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A categorification for the partial-dual genus polynomial

by

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Abstract. The partial-dual genus polynomial ${}^{\partial}\varepsilon_G(z)$ of a ribbon graph G is the generating function that enumerates all partial duals of G. We exhibit a categorification for this polynomial. The key ingredient of the construction is an extended Frobenius algebra related to unoriented topological quantum field theory.

1. Introduction. A ribbon graph is a surface with nonempty boundary, which consists of disjoint disks with some bands connecting them. Ribbon graphs occur in many areas of mathematics, such as topological graph theory, combinatorics, low-dimensional topology and representation theory. As a generalization of the classical Euler–Poincaré duality, Chmutov [8] introduced the notion of partial duality, which is a duality of ribbon graphs relative to a subset of edges. For a given ribbon graph G and a subset $A \subseteq E(G)$, the partial dual of G with respect to A defines a new ribbon graph G^A. This duality has several natural properties. For instance, $(G^A)^A = G$, $(G^A)^B = G^{(A\cup B)\setminus(A\cap B)}$ and partial duality preserves orientability and the number of connected components of ribbon graphs. The readers are referred to [10] for a brief survey of partial duality of ribbon graphs.

However, the genus of a ribbon graph may be changed under partial duality. For this reason, Gross, Mansour and Tucker [11] introduced the partial-dual genus polynomial, which enumerates all possible partial duals of a ribbon graph by genus as represented by its generating function. This polynomial has been intensively studied during the past several years. For example, in [11] a concrete example was given for which the partial-dual genus polynomial is not log-concave. Later in [12, 13], Gross, Mansour and Tucker introduced some other related polynomials and presented a Gray code algo-

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rithm for calculating them. For bouquets, i.e. ribbon graphs with exactly one vertex-disk, it was proved in [30] that the partial-dual genus polynomial only depends on the signed intersection graph. For ribbon graphs derived from chord diagrams, recently Chmutov [9] proved that the partial-dual genus polynomial satisfies the four-term relation, which suggests a potential relation between the partial-dual genus polynomial and finite type invariants of knots. In particular, in this case the partial-dual genus polynomial only depends on the intersection graph of the chord diagram, rather than the chord diagram itself. The reader is referred to [5] for a combinatorial approach to the partial-dual genus polynomial in terms of intersection graphs without referring to chord diagrams.

Roughly speaking, categorification can be regarded as a lifting from an n-category to an $(n+1)$-category. One example is the lifting from the Euler characteristic of a smooth manifold to the homology groups. Since Khovanov's seminal work on the categorification of the Jones polynomial [16], many quantum invariants of knots have been categorified during the past twenty years; see [17, 18, 19, 20] for some examples. These categorifications reinterpreted knot invariants as the graded Euler characteristic of knot homologies. Several major breakthroughs based on categorification have been made. For example, Rasmussen [26] introduced his s-invariant and used it to give a purely combinatorial proof of the Milnor conjecture and the existence of exotic $\mathbb{R}^4$. Recently, Piccirillo [25] indirectly used Rasmussen's invariant to show that the Conway knot is not slice.

On the other hand, Khovanov's work inspired the categorification of polynomials in graph theory. In [14], Helme-Guizon and Rong constructed a categorification of the chromatic polynomial of graphs. For each graph, they defined a bigraded cohomology theory such that the chromatic polynomial can be derived from the graded Euler characteristic of the cohomology groups. In particular, the deletion-contraction formula for the chromatic polynomial is lifted to a long exact sequence. Since then, many other graph polynomials have been categorified [15, 29, 27, 6, 4, 23], which may indicate some approaches to the four coloring theorem. It turns out that usually these categorifications contain more information compared with the original polynomials. For example, the torsion part of homology cannot be read off from the polynomials.

The main aim of this paper is to exhibit a categorification of the partial-dual genus polynomial of ribbon graphs.

This paper is organized as follows. In Section 2, we make a quick review of the definitions and properties of ribbon graphs and partial duality. Then a formula for the partial-dual genus polynomial of ribbon graphs is given. In particular, we introduce the notion of graded partial-dual genus polynomial, which can be regarded as a refined version of the partial-dual genus polynomial. Section 3 introduces a punctured $(1+1)$-TQFT, which plays an important role in the categorification of the partial-dual genus polynomial. The construction of the cochain complex involves four n-cubes, which will be discussed in Section 4. The main result of this paper, Theorem 5.3, shows that the graded partial-dual genus polynomial (hence also the partial-dual genus polynomial) can be obtained from the graded Euler characteristic of the cohomology groups. A concrete example is given in Section 6, which suggests that the cohomology groups contain more information compared with the partial-dual genus polynomial. We discuss some applications of this categorification in Section 7, and consider two ribbon graphs which have the same graded partial-dual genus polynomial and isomorphic cohomology groups. Finally, some algebraic structures related to this categorification are discussed in Section 8.

2. Ribbon graphs and partial-dual genus polynomial

2.1. Ribbon graphs and partial dual. A *ribbon graph* is a surface consisting of finitely many disjoint vertex-disks and some edge-ribbons, such that the vertex-disks and edge-ribbons intersect in disjoint line segments and each line segment lies on the boundary of one vertex-disk and the boundary of one edge-ribbon. In particular, each edge-ribbon contains exactly two such line segments. Equivalently, given a cellular embedding of a graph in a 2-dimensional surface (not necessarily orientable), a regular neighborhood of this graph defines an associated ribbon graph. For a given ribbon graph G, let us use $V(G)$ and $E(G)$ to denote the set of vertex-disks of G and the set of edge-ribbons of G, respectively. For simplicity, we sometimes just call the elements in $V(G)$ and $E(G)$ the *vertices* of G and the *edges* of G. If we regard a ribbon graph G as the regular neighborhood of a graph embedded in a surface, then we call each component of the complement of G a *face-disk* of G and use $F(G)$ to denote the set of all face-disks. Some simple examples of ribbon graphs can be found in Figure 1. Note that a ribbon graph is just an abstract surface with boundary, not an embedded surface in $\mathbb{R}^3$.

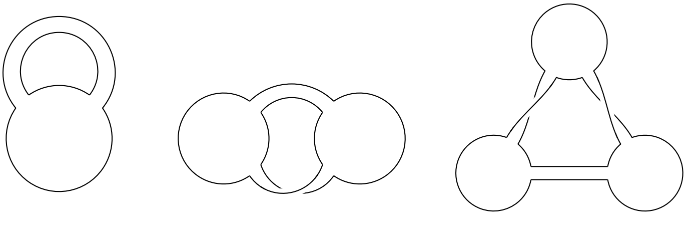


Fig. 1. Three examples of ribbon graphs

Before defining the partial dual of a ribbon graph, we first recall the classical Euler–Poincaré dual. Let G be a ribbon graph. The *Euler–Poincaré* *dual* G^* can be obtained from G by gluing a disk to each boundary component of G and removing the interior of all vertex-disks. Now the newly glued disks form the set of vertex-disks of G^* and the edge-ribbons of G^* are exactly the same as those of G, but the attachments are changed. Notice that the genus of G^* is equal to the genus of G, since they are the dual of each other in the same surface.

Now we turn to the partial dual of G. For a subset $A \subseteq E(G)$, the *partial dual* G^A with respect to the subset A is defined as follows. Consider the spanning subgraph consisting of $V(G)$ and all edges in A, glue a disk to each boundary component of this spanning subgraph and remove the interior of all vertex-disks; then we obtain the *partial dual* G^A. Now the newly glued disks are the vertex-disks of G^A and the edge-ribbons of G^A are the same as those of G, but the attachments of the edges from A are different from those in G. It is easy to observe that if $A = E(G)$, then $G^A = G^*$. The partial dual of a ribbon graph with respect to a subset of $E(G)$ can also be obtained from the so-called *arrow presentation.* The reader is referred to [8, 24] for more details of the construction and some basic properties of the partial dual.

In particular, when the subset A contains exactly one edge, say $A = \{e\} \subseteq E(G)$, the partial dual G^A is shown in Figure 2. We remark that the direction of the arrow in Figure 2 can be reversed, since $(G^{\{e\}})^{\{e\}} = G$. In general, if $|A| \geq 2$, then G^A can be obtained from G by taking the partial dual on each edge in A one by one.

