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Integrability of (ω, m)-subharmonic functions on compact Hermitian manifolds

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Abstract. Let (X, ω) be a compact Hermitian manifold of dimension n. We show that all (ω, m)-subharmonic functions are L^p-integrable on X, for any $p < \frac{n}{n-m}$.

1. Introduction. Let $\Omega \subset \mathbb{C}^n$ be an open set and ω a Hermitian $(1,1)$-form on Ω. Let u be a real $\mathcal{C}^2$ function on Ω such that the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ of the complex Hessian matrix $[u_{i\bar{j}}]_{1 \le i,j \le n}$, belong to the closure of the cone

$$\Gamma_m = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \ldots, S_m(\lambda) > 0\},$$

where $S_k(\lambda)$ denotes the kth elementary symmetric function of λ:

$$S_k(\lambda) = \sum_{0 < j_1 < \cdots < j_k \le n} \lambda_{j_1} \cdots \lambda_{j_k}.$$

Such a function is called *m-subharmonic* (m-sh). As shown by Błocki [B05], the m-sh functions are the right class of admissible solutions to the complex Hessian equations

$$(dd^c u)^m \wedge \omega^{n-m} = f \omega^n,$$

which are the generalization of the Poisson equations ($m = 1$) and the Monge–Ampère equations ($m = n$).

While all plurisubharmonic functions are locally L^p-integrable for all $p \ge 1$, this is not necessarily the case for m-sh functions. A typical example is given by Błocki [B05, Section 1]: the m-subharmonic function $G(z) = -|z|^{2-2n/m}$ is L^p-integrable only when $p < \frac{nm}{n-m}$. Błocki conjectured that all m-sh functions are locally L^p-integrable for $p < \frac{nm}{n-m}$. This

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conjecture is partially confirmed in [DK14, ÅC20], where the former developed a useful tool to study the integrability of m-sh functions, known as volume-capacity inequality

$$\mathrm{Vol}(K) \leq C_p \operatorname{Cap}_m^p(K, \varOmega),$$

with K being a compact subset of $\varOmega$, $\operatorname{Cap}_m(K, \varOmega)$ denoting the Hessian capacity, and $p < \frac{n}{n-m}$.

On compact Hermitian manifolds (X, ω), we consider the m-Hessian operator

$$H_m(u) = (\omega + dd^c u)^m \wedge \omega^{n-m}.$$

A $\mathcal{C}^2$ function is called (ω, m)-*subharmonic* with respect to ω if $H_k(u) \geq 0$ for all $k = 1, \dots, m$. The study of m-Hessian equations on compact manifolds (see for example [S18, Z17, KN16, GL25, KN26]) motivates the development of potential theory for (ω, m)-sh functions on Hermitian manifolds. The main purpose of this note is to extend Dinew–Kołodziej's integrability result to Hermitian manifolds. We now present the precise statement.

Main Theorem 1.1. *Let (X, ω) be a compact Hermitian manifold of dimension n, equipped with a Hermitian form ω. Fix an integer $1 \leq m \leq n$. For any $p < \frac{n}{n-m}$, we have $\mathcal{SH}_m(X, \omega) \subset L^p(X, \omega^n)$.*

When (X, ω) is a Kähler manifold, the theorem is proved in [LN15], using the volume-capacity inequality provided by Dinew and Kołodziej [DK14], and the Chern–Levine–Nirenberg (CLN) inequality. The proof of the latter inequality involves several integrations by parts, which are delicate in the Hermitian setting due to the appearance of torsion terms. In our setting when $m < n$, there is also a lack of positivity, since (ω, m)-subharmonic functions are not in general ω-plurisubharmonic.

In this short note, we observe that in the volume-capacity estimate of Dinew and Kołodziej the candidates defining the capacity can be taken to be ω-plurisubharmonic. This observation simplifies several potential estimates that we carry out in Section 3, where we prove the main result. In Section 2, we review the essential concepts and results required for the proof.

2. Preliminaries.

Throughout this note, we denote by (X, ω) a compact Hermitian manifold of complex dimension $n \in \mathbb{N}^*$, equipped with a Hermitian metric ω. We use the differential operators $d = \partial + \bar\partial$, and $d^c = i(\bar\partial - \partial)$, so that $dd^c = 2i\partial\bar\partial$.

We now recall the definition of (ω, m)-subharmonic functions together with related results for ω-plurisubharmonic and (ω, m)-subharmonic functions.

