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Asymptotic behavior of solutions for the velocity-vorticity model of the three-dimensional generalized Navier–Stokes equations with exponential damping

Le Tran Tinh

Abstract. We propose the velocity-vorticity model of the three-dimensional (3D) generalized Navier–Stokes equations with exponential damping terms and then study the asymptotic behavior of solutions in a periodic bounded domain. First, we study the global well-posedness using the Faedo–Galerkin approximation method. Then, we investigate the asymptotic behavior of weak solutions via attractors and their properties using the theory of the evolutionary system which was recently developed by Cheskidov, Foias and Lu. Finally, we investigate determining wavenumbers.

1. Introduction. In this work, we will study the asymptotic behavior of solutions for the velocity-vorticity model of the three-dimensional (3D) generalized Navier–Stokes equations with exponential damping determined by

$$(1.1)\quad \begin{cases} \partial_t u + \nu(-\Delta)^l u + w\times u + a(e^{b|u|^r}-1)u + \nabla p = f, \\ \partial_t w + \nu(-\Delta)^l w + (u\cdot\nabla)w - (w\cdot\nabla)u + a(e^{b|w|^r}-1)w + \nabla\eta = \nabla\times f, \\ \nabla\cdot u = \nabla\cdot w = 0. \end{cases}$$

Here, $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ represents the fluid velocity vector field, $w = w(t,x) = (w_1(t,x), w_2(t,x), w_3(t,x))$ plays the role of vorticity (but we do not assume $w = \nabla\times u$), $p = p(t,x)$ and $\eta = \eta(t,x)$ denote the scalar pressure and the Lagrange multiplier at the point $(t,x)\in\mathbb{R}^+\times\mathbb{T}$, $(-\Delta)^l$ is the l-fractional Laplacian, $f(t,x)$ is the external body force, $\nu > 0$ is the constant kinematic viscosity, a is the positive damping coefficient, the exponents b and r are nonnegative constants, and the terms $a(e^{b|u|^r}-1)u$

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and $a(e^{b|w|^r} - 1)w$ are damping terms that parameterize the extra dissipation occurring in the planetary boundary layer (see, e.g., [51] and references therein).

It is well-known that the fractional power of the Laplacian and the damping term are very helpful from a mathematical point of view. Dissipation corresponding to the fractional power of the Laplacian can, in principle, arise from modeling real physical phenomena. The fractional diffusion operators can model anomalous diffusion and have now been widely used in turbulence modeling to control the effective range of nonlocal dissipation (see, e.g., [1, 8, 20, 22, 26, 35, 36] and references therein). The damping term can be interpreted as resistance to motion and describes various physical situations such as porous media flow, drag, or friction effects, etc. (see, e.g., [9, 43, 51] and references therein). Recently, exponential damping was first used in [3] by J. Benameur (see also, e.g., [4, 6, 7, 42] for more details).

Recently, researchers have found that the velocity-vorticity formulation of the 3D Navier–Stokes equations yields excellent numerical results for flows with strong rotation. This formulation is as follows:

$$\begin{cases} \partial_t u - \nu \Delta u + w \times u + \nabla p = f, \\ \partial_t w - \nu \Delta w + (u \cdot \nabla) w - (w \cdot \nabla) u + \nabla \eta = \nabla \times f, \\ \nabla \cdot u = \nabla \cdot w = 0. \end{cases} \tag{1.2}$$

If $l = 1$ and $b = 0$, then (1.1) also reduces to (1.2). To the best of our knowledge, the system (1.2) was introduced in [23]. The steady velocity-vorticity system (1.2) was analyzed in [37] in the case of no-slip velocity boundary conditions. However, the global well-posedness and the long-time behavior of solutions for (1.2) are open issues. Therefore, the Voigt-regularization has been used with the velocity-vorticity formulation by A. Larios et al. [30] and they have obtained the velocity-vorticity-Voigt (VVV) model determined by

$$\begin{cases} (I - \alpha^2 \Delta) \partial_t u - \nu \Delta u + w \times u + \nabla p = f, \\ \partial_t w - \nu \Delta w + (u \cdot \nabla) w - (w \cdot \nabla) u + \nabla \eta = \nabla \times f, \\ \nabla \cdot u = \nabla \cdot w = 0. \end{cases}$$

Some results on VVV and related models have been studied in [30, 44, 50]. In [30], A. Larios et al. only proved the global well-posedness and convergence properties of the system. In [50], G. Yue and J. Wang have also considered the VVV model as in [30], but added the damping term λw to the second equation. They proved the existence of global and exponential attractors of the three-dimensional VVV system. In [44], N. D. Toan has also considered the VVV model with damping as in [50], but added the memory terms to both equations.

We are also interested in turbulence, where the fluid dynamic equations may involve nonlocal effects, anomalous diffusion, and strong rotation. To the best of our knowledge, studying the global well-posedness and the asymptotic behavior of solutions of (1.1) are open issues. Motivated by the results on the 3D generalized Navier–Stokes equations and related models (see, e.g., [18, 19, 27, 34, 42, 46, 48, 47, 52]), our goal for studying (1.1) is mainly mathematical and to understand how the nonlinear exponential damping terms affect the global well-posedness and the asymptotic behavior of the weak solutions for the system (1.1).

The paper is organized as follows. In Section 2, we recall the functional setting and preliminaries. In Section 3, we present the main results of the paper. In Sections 4, we prove the global well-posedness results using the Faedo–Galerkin approximation method. In Section 5, the long-time behavior of solutions is investigated via attractors and their properties using the theory of the evolutionary system which was recently developed by Cheskidov, Foias and Lu in [12, 13, 14, 15, 32]. In Section 6, we study the determining wavenumbers.

Throughout this paper, we denote by $A \lesssim \sum_{i=1}^{n} B_i$ an estimate of the form $A \leq \sum_{i=1}^{n} c_i B_i$ with some positive constants c_i. We also use $\|\cdot\|_X$ as the notation of norm in the normed space X.

2. Functional setting and preliminaries. We consider (1.1) on a bounded domain in $\mathbb{R}^3$. In fact, the most practically important boundary conditions are no-slip and no-penetration conditions on the solid boundaries of the fluid domain (see, e.g., [21, 37, 49]). Due to simplicity and lack of natural boundary conditions, we work on the torus, i.e., the spatial variable $x = (x_1, x_2, x_3) \in \mathbb{T} := [-\pi, \pi]^3$ and periodic boundary conditions are assumed. Therefore, all functions are periodic in each x_i, $i = 1, 2, 3$, with period 2π. If we restrict ourselves to dealing with initial data and f with vanishing spatial averages, then the solutions will also have this property.

We will now provide some preliminary background material (see, e.g., [5, 30]) that is used throughout this paper.

(i) Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be divergence-free, periodic, and have zero spatial averages. It can be represented as follows:

$$v := \sum_{k \in J} v_k \phi_k \quad \text{with } v_k \in \mathbb{C}^3,\ v_k^* = v_{-k},\ v_k \cdot k = 0\ \ \forall k \in J,$$

where $\phi_k = e^{ik \cdot x}$ and $J = \mathbb{Z}^3 \setminus \{0\}$.

(ii) Let $\mathcal{F}$ be the space of formal Fourier series

$$\Big\{ v := \sum_{k \in J} \widehat{v}_k \phi_k : \widehat{v}_k \in \mathbb{C}^3,\ \phi_k = e^{ik \cdot x} \Big\}.$$

We then define $\mathcal{V}$ as the space of divergence-free trigonometric polynomials consisting of all $v \in \mathcal{F}$ such that $k \cdot \widehat{v}_k = 0$ for all $k \in J$ and $\widehat{v}_k = 0$ for all but finitely many values of $k \in J$.

(iii) For $s \in \mathbb{R}$, we define

$$H^s := \Big\{ v \in \mathcal{F} : \|v\|_{H^s}^2 := \sum_{k \in J} |v_k|^2 |k|^{2s} < \infty,\ \widehat{v^*}_k = \widehat{v}_{-k} \text{ and } \widehat{v}_0 = 0 \Big\},$$

and

$$V^s := \Big\{ v := \sum_{k \in J} v_k \phi_k : v_k \in \mathbb{C}^3,\ v_k^* = v_{-k},\ v_k \cdot k = 0,\ \phi_k = e^{ik\cdot x} \text{ and } \sum_{k \in J} |v_k|^2 |k|^{2s} < \infty \Big\}.$$

These spaces are Hilbert spaces with the scalar product

$$\langle u, v \rangle_{V^s} := \sum_{k \in J} u_k \cdot v_{-k} |k|^{2s}.$$

We also see that V^s is the closure of $\mathcal{V}$ in H^s with respect to the $\|\cdot\|_{H^s}$ norm. For simplicity, we use the notation $\langle \cdot, \cdot \rangle$ to denote the scalar product in V^0 and also the dual pairing of V^s-V^{-s}, given by $\langle u, v \rangle := \sum_{k \in J} u_k \cdot v_{-k}$.

(iv) We recall the following embeddings, interpolation, and inequalities. We have the compact embedding $V^{s+\varepsilon} \hookrightarrow\hookrightarrow V^s$ for any $\varepsilon > 0$. Let $s_1 \le s_2$ and $v \in V^{s_2}$. We have

$$\|v\|_{V^{s_1}} \lesssim \|v\|_{V^{s_2}}. \tag{2.1}$$

If $s_1 \le s_2$ and $s = \gamma s_1 + (1-\gamma) s_2$, $0 \le \gamma \le 1$, then

$$\|v\|_{V^s} \lesssim \|v\|_{V^{s_1}}^{\gamma} \|v\|_{V^{s_2}}^{1-\gamma} \quad \text{for all } v \in V^{s_2}.$$

If $0 \le s < 3/2$ and $1/p \ge 1/2 - s/3$, then the continuous embedding $V^s \hookrightarrow L^p(\mathbb{T})$ holds and

$$\|v\|_{L^p(\mathbb{T})} \lesssim \|v\|_{V^s} \quad \text{for all } v \in V^s.$$

If $s = 3/2$, then

$$\|v\|_{L^p(\mathbb{T})} \lesssim \|v\|_{V^s} \quad \text{for any finite } p \text{ and all } v \in V^s.$$

If $s > 3/2$, then

$$\|v\|_{L^\infty(\mathbb{T})} \lesssim \|v\|_{V^s} \quad \text{for all } v \in V^s.$$

(v) The linear operator $\varLambda = (-\varDelta)^{1/2}$ is defined as follows:

$$\varLambda v := \sum_{k \in J} |k| v_k \phi_k \quad \text{with } v = \sum_{k \in J} v_k \phi_k,\ \phi_k = e^{ik\cdot x},$$

and its powers Λ^s are determined by

$$\Lambda^s v := \sum_{k\in J} |k|^s v_k \phi_k.$$

Therefore, $(-\Delta)^s = \Lambda^{2s}$ and Λ^s preserves the divergence-free condition. It follows from the construction that Λ^s maps V^α onto $V^{\alpha-s}$ and

$$\|v\|_{V^s} = \|\Lambda^s v\|_{V^0}.$$

In particular, Λ^s maps V^s onto V^0 for all $s > 0$, so $D(\Lambda^s) = V^s$. Denote by $P_\sigma : L^2(\mathbb{T}) \to V^0$ the Leray–Helmholtz orthogonal projection operator. We have $P_\sigma \Lambda^s = \Lambda^s P_\sigma$.

(vi) Let $u, v, w \in \mathcal{V}$. We define the bilinear form

$$B(u,v) := P_\sigma\{(u\cdot\nabla)v\},$$

and the trilinear form

$$b(u,v,w) := \int_{\mathbb{T}} \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.$$

(vii) We denote by Π_n the finite-dimensional projectors onto V^0 which are determined by

$$\Pi_n v := \sum_{0<|k|\le n} v_k\phi_k \quad \text{for } v = \sum_{k\in J} v_k\phi_k \text{ and } \phi_k = e^{ik\cdot x},$$

and

$$B_n(u,v) := \Pi_n B(u,v).$$

In the following results, we recall certain relevant properties and inequalities of b (see, e.g., [5, 16, 28, 29]).

LEMMA 2.1. *Let $u, v, w \in \mathcal{V}$. Then*

(a) $b(u,v,v) = 0$,
(b) $b(u,v,w) = -b(u,w,v)$,
(c) $b(u-v,u,u-v) = b(u,u,u-v) - b(v,v,u-v)$.

This result may be extended to larger spaces by the density of $\mathcal{V}$ in V^σ for the appropriate values of σ such that the trilinear forms are continuous.

The following proposition is taken from [25, Proposition 2.5] (see also, e.g., [2]).

PROPOSITION 2.2. *The trilinear form $b : V^{\sigma_1}\times V^{\sigma_2}\times V^{\sigma_3} \to \mathbb{R}$ is bounded provided that all of the following conditions hold:*

(a) $\sigma_1+\sigma_2+\sigma_3 > 5/2$,
(b) $\sigma_1+\sigma_2 \ge s$,
(c) $\sigma_2+\sigma_3 \ge 1$,
(d) $\sigma_1+\sigma_3 \ge 1-s$,

for some $s \in \{0,1\}$. *If the last three conditions are satisfied and if* σ_i *is a nonpositive integer for some* $i \in \{1,2,3\}$, *then "*$>$*" in* (a) *can be replaced by "*$\geq$*". The nonstrict inequality is also allowed if for some* $s \in \{0,1\}$,

$$\sigma_1 \geq 0, \quad \sigma_2 \geq s, \quad \sigma_3 \geq 1-s.$$

(viii) For an integer $j \geq 0$, we set A_j to be the 2^j-sized block of 3D integer lattice points,

$$A_j := \{k = (k_1,k_2,k_3) \in \mathbb{Z}^3 : |k_m| \leq 2^j,\ m = 1,2,3\}.$$

We define the localized Fourier projection operators by

$$\Delta_0 f(x) := \sum_{k \in A_0} \widehat{f}(k) e^{ik\cdot x}, \tag{2.2}$$

$$\Delta_j f(x) := \sum_{k \in A_j \setminus A_{j-1}} \widehat{f}(k) e^{ik\cdot x}, \quad j \geq 1,\ j \in \mathbb{N}. \tag{2.3}$$

For notational convenience, we also write $\Delta_j = 0$ for $j < 0$. With a slight abuse of notation, we set

$$S_j f(x) := \sum_{m=0}^{j} \Delta_m f(x) = \sum_{k \in A_j} \widehat{f}(k) e^{ik\cdot x}. \tag{2.4}$$

In terms of these operators, we can write the Littlewood–Paley decomposition for any $f \in L^p(\mathbb{T})$ with $1 < p \leq \infty$ as follows:

$$f(x) = \sum_{m=0}^{\infty} \Delta_m f(x).$$

The following lemma presents useful basic properties of the operators defined above (see, e.g., [17, 24, 42, 45] for more details).

Lemma 2.3. *Let* $j \geq 0$ *be an integer. Let* Δ_j *and* S_j *be defined as in* (2.2)–(2.4). *Then the following properties hold:*

(a) *If* $f \in L^p(\mathbb{T})$ *with* $1 < p \leq \infty$, *then*

$$\|\Delta_j f\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})}, \quad \|S_j f\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})},$$

where the constants depend only on p *and* d.

(b) *Let* $h, j \geq 0$ *be integers. Assume* $f \in L^p(\mathbb{T})$ *with* $1 < p \leq \infty$. *Then,*

$$\Delta_h \Delta_j f = 0 \quad \text{if } h \neq j.$$

(c) *Let* $j, m \geq 0$ *and* $n \geq 1$ *be integers. Assume* $f, g \in L^p(\mathbb{T})$ *with* $1 < p \leq \infty$. *Then, if* $|m-j| \geq n$, *we have*

$$\Delta_j(S_{m-n} f \Delta_m g) = 0 \quad \text{and} \quad \Delta_j(\Delta_m f \widetilde{\Delta}_m g) = 0,$$

where

$$\widetilde{\Delta}_m g = \Delta_{m-n+1} g + \Delta_{m-n+2} g + \cdots + \Delta_{m+n-1} g.$$

We also have the following Bernstein-type inequalities for Δ_j (see, e.g., [17, Proposition 2.8]).