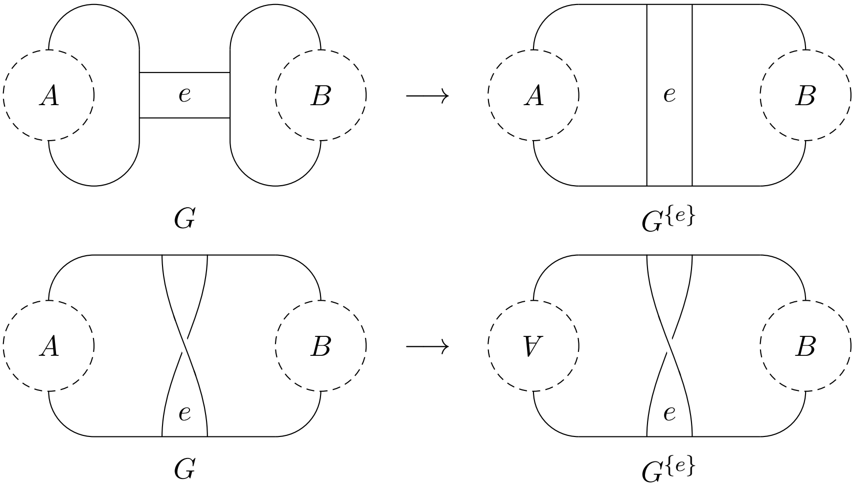


Fig. 2. Partial dual with respect to one edge-ribbon

2.2. Partial-dual genus polynomial. Let G be a ribbon graph. After gluing a disk to each boundary component of G, we obtain a closed surface S.

The *Euler genus* of G, denoted by $\varepsilon(G)$, is defined to be twice the genus of S if S is orientable, and otherwise the crosscap number of S. The Euler genus of G can be calculated from the formula

$$\varepsilon(G) = 2c(G) - |V(G)| + |E(G)| - |F(G)|,$$

where $c(G)$ denotes the number of components of G.

DEFINITION 2.1 ([11]). For a ribbon graph G, denote the set of vertex-disks, edge-ribbons and face-disks of G by $V(G), E(G)$ and $F(G)$ respectively, and let $c(G)$ be the number of components of G. The *partial-dual genus polynomial* of G is defined by

$$^{\partial}\varepsilon_G(z) = \sum_{A\subseteq E(G)} z^{\varepsilon(G^A)}.$$

Consider $A \subseteq E(G)$ as the ribbon subgraph obtained from G by removing from G all edge-ribbons out of A. Then we have the following combinatorial formula for calculating the partial-dual genus polynomial.

LEMMA 2.2. *For every ribbon graph G, we have*

$$^{\partial}\varepsilon_G(z) = z^{2c(G)+|E(G)|} \sum_{A\subseteq E(G)} z^{-|F(A)|-|F(A^c)|},$$

where A^c denotes the ribbon graph $E(G) \setminus A$.

Proof. For a subset $A \subseteq E(G)$, the Euler genus of the partial dual G^A equals

$$\begin{aligned}\varepsilon(G^A) &= 2c(G^A) - |V(G^A)| + |E(G^A)| - |F(G^A)| \\ &= 2c(G) + |E(G)| - |F(A)| - |F(A^c)|.\end{aligned}$$

The second equality is obtained from [11, Theorem 2.1] as well as the facts that $c(G) = c(G^A)$, $|E(G)| = |E(G^A)|$, $|V(G^A)| = |F(A)|$ and $|F(G^A)| = |V((G^A)^*)| = |V(G^{A^c})| = |F(A^c)|$. The result follows immediately. ∎

EXAMPLE 2.3. Consider the ribbon graph indexed by 11 in Figure 3. It has only one vertex-disk, which is the disk twisted to form a figure eight in the center, two edge-ribbons, which we order so that the left one precedes the right one. It has four subgraphs, denoted by 00, 01, 10, and 11 for short, where the index 1 means preserving the corresponding edge and 0 means removing it. Note that $|F(00)| = |F(01)| = |F(11)| = 1$ and $|F(10)| = 2$, thus we have

$$^{\partial}\varepsilon_{11}(z) = z^{2\times1+2} \cdot 2(z^{-1-1} + z^{-1-2}) = 2z^2 + 2z.$$

2.3. Graded partial-dual genus polynomial. For a given ribbon graph G, the component number $c(G)$ and edge number $|E(G)|$ are fixed, thus the essential part of the partial-dual genus polynomial $^{\partial}\varepsilon_G(z)$ is the

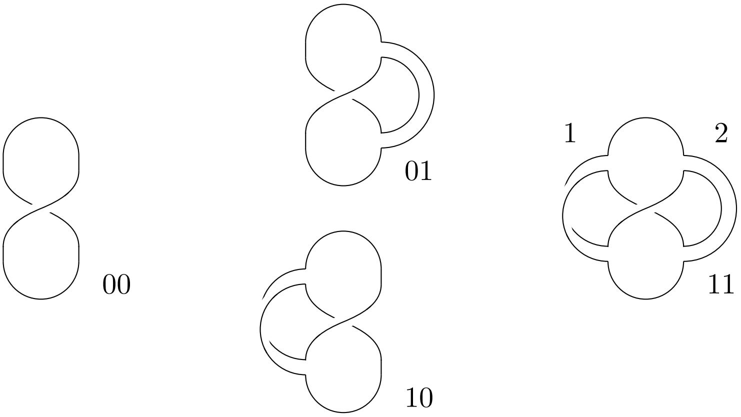


Fig. 3. A ribbon graph (indexed by 11) and its three proper subgraphs

sum $\sum_{A\subseteq E(G)} z^{-|F(A)|-|F(A^c)|}$, denoted by $e_G(z)$. Therefore, the partial-dual genus polynomial can be rewritten as

$$^{\partial}\varepsilon_G(z) = z^{2c(G)+|E(G)|}e_G(z).$$

Now we introduce a polynomial of ribbon graphs in two variables, which can be considered as a graded version of the partial-dual genus polynomial.

Definition 2.4. The *graded partial-dual genus polynomial* of any ribbon graph G is defined by

$$^{\partial}\tilde{\varepsilon}_G(w,z) = \sum_{A\subseteq E(G)} w^{|A|}z^{\varepsilon(G^A)} = z^{2c(G)+|E(G)|}\sum_{A\subseteq E(G)} w^{|A|}z^{-|F(A)|-|F(A^c)|}.$$

The graded partial-dual genus polynomial $^{\partial}\tilde{\varepsilon}_G(w,z)$ is a refined version of the partial-dual genus polynomial $^{\partial}\varepsilon_G(z)$: we split $^{\partial}\varepsilon_G(z)$ into $|E(G)|+1$ parts with respect to the number of edges in the subset A. Just like the partial-dual genus polynomial, this graded version is completely determined by the sum

$$\tilde{e}_G(w,z) = \sum_{A\subseteq E(G)} w^{|A|}z^{-|F(A)|-|F(A^c)|} \quad \text{up to a } z\text{-degree shift.}$$

Then we have

$$^{\partial}\tilde{\varepsilon}_G(w,z) = z^{2c(G)+|E(G)|}\tilde{e}_G(w,z), \tilde{e}_G(1,z) = e_G(z) \text{ and } {}^{\partial}\tilde{\varepsilon}_G(1,z) = {}^{\partial}\varepsilon_G(z).$$

Example 2.5. Consider the ribbon graph G indexed by 11 in Figure 3. Direct calculation shows that

$$\begin{aligned}\tilde{e}_G(w,z) &= z^{-2}+2wz^{-3}+w^2z^{-2},\\ {}^{\partial}\tilde{\varepsilon}_G(w,z) &= z^{2\times 1+2}\tilde{e}_G(w,z) = z^2+2zw+z^2w^2.\end{aligned}$$

In general, the coefficients of a partial-dual genus polynomial ${}^{\partial}\varepsilon_G(z)$ are not symmetric; see [11, Example 3.1] for an example. However, if the graded partial-dual genus polynomial of a ribbon graph G has the form

$$ {}^{\partial}\tilde{\varepsilon}_G(w,z) = \sum_{i=0}^{|E(G)|} f_i(z) w^i, $$

it is easy to conclude from the definition that $f_i(z) = f_{n-i}(z)$.

3. A punctured (1+1)D-TQFT.

The aim of this section is to introduce a punctured (1+1)D-TQFT, which will be used in the construction of the categorification of the graded partial-dual genus polynomial.