2.1. (ω, m)-subharmonic functions.

Fix an integer $1 \leq m \leq n$. Fix an open set $\varOmega$ in $\mathbb{C}^n$. We follow the definition in [GN18].

DEFINITION 2.1. A real $(1,1)$-form α on X is called *m-positive with respect to ω* if at all points in X,

$$\alpha^k \wedge \omega^{n-k} \geq 0, \quad \forall k = 1, \ldots, m.$$

DEFINITION 2.2. Given a Hermitian metric α on Ω, a $\mathcal{C}^2(\Omega)$ function $u : \Omega \to \mathbb{R}$ is called *harmonic with respect to α* if $dd^c u \wedge \alpha^{n-1} = 0$ at all points in Ω.

DEFINITION 2.3. A function $u : \Omega \to \{-\infty\} \cup \mathbb{R}$ is *subharmonic with respect to ω* if

(a) u is upper semicontinuous and $u \in L^1_{\mathrm{loc}}(\Omega)$;
(b) for every relatively compact open set $D \Subset \Omega$ and every function h in $\mathcal{C}^0(D)$ that is harmonic with respect to ω on D, the following implication holds:

$$u \leq h \text{ on } \partial D \implies u \leq h \text{ in } D.$$

DEFINITION 2.4. A function $\varphi : \Omega \to \{-\infty\} \cup \mathbb{R}$ is *quasi-subharmonic with respect to α* if locally $\varphi = u + \rho$, where u is subharmonic with respect to α and ρ is smooth.

A function φ is *ω-subharmonic with respect to α* if φ is quasi-subharmonic with respect to α and $(\omega + dd^c\varphi) \wedge \alpha^{n-1} \geq 0$ in the sense of distributions.

The *positive cone $\Gamma_m(\alpha)$* associated with the metric α is defined by

$$\{\gamma \text{ real } (1,1)\text{-form} : \gamma^k \wedge \alpha^{n-k} > 0,\ k = 1, \ldots, m\}.$$

It follows from Gårding's inequality [G59] that if $\gamma_0, \gamma_1, \ldots, \gamma_{m-1} \in \Gamma_m(\alpha)$, then

$$\gamma_0 \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \alpha^{n-m} > 0.$$

Set $\widetilde{\alpha} = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \alpha$; it is a strictly positive $(n-1, n-1)$-form on Ω.

DEFINITION 2.5. A function $\varphi : \Omega \to \{-\infty\} \cup \mathbb{R}$ is called *(ω, m)-subharmonic with respect to α* if φ is ω-subharmonic with respect to $\widetilde{\alpha}$ in Ω for all $\widetilde{\alpha}$ of the form $\widetilde{\alpha}^{n-1} = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \alpha$, where $\gamma_1, \ldots, \gamma_{m-1} \in \Gamma_m(\alpha)$.

A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called *(ω, m)-subharmonic on X* if u is (ω, m)-subharmonic on each local chart U of X.

The set of all locally integrable functions on U which are (ω, m)-sh with respect to α in U is denoted by $\mathcal{SH}_{\alpha,m}(U, \omega)$. We remark that ω and α are not necessarily the same.

However, in this paper, we focus on the set $\mathcal{SH}_{\omega,m}(U, \omega)$. To simplify the notations, we denote by $\mathcal{SH}_m(U, \omega)$ the set of all (ω, m)-subharmonic functions with respect to ω in U. The set of all ω-plurisubharmonic functions on U is denoted by $\mathcal{PSH}(U, \omega) = \mathcal{SH}_n(U, \omega)$.

For notational simplicity, we also denote $\omega_u := \omega + dd^c u$.

Remark 2.6. By Gårding's inequality [G59], if $u \in \mathcal{C}^2(X)$, then u is (ω, m)-subharmonic with respect to ω on X if and only if the associated form ω_u belongs to the closure of $\Gamma_m(\omega)$.

The integration by parts formula is valid for $\mathcal{C}^2(X)$ functions (by Stokes' theorem). We can see that it still holds for bounded (ω, m)-subharmonic functions by [KN26, Proposition 3.20]. We will need the following proposition.