PROPOSITION 2.4. *Let Δ_j and S_j be defined as in* (2.2)–(2.4). *Then the following properties hold:*

(a) *Let $\sigma \ge 0$ and $1 \le q \le p \le \infty$. Assume $f \in L^p(\mathbb{T})$. Then*
$$\|\Delta_j \Lambda^\sigma f\|_{L^p(\mathbb{T})} \lesssim 2^{\sigma j + 3j(1/q-1/p)}\|\Delta_j f\|_{L^q(\mathbb{T})},$$
$$\|S_j f\|_{L^p(\mathbb{T})} \lesssim 2^{3j(1/q-1/p)}\|S_j f\|_{L^q(\mathbb{T})}.$$

(b) *Let $\sigma \ge 0$, $j \ge 0$, and $1 \le p \le \infty$. Assume $f \in L^p(\mathbb{T})$. Then there exist positive constants C_1 and C_2 (depending on p) such that*
$$C_1 2^{\sigma j}\|\Delta_j f\|_{L^p(\mathbb{T})} \lesssim \|\Delta_j \Lambda^\sigma f\|_{L^p(\mathbb{T})} \lesssim C_2 2^{\sigma j}\|\Delta_j f\|_{L^p(\mathbb{T})}.$$

Appealing to the periodic setting and the projection operator P_σ, we can rewrite (1.1) in the following abstract form:

$$\text{(2.5)}\quad \begin{cases} \partial_t u + \nu\Lambda^{2l}u + P_\sigma\{w\times u\} + aP_\sigma\{(e^{b|u|^r}-1)u\} = P_\sigma f, \\ \partial_t w + \nu\Lambda^{2l}w + B(u,w) - B(w,u) + aP_\sigma\{(e^{b|w|^r}-1)w\} = P_\sigma\{\nabla\times f\}, \\ \nabla\cdot u = \nabla\cdot w = 0. \end{cases}$$

Our system should be supplemented with L^2 initial data:
$$u(\tau,x) = u_\tau(x), \qquad w(\tau,x) = w_\tau(x).$$
Let us define the weak solutions to (2.5) with the L^2 initial data (u_τ, w_τ).

DEFINITION 2.5. Let ν, l, a, b be positive real parameters and let $r \ge 1$. For $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V^{1-l})$, $u_\tau, w_\tau \in V^0$ and a fixed $T > \tau$, a *weak solution* of (2.5) is a pair of functions (u,w) satisfying
$$\text{(2.6)}\quad u \in L^\infty(\tau,T;V^0)\cap L^2(\tau,T;V^l)\cap \mathcal{G}_b^r(\tau,T;L^1(\mathbb{T}))\cap C_w([\tau,T];V^0),$$
$$\text{(2.7)}\quad w \in L^\infty(\tau,T;V^0)\cap L^2(\tau,T;V^l)\cap \mathcal{G}_b^r(\tau,T;L^1(\mathbb{T}))\cap C_w([\tau,T];V^0),$$
where
$$\text{(2.8)}\quad \mathcal{G}_b^r(\tau,T;L^1(\mathbb{T})) := \{u : [\tau,T]\times\mathbb{T}\to\mathbb{R}^3 \text{ measurable}, \\ (e^{b|u|^r}-1)|u|^2 \in L^1(\tau,T;L^1(\mathbb{T}))\}.$$
Moreover, for any $t \in [\tau,T]$ and $\gamma > \max\{5/2-l, l\}$, it satisfies $(u(\tau), w(\tau)) = (u_\tau, w_\tau)$ and

$$\text{(2.9)}\quad \langle u(t),\varphi\rangle + \nu\int_\tau^t \langle \Lambda^l u(s), \Lambda^l\varphi\rangle\,ds + \int_\tau^t \langle w(s)\times u(s),\varphi\rangle\,ds \\ + a\int_\tau^t \langle (e^{b|u(s)|^r}-1)u(s),\varphi\rangle\,ds = \langle u_\tau,\varphi\rangle + \int_\tau^t \langle f(s),\varphi\rangle\,ds$$

for a.e. $t \in [\tau, T]$, $\varphi \in V^\gamma \cap L^\infty(\mathbb{T})$, and

$$(2.10) \qquad \langle w(t), \psi\rangle + \nu \int_\tau^t \langle \Lambda^l w(s), \Lambda^l \psi\rangle\, ds + \int_\tau^t \langle B(u(s), w(s)), \psi\rangle\, ds - \int_\tau^t \langle B(w(s), u(s)), \psi\rangle\, ds + a\int_\tau^t \langle (e^{b|w(s)|^r} - 1) w(s), \psi\rangle\, ds = \langle w_\tau, \psi\rangle + \int_\tau^t \langle \nabla \times f(s), \psi\rangle\, ds,$$

for a.e. $t \in [\tau, T]$, $\psi \in V^\gamma \cap L^\infty(\mathbb{T})$.

Remark 2.6. In the weak formulations above, we see that the trilinear terms are well-defined. Indeed, it is easily deduced that $\gamma > \max\{5/2-l, l\} \geq 5/4 > 1$, and it follows from Lemma 2.1 and Proposition 2.2 that

$$|\langle B(u,w), \psi\rangle| = |b(u,\psi,w)| \lesssim \|u\|_{V^l}\|\psi\|_{V^\gamma}\|w\|_{V^0},$$
$$|\langle B(w,u), \psi\rangle| = |b(w,\psi,u)| \lesssim \|w\|_{V^0}\|\psi\|_{V^\gamma}\|u\|_{V^l}.$$

Lemma 2.7. *Let $l > 0$ and $\gamma = \max\{5/2 - l, l\}$. Assume that $w \in V^0$, $u \in V^l$ and $\varphi \in V^\gamma$. Then*

$$|\langle w \times u, \varphi\rangle| \lesssim \|w\|_{V^0}\|u\|_{V^l}\|\varphi\|_{V^\gamma}.$$

Proof. By employing suitable inequalities, interpolation, and embeddings, the proof is done case by case:

Case 1: $l \geq 3/2$. We deduce that $\gamma = \max\{5/2 - l, l\} = l \geq 3/2$. Therefore,

$$|\langle w \times u, \varphi\rangle| \lesssim \|w\|_{L^2(\mathbb{T})}\|u\|_{L^3(\mathbb{T})}\|\varphi\|_{L^6(\mathbb{T})} \lesssim \|w\|_{V^0}\|u\|_{V^l}\|\varphi\|_{V^\gamma}.$$

Case 2: $5/4 \leq l < 3/2$. We deduce that $\gamma = \max\{5/2 - l, l\} = l \in [5/4, 3/2)$ and $\frac{6}{3-2l} \in (12, +\infty)$. Since $V^l \hookrightarrow L^{\frac{6}{3-2l}}(\mathbb{T})$, we get

$$|\langle w \times u, \varphi\rangle| \lesssim \|w\|_{L^2(\mathbb{T})}\|u\|_{L^3(\mathbb{T})}\|\varphi\|_{L^6(\mathbb{T})} \lesssim \|w\|_{L^2(\mathbb{T})}\|u\|_{L^{\frac{6}{3-2l}}(\mathbb{T})}\|\varphi\|_{L^{\frac{6}{3-2l}}(\mathbb{T})} \lesssim \|w\|_{V^0}\|u\|_{V^l}\|\varphi\|_{V^\gamma}.$$

Case 3: $1 < l < 5/4$. We deduce that $\gamma = \max\{5/2 - l, l\} = 5/2 - l \in (5/4, 3/2)$. Since $V^l \hookrightarrow L^{\frac{6}{3-2l}}(\mathbb{T}) \hookrightarrow L^{\frac{6}{5-2l}}(\mathbb{T})$ and $V^\gamma \hookrightarrow L^{\frac{6}{2l-2}}(\mathbb{T})$, we get

$$|\langle w \times u, \varphi\rangle| \lesssim \|w\|_{L^2(\mathbb{T})}\|u\|_{L^{\frac{6}{5-2l}}(\mathbb{T})}\|\varphi\|_{L^{\frac{6}{2l-2}}(\mathbb{T})} \lesssim \|w\|_{V^0}\|u\|_{V^l}\|\varphi\|_{V^\gamma}.$$

Case 4: $0 < l \leq 1$. We deduce that $\gamma = \max\{5/2 - l, l\} = 5/2 - l \in [3/2, 5/2)$. Since $V^l \hookrightarrow L^{\frac{6}{3-2l}}(\mathbb{T})$ and $V^\gamma \hookrightarrow L^{\frac{6}{2l}}(\mathbb{T})$, we get

$$|\langle w \times u, \varphi\rangle| \lesssim \|w\|_{L^2(\mathbb{T})}\|u\|_{L^{\frac{6}{3-2l}}(\mathbb{T})}\|\varphi\|_{L^{\frac{6}{2l}}(\mathbb{T})} \lesssim \|w\|_{V^0}\|u\|_{V^l}\|\varphi\|_{V^\gamma}. \qquad \blacksquare$$

LEMMA 2.8. *Let b be a positive real parameter and let $r \geq 1$. If (u, w) is a weak solution of (2.5) determined by Definition 2.5, then*

$$(e^{b|u|^r} - 1)u \in L^1(\tau, T; L^1(\mathbb{T})) \quad \text{and} \quad (e^{b|w|^r} - 1)w \in L^1(\tau, T; L^1(\mathbb{T})).$$

Proof. Indeed, we define

$$\begin{aligned} \Omega &:= [\tau, T] \times \mathbb{T}, \\ \Omega_1 &:= \{(t,x) \in [\tau, T] \times \mathbb{T} : 0 < |u(t,x)| < 1\}, \\ \Omega_2 &:= \{(t,x) \in [\tau, T] \times \mathbb{T} : |u(t,x)| \geq 1\}. \end{aligned}$$

We then have

$$\begin{aligned} \int_{\Omega} (e^{b|u(s)|^r} - 1)|u(s)|\,dx\,ds &= \int_{\Omega_1 \cup \Omega_2} (e^{b|u(s)|^r} - 1)|u(s)|\,dx\,ds \\ &= \int_{\Omega_1} (e^{b|u(s)|^r} - 1)|u(s)|\,dx\,ds + \int_{\Omega_2} (e^{b|u(s)|^r} - 1)|u(s)|\,dx\,ds \\ &= \int_{\Omega_1} \frac{e^{b|u(s)|^r} - 1}{|u(s)|}|u(s)|^2\,dx\,ds + \int_{\Omega_2} \frac{1}{|u(s)|}(e^{b|u(s)|^r} - 1)|u(s)|^2\,dx\,ds \\ &\lesssim M_{br} \int_{\Omega_1} |u(s)|^2\,dx\,ds + \int_{\Omega_2} (e^{b|u(s)|^r} - 1)|u(s)|^2\,dx\,ds \\ &\lesssim M_{br}(T - \tau)\|u\|_{L^\infty(\tau,T;V^0)} + \int_{\Omega} (e^{b|u(s)|^r} - 1)|u(s)|^2\,dx\,ds \\ &\lesssim M_{br}(T - \tau)\|u\|_{L^\infty(\tau,T;V^0)} + \|(e^{b|u|^r} - 1)|u|^2\|_{L^1(\tau,T;L^1(\mathbb{T}))}, \end{aligned} \tag{2.11}$$

where

$$M_{br} := \sup_{0<t\leq 1} \frac{e^{bt^r} - 1}{t} < \infty$$

for $r \geq 1$, $b > 0$. This implies that $(e^{b|u|^r} - 1)u \in L^1(\tau, T; L^1(\mathbb{T}))$. Repeating the above arguments, we prove the above result for w. ■

LEMMA 2.9. *Let b be a positive real parameter; let $r \geq 1$. If $(e^{b|u|^r} - 1)|u|^2 \in L^1(\tau, T; L^1(\mathbb{T}))$, then $u \in \bigcap_{k=1}^{\infty} L^{rk+2}(\tau, T; L^{rk+2}(\mathbb{T}))$.*

Proof. Set $s = b^{1/r}|u|$. The proof is completed if we can show that, for each positive integer k, there exists a finite positive constant C_k such that

$$s^{rk+2} \lesssim C_k(e^{s^r} - 1)s^2$$

for all $s \in [0, \infty)$. We will consider two cases.

In the case of $s \in [1, \infty)$, by induction, for each positive integer k, we have

$$\lim_{s \to +\infty} \frac{s^{rk}}{e^{s^r}} = 0.$$

Therefore, there exists a finite positive constant C_k such that $s^{rk} \lesssim C_k e^{s^r}$. On the other hand, if $s \geq 1$, then $\frac{e-1}{e} e^{s^r} s^2 \leq (e^{s^r} - 1)s^2$. Thus,

$$s^{rk+2} \lesssim C_k e^{s^r} s^2 \lesssim C_k (e^{s^r} - 1)s^2.$$

For $s \in [0,1)$, we have

$$e^{s^r} - 1 = \sum_{k=1}^{\infty} \frac{s^{rk}}{k!}. \tag{2.12}$$

Since $s \in [0,1)$, we infer from (2.12) that $s^r \lesssim e^{s^r} - 1$. Therefore, for each positive integer k,

$$s^{rk+2} \lesssim s^{r+2} \lesssim (e^{s^r} - 1)s^2. \quad \blacksquare$$

In particular, we also have the following important inequalities for exponential damping (see, e.g., [6, Lemma 2.3]).

Lemma 2.10. *Assume that $b, r > 0$. Then there exists a positive constant c_1 such that for all $x, y \in \mathbb{R}^3$,*

$$\big((e^{b|x|^r} - 1)x - (e^{b|y|^r} - 1)y\big) \cdot (x - y) \geq c_1 |x - y|^2 \big((e^{b|x|^r} - 1) + (e^{b|y|^r} - 1)\big).$$

Let us recall the following strong continuity result in time (see, e.g., [5, Lemma 6]).

Lemma 2.11. *If $u \in L^2(\tau, T; V^{s+h})$ and $\frac{du}{dt} \in L^2(\tau, T; V^{s-h})$ for $s \in \mathbb{R}$ and $h > 0$, then $u \in C([\tau, T]; V^s)$ and*

$$\frac{d}{dt}\|u(t)\|_{V^s}^2 = 2\left\langle \Lambda^{-h}\frac{du}{dt}(t), \Lambda^h u(t)\right\rangle_{V^s}.$$

Let us recall the following weak continuity result in time (see, e.g., [5, Lemma 7]).

Lemma 2.12. *Let X and Y be Banach spaces such that $Y \hookrightarrow X$ with a continuous injection. Then*

$$L^\infty(\tau, T; Y) \cap C_w([\tau, T]; X) = C_w([\tau, T]; Y).$$

Let us recall some types of functions that we will need when studying the long-time dynamical behavior of solutions via attractors (see, e.g., [10, 11, 33, 31]).

Definition 2.13. Let $\mathcal{B}$ be a reflexive separable Banach space.

(i) A function $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{B})$ is said to be *translation bounded* in $L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{B})$, written $\varphi \in L^2_b(\mathbb{R}; \mathcal{B})$, if

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} \|\varphi(s)\|_{\mathcal{B}}^2 \, ds < \infty.$$

(ii) A function $\varphi \in L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$ is said to be *normal* in $L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$, written $\varphi \in L^2_n(\mathbb{R};\mathcal{B})$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+\delta} \|\varphi(s)\|^2_{\mathcal{B}}\, ds \le \epsilon.$$

(iii) A function $\varphi \in L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$ is said to be *translation compact* in $L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$, written $\varphi \in L^2_c(\mathbb{R};\mathcal{B})$, if the closure of $\{\varphi(s+h) : h \in \mathbb{R}\}$ is compact in $L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$.

THEOREM 2.14 ([33, Section 4]).

(i) *$L^2_n(\mathbb{R};\mathcal{B})$ is a closed subspace of $L^2_b(\mathbb{R};\mathcal{B})$.*
(ii) *$L^2_c(\mathbb{R};\mathcal{B})$ is a closed subspace of $L^2_b(\mathbb{R};\mathcal{B})$.*
(iii) *$L^2_c(\mathbb{R};\mathcal{B})$ is a proper closed subset of $L^2_n(\mathbb{R};\mathcal{B})$.*

We also need the following Grönwall inequality (see, e.g., [38, 39, 41]): Assume that $\varphi : [\tau, T] \to \mathbb{R}$ is differentiable and satisfies the differential inequality

$$\frac{d}{dt}\varphi \le g(t)\varphi + h(t)$$

for g continuous and h locally integrable. Then

$$\varphi(t) \le \varphi(\tau)e^{G(t)} + \int_\tau^t e^{G(t)-G(s)}h(s)\, ds,$$

where $G(t) = \int_\tau^t g(r)\, dr$. In particular, if

$$\frac{d}{dt}\varphi \le \beta\varphi + \gamma,$$

where β and γ are constants, then

$$\varphi(t) \le \varphi(\tau)e^{\beta(t-\tau)} + \frac{\gamma}{\beta}(e^{\beta(t-\tau)} - 1).$$

3. Main results. The first purpose of this paper is to establish the global well-posedness of the weak solutions to (2.5). This is the content of the following theorem.