Let $M = \mathbb{Z}[\sqrt{3}, x]/\langle x^3\rangle$ over the ring $\mathbb{Z}[\sqrt{3}]$, equipped with the unit

$$ u : \mathbb{Z}[\sqrt{3}] \to M, \quad 1 \mapsto 1, $$

natural multiplication, Frobenius trace

$$ \epsilon : M \to \mathbb{Z}[\sqrt{3}], \quad 1, x \mapsto 0;\ x^2 \mapsto 1, $$

and the half-genus map

$$ h : M \to M, \quad 1 \mapsto \sqrt{3}x;\ x \mapsto \sqrt{3}x^2;\ x^2 \mapsto 0, $$

which can be seen as multiplication by $\sqrt{3}x$. It is a commutative Frobenius algebra and the Frobenius trace induces a unique comultiplication

$$ \begin{aligned} \Delta : M &\to M \otimes M, \\ 1 &\mapsto 1 \otimes x^2 + x \otimes x + x^2 \otimes 1, \\ x &\mapsto x \otimes x^2 + x^2 \otimes x, \\ x^2 &\mapsto x^2 \otimes x^2. \end{aligned} $$

Note that throughout this paper, $\otimes$ denotes the tensor product over the ring $\mathbb{Z}[\sqrt{3}]$.

Now we make M into a graded ring by assigning

$$ \deg 1 = 1, \quad \deg x = 0 \quad \text{and} \quad \deg x^2 = -1. $$

Then the multiplication m, the half-genus map h and the comultiplication Δ are all of degree -1, while the unit u and trace ϵ are of degree 1.

Remark 3.1. Here we use a different degree and an opposite trace compared with the Frobenius algebra used by Khovanov in the definition of sl(3) link homology [17]. The reason is that we do not need to realize M as the cohomology ring of some concrete space, nor hope that our half-genus map will introduce $\sqrt{-1}$.

The commutative Frobenius algebra M equipped with the half-genus map h gives rise to a functor $\mathcal{F}$ from the category of 2-dimensional oriented cobordisms equipped with punctured points to the category of graded abelian groups. On objects, $\mathcal{F}$ is given by $\mathcal{F}(\bigsqcup_k S^1) = M^{\otimes k}$. On morphisms, for each basic cobordism occurring in the first row of Figure 4, $\mathcal{F}$ maps it to the map listed below it.

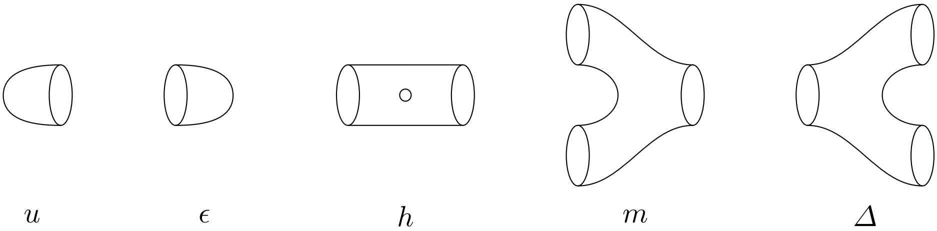
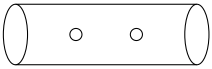
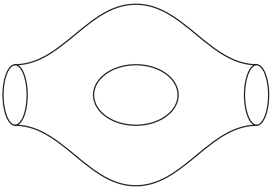


Fig. 4. Basic cobordisms and corresponding maps

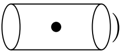
The name for the half-genus map comes from the relation

(3.1) $\mathcal{F}($  $) = \mathcal{F}($  $),$

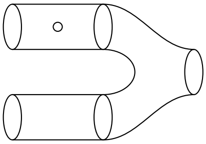
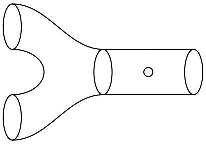
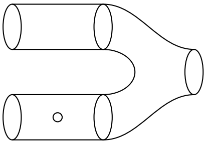
or algebraically,

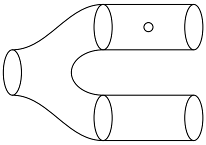
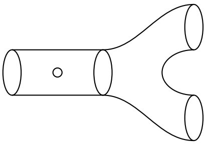
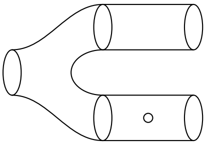
$$h^2 = m \circ \Delta.$$

That is, the addition of two punctures is equal to the addition of one genus.

Remark 3.2. Khovanov's sl(3) link homology uses a similar morphism which is decorated by a dot, and $\mathcal{F}($$)$ denotes multiplication by x, being different from our half-genus map by a coefficient $\sqrt{3}$. To emphasize this difference but inherit the similarity, here we use the notation of punctures.

Naturally, the punctures on cobordisms should be able to move freely, with the morphisms these cobordisms represent preserved. This follows from the following relations:

(3.2) $\mathcal{F}($  $) = \mathcal{F}($  $) = \mathcal{F}($  $),$

(3.3) $\mathcal{F}($  $) = \mathcal{F}($  $) = \mathcal{F}($  $),$

or algebraically,

$$m \circ (h \otimes \mathrm{Id}_M) = h \circ m = m \circ (\mathrm{Id}_M \otimes h),$$
$$(h \otimes \mathrm{Id}_M) \circ \Delta = \Delta \circ h = (\mathrm{Id}_M \otimes h) \circ \Delta.$$

Concisely speaking, h is a self-dual M-module endomorphism, thus it is multiplication by an element. These relations play an important role in our construction of commutative cubes.

Some other relations, especially the Frobenius relation

$$(\mathrm{Id}_M \otimes m) \circ (\Delta \otimes \mathrm{Id}_M) = \Delta \circ m = (m \otimes \mathrm{Id}_M) \circ (\mathrm{Id}_M \otimes \Delta),$$

are needed to prove that our cubes introduced later are commutative; these relations hold for any Frobenius algebra. More details can be found in [21].

REMARK 3.3. The reader who is familiar with topological quantum field theory, especially unoriented topological quantum field theory, may have realized that the algebraic structure we introduced in this section is not new. In order to construct a link homology theory for stable equivalence classes of links in thickened orientable surfaces, the notion of unoriented topological quantum field theory was introduced by Turaev and Turner [28]. Another closely related idea, the Klein topological field theory, was proposed by Alexeevski and Natanzon [2] in the context of open-closed field theory. The isomorphism classes of (1+1)D unoriented topological quantum theories were classified in terms of extended Frobenius algebras. The algebraic structure we discussed here happens to be a special case of *extended Frobenius algebra* in [28] and *structure algebra* in [2]. More precisely, by setting $N = 3$ in [28, Example 2.6], we obtain the Frobenius algebra M above. And the case for general N was used recently in [4] to categorify the *n-color polynomial*, which explains the dot in cobordisms as connected sum of 2-punctured projective planes. We will continue the discussion of other possible choices of algebraic structures in Section 8.

4. Commutative n-cube category and four cubes

4.1. Commutative n-cube category. An n-cube, geometrically, is an n-dimensional unit cube located in an n-dimensional rectangular coordinate system. For each vertex $A \in \{0,1\}^n$, we assign a linear space V_A and call it the *vertex space* at A. As usual, we denote by $|A|$ the number of 1's in A. To each edge l connecting $A, B \in \{0,1\}^n$, where B can be obtained from A by replacing a 0 with 1 (we simply say A, B satisfy the *edge condition*), we assign a linear map $\varphi_A^B : V_A \to V_B$ and call it the *edge map* from A to B. In this way we obtain an algebraic cube (V, φ). If for an arbitrary 2-dimensional face

$$\begin{tikzcd}
 & V_B \arrow[dr, "\varphi_B^D"] & \\
V_A \arrow[ur, "\varphi_A^B"] \arrow[dr, "\varphi_A^C"'] & & V_D \\
 & V_C \arrow[ur, "\varphi_C^D"'] &
\end{tikzcd}$$

of (V, φ) we have $\varphi_B^D \circ \varphi_A^B = \varphi_C^D \circ \varphi_A^C$, then we say (V, φ) is *commutative*. We say (V, φ) is *anti-commutative* if $\varphi_B^D \circ \varphi_A^B = -\varphi_C^D \circ \varphi_A^C$.

If there is a set of linear maps between the corresponding vertex spaces of two n-cubes, commuting with the edge maps, we call it an *n-cube morphism*. The *commutative n-cube category* is a category which consists of a collection of commutative n-cubes as objects and a collection of n-cube morphisms as morphisms. The reader is referred to [3] for more details.