Proposition 2.7 (Integration by parts). *Let $\varphi, \psi \in \mathcal{SH}_m(X,\omega) \cap \mathcal{C}^2(X)$. Let T be a smooth $(n-1, n-1)$-form. Then*

$$\int_X \varphi dd^c\psi \wedge T = \int_X \psi dd^c\varphi \wedge T + 2\int_X \psi d\varphi \wedge d^cT + \int_X \psi\varphi dd^cT.$$

Proof. Observe that $\int_X \varphi\partial\psi \wedge \partial T = \int_X \varphi\bar\partial\psi \wedge \bar\partial T = 0$ for the bidegree reason. Hence,

$$\int_X \varphi d^c\psi \wedge dT = -\int_X \varphi d\psi \wedge d^cT.$$

It thus follows from Stokes' theorem that

$$\begin{aligned}\int_X \varphi dd^c\psi \wedge T &= \int_X \psi dd^c(\varphi T) = \int_X \psi d(d^c\varphi \wedge T + \varphi d^cT)\\ &= \int_X \psi(dd^c\varphi \wedge T - d^c\varphi \wedge dT + d\varphi \wedge d^cT + \varphi dd^cT)\\ &= \int_X \psi dd^c\varphi \wedge T + 2\int_X \psi d\varphi \wedge d^cT + \int_X \psi\varphi dd^cT. \ \blacksquare\end{aligned}$$

We now state the following important result for (ω, m)-subharmonic functions (see [KN16, Lemma 2.3]).

Lemma 2.8. *Let $u \in \mathcal{SH}_m(\Omega,\omega) \cap \mathcal{C}^2(\Omega)$ and T be a smooth $(n-k, n-k)$-form with $1 \le k \le m-1$. Then*

$$|\omega_u^k \wedge T/\omega^n| \le C_{n,k,\|T\|}\omega_u^k \wedge \omega^{n-k}/\omega^n,$$

where $C_{n,k,\|T\|}$ is a uniform constant depending only on n, k and the sup norm of coefficients of T.

We remark that the assumption $k < m$ is crucial.

We conclude this subsection with an L^1-compactness result (see [KN16, Lemma 3.3]).

Lemma 2.9. *Let $u \in \mathcal{SH}_m(X,\omega)$ be normalized by $\sup_X u = 0$. Then there exists a uniform constant $A > 0$ depending only on X and ω such that*

$$\int_X |u|\omega^n \le A.$$

2.2. The Cauchy–Schwarz inequality. Let h be a smooth real-valued function and let u, v be Borel functions. Let T be a positive current of bidegree $(n-2, n-2)$. The following Cauchy–Schwarz inequality will be useful (cf. [N16, Proposition 1.4], [KN26, Lemma 2.3]).

Lemma 2.10. *There exists a uniform constant C depending on ω such that*

$$\left|\int_X uv dh \wedge d^c\omega \wedge T\right|^2 \leq C \int_X |u|^2 dh \wedge d^c h \wedge \omega \wedge T \int_X |v|^2 \omega^2 \wedge T.$$

Although the Cauchy–Schwarz inequality typically requires T to be positive, it still holds for $T = \gamma^{m-1} \wedge \omega^{n-m-1}$, which is not necessarily positive, where γ is m-positive with respect to ω (see [KN26, Lemma 2.4]).

2.3. Capacity. Let $E \subset X$ be a Borel subset.

Definition 2.11. The (ω, m)-*capacity* of E is defined by

$$\mathrm{Cap}_{\omega,m}(E) := \sup\left\{\int_E \omega_\varphi^m \wedge \omega^{n-m} : \varphi \in \mathcal{SH}_m(X,\omega),\ 0 \leq \varphi \leq 1\right\}.$$

It's clear that the (ω, m)-capacity is well defined for $\varphi \in \mathcal{C}^2(X)$. Thanks to [KN26, Theorem 3.3], the Hessian operator for bounded (ω, m)-subharmonic function is well defined, and so is the capacity.

Since we will need an upper bound for the capacity of the sublevel sets $\{\psi < -t\}$, controlling the terms $d\varphi \wedge d^c\omega^p$ when $\varphi \in \mathcal{SH}_m(X,\omega)$ (which arises from integration by parts) can be difficult. We therefore introduce the following version of the Hessian capacity.