THEOREM 3.1. *Let ν, l, a, b and r be positive real parameters with $l, r \ge 1$. Then for $f \in L^2_{\text{loc}}(\mathbb{R}; V^{1-l})$, $u_\tau, w_\tau \in V^0$ and a fixed $T > \tau$, the system* (2.5) *has a global weak solution obeying Definition* 2.5 *with the initial condition (u_τ, w_τ). Furthermore, the global weak solution is unique and depends continuously on the initial data.*

The second purpose of this paper is to investigate the long-time dynamical behavior of solutions to (2.5) via attractors. The main results in this context are provided in the following theorem.

THEOREM 3.2. *Assuming that f_0 satisfies* (5.3) *below and that $l \geq 5/4$, let $\mathcal{E}_\Sigma$ be the evolutionary system of* (2.5) *with forces in Σ and let $\bar{\mathcal{E}}_\Sigma$ be its closure. Denote by $\mathcal{E}_{\bar{\Sigma}}$ the evolutionary system of* (2.5) *with forces in $\bar{\Sigma}$. Thus, $\mathcal{E}_\Sigma$ and $\mathcal{E}_{\bar{\Sigma}}$ are closed evolutionary systems with uniqueness, and then the following results hold:*

(1) *The three weak uniform global attractors $\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ for the evolutionary systems $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\bar{\Sigma}}$, respectively, exist. They are the maximal invariant and maximal quasi-invariant sets with respect to $\mathcal{E}_{\bar{\Sigma}}$ and satisfy*
$$\mathcal{A}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \mathcal{A}_w^{\bar{\Sigma}} = \{(u(0), w(0)) : (u, w) \in \mathcal{E}_{\bar{\Sigma}}((-\infty, \infty))\}.$$

(2) *The three weak trajectory attractors $\mathfrak{A}_w^\Sigma$, $\bar{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\bar{\Sigma}}$ for the evolutionary systems $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\bar{\Sigma}}$, respectively, exist and satisfy*
$$\mathfrak{A}_w^\Sigma = \bar{\mathfrak{A}}_w^\Sigma = \mathfrak{A}_w^{\bar{\Sigma}} = \Pi_+ \bigcup_{\sigma \in \bar{\Sigma}} \mathcal{E}_\sigma((-\infty, \infty)).$$
Hence, they satisfy the finite weak uniform tracking property for all three evolutionary systems and are weakly equicontinuous on $[0, \infty)$.

(3) *$\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ are sections of $\mathfrak{A}_w^\Sigma$, $\bar{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\bar{\Sigma}}$:*
$$\begin{aligned}\mathcal{A}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \mathcal{A}_w^{\bar{\Sigma}} &= \mathfrak{A}_w^\Sigma(t) = \bar{\mathfrak{A}}_w^\Sigma(t) = \mathfrak{A}_w^{\bar{\Sigma}}(t) \\ &= \{(u(t), w(t)) : (u, w) \in \mathfrak{A}_w^{\bar{\Sigma}}\}, \quad \forall t \geq 0.\end{aligned}$$

(4) *If f_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R}; V^{1-l})$, then $\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ are strongly compact strong uniform global attractors. Furthermore, $\mathfrak{A}_w^\Sigma$, $\bar{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\bar{\Sigma}}$ are strongly compact strong trajectory attractors. Moreover, $\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ satisfy the finite strong uniform tracking property and are strongly equicontinuous on* $[0, \infty)$.

Finally, we study the finite uniform tracking property of attractors by determining wavenumbers. Due to the length of the paper, we only consider (2.5). Let (u, w) and (v, h) be two weak solutions to (2.5). We define the determining wavenumber in the following way:

$$(3.1) \qquad \begin{aligned}\mathcal{N}_u^l(t) := \min\Big\{\lambda_q = 2^q : \lambda_p^{-\alpha+1+\delta}\lambda_q^{-\alpha-\delta}\|u_p\|_{L^\infty(\mathbb{T})} \lesssim c_0\nu,\ \forall p > q, \\ \text{and } \lambda_q^{-2\alpha}\sum_{j=0}^{q}\lambda_j\|u_j\|_{L^\infty(\mathbb{T})} \lesssim c_0\nu,\ q \in \mathbb{N}\Big\},\end{aligned}$$

where $0 < \delta < \alpha$ is a fixed (small) parameter, and c_0 is a dimensionless constant that depends only on l and λ_q, and $u_p = \Delta_p u$ is the pth Littlewood–Paley projection of u. We are now ready to state our main results.

THEOREM 3.3. *Let ν, l, a, b and r be positive real parameters with $l, r \geq 1$. Assume that (u, w) and (v, h) are two weak solutions to* (2.5) *on the* *weak global attractor* $\mathcal{A}$. *Let* $\mathcal{N}(t) := \max\{\mathcal{N}_u^l(t), \mathcal{N}_v^l(t), \mathcal{N}_w^l(t), \mathcal{N}_h^l(t)\}$ *and* $Q(t)$ *be such that* $\mathcal{N}(t) = \lambda_{Q(t)}$. *If*

$$u(t)_{\le Q(t)} = v(t)_{\le Q(t)}, \quad w(t)_{\le Q(t)} = h(t)_{\le Q(t)}, \quad \forall t < 0, \tag{3.2}$$

then

$$u(t) = v(t), \quad w(t) = h(t), \quad \forall t \le 0.$$

THEOREM 3.4. *Under the hypotheses of Theorem* 3.3,

$$\lim_{t\to\infty} \|u(t) - v(t)\|_{V^0} = 0,$$
$$\lim_{t\to\infty} \|w(t) - h(t)\|_{V^0} = 0.$$

4. Proof of Theorem 3.1. The existence of a weak solution of (2.5) is obtained via using the Galerkin approximation method. Therefore, we only outline the main points here.

(i) *Existence.* We consider sequences $u_n = \Pi_n u$ and $w_n = \Pi_n w$ solving the following system:

$$\begin{cases} \partial_t u_n + \nu\Lambda^{2l} u_n + \Pi_n P_\sigma\{w_n \times u_n\} + a\Pi_n P_\sigma\{(e^{b|u_n|^r} - 1)u_n\} = \Pi_n P_\sigma f, \\ \partial_t w_n + \nu\Lambda^{2l} w_n + B_n(u_n, w_n) - B_n(w_n, u_n) \\ \qquad\qquad + a\Pi_n P_\sigma\{(e^{b|w_n|^r} - 1)w_n\} = \Pi_n P_\sigma\{\nabla\times f\}, \end{cases} \tag{4.1}$$

with the initial condition

$$u_n(\tau) = \Pi_n u_\tau, \qquad w_n(\tau) = \Pi_n w_\tau.$$

Obviously, $u_n(\tau)$ and $w_n(\tau)$ strongly converge in V^0 to u_τ and w_τ, respectively.

We take the L^2 inner product of equation $(4.1)_1$ with u_n, and keeping in mind that $\langle w_n \times u_n, u_n\rangle = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{V^0}^2 + \nu\|u_n(t)\|_{V^l}^2 + a\|(e^{b|u_n|^r} - 1)|u_n|^2\|_{L^1(\mathbb{T})} = \langle f(t), u_n(t)\rangle$$
$$\lesssim \|f(t)\|_{V^{-l}}\|u_n(t)\|_{V^l} \lesssim \|f(t)\|_{V^{1-l}}\|u_n(t)\|_{V^l} \lesssim \frac{\nu}{2}\|u_n(t)\|_{V^l}^2 + \frac{1}{2\nu}\|f(t)\|_{V^{1-l}}^2,$$

where we have used (2.1) and the Cauchy–Schwarz inequality. Therefore,

$$\frac{d}{dt}\|u_n(t)\|_{V^0}^2 + \nu\|u_n(t)\|_{V^l}^2 + 2a\|(e^{b|u_n(t)|^r} - 1)|u_n(t)|^2\|_{L^1(\mathbb{T})} \lesssim \frac{1}{\nu}\|f(t)\|_{V^{1-l}}^2. \tag{4.2}$$

Note that $\|u_\tau\|_{V^0}^2$ and $\int_\tau^t \|f(s)\|_{V^{1-l}}^2\,ds$ are bounded. Integrating (4.2) in time from τ to t, we obtain

$$\|u_n(t)\|_{V^0}^2 + \nu \int_\tau^t \|u_n(s)\|_{V^l}^2\, ds + 2a \int_\tau^t \|(e^{b|u_n(s)|^r} - 1)|u_n(s)|^2\|_{L^1(\mathbb{T})}\, ds \lesssim \|u_\tau\|_{V^0}^2 + \frac{1}{\nu} \int_\tau^T \|f(s)\|_{V^{1-l}}^2\, ds =: M_T. \tag{4.3}$$

It follows from (4.3) and Lemma 2.9 that the sequence $\{u_n\}$ is uniformly bounded in

$$L^\infty(\tau, T; V^0) \cap L^2(\tau, T; V^l) \cap \bigcap_{k=1}^\infty L^{rk+2}(\tau, T; L^{rk+2}(\mathbb{T})),$$

and

$$\|u_n\|_{L^\infty(\tau,T;L^\infty(\mathbb{T}))} < \infty. \tag{4.4}$$

We take the L^2 inner product of equation $(4.1)_2$ with w_n, and keeping in mind Lemma 2.1 and using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w_n(t)\|_{V^0}^2 + \nu\|w_n(t)\|_{V^l}^2 + a\|(e^{b|w_n(t)|^r} - 1)|w_n(t)|^2\|_{L^1(\mathbb{T})} = b(w_n(t), u_n(t), w_n(t)) + \langle \nabla \times f(t), w_n(t)\rangle. \tag{4.5}$$

We have

$$\begin{aligned} |b(w_n(t), u_n(t), w_n(t))| &= |b(w_n(t), w_n(t), u_n(t))| \\ &\lesssim \|w_n(t)\|_{L^2(\mathbb{T})}\|\nabla w_n(t)\|_{L^2(\mathbb{T})}\|u_n(t)\|_{L^\infty(\mathbb{T})} \\ &\lesssim \|w_n(t)\|_{V^0}\|w_n(t)\|_{V^l}\|u_n(t)\|_{L^\infty(\mathbb{T})} \\ &\lesssim \frac{\nu}{4}\|w_n(t)\|_{V^l}^2 + \frac{1}{\nu}\|u_n(t)\|_{L^\infty(\mathbb{T})}^2\|w_n(t)\|_{V^0}^2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \langle \nabla \times f(t), w_n(t)\rangle &\lesssim \|\nabla \times f(t)\|_{V^{-l}}\|w_n(t)\|_{V^l} \\ &\lesssim \|f(t)\|_{V^{1-l}}\|w_n(t)\|_{V^l} \\ &\lesssim \frac{\nu}{4}\|w_n(t)\|_{V^l}^2 + \frac{1}{\nu}\|f(t)\|_{V^{1-l}}^2. \end{aligned} \tag{4.7}$$

We deduce from (4.5)–(4.7) that

$$\frac{d}{dt}\|w_n(t)\|_{V^0}^2 + \nu\|w_n(t)\|_{V^l}^2 + 2a\|(e^{b|w_n(t)|^r} - 1)|w_n(t)|^2\|_{L^1(\mathbb{T})} \lesssim \frac{2}{\nu}\|f(t)\|_{V^{1-l}}^2 + \frac{2}{\nu}\|u_n(t)\|_{L^\infty(\mathbb{T})}^2\|w_n(t)\|_{V^0}^2. \tag{4.8}$$

Eliminating $\nu\|w_n(t)\|_{V^l}^2$ and $2a\|(e^{b|w_n(t)|^r} - 1)|w_n(t)|^2\|_{L^1(\mathbb{T})}$ in (4.8) yields

$$\frac{d}{dt}\|w_n(t)\|_{V^0}^2 \lesssim \frac{2}{\nu}\|u_n(t)\|_{L^\infty(\mathbb{T})}^2\|w_n(t)\|_{V^0}^2 + \frac{2}{\nu}\|f(t)\|_{V^{1-l}}^2. \tag{4.9}$$

We have

$$G(t) := \frac{2}{\nu}\int_{\tau}^{t} \|u_n(s)\|^2_{L^\infty(\mathbb{T})}\, ds \lesssim \frac{2}{\nu}\|u_n\|^2_{L^\infty(\tau,T;L^\infty(\mathbb{T}))}(T-\tau), \tag{4.10}$$

$$G(t) - G(s) := \frac{2}{\nu}\int_{s}^{t} \|u_n(r)\|^2_{L^\infty(\mathbb{T})}\, dr \lesssim \frac{2}{\nu}\|u_n\|^2_{L^\infty(\tau,T;L^\infty(\mathbb{T}))}(T-\tau). \tag{4.11}$$

Using (4.3), (4.4), (4.10), (4.11) and the Grönwall inequality, we deduce from (4.9) that

$$\begin{aligned}\|w_n(t)\|^2_{V^0} &\lesssim \|w_\tau\|^2_{V^0} e^{\frac{2}{\nu}\|u_n\|^2_{L^\infty(\tau,T;L^\infty(\mathbb{T}))}(T-\tau)} \\ &\quad + \frac{2}{\nu} e^{\frac{2}{\nu}\|u_n\|^2_{L^\infty(\tau,T;L^\infty(\mathbb{T}))}(T-\tau)} \int_{\tau}^{t} \|f(s)\|^2_{V^{1-l}}\, ds =: N_T.\end{aligned}$$

Integrating (4.8) in time from τ to t, we obtain

$$\begin{aligned}&\|w_n(t)\|^2_{V^0} + \nu\int_{\tau}^{t}\|w_n(s)\|^2_{V^l}\, ds + 2a\int_{\tau}^{t}\|(e^{b|w_n(s)|^r}-1)|w_n(s)|^2\|_{L^1(\mathbb{T})}\, ds \\ &\quad\lesssim \|w_\tau\|^2_{V^0} + \frac{2}{\nu}\int_{\tau}^{t}\|f(s)\|^2_{V^{1-l}}\, ds + \frac{2}{\nu}\int_{\tau}^{t}\|u_n(s)\|^2_{L^\infty(\mathbb{T})}\|w_n(s)\|^2_{V^0}\, ds \\ &\quad\lesssim \|w_\tau\|^2_{V^0} + \frac{2}{\nu}\int_{\tau}^{T}\|f(s)\|^2_{V^0}\, ds + \frac{2N_T}{\nu}\int_{\tau}^{T}\|u_n(s)\|^2_{L^\infty(\mathbb{T})}\, ds \\ &\quad\lesssim \|w_\tau\|^2_{V^0} + 2M_T + \frac{2N_T\|u_n\|^2_{L^\infty(\tau,T;L^\infty(\mathbb{T}))}}{\nu}(T-\tau) =: P_T.\end{aligned} \tag{4.12}$$

We infer from (4.12) and Lemma 2.9 that the sequence $\{w_n\}$ is uniformly bounded in

$$L^\infty(\tau,T;V^0) \cap L^2(\tau,T;V^l) \cap \bigcap_{k=1}^{\infty} L^{rk+2}(\tau,T;L^{rk+2}(\mathbb{T})).$$

We deduce from (4.1) that

$$\begin{cases}\partial_t u_n = -\nu\Lambda^{2l}u_n - \Pi_n P_\sigma\{w_n\times u_n\} - a\Pi_n P_\sigma\{(e^{b|u_n|^r}-1)u_n\} + \Pi_n P_\sigma f, \\ \partial_t w_n = -\nu\Lambda^{2l}w_n - B_n(u_n,w_n) + B_n(w_n,u_n) \\ \qquad\quad - a\Pi_n P_\sigma\{(e^{b|w_n|^r}-1)w_n\} + \Pi_n P_\sigma\{\nabla\times f\}.\end{cases}$$