The commutative n-cube category can be equipped with a natural tensor operator $\otimes$, defined below. For any two commutative n-cubes (V, φ) and (W, ϕ), we define

$$\begin{aligned}
&(V, \varphi) \otimes (W, \phi) := (V \otimes W, \varphi \otimes \phi), \\
&\qquad (V \otimes W)_A := V_A \otimes W_A, \ \forall A \in \{0,1\}^n, \\
&\qquad\quad (\varphi \otimes \phi)_A^B := \varphi_A^B \otimes \phi_A^B, \ \forall A, B \in \{0,1\}^n \text{ satisfying the edge condition.}
\end{aligned}$$

It is straightforward to verify that $(V \otimes W, \varphi \otimes \phi)$ is still a commutative n-cube, which means that $\otimes$ is a well defined operator in the commutative n-cube category.

REMARK 4.1. The definition of $\otimes$ can be extended to any two algebraic cubes, including anti-commutative cubes. It is easy to check that for any two algebraic n-cubes (V, φ) and (W, ϕ), if both (V, φ) and (W, ϕ) are anti-commutative, then $(V, \varphi) \otimes (W, \phi)$ is commutative. And, if one of (V, φ) and (W, ϕ) is commutative and the other one is anti-commutative, then $(V, \varphi) \otimes (W, \phi)$ is anti-commutative.

In the remaining part of this section, we construct one anti-commutative cube and three commutative cubes based on a given ribbon graph. According to Remark 4.1, we can take the tensor product of them to obtain an anti-commutative cube. A cochain complex derived from this anti-commutative cube will be given in the next section.

4.2. The anti-commutative n-cube (S, s). From now on, we always use n to denote the number of the edge-ribbons in a given ribbon graph G. For arbitrary $A, B, C, D \in \{0,1\}^n$ such that AB, AC, BD, CD are adjacent, let m_A^B denote the number of 1's in front of the element which is 0 in A and

1 in B. The integers m_A^C, m_B^D and m_C^D can be defined similarly. It is easy to find that

$$m_A^B + m_B^D - m_A^C - m_C^D = \pm 1,$$

which guarantees that the cube (S, s) defined below is anti-commutative.

Definition 4.2. There is an anti-commutative n-cube (S, s), where for each $A \in \{0,1\}^n$, the vertex space $S_A = \mathbb{Z}[\sqrt{3}]$, and for each $A, B \in \{0,1\}^n$ satisfying the edge condition, the edge map s_A^B is multiplication by $(-1)^{m_A^B}$. For brevity, we call it the *S-cube.*

Example 4.3. Let G be the ribbon graph indexed by 11 in Example 3, the S-cube of G is shown in Figure 5.

$$\begin{tikzcd}
 & \mathbb{Z}[\sqrt{3}] \arrow[dr, "1"] & \\
\mathbb{Z}[\sqrt{3}] \arrow[ur, "1"] \arrow[dr, "1"'] & & \mathbb{Z}[\sqrt{3}] \\
 & \mathbb{Z}[\sqrt{3}] \arrow[ur, "-1"'] &
\end{tikzcd}$$

Fig. 5. An S-cube

Here the letter S stands for sign, since if one takes the tensor product of a commutative n-cube with the S-cube, it just adds some signs to the edge maps of the original cube, which translates it to an anti-commutative n-cube.

4.3. The commutative n-cube (F, f). In this subsection, we introduce another graded algebra $N = \mathbb{Z}[\sqrt{3}, y]/\langle y^2 \rangle$, where the degree of 1 and y are equal to 0 and 1, respectively. This makes the edge map of the cube defined below have degree 0.

Definition 4.4. There is a commutative n-cube (F, f), where for each $A \in \{0,1\}^n$ we set $F_A = N^{\otimes |A|}$, and for each $A, B \in \{0,1\}^n$ which satisfy the edge condition, f_A^B is defined by $u \otimes \mathrm{Id}_N^{\otimes |A|}$, that is,

$$f_A^B : N^{\otimes |A|} \to N^{\otimes (|A|+1)}, \qquad \alpha \mapsto 1 \otimes \alpha.$$

We simply call this cube the *F-cube.*

Remark 4.5. The main aim of introducing this cube is to erase the alternating terms of the graded (quantum) Euler characteristic of the cochain complex given later. This method was also used by Vershinin and Vesnin in the categorification of the Yamada polynomial [29]. Actually, F_A can be chosen to be any graded $\mathbb{Z}[\sqrt{3}]$-module and f_A^B can be chosen to be any graded morphism. The construction of this definition is chosen to make the cohomology groups we obtain not only easy to calculate but also preserve as much information as possible.

Example 4.6. For the ribbon graph indexed by 11 in Figure 3, the F-cube is depicted in Figure 6.

$$\begin{tikzcd}
 & N \arrow[dr, "u\otimes \mathrm{Id}_N"] & \\
\mathbb{Z}[\sqrt{3}] \arrow[ur, "u"] \arrow[dr, "u"'] & & N^{\otimes 2} \\
 & N \arrow[ur, "u\otimes \mathrm{Id}_N"] &
\end{tikzcd}$$

Fig. 6. An F-cube

4.4. The commutative n-cube (V, v). Fix an order of the edge-ribbons of the ribbon graph G, so that any $A \in \{0,1\}^n$ can be seen as a subset of $E(G)$, where 0 means rejecting the corresponding edge-ribbon and 1 means including the corresponding one. As before, we also use A to refer to the subgraph obtained from G by removing the edges not in A.

Now we give the definition of the commutative cube (V, v).

Definition 4.7. For each $A \in \{0,1\}^n$, we define $V_A = M^{\otimes F(A)}$ (1). For all $A, B \in \{0,1\}^n$ satisfying the edge condition,

- if $|F(B)| = |F(A)| - 1$, then the addition of the new edge-ribbon merges two circles into one circle, and we assign multiplication m on $M \otimes M$ corresponding to these two circles and identity Id_M to other circles, and take the tensor product of these operators together to obtain the edge map v_A^B;
- if $|F(B)| = |F(A)| + 1$, then the addition of the new edge-ribbon splits one circle into two circles, and we assign a comultiplication Δ on M corresponding to this circle and identity Id_M to other circles, and take the tensor product of these operators together to obtain the edge map v_A^B;
- if $|F(B)| = |F(A)|$, then the addition of the new edge-ribbon translates one circle into another one, and we assign a half-genus map h on M corresponding to this circle and identity Id_M to other circles and take the tensor product of these operators together to obtain the edge map v_A^B.

We call this cube the *V-cube* for short.

---

(1) Here, our definition may seem less rigorous because a set is used as a superscript instead of the number of its elements. This is because we want this tensor product to follow the order of the set.

Remark 4.8. The first and second cases appear in Khovanov's approach to the categorification of the Jones polynomial [16], and the third case occurs because here these circles are no longer restricted to lie in $\mathbb{S}^2$ or $\mathbb{R}^2$. In order to address this new problem, we have to propose the half-genus map h.

Lemma 4.9. *The V-cube is commutative.*

Proof. If the half-genus map h is not involved in a 2-dimensional face of (V, v), the commutativity follows from the associativity of the multiplication, the coassociativity of the comultiplication and the Frobenius relation. If h is involved, the commutativity can be derived from (3.1)–(3.3). ■

Example 4.10. For the ribbon graph indexed by 11 in Figure 3, observing that $|F(00)| = |F(01)| = |F(11)| = 1$ and $|F(10)| = 2$, we know that the spaces assigned to 00, 01, 11 are all M and the space associated to 10 is $M \otimes M$, and the edge map is shown in Figure 7. It is commutative since $m \circ \Delta = h^2$.

$$\begin{tikzcd}
& M \arrow[dr, "h"] & \\
M \arrow[ur, "h"] \arrow[dr, "\Delta"] & & M \\
& M \otimes M \arrow[ur, "m"] &
\end{tikzcd}$$

Fig. 7. A V-cube

4.5. The commutative n-cube (W, w)

Definition 4.11. For each $A \in \{0,1\}^n$, we set $W_A = M^{\otimes F(A^c)}$, where $A^c = E(G) \setminus A$. And for all $A, B \in \{0,1\}^n$ which satisfy the edge condition, depending on the value of $|F(B^c)| - |F(A^c)| \in \{-1, 0, +1\}$, we define w_A^B to be the edge map v introduced in Definition 4.7. Let us call it the *W-cube* for short.

Similarly to the proof of Lemma 4.9, one can show that the cube (W, w) is also commutative.

Example 4.12. The W-cube of the ribbon graph indexed by 11 in Figure 3 can be found in Figure 8.