Definition 2.12. We define

$$\widetilde{\mathrm{Cap}}_{\omega,m}(E) := \sup\left\{\int_E \omega_\varphi^m \wedge \omega^{n-m} : \varphi \in \mathcal{PSH}(X,\omega), 0 \leq \varphi \leq 1\right\}.$$

We observe that $\widetilde{\mathrm{Cap}}_{\omega,m}(E) \leq \mathrm{Cap}_{\omega,m}(E)$, since

$$\mathcal{PSH}(X,\omega) = \mathcal{SH}_n(X,\omega) \subset \mathcal{SH}_m(X,\omega).$$

3. Integrability of (ω, m)-subharmonic functions on Hermitian manifolds

Proposition 3.1 (Volume-capacity estimate). *For $1 < \tau < \frac{n}{n-m}$, there exists a constant C_τ such that for each Borel subset K of X,*

$$V(K) \leq C_\tau \widetilde{\mathrm{Cap}}_{\omega,m}(K)^\tau,$$

where $V(K) = \int_K \omega^n$.

Proof. The argument presented in [KN16, Proposition 3.6] is still valid here, as we use the estimates for ω-plurisubharmonic solutions to the Monge–Ampère equations. For the sake of completeness, we provide a slightly different proof below.

We normalize the volume form so that $\int_X \omega^n = 1$. We can assume that $V(K) > 0$, otherwise the inequality to be proved is trivial. Fixing $p > 1$, we solve the complex Monge–Ampère equation

$$\text{(MA)} \qquad (\omega + dd^c u)^n = a f_K \omega^n, \ \sup_X u = 0, \ \text{where } f_K = (V(K)^{-1/p}\mathbf{1}_K + 1).$$

The existence of $a > 0$ and $u \in \mathcal{PSH}(X,\omega) \cap L^\infty$ follows from [KN15]. We next show how to control the constant $a > 0$ uniformly. Observe first that $(\omega + dd^c u)^n \geq a\omega^n$, hence the domination principle [GL23, Corollary 1.13] gives $a \leq 1$. We also have

$$1 = \|V(K)^{-1/p}\mathbf{1}_K\|_p \leq \|f_K\|_p \leq \|V(K)^{-1/p}\mathbf{1}_K\|_p + \|1\|_p \leq 2.$$

It thus follows from [GL23, Theorem 2.1] that

$$u \geq -C_1,$$

where $C_1 \geq 1$ is a uniform constant, independent of K. We also deduce from Step 1 of [GL23, Theorem 2.1] that there exists a uniform constant $C_2 > 0$, independent of K, and a bounded ω-psh function ψ, $-1 \leq \psi \leq 0$, such that

$$(\omega + dd^c \psi)^n \geq C_2^{-1} f_K \omega^n.$$

We conclude from the domination principle [GL23, Corollary 1.13] that $C_2^{-1} \leq a$.

On the other hand, the mixed Monge–Ampère inequality [N16] yields

$$\omega_u^m \wedge \omega^{n-m} \geq a^{m/n} f_K^{m/n} \omega^n.$$

For $v = C_1^{-1}u$, we then have

$$\begin{aligned}\widetilde{\operatorname{Cap}}_{\omega,m}(K) &\geq \int_K (\omega + dd^c v)^m \wedge \omega^{n-m} \\ &\geq a^{m/n} C_1^{-m} V(K)^{1-\frac{m}{pn}} \\ &\geq C_2^{-m/n} C_1^{-m} V(K)^{1-\frac{m}{pn}}.\end{aligned}$$

Hence, for every $p > 1$, there exists a constant $C_p > 0$ such that

$$V(K) \leq C_p \, \widetilde{\operatorname{Cap}}_{\omega,m}(K)^{\frac{pn}{pn-m}},$$

which completes the proof. ■

Lemma 3.2. *Let φ be a bounded ω-plurisubharmonic function satisfying $-1 \leq \varphi \leq 0$, and ψ be a bounded (ω, m)-subharmonic function with respect to ω, normalized by $\sup_X \psi = 0$. Then there exists a uniform constant $C > 0$,* *depending on ω, n, m, such that*

$$\int_X |\psi|\omega_\varphi^m \wedge \omega^{n-m} \le C.$$

We emphasize here that the constant C does not depend on the L^∞-norm of ψ.