Setting $\gamma_0 := \max\{3, 2l, \gamma\}$ we see that $\gamma \le \gamma_0$. Since $L^1(\mathbb{T}) \hookrightarrow V^{-\gamma_0}$, we deduce that

$$\begin{aligned}(4.13)\qquad &\int_\tau^t \|(e^{b|u_n(s)|^r}-1)|u_n(s)|\|_{V^{-\gamma_0}}\,ds\\ &\lesssim \int_\tau^t \|(e^{b|u_n(s)|^r}-1)|u_n(s)|\|_{L^1(\mathbb{T})}\,ds\\ &\le M_{br}(T-\tau)\|u_n\|_{L^\infty(\tau,T;V^0)} + \|(e^{b|u_n|^r}-1)|u_n|^2\|_{L^1(\tau,T;L^1(\mathbb{T}))},\end{aligned}$$

where we have used formula (2.11). We infer from (4.13) that $(e^{b|u_n|^r}-1)u_n \in L^1(\tau,T;V^{-\gamma_0})$. Hence, $(e^{b|w_n|^r}-1)w_n \in L^1(\tau,T;V^{-\gamma_0})$. We infer from the construction that $\Lambda^{2l}u_n \in L^2(\tau,T;V^{-l}) \subset L^1(\tau,T;V^{-\gamma_0})$ and $\Lambda^{2l}w_n \in L^2(\tau,T;V^{-l}) \subset L^1(\tau,T;V^{-\gamma_0})$. We find from Lemma 2.7 that $\Pi_n P_\sigma\{w_n\times u_n\} \in L^1(\tau,T;V^{-\gamma_0})$. It then follows from Remark 2.6 that $B_n(u_n,w_n)$ and $B_n(w_n,u_n)$ belong to $L^1(\tau,T;V^{-\gamma_0})$. Since $f\in L^2_{\mathrm{loc}}(\mathbb{R};V^0)$, we infer that $\Pi_n P_\sigma f$ and $\Pi_n P_\sigma\{\nabla\times f\}$ also belong to $L^1(\tau,T;V^{-\gamma_0})$. Therefore, $\partial_t u_n$ and $\partial_t w_n$ are bounded uniformly in $L^1(\tau,T;V^{-\gamma_0})$. Since

$$V^l \cap \bigcap_{k=1}^\infty L^{rk+2}(\mathbb{T}) \hookrightarrow\hookrightarrow V^0 \hookrightarrow V^{-\gamma_0},$$

and

$$V^l \cap \bigcap_{k=1}^\infty L^{rk+2}(\mathbb{T}) \hookrightarrow\hookrightarrow V^{\bar l} \hookrightarrow V^{-\gamma_0},$$

for some $\bar l \in (0,l)$ such that $\bar l + \gamma \ge 5/2$. We deduce from the Aubin–Lions Lemma (see [40]) that the sequences $\{u_n\}$ and $\{w_n\}$ are compact in $L^2(\tau,T;V^0)$ and so we can extract subsequences, still denoted by u_n and w_n, respectively, such that

(4.14) $u_n \rightharpoonup u$ weakly in $L^2(\tau,T;V^l)$,

(4.15) $u_n \rightharpoonup u$ weakly in $L^{rk+2}(\tau,T;L^{rk+2}(\mathbb{T}))$ for any positive integer k,

(4.16) $w_n \rightharpoonup w$ weakly in $L^2(\tau,T;V^l)$,

(4.17) $w_n \rightharpoonup w$ weakly in $L^{rk+2}(\tau,T;L^{rk+2}(\mathbb{T}))$ for any positive integer k,

(4.18) $u_n \rightharpoonup^* u$ weakly star in $L^\infty(\tau,T;V^0)$,

(4.19) $w_n \rightharpoonup^* w$ weakly star in $L^\infty(\tau,T;V^0)$,

(4.20) $u_n \to u$ strongly in $L^2(\tau,T;V^0)$,

(4.21) $u_n \to u$ strongly in $L^2(\tau,T;V^{\bar l})$

(4.22) $w_n \to w$ strongly in $L^2(\tau,T;V^0)$, (4.23) $\qquad w_n \to w$ strongly in $L^2(\tau, T; V^{\bar{l}})$,

(4.24) $\qquad P_\sigma\{w_n \times u_n\} \rightharpoonup P_\sigma\{w \times u\}$ weakly in $L^2(\tau, T; V^{-\gamma})$,

(4.25) $\qquad B(u_n, w_n) \rightharpoonup B(u, w)$ weakly in $L^2(\tau, T; V^{-\gamma})$,

(4.26) $\qquad B(w_n, u_n) \rightharpoonup B(w, u)$ weakly in $L^2(\tau, T; V^{-\gamma})$.

Using all convergences (4.14)–(4.26), it is a classical result to pass to the limit in the variational formulations (2.9) and (2.10), and prove that (u, w) is the solution of (2.5) and inherits all the regularity from (u_n, w_n), i.e.,

$$u \in L^\infty(\tau, T; V^0) \cap L^2(\tau, T; V^l) \cap \mathcal{G}_b^r(\tau, T; L^1(\mathbb{T})),$$
$$w \in L^\infty(\tau, T; V^0) \cap L^2(\tau, T; V^l) \cap \mathcal{G}_b^r(\tau, T; L^1(\mathbb{T})),$$

where $\mathcal{G}_b^r(\tau, T; L^1(\mathbb{T}))$ is as in (2.8). In addition, we integrate in time and we obtain

$$u(t) = u_\tau + \int_\tau^t [-\nu \Lambda^{2l} u(s) - P_\sigma\{w(s) \times u(s)\} - a P_\sigma\{e^{b|u(s)|^r} - 1)u(s)\} + P_\sigma f(s)]\, ds,$$

and

$$w(t) = w_\tau + \int_\tau^t [-\nu \Lambda^{2l} w(s) - B(u(s), w(s)) + B(w(s), u(s))]\, ds + \int_\tau^t [-a P_\sigma\{e^{b|w(s)|^r} - 1)w(s)\} + P_\sigma\{\nabla \times f(s)\}]\, ds.$$

This implies that $u \in C([\tau, T]; V^{-\gamma_0})$ and $w \in C([\tau, T]; V^{-\gamma_0})$. In addition, since $u \in L^\infty(\tau, T; V^0)$ and $w \in L^\infty(\tau, T; V^0)$, we deduce from Lemma 2.12 that (2.6) and (2.7) are satisfied.

(ii) *Continuous dependence on the initial data.* We will consider the continuous dependence of weak solutions on the initial data to (2.5), in particular their uniqueness. Let (u_1, w_1) and (u_2, w_2) be solutions to (2.5) with the same initial data (u_τ, w_τ) and the same forcing f on their common time interval of existence (τ, T). We define $U = u_1 - u_2$ and $W = w_1 - w_2$, and thus (U, W) satisfies

$$\begin{cases} \partial_t U + \nu \Lambda^{2l} U + P_\sigma\{W \times u_1 + w_2 \times U\} \\ \qquad\qquad + a P_\sigma\{(e^{b|u_1|^r} - 1)u_1 - (e^{b|u_2|^r} - 1)u_2\} = 0, \\ \partial_t W + \nu \Lambda^{2l} W + B(U, w_1) + B(u_2, W) - B(W, u_1) - B(w_2, U) \\ \qquad\qquad + a P_\sigma\{(e^{b|w_1|^r} - 1)w_1 - (e^{b|w_2|^r} - 1)w_2\} = 0, \\ \nabla \cdot U = \nabla \cdot W = 0, \quad U(\cdot, \tau) = W(\cdot, \tau) = 0. \end{cases} \tag{4.27}$$

We take the L^2 inner product of equation $(4.27)_1$ with U, and keeping in mind that $\langle w_2 \times U, U\rangle = 0$, we have

$$\begin{aligned}(4.28)\quad &\frac{1}{2}\frac{d}{dt}\|U(t)\|_{V^0}^2 + \nu\|U(t)\|_{V^l}^2 + \langle W(t)\times u_1(t), U(t)\rangle \\ &+ a\langle P_\sigma\{(e^{b|u_1(t)|^r}-1)u_1(t) - (e^{b|u_2(t)|^r}-1)u_2(t)\}, u_1(t)-u_2(t)\rangle = 0.\end{aligned}$$

Note that $l \geq 1$. Using embeddings and Lemmas 2.9 and 2.10, we have

$$(4.29)\quad \langle P_\sigma\{(e^{b|u_1(t)|^r}-1)u_1(t) - (e^{b|u_2(t)|^r}-1)u_2(t)\}, u_1(t)-u_2(t)\rangle \geq 0,$$

and we can take $rk \in [0,\infty]$ so that

$$\begin{aligned}(4.30)\quad |\langle W(t)\times u_1(t), U(t)\rangle| &\lesssim \|W(t)\|_{V^0}\|u_1(t)\|_{L^{rk+2}(\mathbb{T})}\|U(t)\|_{V^l} \\ &\lesssim \frac{\nu}{4}\|U(t)\|_{V^l}^2 + \frac{1}{\nu}\|u_1(t)\|_{L^{rk+2}(\mathbb{T})}^2\|W(t)\|_{V^0}^2.\end{aligned}$$

It follows from (4.28)–(4.30) that

$$(4.31)\quad \frac{d}{dt}\|U(t)\|_{V^0}^2 + \frac{3\nu}{2}\|U(t)\|_{V^l}^2 \lesssim \frac{2}{\nu}\|u_1(t)\|_{L^{rk+2}(\mathbb{T})}^2\|W(t)\|_{V^0}^2.$$

We take the L^2 inner product of equation $(4.27)_2$ with W, and keeping in mind Lemma 2.1, we obtain

$$\begin{aligned}(4.32)\quad &\frac{1}{2}\frac{d}{dt}\|W(t)\|_{V^0}^2 + \nu\|W(t)\|_{V^l}^2 \\ &+ a\langle P_\sigma\{(e^{b|w_1(t)|^r}-1)w_1(t) - (e^{b|w_2(t)|^r}-1)w_2(t)\}, w_1(t)-w_2(t)\rangle \\ &= -b(U(t), w_1(t), W(t)) + b(W(t), u_1(t), W(t)) + b(w_2(t), U(t), W(t)).\end{aligned}$$

Using embeddings and Lemmas 2.9 and 2.10 again, we have

$$(4.33)\quad \langle P_\sigma\{(e^{b|w_1(t)|^r}-1)w_1(t) - (e^{b|w_2(t)|^r}-1)w_2(t)\}, w_1(t)-w_2(t)\rangle \geq 0,$$

and we can take $rk \in [0,\infty]$ so that

$$\begin{aligned}(4.34)\quad |b(U(t), w_1(t), W(t))| &= |b(U(t), W(t), w_1(t))| \\ &\lesssim \|U(t)\|_{V^0}\|W(t)\|_{V^l}\|w_1(t)\|_{L^{rk+2}(\mathbb{T})} \\ &\lesssim \frac{\nu}{4}\|W(t)\|_{V^l}^2 + \frac{1}{\nu}\|w_1(t)\|_{L^{rk+2}(\mathbb{T})}^2\|U(t)\|_{V^0}^2,\end{aligned}$$

$$\begin{aligned}(4.35)\quad |b(W(t), u_1(t), W(t))| &= |b(W(t), W(t), u_1(t))| \\ &\lesssim \|W(t)\|_{V^0}\|W(t)\|_{V^l}\|u_1(t)\|_{L^{rk+2}(\mathbb{T})} \\ &\lesssim \frac{\nu}{4}\|W(t)\|_{V^l}^2 + \frac{1}{\nu}\|u_1(t)\|_{L^{rk+2}(\mathbb{T})}^2\|W(t)\|_{V^0}^2,\end{aligned}$$

$$\begin{aligned}(4.36)\quad |b(w_2(t), U(t), W(t))| &\lesssim \|w_2(t)\|_{L^{rk+2}(\mathbb{T})}\|U(t)\|_{V^l}\|W(t)\|_{V^0} \\ &\lesssim \frac{\nu}{4}\|U(t)\|_{V^l}^2 + \frac{1}{\nu}\|w_2(t)\|_{L^{rk+2}(\mathbb{T})}^2\|W(t)\|_{V^0}^2.\end{aligned}$$

It follows from (4.32)–(4.36) that

$$\frac{d}{dt}\|W(t)\|_{V^0}^2+\nu\|W(t)\|_{V^l}^2 \lesssim \frac{\nu}{2}\|U(t)\|_{V^l}^2 + \frac{2}{\nu}(\|w_1(t)\|^2_{L^{rk+2}(\mathbb{T})}+\|u_1(t)\|^2_{L^{rk+2}(\mathbb{T})}+\|w_2(t)\|^2_{L^{rk+2}(\mathbb{T})})\|W(t)\|^2_{V^0}. \tag{4.37}$$

We deduce from (4.31) and (4.37) that

$$\begin{aligned}\frac{d}{dt}(\|U(t)\|_{V^0}^2+\|W(t)\|_{V^0}^2)+\nu(\|U(t)\|_{V^l}^2+\|W(t)\|_{V^l}^2)\\ \lesssim \frac{3}{\nu}(\|w_1(t)\|^2_{L^{rk+2}(\mathbb{T})}+\|u_1(t)\|^2_{L^{rk+2}(\mathbb{T})}+\|w_2(t)\|^2_{L^{rk+2}(\mathbb{T})})\\ \times(\|U(t)\|_{V^0}^2+\|W(t)\|_{V^0}^2).\end{aligned}$$

Applying the Grönwall inequality implies that $\|U(t)\|_{V^0}^2+\|W(t)\|_{V^0}^2=0$ since $U(\tau)=W(\tau)=0$. Therefore, the proof of Theorem 3.1 is now finished.

5. Proof of Theorem 3.2. In this section, we use the theory of evolutionary systems to study the long-time dynamical behavior of our systems. Following the ideas in [32, Section 4], [12, Section 8], [15, Sections 5-6], and [13], we first define the *strong* and *weak distances* as follows for $u_1,u_2,w_1,w_2\in V^0$:

$$d_s((u_1,w_1);(u_2,w_2)) := \|u_1-u_2\|_{V^0}+\|w_1-w_2\|_{V^0},$$
$$d_w((u_1,w_1);(u_2,w_2)) := \sum_{k\in J}\frac{1}{2^{|k|}}\left(\frac{|u_{1k}-u_{2k}|}{1+|u_{1k}-u_{2k}|}+\frac{|w_{1k}-w_{2k}|}{1+|w_{1k}-w_{2k}|}\right),$$

where u_{ik} and w_{ik} are the Fourier coefficients of u_i and w_i, $i=1,2$, respectively. Note that the weak metric d_w induces the weak topology in any ball in $V^0\times V^0$.

We now fix an external force $f_0\in L^2_{\text{loc}}(\mathbb{R};V^{1-l})$ that is translation bounded in $L^2_{\text{loc}}(\mathbb{R};V^{1-l})$, i.e.,

$$\|f_0\|_b^2 := \|f_0\|^2_{L^2_b(\mathbb{R};V^{1-l})} := \sup_{t\in\mathbb{R}}\int_t^{t+1}\|f_0(s)\|^2_{V^{1-l}}\,ds<\infty.$$

Let $L^{2,w}_{\text{loc}}(\mathbb{R};V^{1-l})$ be the space $L^2_{\text{loc}}(\mathbb{R};V^{1-l})$ endowed with the local weak convergence topology. Then f_0 is translation compact in $L^{2,w}_{\text{loc}}(\mathbb{R};V^{1-l})$, i.e., the translation family of f_0,

$$\Sigma := \{f_0(\cdot+h): h\in\mathbb{R}\},$$

is precompact in $L^{2,w}_{\text{loc}}(\mathbb{R};V^{1-l})$ (see, e.g., [11]). Moreover, for all $f\in\Sigma$, we get

$$\|f\|_b^2\le\|f_0\|_b^2, \tag{5.1}$$

and, for any positive constant θ and $t \geq \tau$, we also have

$$\begin{aligned}(5.2)\qquad &\int_{\tau}^{t} \|f(s)\|_{V^{1-l}}^2 e^{\theta s}\, ds \\ &\leq \int_{t-1}^{t} \|f(s)\|_{V^{1-l}}^2 e^{\theta s}\, ds + \int_{t-2}^{t-1} \|f(s)\|_{V^{1-l}}^2 e^{\theta s}\, ds + \cdots \\ &\leq \|f\|_b^2 (1 + e^{-\theta} + \cdots) e^{\theta t} \leq \frac{e^{\theta}}{e^{\theta}-1} \|f\|_b^2 e^{\theta t} \leq \frac{e^{\theta}}{e^{\theta}-1} \|f_0\|_b^2 e^{\theta t}.\end{aligned}$$

In this section, we also assume that f_0 satisfies the stronger condition

$$(5.3)\qquad \sup_{t\in\mathbb{R},\, h\in\mathbb{R}_+} \int_{t}^{t+h} \|f_0(s)\|_{V^{1-l}}^2\, ds \leq L_0 < +\infty$$

for some positive constant L_0. Obviously, if f_0 satisfies (5.3), then f_0 is translation bounded in $L^2_{\mathrm{loc}}(\mathbb{R}; V^{1-l})$, i.e., $h = 1$. Moreover, for all $f \in \Sigma$, we get

$$(5.4)\qquad \sup_{t\in\mathbb{R},\, h\in\mathbb{R}_+} \int_{t}^{t+h} \|f(s)\|_{V^{1-l}}^2\, ds \leq L_0 < +\infty.$$

If f_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R}; V^{1-l})$, then there exists $\delta > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_{t}^{t+\delta} \|\varphi(s)\|_{V^{1-l}}^2\, ds \leq \epsilon \quad \text{for any } \epsilon > 0.$$

It follows from Theorem 2.14 that $L^2_n(\mathbb{R}; V^{1-l})$ is a closed subspace of $L^2_b(\mathbb{R}; V^{1-l})$.