Remark 4.13. In any Frobenius algebra (A, m, Δ, ϵ) on a coefficient ring R, the map

$$A \otimes A \xrightarrow{m} A \xrightarrow{\epsilon} R$$

induces a self-dual isomorphism $A^* \cong A$. For M, this isomorphism maps x^i to x^{2-i} for $i \in \{0, 1, 2\}$. It is easy to check that multiplication m and

$$\begin{tikzcd}
& M \otimes M \arrow[dr, "m"] & \\
M \arrow[ur, "\Delta"] \arrow[dr, "h"'] & & M \\
& M \arrow[ur, "h"'] &
\end{tikzcd}$$

Fig. 8. An W-cube

comultiplication Δ are dual to each other with respect to this isomorphism, as also are the trace map ϵ and the unit map u. Hence the W-cube actually can be seen as the dual cube of the V-cube. The concept of dual cube is of less concern in this paper so we omit the details.

5. Cochain complex. In this section, we introduce a cochain complex such that the partial-dual genus polynomial ${}^{\partial}\varepsilon_G(z)$ can be recovered from the graded Euler characteristic of this cochain complex. Actually, the polynomial being categorified is not the partial-dual genus polynomial, but the graded partial-dual genus polynomial. More precisely, the graded Euler characteristic of this cochain complex equals the 2-variable polynomial $\tilde{e}_G(w,z)$. In order to do this, for any ribbon graph G, we construct an anti-commutative cube $(\mathrm{Cube}(G), d)$ by taking the tensor product of its S-, F-, V-, W-cubes together. We name it the *partial dual cube* of the ribbon graph G. Based on this cube, we construct a cochain complex and prove that its cohomology groups categorify the polynomial $\tilde{e}_G(w,z)$.

Example 5.1. The partial dual cube of the graph indexed by 11 in Figure 3 is shown in Figure 9.

$$\begin{tikzcd}
& N \otimes M^{\otimes 3} \arrow[dr, "u\otimes \mathrm{Id}_N \otimes h \otimes m"] & \\
M^{\otimes 2} \arrow[ur, "u\otimes h \otimes \Delta"] \arrow[dr, "u\otimes \Delta \otimes h"] & & N^{\otimes 2} \otimes M^{\otimes 2} \\
& N \otimes M^{\otimes 3} \arrow[ur, "-u\otimes \mathrm{Id}_N \otimes m \otimes h"] &
\end{tikzcd}$$

Fig. 9. The partial dual cube of the rightmost ribbon graph in Figure 3

The grading structures of M and N induce a bigrading structure for each vertex space of $(\mathrm{Cube}(G), d)$. That is, for each monoid in a vertex space, its bidegree is a couple of integers, where the first (second, resp.) is the degree induced by the elements in N (M, resp.). For example, the bidegree of $1_N \otimes y \otimes 1_M \otimes x \otimes x^2 \in N^{\otimes 2} \otimes M^{\otimes 3}$ is $(0+1, 1+0+(-1)) = (1,0)$, where 1_N and 1_M are the identity elements in N and M, respectively.

A $\mathbb{Z}\oplus\mathbb{Z}$-graded, or a bigraded $\mathbb{Z}[\sqrt{3}]$-module is a $\mathbb{Z}[\sqrt{3}]$-module $\mathcal{M}$ with a family of submodules $\mathcal{M}_{m,k}$, where $m, k \in \mathbb{Z}$, such that $\mathcal{M} = \bigoplus_{m,k\in\mathbb{Z}} \mathcal{M}_{m,k}$, and the elements of $\mathcal{M}_{m,k}$ are homogeneous elements of bidegree (m, k). The *bigraded dimension* (or *biquantum dimension*) of $\mathcal{M}$ over $\mathbb{Z}[\sqrt{3}]$ is a 2-variable power series

$$\operatorname{qdim} M = \sum_{m,k\in\mathbb{Z}} p^m q^k \cdot \dim_{\mathbb{R}}(\mathcal{M}_{m,k} \otimes \mathbb{R}).$$

Following [16], we use the notation $\{l\}$ $(l \in \mathbb{Z})$ to denote the shift operator on the second grading, which decreases the second grading of each monoid by l. In other words, we have

$$\mathcal{M}\{l\}_{m,k} = \mathcal{M}_{m,k+l},$$

where $\mathcal{M}$ is a bigraded $\mathbb{Z}[\sqrt{3}]$-module. Now we can define our cochain complex.

Definition 5.2. For a given ribbon graph G, we define the *partial dual cochain complex* to be $(C(G), \partial)$, where the cochain groups are defined by

$$C^i(G) = \bigoplus_{|A|=i} \operatorname{Cube}(G)_A\{-2i\},$$

and the coboundary map $\partial^i : C^i(G) \to C^{i+1}(G)$ is given by $\partial^i = \sum d_A^B$, where the sum is taken over all pairs $\{A, B\}$ such that $|A| = i, |B| = i+1$ and A, B satisfy the edge condition.

Theorem 5.3. *For an arbitrary ribbon graph G, $(C(G), \partial)$ is a bigraded cochain complex and the cohomology groups $H^*(G)$ are well defined ribbon graph invariants. In particular, the graded Euler characteristic of the cohomology groups $H^*(G)$ is equal to $\tilde{e}_G(w, z)$ evaluated at $w = -q^2 - pq^2$ and $z = (q^{-1} + 1 + q)^{-1}$.*

Proof. The reason why the coboundary maps satisfy $\partial^{i+1} \circ \partial^i = 0$ is that $(\operatorname{Cube}(G), d)$ is anti-commutative, as it is the tensor product of one anti-commutative cube and three commutative cubes. On the other hand, since m, Δ, h are all of degree $(0,-1)$ and f is of degree $(0,0)$, thanks to the shift operator $\{-2i\}$, now the differential ∂ preserves the bidegree. More precisely, each edge map in the V-cube and W-cube is given by tensor of identities and one of m, Δ and h, and is of degree $(0,-1)$. As the edge maps s, f of the S, F-cubes preserve the bidegree, the tensor product of these four cubes $\operatorname{Cube}(G)$ is equipped with edge maps of degree $(0,-2)$. As a linear combination of these maps, the differential ∂ preserves the degree after the shift $\{-2i\}$. Hence $(C(G), \partial)$ is a bigraded cochain complex. That is, decomposing elements by bidegree yields

$$(C(G),\partial)=\bigoplus_{m,k\in\mathbb{Z}}(C(G)_{m,k},\partial_{m,k}),$$

where $C(G)_{m,k}$ is the submodule of $C(G)$ generated by monoids with degree (m,k) and $\partial_{m,k}$ is the restriction of ∂ on $C(G)_{m,k}$. Furthermore, if we use $H^*_{m,k}(G)$ to denote the cohomology groups of $(C(G)_{m,k},\partial_{m,k})$, then we have

$$H^*(G)=\bigoplus_{m,k\in\mathbb{Z}}H^*_{m,k}(G).$$

Since the definition of the cochain complex needs a fixed order of the edges of G, in order to show the cohomology groups $H^*(G)$ are well defined, we need to prove that $H^*(G)$ actually are independent of the choice of this ordering.

Assume $E(G)=\{e_1,\dots,e_i,e_{i+1},\dots,e_n\}$ and G' is the same ribbon graph as G but the edges are reordered as $E(G')=\{e_1,\dots,e_{i+1},e_i,\dots,e_n\}$, i.e. the positions of e_i and e_{i+1} are switched. It is sufficient to show that $H^*(G)$ and $H^*(G')$ are isomorphic, since any other order can be obtained from the standard one via finitely many permutations of this kind. Since $C^i(G)=\bigoplus_{|A|=i}\mathrm{Cube}(G)_A\{-2i\}$, it is enough to define an isomorphism f_A restricted on each submodule $\mathrm{Cube}(G)_A\{-2i\}$. Consider the map $f_A:\mathrm{Cube}(G)_A\{-2i\}\to\mathrm{Cube}(G')_A\{-2i\}$, which is defined by

$$f_A=\begin{cases}-\mathrm{id} & \text{if } \{e_i,e_{i+1}\}\subseteq A,\\ \mathrm{id} & \text{otherwise.}\end{cases}$$

Now we define the map $f:C^i(G)\to C^i(G')$ by $f=\bigoplus_{A\subseteq E(G),\,|A|=i}f_A$. It is a routine exercise to check that f is a cochain map which induces the desired isomorphism from $H^i(G)$ to $H^i(G')$.