Proof. By approximation, we can also assume that φ and ψ are smooth. We will prove the bound

$$\int_X |\psi|\omega_\varphi^k \wedge \omega^{n-k} \le C_k$$

for all $k \in \{1, \dots, m\}$, by induction on k. Here C_k are uniform constants independent of ψ and φ. We denote by $D_1, D_2, \dots$ uniform positive constants. We fix A such that

$$\int_X |u|\omega^n \le A \qquad \text{for all } u \in \mathcal{SH}_m(X,\omega) \cap L^\infty(X) \text{ with } \sup_X u = 0.$$

We remark that there exists $B \ge 0$ such that

$$-B\omega^2 \le dd^c\omega \le B\omega^2 \quad \text{and} \quad -B\omega^3 \le d\omega \wedge d^c\omega \le B\omega^3.$$

Up to enlarging B, we can also assume that

$$\text{(B)} \qquad -B\omega^{k+1} \le dd^c(\omega^k) \le B\omega^{k+1}, \qquad \forall k.$$

First we prove the bound for $k = 1$. Using integration by parts (Proposition 2.7), and the compactness result (Lemma 2.9), we obtain

$$\begin{aligned}\int_X |\psi|\omega_\varphi \wedge \omega^{n-1} &= \int_X (-\psi)\omega^n + \int_X (-\psi) dd^c\varphi \wedge \omega^{n-1} \\ &\le A + \int_X (-\varphi) dd^c(\psi \wedge \omega^{n-1}) \\ &= A + \int_X (-\varphi) dd^c\psi \wedge \omega^{n-1} \\ &\quad + 2\int_X (-\varphi) d\psi \wedge d^c\omega^{n-1} + \int_X (-\varphi)\psi dd^c\omega^{n-1}.\end{aligned}$$

A basic computation gives

$$\int_X (-\varphi)\, dd^c\psi \wedge \omega^{n-1} = \int_X (-\varphi)\omega_\psi \wedge \omega^{n-1} + \int_X \varphi\omega^n.$$

The assumption $-1 \le \varphi \le 0$ implies that $\int_X \varphi\omega^n \le 0$. Moreover, due to the positivity of $\omega_\psi \wedge \omega^{n-1}$, it follows that

$$\int_X (-\varphi)\omega_\psi \wedge \omega^{n-1} \le \int_X \Big(\sup_X |\varphi|\Big)\omega_\psi \wedge \omega^{n-1} \le \int_X \omega_\psi \wedge \omega^{n-1}.$$

Moreover,

$$\begin{aligned}\int_X \omega_\psi \wedge \omega^{n-1} &= \int_X \omega^n + \int_X dd^c\psi\omega^{n-1} \\ &= \int_X \omega^n + \int_X \psi\, dd^c(\omega^{n-1}).\end{aligned}$$

Together with (B) and the compactness result, we thus obtain

$$\int_X (-\varphi)\, dd^c\psi \wedge \omega^{n-1} \le V + BA.$$

Furthermore, by the same computation as above,

$$\int_X (-\varphi)\psi\, dd^c\omega^{n-1} \le\ B\int_X (-\varphi)(-\psi)\omega^n \le BA. \tag{3.1}$$

We need to obtain an upper bound for $\int_X (-\varphi) d\psi \wedge d^c\omega^{n-1}$. After an elementary computation, Stokes' theorem and (3.1) yield

$$\begin{aligned}\int_X (-\varphi)\, d\psi \wedge d^c\omega^{n-1} &= \int_X \psi\, d\varphi \wedge d^c\omega^{n-1} + \int_X \psi\varphi dd^c\omega^{n-1} \\ &\le \int_X \psi\, d\varphi \wedge d^c\omega^{n-1} + BA.\end{aligned}$$

The Cauchy–Schwarz inequality (Lemma 2.10) yields

$$\begin{aligned}-\int_X (-\psi)\, d\varphi \wedge d^c\omega^{n-1} &= -(n-1)\int_X (-\psi) d\varphi \wedge d^c\omega \wedge \omega^{n-2} \\ &\le D_1\Big(\int_X (-\psi) d\varphi \wedge d^c\varphi \wedge \omega^{n-1}\Big)^{1/2}\Big(\int_X (-\psi)\omega^n\Big)^{1/2} \\ &\le D_1 A^{1/2}\Big(\int_X (-\psi) d\varphi \wedge d^c\varphi \wedge \omega^{n-1}\Big)^{1/2}.\end{aligned}$$

We claim that

$$d\varphi \wedge d^c\varphi \le \omega + \tfrac{1}{2} dd^c(\varphi+1)^2. \tag{3.2}$$

Indeed, one has

$$\omega + \tfrac{1}{2} dd^c(\varphi+1)^2 - d\varphi \wedge d^c\varphi = \omega + (\varphi+1)dd^c\varphi = (\varphi+1)\omega_\varphi - \varphi\omega,$$

which is positive because of the ω-plurisubharmonicity of φ and the assumption $-1 \le \varphi \le 0$.