Let $\bar{\Sigma}$ be the closure of the translation family Σ of f_0 in $L^{2,w}_{\mathrm{loc}}(\mathbb{R}; V^{1-l})$. In this case, $\bar{\Sigma}$ is compact in $L^{2,w}_{\mathrm{loc}}(\mathbb{R}; V^{1-l})$. Moreover, $\bar{\Sigma}$ is metrizable in the weak topology, and it is compact with respect to this metric. Hence, it is weakly sequentially compact (see, e.g., [11]).

We take the L^2 inner product of equation $(2.5)_1$ with u, and keeping in mind that $\langle w \times u, u\rangle = 0$, we obtain

$$(5.5)\qquad \frac{1}{2}\frac{d}{dt}\|u(t)\|_{V^0}^2 + \nu\|u(t)\|_{V^l}^2 + a\|(e^{b|u(t)|^r} - 1)|u(t)|^2\|_{L^1(\mathbb{T})} = \langle f(t), u(t)\rangle.$$

By using the Cauchy–Schwarz inequality, we deduce from (5.5) that

$$(5.6)\qquad \frac{d}{dt}\|u(t)\|_{V^0}^2 + \nu\|u(t)\|_{V^l}^2 + 2a\|(e^{b|u(t)|^r} - 1)|u(t)|^2\|_{L^1(\mathbb{T})} \lesssim \frac{1}{\nu}\|f(t)\|_{V^{1-l}}^2.$$

By using (2.1) we find that

$$\frac{d}{dt}\|u(t)\|_{V^0}^2 + \nu\|u(t)\|_{V^0}^2 \lesssim \frac{1}{\nu}\|f(t)\|_{V^{1-l}}^2.$$

Therefore,

$$\frac{d}{dt}\{\|u(t)\|_{V^0}^2 e^{\nu t}\} \lesssim \frac{e^{\nu t}}{\nu}\|f(t)\|_{V^{1-l}}^2.$$

Integrating in time from τ to t, we deduce that

$$\|u(t)\|_{V^0}^2 e^{\nu t} - \|u(\tau)\|_{V^0}^2 e^{\nu\tau} \lesssim \frac{1}{\nu}\int_\tau^t e^{\nu s}\|f(s)\|_{V^{1-l}}^2\, ds.$$

Applying (5.2) and (5.3) implies that

$$\|u(t)\|_{V^0}^2 \lesssim \|u(\tau)\|_{V^0}^2 e^{-\nu(t-\tau)} + \frac{L_0 e^\nu}{\nu(e^\nu - 1)}.$$

This in turn implies that for any bounded initial condition $u(\tau)$, there exist a positive constant $R_1 := R_1\big(\frac{L_0 e^\nu}{\nu(e^\nu-1)}\big)$ and a time $T_1 \geq 0$ independent of the initial time τ such that

$$\|u(t)\|_{V^0}^2 \lesssim R_1, \text{ for all } t \geq \tau_1 = \tau + T_1. \tag{5.7}$$

Integrating over any interval $[s,t] \subset [\tau_1, \infty)$, we deduce from (5.4), (5.6) and (5.7) that

$$\begin{aligned}\|u(t)\|_{V^0}^2 + \nu\int_s^t \|u(\xi)\|_{V^l}^2\, d\xi + 2a\int_s^t \|(e^{b|u(\xi)|^r} - 1)|u(\xi)|^2\|_{L^1(\mathbb{T})}\, d\xi \\ \lesssim \|u(s)\|_{V^0}^2 + \frac{1}{\nu}\int_s^t \|f(\xi)\|_{V^{1-l}}^2\, d\xi \lesssim R_1 + \frac{L_0}{\nu}.\end{aligned}$$

This implies that, for any interval $[s,t] \subset [\tau_1, \infty)$, we have

$$\int_s^t \|u(\xi)\|_{V^l}^2\, d\xi \lesssim \frac{\nu R_1 + L_0}{\nu^2}. \tag{5.8}$$

We take the L^2 inner product of equation $(2.5)_2$ with w, and keeping in mind Lemma 2.1 and using integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_{V^0}^2 + \nu\|w(t)\|_{V^l}^2 + a\|(e^{b|w(t)|^r} - 1)|w(t)|^2\|_{L^1(\mathbb{T})} \\ = b(w(t), u(t), w(t)) - \langle f(t), \nabla \times w(t)\rangle.\end{aligned} \tag{5.9}$$

We use Proposition 2.2 and $l \geq 5/4$ to obtain

$$|b(w(t), u(t), w(t))| = |b(w(t), w(t), u(t))| \lesssim \|w(t)\|_{V^0}\|w(t)\|_{V^l}\|u(t)\|_{V^l}. \tag{5.10}$$

By using the Cauchy–Schwarz inequality again, we deduce from (5.9) and

(5.10) that

$$\frac{d}{dt}\|w(t)\|_{V^0}^2+\nu\|w(t)\|_{V^l}^2+2a\|(e^{b|w(t)|^r}-1)|w(t)|^2\|_{L^1(\mathbb{T})} \lesssim \frac{2}{\nu}\|f(t)\|_{V^{1-l}}^2+\frac{2}{\nu}\|u(t)\|_{V^l}^2\|w(t)\|_{V^0}^2. \tag{5.11}$$

Summing up (5.6) and (5.11), we infer that

$$\frac{d}{dt}\{\|u(t)\|_{V^0}^2+\|w(t)\|_{V^0}^2\}+\nu\{\|u(t)\|_{V^l}^2+\|w(t)\|_{V^l}^2\} \lesssim \frac{3}{\nu}\|f(t)\|_{V^{1-l}}^2+\frac{2}{\nu}\|u(t)\|_{V^l}^2\{\|u(t)\|_{V^0}^2+\|w(t)\|_{V^0}^2\}. \tag{5.12}$$

By using (2.1), we deduce from (5.12) that

$$\frac{d}{dt}\{\|u(t)\|_{V^0}^2+\|w(t)\|_{V^0}^2\} \lesssim \left\{\frac{2}{\nu}\|u(t)\|_{V^l}^2-\nu\right\}\{\|u(t)\|_{V^0}^2+\|w(t)\|_{V^0}^2\}+\frac{3}{\nu}\|f(t)\|_{V^{1-l}}^2. \tag{5.13}$$

Applying the Grönwall inequality implies that

$$\|u(t)\|_{V^0}^2+\|w(t)\|_{V^0}^2 \lesssim \{\|u(\tau_1)\|_{V^0}^2+\|w(\tau_1)\|_{V^0}^2\}e^{G(t)}+\frac{3}{\nu}\int_{\tau_1}^{t} e^{G(t)-G(r)}\|f(r)\|_{V^{1-l}}^2\,dr, \tag{5.14}$$

where $[\tau_1,t]\subset[\tau_1,\infty)$ and

$$G(t)=\frac{2}{\nu}\int_{\tau_1}^{t}\|u(r)\|_{V^l}^2\,dr-\nu(t-\tau_1).$$

We use (5.8) to obtain the following estimates:

$$e^{G(t)}\lesssim e^{-\nu t+\nu\tau_1+\frac{2(\nu R_1+L_0)}{\nu^3}},$$

and

$$e^{G(t)-G(r)}=e^{\frac{2}{\nu}\int_r^t\|u(s)\|_{V^l}^2\,ds-\nu(t-r)}\lesssim e^{-\nu t+\nu r+\frac{2(\nu R_1+L_0)}{\nu^3}}.$$

Therefore, we deduce from (5.14) that

$$\|u(t)\|_{V^0}^2+\|w(t)\|_{V^0}^2\lesssim\{\|u(\tau_1)\|_{V^0}^2+\|w(\tau_1)\|_{V^0}^2\}e^{-\nu t+\nu\tau_1+\frac{2(\nu R_1+L_0)}{\nu^3}} +\frac{3}{\nu}e^{-\nu t+\frac{2(\nu R_1+L_0)}{\nu^3}}\int_{\tau_1}^{t}e^{\nu r}\|f(r)\|_{V^{1-l}}^2\,dr.$$

Applying (5.2) and (5.3) implies that

$$\|u(t)\|_{V^0}^2 + \|w(t)\|_{V^0}^2 \lesssim \{\|u(\tau_1)\|_{V^0}^2 + \|w(\tau_1)\|_{V^0}^2\} e^{-\nu t + \nu\tau_1 + \frac{2(\nu R_1 + L_0)}{\nu^3}} + \frac{3L_0}{\nu(e^\nu - 1)} e^{\nu + \frac{2(\nu R_1 + L_0)}{\nu^3}}. \tag{5.15}$$

We infer from (5.7) and (5.15) that there exists a uniformly absorbing ball $B_s(0,R) \subset V^0 \times V^0$, where the radius R depends on ν and L_0. Let us denote X_{cab} a closed absorbing ball

$$X_{\mathrm{cab}} := \{(u,w) \in V^0 \times V^0 : \|u\|_{V^0}^2 + \|w\|_{V^0}^2 \le R^2\}. \tag{5.16}$$

This means that for any bounded set $B \subset V^0 \times V^0$, there exists a time $\bar{t} \ge 0$, independent of the initial time τ, such that

$$(u(t), w(t)) \in X_{\mathrm{cab}}, \qquad \forall t \ge t_1 := \tau + \bar{t}, \tag{5.17}$$

for all weak solutions $(u(t), w(t))$ with $f \in \Sigma$ and the initial time $(u(\tau), w(\tau)) \in B$. It is known that X_{cab} is weakly compact in $V^0 \times V^0$ and metrizable with the weak metric d_w. The weak metric d_w induces the weak topology in X_{cab}. For any sequence (u_n, w_n) of weak solutions of (2.5) obeying Theorem 3.1 the following result holds.

LEMMA 5.1. *Assume that* (u_n, v_n) *is a sequence of weak solutions of* (2.5) *with* $f_n \in \Sigma$ *satisfying Theorem* 3.1 *and* $(u_n(t), w_n(t)) \in X_{\mathrm{cab}}$ *for all* $t \ge t_1$. *Then*

- *the sequences* u_n *and* w_n *are bounded in* $L^2(t_1, t_2; V^l)$, $\mathcal{G}_b^r(t_1, t_2; L^1(\mathbb{T}))$ *and* $L^\infty(t_1, t_2; V^0)$,
- $\frac{d}{dt}u_n$ *and* $\frac{d}{dt}w_n$ *are bounded in* $L^1(t_1, t_2; V^{-\gamma_0})$,

for all $t_2 \ge t_1$, *where* $\gamma_0 := \max\{3, 2l\}$.

Moreover, there exists a subsequence (u_{n_j}, w_{n_j}) *that converges to some solution* (u, w) *in* $C_w([t_1, t_2]; V^0) \times C_w([t_1, t_2]; V^0)$, *i.e.*,

- $\langle u_{n_j}, \psi\rangle \to \langle u, \psi\rangle$ *uniformly on* $[t_1, t_2]$, *as* $n_j \to \infty$, *for all* $\psi \in V^0$.
- $\langle w_{n_j}, \psi\rangle \to \langle w, \psi\rangle$ *uniformly on* $[t_1, t_2]$, *as* $n_j \to \infty$, *for all* $\psi \in V^0$.

Proof. The proof can be completed by modifying that of Theorem 3.1. Therefore, we omit the details here (the readers can consult [15, Lemma 5.4], [32, Lemma 5.3]). ■

We now consider the following evolutionary system of (2.5):

$$\mathcal{E}_\Sigma([\tau,\infty)) := \{(u(\cdot), w(\cdot)) : (u(\cdot), w(\cdot)) \text{ is a weak solution on } [\tau,\infty) \text{ with } f \in \Sigma \text{ and } (u(t), w(t)) \in X_{\mathrm{cab}}, \forall t \in [\tau,\infty)\}, \tau \in \mathbb{R},$$

$$\mathcal{E}_\Sigma((-\infty,\infty)) := \{(u(\cdot), w(\cdot)) : (u(\cdot), w(\cdot)) \text{ is a weak solution on } (-\infty,\infty) \text{ with} f \in \Sigma \text{ and } (u(t), w(t)) \in X_{\mathrm{cab}}, \forall t \in (-\infty,\infty)\}.$$

Clearly, all conditions of an evolutionary system $\mathcal{E}_\Sigma$ hold because of the translation identity, i.e., a weak solution of (2.5) with $f \in \Sigma$ starting at time $\tau + h$ is also a weak solution of (2.5) with $f(\cdot + h) \in \Sigma$ starting at time τ.

We also define

$$\bar{\mathcal{E}}_\Sigma([\tau,\infty)) := \overline{\mathcal{E}([\tau,\infty))}^{C([\tau,\infty);X_{\mathrm{cab},w})}, \quad \forall \tau \in \mathbb{R},$$
$$\bar{\mathcal{E}}_\Sigma((-\infty,\infty)) := \{(u(\cdot),w(\cdot)) : (u(\cdot),w(\cdot))|_{[\tau,\infty)} \in \bar{\mathcal{E}}([\tau,\infty)),\ \forall \tau \in \mathbb{R}\}.$$

Then $\bar{\mathcal{E}}_\Sigma$ is also an evolutionary system, called the closure of $\mathcal{E}_\Sigma$.

We denote $\mathcal{K} := \mathcal{E}_\Sigma((-\infty,\infty))$ and $\bar{\mathcal{K}} := \bar{\mathcal{E}}_\Sigma((-\infty,\infty))$; these are called the kernels of $\mathcal{E}_\Sigma$ and $\bar{\mathcal{E}}_\Sigma$, respectively. Set

$$\Pi_+\mathcal{K} := \{(u(\cdot),w(\cdot))|_{[0,\infty)} : (u,w) \in \mathcal{K}\},$$
$$\Pi_+\bar{\mathcal{K}} := \{(u(\cdot),w(\cdot))|_{[0,\infty)} : (u,w) \in \bar{\mathcal{K}}\}.$$

Following the ideas in [32, Section 4], [12, Section 8], [15, Sections 5–6] and [13], we will check that the evolutionary system $\mathcal{E}_\Sigma$ of (2.5) satisfies the following properties:

(A1) $\mathcal{E}_\Sigma([0,\infty))$ is a precompact set in $C([0,\infty);X_{\mathrm{cab},w})$.

(A2) (Energy inequality) For any $\varepsilon > 0$, there exists $\delta > 0$, such that for every $(u,w) \in \mathcal{E}_\Sigma([0,\infty))$ and $t > 0$,

$$\|u(t)\|_{V^0}^2 + \|w(t)\|_{V^0}^2 \le \|u(t_0)\|_{V^0}^2 + \|w(t_0)\|_{V^0}^2 + \varepsilon$$

for a.e. t_0 in $(t-\delta,t)$.

(A3) (Strong a.e. convergence) Let $(u_n,w_n) \in \mathcal{E}_\Sigma([0,\infty))$ be such that (u_n,w_n) is a $d_{C([0,T];X_{\mathrm{cab},w})}$-Cauchy sequence in $C([0,T];X_{\mathrm{cab},w})$ for some $T > 0$. Then $(u_n(t),w_n(t))$ is a d_s-Cauchy sequence a.e. in $[0,T]$.

We have the following lemma.

Lemma 5.2. *Assume that f_0 satisfies* (5.3). *Then the evolutionary system $\mathcal{E}_\Sigma$ of* (2.5) *with the forces f_0 satisfies* (A1) *and* (A3). *Moreover, if f_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R};V^{1-l})$, then $\mathcal{E}_\Sigma$ of* (2.5) *also satisfies* (A2).

Proof. First, we verify that (A1) holds. Indeed, we deduce from Definition 2.5, Theorem 3.1 and (5.17) that $\mathcal{E}_\Sigma([0,\infty)) \subset C([0,\infty);X_{\mathrm{cab},w})$. Let $\{(u_n,w_n)\}$ be a sequence in $\mathcal{E}_\Sigma([0,\infty))$. It follows from Lemma 5.1 that there exists a subsequence, still denoted by $\{(u_n,w_n)\}$, which converges in $C([0,1];X_{\mathrm{cab},w})$ to some $(u^1,w^1) \in C([0,1];X_{\mathrm{cab},w})$ as $n \to \infty$. Passing to a subsequence and dropping a subindex once more, we find that this subsequence converges in $C([0,2];X_{\mathrm{cab},w})$ to some $(u^2,w^2) \in C([0,2];X_{\mathrm{cab},w})$ as $n \to \infty$. Note that $(u^1(t),w^1(t)) = (u^2(t),w^2(t))$ on $[0,1]$. Continuing this diagonalization process, we obtain a subsequence $\{(u_{n_j},w_{n_j})\}$ of $\{(u_n,w_n)\}$ that converges in $C([0,\infty);X_{\mathrm{cab},w})$ to some $(u,w) \in C([0,\infty);X_{\mathrm{cab},w})$ as $n_j \to \infty$. Therefore, (A1) holds.

