}_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Let $\tilde{q}$ be a projection in the orbit of $\tilde{p}_0$, and $\gamma(t) = e^{itX}\tilde{p}_0 e^{-itX}$ a geodesic with $\gamma(1) = \tilde{q}$, with $X^* = X$ $\tilde{p}_0$-codiagonal and $\|X\| \leq \pi/2$. It is known that (see [17])

$$\|\gamma(t) - \gamma(s)\| = \sin(|t-s|\,\|x\|).$$

Therefore, given $\epsilon > 0$, we can choose $t_0 < 1$ such that $\tilde{q}_0 =: \gamma(t_0)$ satisfies

$$\|\tilde{q}_0 - \tilde{q}\| = \|\gamma(t_0) - \gamma(1)\| = \sin((1-t_0)\|X\|) < \epsilon.$$

Clearly also $\|\tilde{p}_0 - \tilde{q}_0\| = \|\gamma(0) - \gamma(t_0)\| < 1$. That is, $\{\tilde{q} \in P_2(\mathcal{A}) : \|\tilde{q} - \tilde{p}_0\| < 1\}$ is dense in $\mathcal{R}$, the orbit of $\tilde{p}_0$.

The next example shows that this is no longer the case if $\mathcal{A} = \mathcal{B}(H)$ for H infinite-dimensional. To present the specific subspaces, first we need to recall results on the theory of common complements of pairs of subspaces, as presented by M. Lauzon and S. Treil [12], and continued by J. Giol [8].

Remark 5.20. In [12], necessary and sufficient conditions were given for a pair of closed subspaces $\mathcal{S}, \mathcal{T}$ of an infinite-dimensional Hilbert space $\mathcal{L}$ to have (or not to have) a common complement, i.e., that there exists (or not) a closed subspace $\mathcal{Z} \subset \mathcal{L}$ such that $\mathcal{S} \dot{+} \mathcal{Z} = \mathcal{L}$ and $\mathcal{T} \dot{+} \mathcal{Z} = \mathcal{L}$, where the symbol $\dot{+}$ stands for direct but not necessarily orthogonal sum. For instance, in [12] it was shown that, if $\mathcal{L}$ is separable, $\mathcal{S}, \mathcal{T} \subset \mathcal{L}$ do not have a common complement if and only if $\dim \mathcal{S} \cap \mathcal{T}^\perp \neq \dim \mathcal{S}^\perp \cap \mathcal{T}$ and

$$1_{\mathcal{S}} - G^*G : \mathcal{S} \to \mathcal{S} \text{ is compact when restricted to } N(G)^\perp,$$

where $G := P_{\mathcal{T}}|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}$. Here $N(G) = \mathcal{S} \cap \mathcal{T}^\perp$.

Later on, J. Giol [8] proved that $\mathcal{S}$ and $\mathcal{T}$ do have a common complement if and only if there exists an intermediate orthogonal projection Q such that $\|P_{\mathcal{S}} - Q\| < 1$ and $\|Q - P_{\mathcal{T}}\| < 1$.

Building on these facts, it is easy to see that a pair of subspaces $\mathcal{S}$, $\mathcal{T}$ of $\mathcal{L}$, with infinite dimension and infinite codimension, and without a common complement, provides an example where $\{P \in P(\mathcal{L}) : \|P_\mathcal{S} - P\| < 1\}$ is not dense in the unitary orbit of $P_\mathcal{S}$: indeed, note that in this case

$$\{P \in P(\mathcal{L}) : \|P_\mathcal{S} - P\| < 1\} \cap \{Q \in P(\mathcal{L}) : \|P_\mathcal{T} - Q\| < 1\} = \emptyset.$$

Clearly, an element Q in this intersection would provide an intermediate projection with $\|P_\mathcal{S} - Q\| < 1$ and $\|P_\mathcal{T} - Q\| < 1$, and this would imply, by Giol's result, that $\mathcal{S}$ and $\mathcal{T}$ have a common complement.