For the graded Euler characteristic of $H^*(G)$, since the differential is bidegree preserving and all cochain groups are finite dimensional, one observes that the graded (quantum) Euler characteristic of $(C(G),\partial)$ equals

$$\begin{aligned}\chi_q(C)&=\sum_{i=0}^{|E(G)|}(-1)^i\,\mathrm{qdim}\,H^i(G)=\sum_{i=0}^{|E(G)|}(-1)^i\,\mathrm{qdim}\,C^i(G)\\ &=\sum_{i=0}^{|E(G)|}(-1)^i\sum_{|A|=i}\mathrm{qdim}\,\mathrm{Cube}(G)_A\{-2i\}\\ &=\sum_{A\subseteq E(G)}(-1)^{|A|}q^{2|A|}\,\mathrm{qdim}(S_A\otimes F_A\otimes V_A\otimes W_A)\\ &=\sum_{A\subseteq E(G)}(-q^2)^{|A|}(\mathrm{qdim}\,N)^{|A|}(\mathrm{qdim}\,M)^{|F(A)|+|F(A^c)|}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{A\subseteq E(G)} (-q^2)^{|A|}(1+p)^{|A|}(q^{-1}+1+q)^{|F(A)|+|F(A^c)|}\\
&= \sum_{A\subseteq E(G)} (-q^2-pq^2)^{|A|}(q^{-1}+1+q)^{|F(A)|+|F(A^c)|}\\
&\xlongequal{w=-q^2-pq^2,\, z=(q^{-1}+1+q)^{-1}} \sum_{A\subseteq E(G)} w^{|A|}z^{-|F(A)|-|F(A^c)|}\\
&= \tilde{e}_G(w,z). \blacksquare
\end{aligned}
$$

Corollary 5.4. *For a given ribbon graph G, both the graded partial-dual genus polynomial and the partial-dual genus polynomial can be recovered from the graded Euler characteristic of the cohomology groups $H^*(G)$.*

Proof. According to Theorem 5.3, we have

$$
\begin{aligned}
{}^{\partial}\tilde{\varepsilon}_G(w,z) &= z^{2c(G)+|E(G)|}\chi_q(C)\Big|_{-q^2-pq^2=w,\, q^{-1}+1+q=z^{-1}},\\
{}^{\partial}\varepsilon_G(z) &= z^{2c(G)+|E(G)|}\chi_q(C)\Big|_{-q^2-pq^2=1,\, q^{-1}+1+q=z^{-1}}. \blacksquare
\end{aligned}
$$

6. An example. In this section, we compute the cohomology groups of the rightmost ribbon graph shown in Figure 3. The cochain complex is as the bottom row of Figure 10, where the other cochain groups vanish. On the other hand, given that this complex is bigraded, we use the subscript (j,k) to represent corresponding objects with bidegree (j,k). Additionally, with the degree shift, these cohomology groups are nonzero only if the first degree j is $0,1,2$ and the second degree k is $-2,-1,\ldots,6$. Hence we can calculate the cohomology groups of these $3\times 9 = 27$ subcomplexes one by one and the desired cohomology is the direct sum of them.

$$\begin{tikzcd}
 & N\otimes M^{\otimes 3}\{-2\} \arrow[dr, "-u\otimes \mathrm{Id}_N\otimes h\otimes m"] & \\
M^{\otimes 2} \arrow[ur, "u\otimes h\otimes \Delta"] \arrow[dr, "u\otimes \Delta\otimes h"'] \arrow[dd, equal] & \oplus & N^{\otimes 2}\otimes M^{\otimes 2}\{-4\} \arrow[dd, equal] \\
 & N\otimes M^{\otimes 3}\{-2\} \arrow[ur, "u\otimes \mathrm{Id}_N\otimes m\otimes h"] \arrow[d, equal] & \\
C^0 \arrow[r, "\partial^0"] & C^1 \arrow[r, "\partial^1"] & C^2
\end{tikzcd}$$

Fig. 10. The cochain complex

Due to the special structure of the map f, the cases with the first degree equal to 0 is relatively intricate. We list our calculation process as follows, where the subscript of the unit element in M is omitted for short.

- $(j,k)=(0,-2)$:

$$C^0_{(0,-2)}=\langle x^2\otimes x^2\rangle=\mathbb{Z}[\sqrt{3}],\qquad C^1_{(0,-2)}=0,\qquad C^2_{(0,-2)}=0;$$
$$H^0_{(0,-2)}=\mathbb{Z}[\sqrt{3}],\qquad H^1_{(0,-2)}=0,\qquad H^2_{(0,-2)}=0.$$

- $(j,k)=(0,-1)$:

$$C^0_{(0,-1)}=\langle x\otimes x^2, x^2\otimes x\rangle=\bigoplus_2\mathbb{Z}[\sqrt{3}],$$
$$C^1_{(0,-1)}=\left\langle\begin{pmatrix}1_N\otimes x^2\otimes x^2\otimes x^2\\0\end{pmatrix},\begin{pmatrix}0\\1_N\otimes x^2\otimes x^2\otimes x^2\end{pmatrix}\right\rangle=\bigoplus_2\mathbb{Z}[\sqrt{3}],$$
$$C^2_{(0,-1)}=0;$$
$$\partial^0: x\otimes x^2\mapsto\sqrt{3}\begin{pmatrix}1_N\otimes x^2\otimes x^2\otimes x^2\\0\end{pmatrix},$$
$$x^2\otimes x\mapsto\sqrt{3}\begin{pmatrix}0\\1_N\otimes x^2\otimes x^2\otimes x^2\end{pmatrix};$$
$$H^0_{(0,-1)}=0,\quad H^1_{(0,-1)}=\bigoplus_2\mathbb{Z}[\sqrt{3}]/\sqrt{3}\mathbb{Z}[\sqrt{3}]=\bigoplus_2\mathbb{Z}_3,\quad H^2_{(0,-1)}=0.$$

- $(j,k)=(0,0)$:

$$C^0_{(0,0)}=\langle 1\otimes x^2, x\otimes x, x^2\otimes 1\rangle=\bigoplus_3\mathbb{Z}[\sqrt{3}],$$
$$C^1_{(0,0)}=\left\langle\begin{pmatrix}1_N\otimes x\otimes x^2\otimes x^2\\0\end{pmatrix},\begin{pmatrix}1_N\otimes x^2\otimes x\otimes x^2\\0\end{pmatrix},\right.$$
$$\begin{pmatrix}1_N\otimes x^2\otimes x^2\otimes x\\0\end{pmatrix},\begin{pmatrix}0\\1_N\otimes x\otimes x^2\otimes x^2\end{pmatrix},$$
$$\left.\begin{pmatrix}0\\1_N\otimes x^2\otimes x\otimes x^2\end{pmatrix},\begin{pmatrix}0\\1_N\otimes x^2\otimes x^2\otimes x\end{pmatrix}\right\rangle=\bigoplus_6\mathbb{Z}[\sqrt{3}],$$
$$C^2_{(0,0)}=0;$$
$$\partial^0:\ 1\otimes x^2\mapsto\sqrt{3}\begin{pmatrix}1_N\otimes x\otimes x^2\otimes x^2\\0\end{pmatrix},$$
$$x\otimes x\mapsto\sqrt{3}\begin{pmatrix}1_N\otimes x^2\otimes(x\otimes x^2+x^2\otimes x)\\1_N\otimes(x\otimes x^2+x^2\otimes x)\otimes x^2\end{pmatrix},$$
$$x^2\otimes 1\mapsto\sqrt{3}\begin{pmatrix}0\\1_N\otimes x^2\otimes x^2\otimes x\end{pmatrix};$$
$$H^0_{(0,0)}=0,\quad H^1_{(0,0)}=\Big(\bigoplus_3\mathbb{Z}[\sqrt{3}]\Big)\oplus\Big(\bigoplus_3\mathbb{Z}_3\Big),\quad H^2_{(0,0)}=0.$$