Next, we prove that (A3) is valid. Take a $d_{C([0,T];X_{\mathrm{cab},w})}$-Cauchy sequence $\{(u_n, w_n)\} \subset \mathcal{E}_\Sigma([0,\infty))$ in $C([0,T]; X_{\mathrm{cab},w})$ for some $T > 0$. Thanks to Lemma 5.1 again, $\{(u_n, w_n)\}$ is bounded in $L^2(0,T;V^l) \times L^2(0,T;V^1)$. Hence, there exists some $(u(t), w(t)) \in C([0,T]; X_{\mathrm{cab},w})$ such that, as $n \to \infty$,

$$\int_0^T \|u_n(s) - u(s)\|_{V^0}^2\, ds \to 0, \qquad \int_0^T \|w_n(s) - w(s)\|_{V^0}^2\, ds \to 0.$$

In particular, by using the Radon–Riesz property, we get $\|u_n(t)\|_{V^0} \to \|u(t)\|_{V^0}$ and $\|w_n(t)\|_{V^0} \to \|w(t)\|_{V^0}$ as $n \to \infty$ a.e. on $[0,T]$, which means that $\{(u(t), w(t))\}$ is a d_s-Cauchy sequence a.e. on $[0,T]$. Thus, (A3) is valid.

Finally, for any $(u,w) \in \mathcal{E}_\Sigma([0,\infty))$ and $t > 0$, using the property of normal functions, we can infer from (5.1), (5.4), (5.8) and (5.13) that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|u(t)\|_{V^0}^2 + \|w(t)\|_{V^0}^2 \le \|u(t_0)\|_{V^0}^2 + \|w(t_0)\|_{V^0}^2 + \epsilon,$$

for almost every t_0 in $(t-\delta, t)$. This implies that (A2) holds.

The readers can consult the analogous results in [15, Lemma 5.7]. ∎

Using Lemma 5.2 and [15, Theorem 3.7], [32, Theorems 3.6 and 3.12], we get the following result.

THEOREM 5.3. *Assume that f_0 satisfies* (5.3) *and $l \ge 5/4$. Let $\mathcal{E}_\Sigma$ be the evolutionary system of* (2.5) *with forces in Σ and let $\bar{\mathcal{E}}_\Sigma$ is its closure. Then:*

(i) *The weak uniform global attractor $\mathcal{A}_w^\Sigma$ and the weak trajectory attractor $\mathfrak{A}_w^\Sigma$ for* (2.5) *with forces f_0 exist. $\mathcal{A}_w^\Sigma$ is the maximal invariant and maximal quasi-invariant set with respect to the closure $\bar{\mathcal{E}}_\Sigma$ of the corresponding evolutionary system $\mathcal{E}_\Sigma$. Additionally,*

$$\begin{aligned}\mathcal{A}_w^\Sigma &= \omega_w(X_{\mathrm{cab}}) = \omega_s(X_{\mathrm{cab}}) = \{(u(0), w(0)) : (u,w) \in \bar{\mathcal{K}}\},\\ \mathfrak{A}_w^\Sigma &= \Pi_+ \bar{\mathcal{K}} = \{(u(\cdot), w(\cdot))|_{[0,\infty)} : (u,w) \in \bar{\mathcal{K}}\},\\ \mathcal{A}_w^\Sigma &= \mathfrak{A}_w^\Sigma(t) = \{(u(t), w(t)) : (u,w) \in \mathfrak{A}_w^\Sigma\}, \qquad \forall t \ge 0.\end{aligned}$$

Moreover, $\mathfrak{A}_w^\Sigma$ satisfies the finite weak uniform tracking property and is weakly equicontinuous on $[0,\infty)$.

(ii) *Furthermore, if f_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R}; V^{1-l})$ and every complete trajectory of $\bar{\mathcal{E}}_\Sigma$ is strongly continuous, then the weak global attractor $\mathcal{A}_w^\Sigma$ is a strongly compact strong global attractor $\mathcal{A}_s^\Sigma$, and the weak trajectory attractor $\mathfrak{A}_w^\Sigma$ is a strongly compact strong trajectory attractor $\mathfrak{A}_s^\Sigma$. Moreover, $\mathfrak{A}_s^\Sigma = \Pi_+ \bar{\mathcal{K}}$ satisfies the finite strong uniform tracking property and is strongly equicontinuous on $[0,\infty)$.*

We will now consider another evolutionary system of (2.5) with forces in $\bar{\Sigma}$ as follows:

$$\mathcal{E}_{\bar{\Sigma}}([\tau,\infty)) := \{(u(\cdot),w(\cdot)) : (u(\cdot),w(\cdot)) \text{ is a weak solution on } [\tau,\infty) \\ \text{with } f \in \bar{\Sigma} \text{ and } (u(t),w(t)) \in X_{\mathrm{cab}},\ \forall t \in [\tau,\infty)\}, \quad \tau \in \mathbb{R},$$

$$\mathcal{E}_{\bar{\Sigma}}((-\infty,\infty)) := \{(u(\cdot),w(\cdot)) : (u(\cdot),w(\cdot)) \text{ is a weak solution on } (-\infty,\infty) \\ \text{with } f \in \bar{\Sigma} \text{ and } (u(t),w(t)) \in X_{\mathrm{cab}}, \quad \forall t \in (-\infty,\infty)\}.$$

We can also define the evolutionary system $\bar{\mathcal{E}}_{\bar{\Sigma}}$ which is called the closure of $\mathcal{E}_{\bar{\Sigma}}$. We see that $\mathcal{E}_{\Sigma} \subset \mathcal{E}_{\bar{\Sigma}}$ and $\mathcal{E}_{\bar{\Sigma}}$ is closed.

Due to the closedness of $\bar{\Sigma}$, the arguments in Lemma 5.1 are still valid if we substitute $\bar{\Sigma}$ for Σ. The arguments in Lemma 5.2 are still valid if we substitute $\mathcal{E}_{\bar{\Sigma}}$ for $\mathcal{E}_{\Sigma}$ (see also [12, Lemma 8.6], [15, Lemmas 5.7 and 6.2] and [32]). This implies that we can get analogous results to Theorem 5.3 for the evolutionary system $\mathcal{E}_{\bar{\Sigma}}$. We obtain the existence and the tracking properties of the weak uniform global attractor $\mathcal{A}_w^{\bar{\Sigma}}$, the weak trajectory attractor $\mathfrak{A}_w^{\bar{\Sigma}}$, the strongly compact strong global attractor $\mathcal{A}_s^{\bar{\Sigma}}$ and the strongly compact strong trajectory attractor $\mathfrak{A}_s^{\bar{\Sigma}}$.

As indicated in [15, 32], the attractors may not satisfy the minimality property. So, the attractors for $\mathcal{E}_{\bar{\Sigma}}$ might be bigger than those for $\mathcal{E}_{\Sigma}$ in Theorem 5.3. Therefore, we have an interesting question: *Are the attractors* $\mathcal{A}_\bullet^{\Sigma}$, $\mathfrak{A}_\bullet^{\Sigma}$ *and* $\mathcal{A}_\bullet^{\bar{\Sigma}}$, $\mathfrak{A}_\bullet^{\bar{\Sigma}}$ *identical?* The answer may be negative if the weak solution of (2.5) is not unique. The answer is affirmative if (2.5) is well-posed. This means that the attractors satisfy the minimality property if uniqueness does hold. This is the content of Theorem 3.2.

Following Theorem 3.1, we deduce that the system (2.5) is unique and $u \in C_{\mathrm{loc}}([\tau,\infty); V^0)$ for any $\tau \in \mathbb{R}$. The proof of Theorem 3.2 is immediate by applying [32, Theorems 3.24 and 3.25].

6. Proofs of Theorems 3.3 and 3.4

6.1. Proof of Theorem 3.3. Denote $\mathrm{d} := u - v$ and $g := w - h$. These satisfy the following system:

$$\begin{cases} \partial_t \mathrm{d} + \nu \Lambda^{2l}\mathrm{d} + w \times \mathrm{d} + g \times v + aP_\sigma\{(e^{b|u|^r}-1)u - (e^{b|v|^r}-1)v\} = 0, \\ \partial_t g + \nu\Lambda^{2l} g + B(\mathrm{d},w) + B(v,g) - B(w,\mathrm{d}) - B(g,v) \\ \qquad\qquad + aP_\sigma\{(e^{b|w|^r}-1)w - (e^{b|h|^r}-1)h\} = 0. \end{cases} \tag{6.1}$$

We deduce from (3.2) that $\mathrm{d}(t)_{\leq Q(t)} \equiv 0$ and $g(t)_{\leq Q(t)} \equiv 0$. By applying Δ_q to (6.1) we see that

$$\begin{cases} \partial_t \Delta_q \mathrm{d} + \nu\Lambda^{2l}\Delta_q \mathrm{d} + \Delta_q(w \times \mathrm{d}) + \Delta_q(g \times v) \\ \qquad\qquad + a\Delta_q\{(e^{b|u|^r}-1)u - (e^{b|v|^r}-1)v\} = 0, \\ \partial_t \Delta_q g + \nu\Lambda^{2l}\Delta_q g + \Delta_q(\mathrm{d}\cdot\nabla w) + \Delta_q(v\cdot\nabla g) - \Delta_q(w\cdot\nabla \mathrm{d}) \\ \qquad\qquad - \Delta_q(g\cdot\nabla v) + a\Delta_q\{(e^{b|w|^r}-1)w - (e^{b|h|^r}-1)h\} = 0. \end{cases} \tag{6.2}$$

We take the L^2 inner product of equation $(6.2)_1$ with $\Delta_q \mathrm{d}$ and equation $(6.2)_2$ with $\Delta_q g$, and integrating by parts and using

$$\nabla \cdot u = \nabla \cdot v = \nabla \cdot w = \nabla \cdot h = 0,$$

we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}^2 + \nu\|\Lambda^l \mathrm{d}_q\|_{L^2(\mathbb{T})}^2 + \int_{\mathbb{T}} \Delta_q(w \times \mathrm{d})\mathrm{d}_q\,dx \\ + \int_{\mathbb{T}} \Delta_q(g \times v)\mathrm{d}_q\,dx + a\int_{\mathbb{T}} \Delta_q\big((e^{b|u|^r}-1)u - (e^{b|v|^r}-1)v\big)\mathrm{d}_q\,dx = 0, \end{aligned} \tag{6.3}$$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|g_q\|_{L^2(\mathbb{T})}^2 + \nu\|\Lambda^l g_q\|_{L^2(\mathbb{T})}^2 + \int_{\mathbb{T}} \Delta_q(\mathrm{d}\cdot\nabla w)g_q\,dx \\ + \int_{\mathbb{T}} \Delta_q(v\cdot\nabla g)g_q\,dx - \int_{\mathbb{T}} \Delta_q(w\cdot\nabla \mathrm{d})g_q\,dx - \int_{\mathbb{T}} \Delta_q(g\cdot\nabla v)g_q\,dx \\ + a\int_{\mathbb{T}} \Delta_q\big((e^{b|w|^r}-1)w - (e^{b|h|^r}-1)h\big)g_q\,dx = 0. \end{aligned} \tag{6.4}$$

Integrating in time, taking the ℓ^2-norm of the sequence in (6.3) and using Lemma 2.10, we deduce that

$$\begin{aligned} &\frac{1}{2}\|\mathrm{d}(t)\|_{V^0}^2 - \frac{1}{2}\|\mathrm{d}(t_0)\|_{V^0}^2 + \nu\int_{t_0}^{t} \|\Lambda^l \mathrm{d}(\tau)\|_{V^0}^2\,d\tau \\ &\lesssim \int_{t_0}^{t}\sum_{q\ge 0}\Big|\int_{\mathbb{T}} \Delta_q(w\times \mathrm{d})\mathrm{d}_q\,dx\Big|\,d\tau + \int_{t_0}^{t}\sum_{q\ge 0}\Big|\int_{\mathbb{T}} \Delta_q(g\times v)\mathrm{d}_q\,dx\Big|\,d\tau \\ &:= \int_{t_0}^{t} I_1\,d\tau + \int_{t_0}^{t} I_2\,d\tau. \end{aligned} \tag{6.5}$$

We first estimate the term I_1. Using Bony's paraproduct implies

$$w \times \mathrm{d} = \sum_{m=0}^{\infty} w_{\le m-2} \times \mathrm{d}_m + \sum_{m=0}^{\infty} w_m \times \mathrm{d}_{\le m-2} + \sum_{m=0}^{\infty} \widetilde{w}_m \times \mathrm{d}_m,$$

where $\widetilde{w}_m = w_{m-1} + w_m + w_{m+1}$. Therefore,

$$\begin{aligned} \Delta_q(w\times \mathrm{d}) = &\sum_{m=0}^{\infty} \Delta_q(w_{\le m-2}\times \mathrm{d}_m) \\ &+ \sum_{m=0}^{\infty} \Delta_q(w_m \times \mathrm{d}_{\le m-2}) + \sum_{m=0}^{\infty} \Delta_q(\widetilde{w}_m \times \mathrm{d}_m). \end{aligned}$$

We use the triangle inequality and Lemma 2.3 to decompose I_1 as follows:

$$\begin{aligned} I_1 \lesssim & \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(w_{\leq m-2}\times \mathrm{d}_m)\mathrm{d}_q\,dx\Big| \\ & +\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(w_m\times \mathrm{d}_{\leq m-2})\mathrm{d}_q\,dx\Big| \\ & +\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(\widetilde{w}_m\times \mathrm{d}_m)\mathrm{d}_q\,dx\Big| \\ =: & \, I_{11}+I_{12}+I_{13}. \end{aligned}$$

We will estimate the above terms in turn. We use the convention that $(Q, m-2]$ is empty if $m-2\leq Q$. It follows from (3.1) that

$$\|w_m\|_{L^\infty(\mathbb{T})} \lesssim c_0\nu\lambda_Q^{l+\delta}\lambda_m^{l-1-\delta}, \qquad \forall m>Q. \tag{6.6}$$

By using Proposition 2.4, (6.6), Hölder's inequality, and Young's inequality, we can estimate I_{11} as follows:

$$\begin{aligned} I_{11} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(w_{\leq m-2}\times \mathrm{d}_m)\mathrm{d}_q\,dx\Big| \\ &= \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(w_{\leq m-2}\times \mathrm{d}_m)\mathrm{d}_q\,dx\Big| \\ &\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1} \|w_{(Q,m-2]}\|_{L^\infty(\mathbb{T})}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})} \\ &\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} c_0\nu\lambda_Q^{l+\delta}\lambda_p^{l-1-\delta}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})} \\ &\lesssim c_0\nu \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} \lambda_m^l\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\lambda_p^{-1}\lambda_{p-Q}^{-\delta}\lambda_{m-Q}^{-l}\lambda_{q-p}^{-l} \\ &\lesssim c_0\nu \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} \lambda_m^l\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})} \\ &\lesssim c_0\nu \sum_{m\geq Q+1}\lambda_m^{2l}\|\mathrm{d}_m\|^2_{L^2(\mathbb{T})} + c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|\mathrm{d}_q\|^2_{L^2(\mathbb{T})}. \end{aligned}$$