So we obtain

$$\begin{aligned}&\int_X (-\varphi) d\psi \wedge d^c\omega^{n-1} \\ &\qquad \le BA + D_1 A\Big(\int_X (-\psi)\big(\omega + \tfrac{1}{2} dd^c(\varphi+1)^2\big) \wedge \omega^{n-1}\Big)^{1/2}.\end{aligned}$$

To continue let

$$A(\psi) = \sup\Big\{\int_X (-\psi)\omega_u \wedge \omega^{n-1} : 0 \le u \le 1,\, u \in \mathcal{PSH}(X,\omega) \cap \mathcal{C}^2(X)\Big\}.$$

From all the above computations we obtain

$$\int_X (-\psi)\omega_\varphi \wedge \omega^{n-1} \le (A + V + 3BA) + 2D_1 A^{1/2} A(\psi)^{1/2}.$$

Taking the supremum over all such φ we arrive at

$$A(\psi) \le (A + V + 3BA) + 2K_1 A^{1/2} A(\psi)^{1/2},$$

which implies that $A(\psi)$ is uniformly bounded from above, independently of ψ. Thus, the bound holds for $k = 1$. Suppose now that for all $j \in \mathbb{N}^+$ with $1 \le j \le k-1$, and for all $\psi \in \mathcal{SH}_m(X,\omega)$ and $\varphi \in \mathcal{PSH}(X,\omega)$ with normalizing condition as above, the following inequality holds:

$$\int_X |\psi|\omega_\varphi^j \wedge \omega^{n-j} \le C_j.$$

We need to infer the following inequality:

$$\int_X |\psi|\omega_\varphi^k \wedge \omega^{n-k} \le C_k.$$

Since

$$\begin{aligned}\int_X |\psi|\omega_\varphi^k \wedge \omega^{n-k} = &\int_X |\psi|\omega \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k}\\ &+ \int_X |\psi|(dd^c\varphi) \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k},\end{aligned}$$

using the induction hypothesis, it is enough to estimate the second term. Integration by parts (Proposition 2.7) yields

$$\begin{aligned}\int_X (-\psi)(dd^c\varphi) \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} &= \int_X (-\varphi)\, dd^c(\psi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k})\\ &= \int_X (-\varphi) dd^c\psi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k}\\ &\quad + 2\int_X (-\varphi) d\psi \wedge d^c(\omega_\varphi^{k-1} \wedge \omega^{n-k})\\ &\quad + \int_X (-\varphi)\psi dd^c(\omega_\varphi^{k-1} \wedge \omega^{n-k}).\end{aligned} \tag{3.3}$$

We now deal with the first term of (3.3). We first observe that

$$\begin{aligned}\int_X (-\varphi) dd^c\psi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} &= \int_X (-\varphi)\omega_\psi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\ &\quad - \int_X (-\varphi)\omega \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\ &\leq \int_X (-\varphi)\omega_\psi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\ &\leq \int_X \omega_\psi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\ &= \int_X \omega \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\ &\quad + \int_X dd^c\psi \wedge (\omega_\varphi^{k-1} \wedge \omega^{n-k}),\end{aligned}$$

where the first inequality follows from the assumption that $\varphi \leq 0$, and that $\omega_\varphi^{k-1} \wedge \omega^{n-k+1}$ is positive since $\varphi \in \mathcal{PSH}(X, \omega)$, while the second inequality holds because of the normalization of φ and the positivity of $\omega_\varphi^{k-1} \wedge \omega^{n-k}$. Note that

$$\int_X \omega \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} = \sum_{j=0}^{k-1} a_{k,j} \int_X \omega^{j+1} \wedge (dd^c\varphi)^{k-1-j} \wedge \omega^{n-k}.$$

For each term in the sum, integration by parts together with the bound on φ leads to

$$\int_X \omega \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \leq D_2.$$