- $(j,k) = (0,1)$:

$$C^0_{(0,1)} = \langle 1 \otimes x, x \otimes 1 \rangle = \bigoplus_2 \mathbb{Z}[\sqrt{3}],$$

$$C^1_{(0,1)} = \left\langle \begin{pmatrix} 1_N \otimes 1 \otimes x^2 \otimes x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1_N \otimes x^2 \otimes 1 \otimes x^2 \\ 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1_N \otimes x^2 \otimes x^2 \otimes 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1_N \otimes x^2 \otimes x \otimes x \\ 0 \end{pmatrix},$$

$$\left. \begin{pmatrix} 1_N \otimes x \otimes x^2 \otimes x \\ 0 \end{pmatrix}, \begin{pmatrix} 1_N \otimes x \otimes x \otimes x^2 \\ 0 \end{pmatrix}, \dots \right\rangle = \bigoplus_{12} \mathbb{Z}[\sqrt{3}],$$

$$C^2_{(0,1)} = 0;$$

where the ellipsis means to repeat the elements before it with the two rows switched;

$$\partial^0 : 1 \otimes x \mapsto \sqrt{3} \begin{pmatrix} 1_N \otimes x \otimes (x^2 \otimes x + x \otimes x^2) \\ 1_N \otimes (1 \otimes x^2 + x \otimes x + x^2 \otimes 1) \otimes x^2 \end{pmatrix},$$

$$x \otimes 1 \mapsto \sqrt{3} \begin{pmatrix} 1_N \otimes x^2 \otimes (1 \otimes x^2 + x \otimes x + x^2 \otimes 1) \\ 1_N \otimes (x \otimes x^2 + x^2 \otimes x) \otimes x \end{pmatrix};$$

$$H^0_{(0,1)} = 0, \quad H^1_{(0,1)} = \Big(\bigoplus_{10} \mathbb{Z}[\sqrt{3}] \Big) \oplus \Big(\bigoplus_2 \mathbb{Z}_3 \Big), \quad H^2_{(0,1)} = 0.$$

- $(j,k) = (0,2)$:

$$C^0_{(0,2)} = \langle 1 \otimes 1 \rangle = \mathbb{Z}[\sqrt{3}],$$

$$C^1_{(0,2)} = \left\langle \begin{pmatrix} 1_N \otimes 1 \otimes x \otimes x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1_N \otimes x \otimes 1 \otimes x^2 \\ 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1_N \otimes x^2 \otimes 1 \otimes x \\ 0 \end{pmatrix}, \begin{pmatrix} 1_N \otimes 1 \otimes x^2 \otimes x \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 1_N \otimes x \otimes x^2 \otimes 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1_N \otimes x^2 \otimes x \otimes 1 \\ 0 \end{pmatrix},$$

$$\left. \begin{pmatrix} 1_N \otimes x \otimes x \otimes x \\ 0 \end{pmatrix}, \dots \right\rangle = \bigoplus_{14} \mathbb{Z}[\sqrt{3}],$$

$$C^2_{(0,2)} = \langle 1_N \otimes 1_N \otimes x^2 \otimes x^2 \rangle = \mathbb{Z}[\sqrt{3}];$$

$$H^0_{(0,2)} = 0, \quad H^1_{(0,2)} = \Big(\bigoplus_{12} \mathbb{Z}[\sqrt{3}] \Big) \oplus \mathbb{Z}_3, \quad H^2_{(0,2)} = \mathbb{Z}_3.$$

- $(j,k)=(0,3)$:

$$C^0_{(0,3)}=0,$$

$$C^1_{(0,3)}=\left\langle\begin{pmatrix}1_N\otimes 1\otimes x\otimes x\\ 0\end{pmatrix},\begin{pmatrix}1_N\otimes x\otimes 1\otimes x\\ 0\end{pmatrix},\right.$$

$$\begin{pmatrix}1_N\otimes x\otimes x\otimes 1\\ 0\end{pmatrix},\begin{pmatrix}1_N\otimes x^2\otimes 1\otimes 1\\ 0\end{pmatrix},$$

$$\left.\begin{pmatrix}1_N\otimes 1\otimes x^2\otimes 1\\ 0\end{pmatrix},\begin{pmatrix}1_N\otimes 1\otimes 1\otimes x^2\\ 0\end{pmatrix},\dots\right\rangle=\bigoplus_{12}\mathbb{Z}[\sqrt{3}],$$

$$C^3_{(0,3)}=\langle 1_N\otimes 1_N\otimes x\otimes x^2, 1_N\otimes 1_N\otimes x^2\otimes x\rangle=\bigoplus_{2}\mathbb{Z}[\sqrt{3}];$$

$$H^0_{(0,3)}=0,\quad H^1_{(0,3)}=\bigoplus_{10}\mathbb{Z}[\sqrt{3}],\quad H^3_{(0,3)}=\bigoplus_{2}\mathbb{Z}_3.$$

- $(j,k)=(0,4)$:

$$C^0_{(0,4)}=0,$$

$$C^1_{(0,4)}=\left\langle\begin{pmatrix}1_N\otimes 1\otimes 1\otimes x\\ 0\end{pmatrix},\begin{pmatrix}1_N\otimes 1\otimes x\otimes 1\\ 0\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}1_N\otimes x\otimes 1\otimes 1\\ 0\end{pmatrix},\dots\right\rangle=\bigoplus_{6}\mathbb{Z}[\sqrt{3}],$$

$$C^2_{(0,4)}=\langle 1_N\otimes 1_N\otimes x\otimes x, 1_N\otimes 1_N\otimes 1\otimes x^2,$$

$$1_N\otimes 1_N\otimes x^2\otimes 1\rangle=\bigoplus_{3}\mathbb{Z}[\sqrt{3}];$$

$$H^0_{(0,4)}=0,\quad H^1_{(0,4)}=\bigoplus_{3}\mathbb{Z}[\sqrt{3}],\quad H^2_{(0,4)}=\bigoplus_{3}\mathbb{Z}_3.$$

- $(j,k)=(0,5)$:

$$C^0_{(0,5)}=0,$$

$$C^1_{(0,5)}=\left\langle\begin{pmatrix}1_N\otimes 1\otimes 1\otimes 1\\ 0\end{pmatrix},\begin{pmatrix}0\\ 1_N\otimes 1\otimes 1\otimes 1\end{pmatrix}\right\rangle=\bigoplus_{2}\mathbb{Z}[\sqrt{3}],$$

$$C^2_{(0,5)}=\langle 1_N\otimes 1_N\otimes 1\otimes x, 1_N\otimes 1_N\otimes x\otimes 1\rangle=\bigoplus_{2}\mathbb{Z}[\sqrt{3}].$$

$$H^0_{(0,5)}=0,\quad H^1_{(0,5)}=0,\quad H^2_{(0,5)}=\bigoplus_{2}\mathbb{Z}_3.$$

- $(j,k)=(0,6)$:

$$C^0_{(0,6)}=0,\quad C^1_{(0,6)}=0,\quad C^2_{(0,6)}=\langle 1_N\otimes 1_N\otimes 1\otimes 1\rangle=\mathbb{Z}[\sqrt{3}];$$

$$H^0_{(0,6)}=0,\quad H^1_{(0,6)}=0,\quad H^2_{(0,6)}=\mathbb{Z}[\sqrt{3}].$$

When the first degree is greater than 1, we have

$$C^0_{(\geq 1,*)} = H^0_{(\geq 1,*)} = 0,$$

and we will not repeat it in the process.

- $(j,k) = (1,-2)$:

$$C^1_{(0,-2)} = C^2_{(0,-2)} = 0, \qquad H^1_{(0,-2)} = H^2_{(0,-2)} = 0.$$

- $(j,k) = (1,-1)$:

$$C^1_{(1,-1)} = \bigoplus_2 \mathbb{Z}[\sqrt{3}], \quad C^2_{(1,-1)} = 0; \qquad H^1_{(1,-1)} = \bigoplus_2 \mathbb{Z}[\sqrt{3}], \quad H^2_{(1,-1)} = 0.$$

- $(j,k) = (1,0)$:

$$C^1_{(1,0)} = \bigoplus_6 \mathbb{Z}[\sqrt{3}], \quad C^2_{(1,0)} = 0; \qquad H^1_{(1,0)} = \bigoplus_6 \mathbb{Z}[\sqrt{3}], \quad H^2_{(1,0)} = 0.$$

- $(j,k) = (1,1)$:

$$C^1_{(1,1)} = \bigoplus_{12} \mathbb{Z}[\sqrt{3}], \quad C^2_{(1,1)} = 0; \qquad H^1_{(1,1)} = \bigoplus_{12} \mathbb{Z}[\sqrt{3}], \quad H^2_{(1,1)} = 0.$$

- $(j,k) = (1,2)$:

$$\begin{aligned} &C^1_{(1,2)} = \bigoplus_{14} \mathbb{Z}[\sqrt{3}], \\ &C^2_{(1,2)} = \langle 1_N \otimes y \otimes x^2 \otimes x^2, y \otimes 1_N \otimes x^2 \otimes x^2 \rangle = \bigoplus_2 \mathbb{Z}[\sqrt{3}]; \\ &H^1_{(1,2)} = \bigoplus_{13} \mathbb{Z}[\sqrt{3}], \quad H^2_{(1,2)} = \mathbb{Z}[\sqrt{3}] \oplus \mathbb{Z}_3. \end{aligned}$$