The terms I_{12} and I_{13} are now estimated using a similar strategy:

$$\begin{aligned} I_{12} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(w_m\times \mathrm{d}_{\leq m-2})\mathrm{d}_q\,dx\Big| \\ &= \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(w_m\times \mathrm{d}_{\leq m-2})\mathrm{d}_q\,dx\Big| \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|w_m\|_{L^\infty(\mathbb{T})}\|\mathrm{d}_{(Q,m-2]}\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} c_0\nu\lambda_Q^{l+\delta}\lambda_m^{l-1-\delta}\|\mathrm{d}_p\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim c_0\nu\sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}\lambda_p^l\|\mathrm{d}_p\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}2^l\lambda_m^{-1}\lambda_{p-Q}^{-l}\lambda_{q+1-m}^{-l}\lambda_{m-Q}^{-\delta}\\
&\lesssim c_0\nu\sum_{q\geq Q+1}\sum_{Q<p\leq q-1}\lambda_p^l\|\mathrm{d}_p\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim c_0\nu\sum_{p\geq Q+1}\lambda_p^{2l}\|\mathrm{d}_p\|_{L^2(\mathbb{T})}^2+c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{13}&=\sum_{q\geq 0}\sum_{|q-m|\leq 1}\left|\int_{\mathbb{T}}\Delta_q(\widetilde{w}_m\times\mathrm{d}_m)\mathrm{d}_q\,dx\right|\\
&=\sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\left|\int_{\mathbb{T}}\Delta_q(\widetilde{w}_m\times\mathrm{d}_m)\mathrm{d}_q\,dx\right|\\
&\lesssim\sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|\widetilde{w}_m\|_{L^\infty(\mathbb{T})}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim\sum_{q\geq Q+1}\sum_{Q<m-2\leq q-1}c_0\nu\lambda_Q^{l+\delta}\lambda_{m+1}^{l-1-\delta}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim c_0\nu\sum_{q\geq Q+1}\sum_{Q<m-2\leq q-1}\lambda_m^l\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}2^{l-1-\delta}\lambda_m^{-1}\lambda_{m-Q}^{-\delta}\lambda_{q-Q}^{-l}\\
&\lesssim c_0\nu\sum_{q\geq Q+1}\sum_{Q<m-2\leq q-1}\lambda_m^l\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}^2+c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}^2.
\end{aligned}
$$

Next, we estimate the term I_2. Using Bony's paraproduct implies

$$
g\times v=\sum_{m=0}^{\infty}g_{\leq m-2}\times v_m+\sum_{m=0}^{\infty}g_m\times v_{\leq m-2}+\sum_{m=0}^{\infty}g_m\times\widetilde{v}_m,
$$

where $\widetilde{v}_m=v_{m-1}+v_m+v_{m+1}$. Therefore,

$$
\begin{aligned}
\Delta_q(g\times v)=&\sum_{m=0}^{\infty}\Delta_q(g_{\leq m-2}\times v_m)\\
&+\sum_{m=0}^{\infty}\Delta_q(g_m\times v_{\leq m-2})+\sum_{m=0}^{\infty}\Delta_q(g_m\times\widetilde{v}_m).
\end{aligned}
$$

We use the triangle inequality and Lemma 2.3 to decompose I_2 as follows:

$$\begin{aligned}
I_2 &\lesssim \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(g_{\leq m-2}\times v_m)\mathrm{d}_q\,dx\Big| \\
&\quad+\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(g_m\times v_{\leq m-2})\mathrm{d}_q\,dx\Big| \\
&\quad+\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(g_m\times \widetilde{v}_m)\mathrm{d}_q\,dx\Big| \\
&=: I_{21}+I_{22}+I_{23}.
\end{aligned}$$

We will estimate the above terms in turn. Again we use the convention that $(Q, m-2]$ is empty if $m-2\leq Q$. It follows from (3.1) that

$$\|v_m\|_{L^\infty(\mathbb{T})}\lesssim c_0\nu\lambda_Q^{l+\delta}\lambda_m^{l-1-\delta},\qquad \forall m>Q. \tag{6.7}$$

By using Proposition 2.4, (6.7), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
I_{21} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(g_{\leq m-2}\times v_m)\mathrm{d}_q\,dx\Big| \\
&= \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(g_{\leq m-2}\times v_m)\mathrm{d}_q\,dx\Big| \\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|g_{(Q,m-2]}\|_{L^2(\mathbb{T})}\|v_m\|_{L^\infty(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})} \\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} c_0\nu\lambda_Q^{l+\delta}\lambda_m^{l-1-\delta}\|g_p\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})} \\
&\lesssim c_0\nu\sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}\lambda_p^l\|g_p\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\lambda_m^{-1}\lambda_{p-Q}^{-l}\lambda_{q-m}^{-l}\lambda_{m-Q}^{-\delta} \\
&\lesssim c_0\nu\sum_{q\geq Q+1}\sum_{Q<p\leq q-1}\lambda_p^l\|g_p\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})} \\
&\lesssim c_0\nu\sum_{p\geq Q+1}\lambda_p^{2l}\|g_p\|^2_{L^2(\mathbb{T})}+c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|\mathrm{d}_q\|^2_{L^2(\mathbb{T})},
\end{aligned}$$

and

$$\begin{aligned}
I_{22} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(g_m\times v_{\leq m-2})\mathrm{d}_q\,dx\Big| \\
&= \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(g_m\times v_{\leq m-2})\mathrm{d}_q\,dx\Big| \\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|g_m\|_{L^2(\mathbb{T})}\|v_{(Q,m-2]}\|_{L^\infty(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} \|v_p\|_{L^\infty(\mathbb{T})}\|g_m\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} c_0\nu\lambda_Q^{l+\delta}\lambda_p^{l-1-\delta}\|g_m\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim c_0\nu \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} \lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\lambda_p^{-1}\lambda_{p-Q}^{-\delta}\lambda_{q-p}^{-l}\lambda_{m-Q}^{-l}\\
&\lesssim c_0\nu \sum_{q\geq Q+1}\sum_{Q<m-2\leq q-1} \lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim c_0\nu \sum_{m\geq Q+1}\lambda_m^{2l}\|g_m\|_{L^2(\mathbb{T})}^2 + c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{23} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(g_m\times \widetilde{v}_m)\mathrm{d}_q\,dx\Big|\\
&= \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(g_m\times \widetilde{v}_m)\mathrm{d}_q\,dx\Big|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1} \|\widetilde{v}_m\|_{L^\infty(\mathbb{T})}\|g_m\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<m-2\leq q-1} c_0\nu\lambda_Q^{l+\delta}\lambda_{m+1}^{l-1-\delta}\|g_m\|_{L^2(\mathbb{T})}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim c_0\nu \sum_{q\geq Q+1}\sum_{Q<m-2\leq q-1} \lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}2^{l-1-\delta}\lambda_m^{-1}\lambda_{m-Q}^{-\delta}\lambda_{q-Q}^{-l}\\
&\lesssim c_0\nu \sum_{q\geq Q+1}\sum_{Q<m-2\leq q-1} \lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|\mathrm{d}_q\|_{L^2(\mathbb{T})}\\
&\lesssim c_0\nu \sum_{m\geq Q+1}\lambda_m^{2l}\|g_m\|_{L^2(\mathbb{T})}^2 + c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|\mathrm{d}_q\|_{L^2(\mathbb{T})}^2.
\end{aligned}
$$

Integrating in time, taking the ℓ^2-norm of the sequence in (6.4) and using Lemma 2.10, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\|g(t)\|_{V^0}^2 - \frac{1}{2}\|g(t_0)\|_{V^0}^2 + \nu\int_{t_0}^t \|\Lambda^l g(\tau)\|_{V^0}^2\,d\tau \\
&\quad\lesssim \int_{t_0}^t \sum_{q\geq 0}\Big|\int_{\mathbb{T}}\Delta_q(\mathrm{d}\cdot\nabla w)g_q\,dx\Big|\,d\tau + \int_{t_0}^t \sum_{q\geq 0}\Big|\int_{\mathbb{T}}\Delta_q(v\cdot\nabla g)g_q\,dx\Big|\,d\tau\\
&\qquad + \int_{t_0}^t \sum_{q\geq 0}\Big|\int_{\mathbb{T}}\Delta_q(w\cdot\nabla \mathrm{d})g_q\,dx\Big|\,d\tau + \int_{t_0}^t \sum_{q\geq 0}\Big|\int_{\mathbb{T}}\Delta_q(g\cdot\nabla v)g_q\,dx\Big|\,d\tau\\
&\quad =: \int_{t_0}^t J_1\,d\tau + \int_{t_0}^t J_2\,d\tau + \int_{t_0}^t J_3\,d\tau + \int_{t_0}^t J_4\,d\tau.
\end{aligned}
\tag{6.8}
$$

Estimating J_1. We use Bony's paraproduct to obtain

$$\mathrm{d}\cdot\nabla w=\sum_{m=0}^{\infty}\mathrm{d}_{\le m-2}\cdot\nabla w_m+\sum_{m=0}^{\infty}\mathrm{d}_m\cdot\nabla w_{\le m-2}+\sum_{m=0}^{\infty}\widetilde{\mathrm{d}}_m\cdot\nabla w_m,$$

where $\widetilde{\mathrm{d}}_m=\mathrm{d}_{m-1}+\mathrm{d}_m+\mathrm{d}_{m+1}$. Thus,

$$\begin{aligned}\Delta_q(\mathrm{d}\cdot\nabla w)=\sum_{m=0}^{\infty}&\Delta_q(\mathrm{d}_{\le m-2}\cdot\nabla w_m)\\&+\sum_{m=0}^{\infty}\Delta_q(\mathrm{d}_m\cdot\nabla w_{\le m-2})+\sum_{m=0}^{\infty}\Delta_q(\widetilde{\mathrm{d}}_m\cdot\nabla w_m).\end{aligned}$$

Using the triangle inequality and Lemma 2.3, we can estimate J_1 as follows:

$$\begin{aligned}J_1&=\sum_{q\ge0}\Big|\int_{\mathbb{T}}\Delta_q(\mathrm{d}\cdot\nabla w)g_q\,dx\Big|\\&\lesssim\sum_{q\ge0}\sum_{|q-m|\le1}\Big|\int_{\mathbb{T}}\Delta_q(\mathrm{d}_{\le m-2}\cdot\nabla w_m)g_q\,dx\Big|\\&\quad+\sum_{q\ge0}\sum_{|q-m|\le1}\Big|\int_{\mathbb{T}}\Delta_q(\mathrm{d}_m\cdot\nabla w_{\le m-2})g_q\,dx\Big|\\&\quad+\sum_{q\ge0}\sum_{|q-m|\le1}\Big|\int_{\mathbb{T}}\Delta_q(\widetilde{\mathrm{d}}_m\cdot\nabla w_m)g_q\,dx\Big|\\&=:J_{11}+J_{12}+J_{13}.\end{aligned}$$

We will estimate the above terms in turn. By using Proposition 2.4, (6.6), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}J_{11}&=\sum_{q\ge0}\sum_{|q-m|\le1}\Big|\int_{\mathbb{T}}\Delta_q(\mathrm{d}_{\le m-2}\cdot\nabla w_m)g_q\,dx\Big|\\&\lesssim\sum_{m\ge Q+2}\sum_{|q-m|\le1}\Big|\int_{\mathbb{T}}\Delta_q(\mathrm{d}_{\le m-2}\cdot\nabla w_m)g_q\,dx\Big|\\&\lesssim\sum_{m\ge Q+2}\sum_{|q-m|\le1}\|\mathrm{d}_{(Q,m-2]}\|_{L^2(\mathbb{T})}\lambda_m\|w_m\|_{L^\infty(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\&\lesssim c_0\nu\sum_{m\ge Q+2}\sum_{|q-m|\le1}\lambda_Q^{l+\delta}\lambda_m^{l-\delta}\|g_q\|_{L^2(\mathbb{T})}\sum_{Q<p\le m-2}\|\mathrm{d}_p\|_{L^2(\mathbb{T})}\\&\lesssim c_0\nu\sum_{m\ge Q+2}\sum_{|q-m|\le1}\lambda_q^{l}\|g_q\|_{L^2(\mathbb{T})}\lambda_{q-m}^{-l}\lambda_{m-Q}^{-\delta}\sum_{Q<p\le m-2}\lambda_p^{l}\|\mathrm{d}_p\|_{L^2(\mathbb{T})}\lambda_{p-Q}^{-l}\\&\lesssim c_0\nu\sum_{q\ge Q+1}\lambda_q^{l}\|g_q\|_{L^2(\mathbb{T})}\sum_{Q<p\le q-1}\lambda_p^{l}\|\mathrm{d}_p\|_{L^2(\mathbb{T})}\\&\lesssim c_0\nu\sum_{q\ge Q+1}\lambda_q^{2l}\|g_q\|^2_{L^2(\mathbb{T})}+c_0\nu\sum_{q\ge Q+1}\lambda_p^{2l}\|\mathrm{d}_p\|^2_{L^2(\mathbb{T})},\end{aligned}$$

and

$$
\begin{aligned}
J_{12} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(\mathrm{d}_m\cdot\nabla w_{\leq m-2})g_q\,dx\Big| \\
&\lesssim \sum_{m\geq Q+2}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(\mathrm{d}_m\cdot\nabla w_{\leq m-2})g_q\,dx\Big| \\
&\lesssim \sum_{m\geq Q+2}\sum_{|q-m|\leq 1}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|\nabla w_{(Q,m-2]}\|_{L^\infty(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})} \\
&\quad + \sum_{m\geq Q+2}\sum_{|q-m|\leq 1}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|\nabla w_{\leq Q}\|_{L^\infty(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})} \\
&\lesssim \sum_{m\geq Q+2}\sum_{|q-m|\leq 1}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\sum_{Q<p\leq m-2}\lambda_p\|w_p\|_{L^\infty(\mathbb{T})} \\
&\quad + \sum_{m\geq Q+2}\sum_{|q-m|\leq 1}\lambda_m^l\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_m^{-l}\lambda_q^{-l}\|\nabla w_{\leq Q}\|_{L^\infty(\mathbb{T})} \\
&\lesssim \sum_{m\geq Q+2}\sum_{|q-m|\leq 1}\lambda_m^l\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_m^{-l}\lambda_q^{-l}\sum_{Q<p\leq m-2}c_0\nu\lambda_Q^{l+\delta}\lambda_p^{l-\delta} \\
&\quad + \sum_{m\geq Q+2}\lambda_m^{2l}\|\mathrm{d}_m\|^2_{L^2(\mathbb{T})}\lambda_m^{-2l}\|\nabla w_{\leq Q}\|_{L^\infty(\mathbb{T})} \\
&\quad + \sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|^2_{L^2(\mathbb{T})}\lambda_q^{-2l}\|\nabla w_{\leq Q}\|_{L^\infty(\mathbb{T})} \\
&\lesssim c_0\nu\sum_{m\geq Q+2}\sum_{|q-m|\leq 1}\lambda_m^l\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\sum_{Q<p\leq m-2\leq q-1}\lambda_{m-Q}^{-l}\lambda_{q-p}^{-l}\lambda_{p-Q}^{-\delta} \\
&\quad + \sum_{m\geq Q+2}\lambda_m^{2l}\|\mathrm{d}_m\|^2_{L^2(\mathbb{T})}\lambda_Q^{-2l}\|\nabla w_{\leq Q}\|_{L^\infty(\mathbb{T})} \\
&\quad + \sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|^2_{L^2(\mathbb{T})}\lambda_Q^{-2l}\|\nabla w_{\leq Q}\|_{L^\infty(\mathbb{T})} \\
&\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|\mathrm{d}_m\|^2_{L^2(\mathbb{T})} + c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|^2_{L^2(\mathbb{T})},
\end{aligned}
$$

and

$$
\begin{aligned}
J_{13} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}} \Delta_q(\widetilde{\mathrm{d}}_m\cdot\nabla w_m)g_q\,dx\Big| \\
&\lesssim \sum_{m\geq Q+1}\sum_{Q<q\leq m+1}\|\widetilde{\mathrm{d}}_m\|_{L^2(\mathbb{T})}\|\nabla w_m\|_{L^\infty(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})} \\
&\lesssim \sum_{m\geq Q+1}\sum_{Q<q\leq m+1}\sum_{m-1\leq p\leq m+1}\|\mathrm{d}_p\|_{L^2(\mathbb{T})}c_0\nu\lambda_Q^{l+\delta}\lambda_m^{l-\delta}\|g_q\|_{L^2(\mathbb{T})}
\end{aligned}
$$

$$\lesssim c_0\nu \sum_{m\geq Q+1}\sum_{Q<q\leq m+1}\lambda_{q-Q}^{-l}\lambda_{m-Q}^{-\delta}\lambda_m^l\sum_{m-1\leq p\leq m+1}\|\mathrm{d}_p\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}$$
$$\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}^2 + c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|_{L^2(\mathbb{T})}^2.$$

Estimating J_2. We use Bony's paraproduct to obtain

$$v\cdot\nabla g = \sum_{m=0}^{\infty} v_{\leq m-2}\cdot\nabla g_m + \sum_{m=0}^{\infty} v_m\cdot\nabla g_{\leq m-2} + \sum_{m=0}^{\infty}\widetilde{v}_m\cdot\nabla g_m,$$

where $\widetilde{v}_m = v_{m-1}+v_m+v_{m+1}$. Thus,

$$\begin{aligned}\Delta_q(v\cdot\nabla g) &= \sum_{m=0}^{\infty}\Delta_q(v_{\leq m-2}\cdot\nabla g_m)\\ &\quad+\sum_{m=0}^{\infty}\Delta_q(v_m\cdot\nabla g_{\leq m-2})+\sum_{m=0}^{\infty}\Delta_q(\widetilde{v}_m\cdot\nabla g_m).\end{aligned}$$