Moreover, an elementary computation yields

(3.4)

$$\begin{aligned}dd^c(\omega_\varphi^{k-1} \wedge \omega^{n-k}) &= (k-1)(k-2)\omega_\varphi^{k-3} \wedge d\omega \wedge d^c\omega \wedge \omega^{n-k} \\ &\quad + (k-1)\omega_\varphi^{k-2} dd^c\omega \wedge \omega^{n-k} \\ &\quad + 2(k-1)\omega_\varphi^{k-2} \wedge d\omega \wedge d^c\omega^{n-k} + \omega_\varphi^{k-1} \wedge dd^c\omega^{n-k}.\end{aligned}$$

Thus, performing integration by parts, it follows from (B) and the induction hypothesis that

$$\begin{aligned}\int_X dd^c\psi \wedge (\omega_\varphi^{k-1} \wedge \omega^{n-k}) &= \int_X \psi dd^c(\omega_\varphi^{k-1} \wedge \omega^{n-k}) \\ &\leq D_3 \int_X (-\psi)(\omega_\varphi^{k-1} \wedge \omega^{n-k+1} + \omega_\varphi^{k-2} \wedge \omega^{n-k+2} \\ &\qquad + \omega_\varphi^{k-3} \wedge \omega^{n-k+3}) \\ &\leq D_3(C_{k-1} + C_{k-2} + C_{k-3}).\end{aligned}$$

We can also infer from (3.4) and the induction hypothesis that the third term of (3.3) is uniformly bounded:

$$
\begin{aligned}
\int_X (-\varphi)\psi dd^c(\omega_\varphi^{k-1}\wedge\omega^{n-k}) &\le D_4 \int_X (-\varphi)\psi(\omega_\varphi^{k-1}\wedge\omega^{n-k+1} \\
&\qquad + \omega_\varphi^{k-2}\wedge\omega^{n-k+2} + \omega_\varphi^{k-3}\wedge\omega^{n-k+3}) \\
&\le D_4 \sup_X |\varphi|(C_{k-1} + C_{k-2} + C_{k-3}) \\
&\le D_5.
\end{aligned}
\tag{3.5}
$$

We now control the second term of (3.3). Stokes' theorem yields

$$
\begin{aligned}
\int_X (-\varphi)\, d\psi\wedge d^c(\omega_\varphi^{k-1}\wedge\omega^{n-k}) &= \int_X \psi d\varphi \wedge d^c(\omega_\varphi^{k-1}\wedge\omega^{n-k}) \\
&\quad + \int_X \psi\varphi dd^c(\omega_\varphi^{k-1}\wedge\omega^{n-k}),
\end{aligned}
$$

where the second term is bounded by D_5, for the same reason as in (3.5). Using the Cauchy–Schwarz inequality (Lemma 2.10) for the first term, we obtain

$$
\begin{aligned}
&-\int_X (-\psi)\, d\varphi\wedge d^c(\omega_\varphi^{k-1}\wedge\omega^{n-k}) \\
&\qquad \le \Big|\int_X (-\psi)\, d\varphi\wedge(d^c\omega\wedge T)\Big| \\
&\qquad \le \Big(D_6\int_X (-\psi)\, d\varphi\wedge d^c\varphi\wedge\omega\wedge T\Big)^{1/2}\Big(\int_X (-\psi)\omega^2\wedge T\Big)^{1/2},
\end{aligned}
\tag{3.6}
$$

where $T = (k-1)\omega_\varphi^{k-2}\wedge\omega^{n-k} + (n-k)\omega_\varphi^{k-1}\wedge\omega^{n-k-1}$ is positive. Applying the induction hypothesis, the integral $\int_X (-\psi)\omega^2\wedge T$ can be controlled by

$$(k-1)C_{k-2} + (n-k)C_{k-1}.$$

By the induction hypothesis, and applying (3.2) to (3.6), we arrive at

$$
\begin{aligned}
&\int_X (-\psi) d\varphi\wedge d^c(\omega_\varphi^{k-1}\wedge\omega^{n-k}) \\
&\qquad \le D_6^{1/2}((k-1)C_{k-1} + (n-k)C_{k-1})^{1/2} \\
&\qquad\quad \cdot\Big(\int_X (-\psi)\big(\omega + \tfrac{1}{2}dd^c(\varphi+1)^2\big)\wedge\omega\wedge T\Big)^{1/2} \\
&\qquad \le D_7\Big(\int_X (-\psi)\big(\omega + \tfrac{1}{2}dd^c(\varphi+1)^2\big)\wedge\omega\wedge T\Big)^{1/2}.
\end{aligned}
$$