Moreover, it is clear how to adapt this example to our situation (where one of the subspaces is $H \times \{0\}$). Pick a unitary isomorphism $U : \mathcal{L} \to H \times H$ which maps $\mathcal{S}$ onto $H \times \{0\}$. This is done by choosing orthonormal bases of $\mathcal{S}$ and $H \times \{0\}$, and completing them to orthonormal bases of $\mathcal{L}$ and $H \times H$, respectively, and is possible because $\mathcal{S}$ has infinite dimension and infinite codimension. Since $\mathcal{S}$ and $\mathcal{T}$ do not have a common complement in $\mathcal{L}$, it is clear that $H \times \{0\} = U\mathcal{S}$ and $U\mathcal{T}$ do not have common complement in $H \times H$.

Therefore $\{P \in \mathcal{R} : \|\tilde{p}_0 - P\| < 1\}$ is not dense in $\mathcal{R}$ in this case.

Example 5.21. This example was discussed in [3] in connection with existence and non-existence of geodesics between subspaces, and it is related to the so called Uncertainty Principle in harmonic analysis.

Let $I, J \subset \mathbb{R}^n$ be Lebesgue measurable subsets with finite positive measure. Consider

$$\mathbf{S}_I = \{f \in L^2(\mathbb{R}^n) : \mathrm{supp}(f) \subset I\}, \qquad \mathbf{T}_J = \{g \in L^2(\mathbb{R}^n) : \mathrm{supp}(\hat{g}) \subset J\},$$

where supp stands for the (essential) support, and $\hat{g}$ is the Fourier–Plancherel transform of g. Put $\mathcal{S} = \mathbf{S}_I$ and $\mathcal{T} = \mathbf{T}_J^\perp$. We claim that $\mathcal{S}$ and $\mathcal{T}$ do not have a common complement.

Indeed, it is known that (see [13] or the survey article [7])

$$\mathcal{S} \cap \mathcal{T}^\perp = \mathbf{S}_I \cap \mathbf{T}_J = \{0\} \text{ and } \mathcal{S}^\perp \cap \mathcal{T} = \mathbf{S}_I^\perp \cap \mathbf{T}_J^\perp \text{ is infinite-dimensional.}$$

Also, it is known that $P_{\mathbf{S}_I} P_{\mathbf{T}_J}$ is compact (see [7]). This clearly means that

$$P_\mathcal{S} - P_\mathcal{S} P_\mathcal{T} P_\mathcal{S} = P_\mathcal{S} P_\mathcal{T}^\perp P_\mathcal{S} = P_{\mathbf{S}_I} P_{\mathbf{T}_J} P_{\mathbf{S}_I}$$

is compact, i.e., $1_\mathcal{S} - G^*G$ is compact in the whole $\mathcal{S}$ (here $N(G) = \mathcal{S} \cap \mathcal{T}^\perp = \{0\}$). Therefore, by the result of Lauzon and Treil [12] transcribed in Remark 5.20, $\mathcal{S}$ and $\mathcal{T}$ do not have a common complement.

Remark 5.22. Example 5.21 tells us that the classical Hopf–Rinow Theorem is not valid when $\mathcal{A} = \mathcal{B}(H)$ for H infinite-dimensional. There are points in $\mathcal{R}$ which cannot be reached by a geodesic starting at $\tilde{p}_0$, not even approximated by points in the range of the exponential based at $\tilde{p}_0$. Moreover, elaborating on this example, we also infer that there exist in $\mathcal{R}$ infinitely many disjoint open subsets which are ranges of the exponential map at different points in $\mathcal{R}$.

Example 5.19 suggests that density of the range of the exponential at $\tilde{p}_0$ requires some sort of finiteness (for instance, that the algebra is finite, as shown in this example).

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Esteban Andruchow, Alejandro Varela
Instituto Argentino de Matemática "Alberto P. Calderón"
(C1083ACA) Buenos Aires, Argentina
and
Instituto de Ciencias, Universidad Nacional de General Sarmiento
(B1613GSX) Los Polvorines, Argentina
E-mail: eandruch@campus.ungs.edu.ar
avarela@campus.ungs.edu.ar

Gustavo Corach, Lázaro Recht
Instituto Argentino de Matemática "Alberto P. Calderón"
(C1083ACA) Buenos Aires, Argentina
E-mail: gcorach@gmail.com
lrecht@gmail.com