Using the triangle inequality and Lemma 2.3, we can estimate J_2 as follows:

$$\begin{aligned}J_2 &= \sum_{q\geq 0}\Big|\int_{\mathbb{T}}\Delta_q(v\cdot\nabla g)g_q\,dx\Big|\\ &\lesssim \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(v_{\leq m-2}\cdot\nabla g_m)g_q\,dx\Big|\\ &\quad+\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(v_m\cdot\nabla g_{\leq m-2})g_q\,dx\Big|\\ &\quad+\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(\widetilde{v}_m\cdot\nabla g_m)g_q\,dx\Big|\\ &=: J_{21}+J_{22}+J_{23}.\end{aligned}$$

We will estimate the above terms in turn. By using Proposition 2.4, (6.7), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}J_{21} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(v_{\leq m-2}\cdot\nabla g_m)g_q\,dx\Big|\\ &\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(v_{\leq m-2}\cdot\nabla g_m)g_q\,dx\Big|\\ &\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|v_{(Q,m-2]}\|_{L^\infty(\mathbb{T})}\lambda_m\|g_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\ &\quad+\sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|v_{\leq Q}\|_{L^\infty(\mathbb{T})}\lambda_m\|g_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\lambda_m\|g_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\sum_{Q<p\leq m-2}\|v_p\|_{L^\infty(\mathbb{T})}\\
&\quad+\sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|v_{\leq Q}\|_{L^\infty(\mathbb{T})}\lambda_m\|g_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_m^{-l+1}\lambda_q^{-l}\lambda_Q^{l+\delta}\lambda_p^{l-1-\delta}\\
&\quad+\sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_m^{-l+1}\lambda_q^{-l}\lambda_Q^{2l}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-l+1}\lambda_{q-p}^{-l}\lambda_{p-Q}^{-\delta}\lambda_Q\\
&\quad+\sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-l+1}\lambda_{q-Q}^{-l}\lambda_Q\\
&\lesssim c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|_{L^2(\mathbb{T})}^2+c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|g_m\|_{L^2(\mathbb{T})}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{22}&=\sum_{q\geq 0}\sum_{|q-m|\leq 1}\left|\int_{\mathbb{T}}\Delta_q(v_m\cdot\nabla g_{\leq m-2})g_q\,dx\right|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\left|\int_{\mathbb{T}}\Delta_q(v_m\cdot\nabla g_{\leq m-2})g_q\,dx\right|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|v_m\|_{L^\infty(\mathbb{T})}\|\nabla g_{(Q,m-2]}\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_m^{l-1-\delta}\lambda_Q^{l+\delta}\lambda_p\|g_p\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_p^l\|g_p\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-1-\delta}\lambda_{p-Q}^{-l+1}\lambda_{q-m}^{-l}\\
&\lesssim c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|_{L^2(\mathbb{T})}^2+c_0\nu\sum_{p\geq Q+1}\lambda_p^{2l}\|g_p\|_{L^2(\mathbb{T})}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{23}&=\sum_{q\geq 0}\sum_{|q-m|\leq 1}\left|\int_{\mathbb{T}}\Delta_q(\widetilde{v}_m\cdot\nabla g_m)g_q\,dx\right|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\left|\int_{\mathbb{T}}\Delta_q(\widetilde{v}_m\cdot\nabla g_m)g_q\,dx\right|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|\widetilde{v}_m\|_{L^\infty(\mathbb{T})}\|\nabla g_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \sum_{q\geq Q+1}\sum_{Q<m-2\leq q-1} c_0\nu\lambda_{m+1}^{l-1-\delta}\lambda_Q^{l+\delta}\lambda_m\|g_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<m-2\leq q-1} c_0\nu\lambda_m^{l}\|g_m\|_{L^2(\mathbb{T})}\lambda_q^{l}\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-\delta}\lambda_{q-Q}^{-l}\\
&\lesssim c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|_{L^2(\mathbb{T})}^2 + c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|g_m\|_{L^2(\mathbb{T})}^2.
\end{aligned}$$

Estimating J_3. We use Bony's paraproduct to obtain

$$w\cdot\nabla \mathrm{d} = \sum_{m=0}^{\infty} w_{\leq m-2}\cdot\nabla \mathrm{d}_m + \sum_{m=0}^{\infty} w_m\cdot\nabla \mathrm{d}_{\leq m-2} + \sum_{m=0}^{\infty}\widetilde{w}_m\cdot\nabla \mathrm{d}_m,$$

where $\widetilde{w}_m = w_{m-1}+w_m+w_{m+1}$. Thus,

$$\begin{aligned}
\Delta_q(w\cdot\nabla \mathrm{d}) = &\sum_{m=0}^{\infty}\Delta_q(w_{\leq m-2}\cdot\nabla \mathrm{d}_m)\\
&+\sum_{m=0}^{\infty}\Delta_q(w_m\cdot\nabla \mathrm{d}_{\leq m-2}) + \sum_{m=0}^{\infty}\Delta_q(\widetilde{w}_m\cdot\nabla \mathrm{d}_m).
\end{aligned}$$

Using the triangle inequality and Lemma 2.3, we can estimate J_3 as follows:

$$\begin{aligned}
J_3 &= \sum_{q\geq 0}\Big|\int_{\mathbb{T}}\Delta_q(w\cdot\nabla \mathrm{d})g_q\,dx\Big|\\
&\lesssim \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(w_{\leq m-2}\cdot\nabla \mathrm{d}_m)g_q\,dx\Big|\\
&\quad+\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(w_m\cdot\nabla \mathrm{d}_{\leq m-2})g_q\,dx\Big|\\
&\quad+\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(\widetilde{w}_m\cdot\nabla \mathrm{d}_m)g_q\,dx\Big|\\
&=: J_{31}+J_{32}+J_{33}.
\end{aligned}$$

We will estimate the above terms in turn. By using Proposition 2.4, (6.6), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
J_{31} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(w_{\leq m-2}\cdot\nabla \mathrm{d}_m)g_q\,dx\Big|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(w_{\leq m-2}\cdot\nabla \mathrm{d}_m)g_q\,dx\Big|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|w_{(Q,m-2]}\|_{L^\infty(\mathbb{T})}\|\nabla \mathrm{d}_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\quad+\sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|w_{\leq Q}\|_{L^\infty(\mathbb{T})}\|\nabla \mathrm{d}_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}\|w_p\|_{L^\infty(\mathbb{T})}\lambda_m\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\quad+\sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|w_{\leq Q}\|_{L^\infty(\mathbb{T})}\lambda_m\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_p^{l-1-\delta}\lambda_Q^{l+\delta}\lambda_m\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\quad+\sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\lambda_Q^{2l}\lambda_m\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_m^{l}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^{l}\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-l+1}\lambda_{q-p}^{-l}\lambda_{p-Q}^{-1-\delta}\\
&\quad+\sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\lambda_m^{l}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^{l}\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-l+1}\lambda_{q-Q}^{-l}\lambda_Q\\
&\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}^2+c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|_{L^2(\mathbb{T})}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{32}&=\sum_{q\geq 0}\sum_{|q-m|\leq 1}\left|\int_{\mathbb{T}}\Delta_q(w_m\cdot\nabla \mathrm{d}_{\leq m-2})g_q\,dx\right|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\left|\int_{\mathbb{T}}\Delta_q(w_m\cdot\nabla \mathrm{d}_{\leq m-2})g_q\,dx\right|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|w_m\|_{L^\infty(\mathbb{T})}\|\nabla \mathrm{d}_{(Q,m-2]}\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_m^{l-1-\delta}\lambda_Q^{l+\delta}\lambda_p\|\mathrm{d}_p\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1}c_0\nu\lambda_p^{l}\|\mathrm{d}_p\|_{L^2(\mathbb{T})}\lambda_q^{l}\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-1-\delta}\lambda_{p-Q}^{-l+1}\lambda_{q-m}^{-l}\\
&\lesssim c_0\nu\sum_{p\geq Q+1}\lambda_p^{2l}\|\mathrm{d}_p\|_{L^2(\mathbb{T})}^2+c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|_{L^2(\mathbb{T})}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{33}&=\sum_{q\geq 0}\sum_{|q-m|\leq 1}\left|\int_{\mathbb{T}}\Delta_q(\widetilde{w}_m\cdot\nabla \mathrm{d}_m)g_q\,dx\right|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|\widetilde{w}_m\|_{L^\infty(\mathbb{T})}\|\nabla \mathrm{d}_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \sum_{q\geq Q+1} \sum_{Q<m-2\leq q-1} c_0\nu\lambda_{m+1}^{l-1-\delta}\lambda_Q^{l+\delta}\lambda_m\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1} \sum_{Q<m-2\leq q-1} c_0\nu\lambda_m^{l}\|\mathrm{d}_m\|_{L^2(\mathbb{T})}\lambda_q^{l}\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-\delta}\lambda_{q-Q}^{-l}\\
&\lesssim c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|^2_{L^2(\mathbb{T})} + c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|\mathrm{d}_m\|^2_{L^2(\mathbb{T})}.
\end{aligned}$$

Estimating J_4. We use Bony's paraproduct to obtain

$$g\cdot\nabla v = \sum_{m=0}^{\infty} g_{\leq m-2}\cdot\nabla v_m + \sum_{m=0}^{\infty} g_m\cdot\nabla v_{\leq m-2} + \sum_{m=0}^{\infty}\widetilde{g}_m\cdot\nabla v_m,$$

where $\widetilde{g}_m = g_{m-1}+g_m+g_{m+1}$. Thus,

$$\begin{aligned}
\Delta_q(g\cdot\nabla v) = &\sum_{m=0}^{\infty}\Delta_q(g_{\leq m-2}\cdot\nabla v_m)\\
&+\sum_{m=0}^{\infty}\Delta_q(g_m\cdot\nabla v_{\leq m-2}) + \sum_{m=0}^{\infty}\Delta_q(\widetilde{g}_m\cdot\nabla v_m).
\end{aligned}$$

Using the triangle inequality and Lemma 2.3, we can estimate J_4 as follows:

$$\begin{aligned}
J_4 &= \sum_{q\geq 0}\Big|\int_{\mathbb{T}}\Delta_q(g\cdot\nabla v)g_q\,dx\Big|\\
&\lesssim \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(g_{\leq m-2}\cdot\nabla v_m)g_q\,dx\Big|\\
&\quad+\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(g_m\cdot\nabla v_{\leq m-2})g_q\,dx\Big|\\
&\quad+\sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(\widetilde{g}_m\cdot\nabla v_m)g_q\,dx\Big|\\
&=: J_{41}+J_{42}+J_{43}.
\end{aligned}$$

We will estimate the above terms in turn. By using Proposition 2.4, (6.7), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
I_{41} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(g_{\leq m-2}\cdot\nabla v_m)g_q\,dx\Big|\\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|g_{(Q,m-2]}\|_{L^2(\mathbb{T})}\|\nabla v_m\|_{L^\infty(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}\\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} c_0\nu\lambda_m^{l-\delta}\lambda_Q^{l+\delta}\|g_p\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})}
\end{aligned}$$

$$\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} c_0\nu\lambda_p^l\|g_p\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-\delta}\lambda_{p-Q}^{-l}\lambda_{q-m}^{-l}$$
$$\lesssim c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|_{L^2(\mathbb{T})}^2 + c_0\nu\sum_{p\geq Q+1}\lambda_p^{2l}\|g_p\|_{L^2(\mathbb{T})}^2,$$

and

$$\begin{aligned}
J_{42} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(g_m\cdot\nabla v_{\leq m-2})g_q\,dx\Big| \\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(g_m\cdot\nabla v_{\leq m-2})g_q\,dx\Big| \\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|g_m\|_{L^2(\mathbb{T})}\|\nabla v_{(Q,m-2]})\|_{L^\infty(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})} \\
&\quad + \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|g_m\|_{L^2(\mathbb{T})}\|\nabla v_{\leq Q})\|_{L^\infty(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})} \\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} c_0\nu\lambda_p^{l-\delta}\lambda_Q^{l+\delta}\|g_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})} \\
&\quad + \sum_{q\geq Q+1}\sum_{|q-m|\leq 1} c_0\nu\lambda_Q^{2l}\|g_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})} \\
&\lesssim \sum_{q\geq Q+1}\sum_{Q<p\leq m-2\leq q-1} c_0\nu\lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_{p-Q}^{-\delta}\lambda_{m-p}^{-l}\lambda_{q-Q}^{-l} \\
&\quad + \sum_{q\geq Q+1}\sum_{|q-m|\leq 1} c_0\nu\lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-l}\lambda_{q-Q}^{-l} \\
&\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|g_m\|_{L^2(\mathbb{T})}^2 + c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|_{L^2(\mathbb{T})}^2,
\end{aligned}$$

and

$$\begin{aligned}
J_{43} &= \sum_{q\geq 0}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(\widetilde{g}_m\cdot\nabla v_m)g_q\,dx\Big| \\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\Big|\int_{\mathbb{T}}\Delta_q(\widetilde{g}_m\cdot\nabla v_m)g_q\,dx\Big| \\
&\lesssim \sum_{q\geq Q+1}\sum_{|q-m|\leq 1}\|\widetilde{g}_m\|_{L^2(\mathbb{T})}\|\nabla v_m\|_{L^\infty(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})} \\
&\lesssim \sum_{Q+1\leq m\leq q+1} c_0\nu\lambda_m^{l-\delta}\lambda_Q^{l+\delta}\|g_m\|_{L^2(\mathbb{T})}\|g_q\|_{L^2(\mathbb{T})} \\
&\lesssim \sum_{Q+1\leq m\leq q+1} c_0\nu\lambda_m^l\|g_m\|_{L^2(\mathbb{T})}\lambda_q^l\|g_q\|_{L^2(\mathbb{T})}\lambda_{m-Q}^{-\delta}\lambda_{q-Q}^{-l} \\
&\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|g_m\|_{L^2(\mathbb{T})}^2 + c_0\nu\sum_{q\geq Q+1}\lambda_q^{2l}\|g_q\|_{L^2(\mathbb{T})}^2.
\end{aligned}$$

We deduce from the above estimates that

$$(6.9)\qquad \begin{aligned} &I_1+I_2+J_1+J_2+J_3+J_4\\ &\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|\mathrm{d}_m\|^2_{L^2(\mathbb{T})}+c_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|g_m\|^2_{L^2(\mathbb{T})}\\ &\leq Cc_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|\mathrm{d}_m\|^2_{L^2(\mathbb{T})}+Cc_0\nu\sum_{m\geq Q+1}\lambda_m^{2l}\|g_m\|^2_{L^2(\mathbb{T})}.\end{aligned}$$

It follows from (6.5), (6.8) and (6.9) that if we take $c_0:=\frac{1}{2C}$, then

$$\begin{aligned} &\|\mathrm{d}(t)\|^2_{V^0}+\|g(t)\|^2_{V^0}\\ &\lesssim \|\mathrm{d}(t_0)\|^2_{V^0}+\|g(t_0)\|^2_{V^0}-\nu\int_{t_0}^{t}\left(\|\Lambda^\alpha\mathrm{d}(\tau)\|^2_{V^0}+\|\Lambda^\alpha g(\tau)\|^2_{V^0}\right)d\tau\\ &\lesssim \|\mathrm{d}(t_0)\|^2_{V^0}+\|g(t_0)\|^2_{V^0}-\nu\int_{t_0}^{t}\left(\|\mathrm{d}(\tau)\|^2_{V^0}+\|g(\tau)\|^2_{V^0}\right)d\tau\end{aligned}$$

for all $t_0\leq t$. Thus

$$\|\mathrm{d}(t)\|^2_{V^0}+\|g(t)\|^2_{V^0}\lesssim\{\|\mathrm{d}(t_0)\|^2_{V^0}+\|g(t_0)\|^2_{V^0}\}\|e^{-\nu(t-t_0)}$$

for all $t_0\leq t$. Let $t_0\to-\infty$ to complete the proof of Theorem 3.3.

6.2. Proof of Theorem 3.4. The proof can be obtained by repeating the same arguments as in the proof of Theorem 3.3 and taking the limit as t tends to infinity. Due to the length of the paper, we omit the details.

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Le Tran Tinh
Faculty of Natural Sciences
Hong Duc University
Thanh Hoa, Vietnam
E-mail: letrantinh@hdu.edu.vn