To simplify the notation, we write

$$T_1 = (k-1)\omega_\varphi^{k-2}\wedge\omega^{n-k+1}, \qquad T_2 = (n-k)\omega_\varphi^{k-1}\wedge\omega^{n-k},$$

and

$$A_1 := \int_X (-\psi)\omega_u \wedge T_1, \qquad A_2 := \int_X (-\psi)\omega_u \wedge T_2, \qquad u = \frac{(\varphi+1)^2}{2} - 1.$$

Set $v_1 := \frac{u+(k-2)\varphi}{k-1}$, and $v_2 := \frac{u+(k-1)\varphi}{k}$. We remark that $-1 \le v_1, v_2 \le 0$, and that

$$\omega_u, \omega_\varphi \le (k-1)\omega + dd^c(u+(k-2)\varphi),$$
$$\omega_u, \omega_\varphi \le k\omega + dd^c(u+(k-1)\varphi).$$

By the induction hypothesis, it follows that

$$A_1 \le (k-1)^{k-1}(k-1)\int_X (-\psi)\omega_{v_1}^{k-1} \wedge \omega^{n-k+1} \le (k-1)^{k-1}(k-1)C_{k-1},$$
$$A_2 \le k^k(n-k)\int_X (-\psi)\omega_{v_2}^{k} \wedge \omega^{n-k}.$$

Set $S(\psi) = \sup\{\int_X (-\psi)\omega_v^k \wedge \omega^{n-k} : -1 \le v \le 0,\ v \in \mathcal{PSH}(X,\omega)\}$. From the preceding computations, we conclude that

$$\int_X (-\psi)\omega_\varphi^k \wedge \omega^{n-k} \le D_8 + (D_9 S(\psi))^{1/2},$$

where D_8, D_9 are independent of ψ. Taking the supremum over all $-1 \le \varphi \le 0$, we obtain

$$S(\psi) \le D_8 + (D_9 S(\psi))^{1/2},$$

which leads to a uniform upper bound for $\int_X (-\psi)\omega_v^k \wedge \omega^{n-k}$ for $-1 \le v \le 0$. ∎

Corollary 3.3. *There exists a constant $C > 0$ such that for all $\psi \in \mathcal{SH}_m(X,\omega)$ satisfying $\sup_X \psi = -1$ and for every $t > 0$ we have*

$$\widetilde{\mathrm{Cap}}_{\omega,m}(\psi < -t) \le C/t.$$

Proof. We fix $\varphi \in \mathcal{PSH}(X,\omega)$ satisfying $-1 \le \varphi \le 0$.

By Lemma 3.2, we deduce that

$$\int_{(\psi<-t)} (-\psi)\omega_\varphi^m \wedge \omega^{n-m} \le C.$$

We also observe that

$$\int_{(\psi<-t)} (-\psi)\omega_\varphi^m \wedge \omega^{n-m} \ge \int_{(\psi<-t)} t\omega_\varphi^m \wedge \omega^{n-m},$$

and this completes the proof. ∎

Proof of the main theorem. Without loss of generality, we may assume that $\sup_X \varphi = -1$. Fix $p < \frac{n}{n-m}$ and q such that $p < q < \frac{n}{n-m}$. A fundamental calculation yields

$$\begin{aligned}\int_X (-\varphi)^p \omega^n &= \int_0^{+\infty} \mathrm{Vol}(\{(-\varphi)^p > t\})\,dt \\ &= \int_0^1 \mathrm{Vol}(X) dt + \int_1^{+\infty} \mathrm{Vol}(\{(-\varphi)^p > t\})\,dt \\ &= \int_X \omega^n + \int_1^{+\infty} p\,\mathrm{Vol}(\{-\varphi > s\}) s^{p-1}\,ds.\end{aligned}$$

It follows from the volume-capacity estimate (Proposition 3.1) and Corollary 3.3 that

$$\begin{aligned}\int_1^{+\infty} p\,\mathrm{Vol}(\{-\varphi > s\}) s^{p-1}\,ds &\le C_q p \int_1^{+\infty} s^{p-1}\,\widetilde{\mathrm{Cap}}_{\omega,m}(\{\varphi < -s\})^q\,ds \\ &\le C_{p,q} \int_1^{+\infty} s^{p-1-q}\,ds < +\infty,\end{aligned}$$

where $C_{p,q}$ depends only on p, q. ■

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