- $(j,k) = (1,3)$:

$$\begin{aligned} &C^1_{(1,3)} = \bigoplus_{12} \mathbb{Z}[\sqrt{3}], \quad C^2_{(1,3)} = \bigoplus_4 \mathbb{Z}[\sqrt{3}]; \\ &H^1_{(1,3)} = \bigoplus_{10} \mathbb{Z}[\sqrt{3}], \quad H^2_{(1,3)} = \Big(\bigoplus_2 \mathbb{Z}[\sqrt{3}] \Big) \oplus (\bigoplus_2 \mathbb{Z}_3). \end{aligned}$$

- $(j,k) = (1,4)$:

$$\begin{aligned} &C^1_{(1,4)} = \bigoplus_6 \mathbb{Z}[\sqrt{3}], \quad C^2_{(1,4)} = \bigoplus_6 \mathbb{Z}[\sqrt{3}]; \\ &H^1_{(1,4)} = \bigoplus_3 \mathbb{Z}[\sqrt{3}], \quad H^2_{(1,4)} = \Big(\bigoplus_3 \mathbb{Z}[\sqrt{3}] \Big) \oplus (\bigoplus_3 \mathbb{Z}_3). \end{aligned}$$

- $(j,k) = (1,5)$:

$$\begin{aligned} &C^1_{(1,5)} = \bigoplus_2 \mathbb{Z}[\sqrt{3}], \quad C^2_{(1,5)} = \bigoplus_4 \mathbb{Z}[\sqrt{3}]; \\ &H^1_{(1,5)} = 0, \quad H^2_{(1,5)} = \Big(\bigoplus_2 \mathbb{Z}[\sqrt{3}] \Big) \oplus (\bigoplus_2 \mathbb{Z}_3). \end{aligned}$$

- $(j,k) = (1,6)$:

$$C^1_{(1,6)} = 0, \quad C^2_{(1,6)} = \bigoplus_2 \mathbb{Z}[\sqrt{3}]; \qquad H^1_{(1,6)} = 0, \quad H^2_{(1,6)} = \bigoplus_2 \mathbb{Z}[\sqrt{3}].$$

If the first degree is 2, the situation is trivial, since

$$C^0_{(2,*)} = C^1_{(2,*)} = H^0_{(2,*)} = H^1_{(2,*)} = 0, \quad C^2_{(2,*)} = H^2_{(2,*)} = y\otimes y\otimes M\otimes M\{-4\}.$$

In particular, we have

$$H^2_{(2,2)} = H^2_{(2,6)} = \mathbb{Z}[\sqrt{3}], \quad H^2_{(2,3)} = H^2_{(2,5)} = \bigoplus_2 \mathbb{Z}[\sqrt{3}], \quad H^2_{(2,4)} = \bigoplus_3 \mathbb{Z}[\sqrt{3}].$$

The graded Euler characteristic of the homology groups can be written as follows:

$$\begin{aligned}
\chi_q(C(G)) &= \sum_{i,j,k} (-1)^i p^j q^k \dim H^i_{(j,k)}(G) \\
&= p^0(q^{-2} - 3 - 10q - 12q^2 - 10q^3 - 3q^4 + q^6) \\
&\quad + p^1(-2q^{-1} - 6 - 12q - 12q^2 - 8q^3 + 2q^5 + 2q^6) \\
&\quad + p^2(q^2 + 2q^3 + 3q^4 + 2q^5 + q^6) \\
&\overset{p=-w/q^2-1}{=\!=\!=\!=\!=} q^{-2} + 2q^{-1} + 3 + 2q + q^2 \\
&\quad + 2w(q^{-3} + 3q^{-2} + 6q^{-1} + 7 + 6q + 3q^2 + q^3) \\
&\quad + w^2(q^{-2} + 2q^{-1} + 3 + 2q + q^2) \\
&\overset{q+1+q^{-1}=z^{-1}}{=\!=\!=\!=\!=\!=} z^{-2} + 2wz^{-3} + w^2 z^{-2} \\
&= \tilde{e}_G(w,z).
\end{aligned}$$

It follows that

$${}^{\partial}\tilde{\varepsilon}_G(w,z) = z^{2c(G)+|E(G)|}\tilde{e}_G(w,z) = z^2 + 2wz + w^2z^2$$

and

$${}^{\partial}\varepsilon_G(z) = {}^{\partial}\tilde{\varepsilon}_G(1,z) = 2z^2 + 2z,$$

which coincides with the results we obtained in Examples 2.3 and 2.5. On the other hand, notice that some terms in the cohomology groups cancel out when one calculates the graded Euler characteristic. Consider $H^1_{(1,2)}$ and $H^2_{(1,2)}$, for instance. In addition, torsions also come out when calculating the cohomology groups. This means that the cohomology indeed contains more information compared with the graded partial-dual genus polynomial.

7. Some applications. In this section, we review some basic operations on ribbon graphs discussed in [11]. The behaviors of the partial-dual genus polynomial under these operations are given in [11] based on the topological definition of the partial-dual genus polynomial. In the first subsection, we first extend these results to the graded partial-dual genus polynomial. Then in the remaining subsections, we show that these properties can be categorified in the cohomology theory defined in Section 5.

7.1. Some properties of the graded partial-dual genus polynomial. In [24], Moffatt defines the so-called *ribbon-join* operation on two disjoint ribbon graphs G_1 and G_2, denoted by $G_1 \vee G_2$, in two steps:

- Choose an arc p_1 on the boundary of a vertex-disk v_1 of G_1 that lies between two consecutive ribbon ends, and choose another arc p_2 on the boundary of a vertex-disk v_2 of G_2.
- Paste vertex-disks v_1 and v_2 together by identifying the arcs p_1 and p_2.

The following proposition extends the result of [11, Proposition 3.2(a)] from the partial-dual genus polynomial to the graded partial-dual genus polynomial.

PROPOSITION 7.1. *Let G_1 and G_2 be disjoint ribbon graphs. Then*

$$^{\partial}\tilde{\varepsilon}_{G_1\cup G_2}(w,z) = {}^{\partial}\tilde{\varepsilon}_{G_1\vee G_2}(w,z) = {}^{\partial}\tilde{\varepsilon}_{G_1}(w,z)\,{}^{\partial}\tilde{\varepsilon}_{G_2}(w,z).$$

Proof. According to our combinatorial formula of the partial-dual genus polynomial given in Lemma 2.2, together with the definition of the graded partial-dual genus polynomial, one computes

$$\begin{aligned}
^{\partial}\tilde{\varepsilon}_{G_1\cup G_2}(w,z) &= Z_{\cup} \sum_{A\subseteq E(G_1\cup G_2)} w^{|A|} z^{-|F_{\cup}(A)|-|F_{\cup}(A^c)|} \\
&= Z_{\cup} \sum_{\substack{A_1\subseteq E(G_1)\\ A_2\subseteq E(G_2)}} w^{|A_1|+|A_2|} z^{-|F_1(A_1)|-|F_2(A_2)|-|F_1(A_1^c)|-|F_2(A_2^c)|} \\
&= \Big(z^{2c(G_1)+|E(G_1)|} \sum_{A_1\subseteq E(G_1)} w^{|A_1|} z^{-|F_1(A_1)|-|F_1(A_1^c)|}\Big) \\
&\quad\times \Big(z^{2c(G_2)+|E(G_2)|} \sum_{A_2\subseteq E(G_2)} w^{|A_2|} z^{-|F_2(A_2)|-|F_2(A_2^c)|}\Big) \\
&= {}^{\partial}\tilde{\varepsilon}_{G_1}(w,z){}^{\partial}\tilde{\varepsilon}_{G_2}(w,z),
\end{aligned}$$

where $Z_{\cup} = z^{2c(G_1\cup G_2)+|E(G_1\cup G_2)|} = z^{2c(G_1)+2c(G_2)+|E(G_1)|+|E(G_2)|}$ and

$$\begin{aligned}
^{\partial}\tilde{\varepsilon}_{G_1\vee G_2}(w,z) &= Z_{\vee} \sum_{A\subseteq E(G_1\vee G_2)} w^{|A|} z^{-|F_{\vee}(A)|-|F_{\vee}(A^c)|} \\
&= Z_{\vee} \sum_{\substack{A_1\subseteq E(G_1)\\ A_2\subseteq E(G_2)}} w^{|A_1|+|A_2|} z^{-|F_1(A_1)|-|F_2(A_2)|+1-|F_1(A_1^c)|-|F_2(A_2^c)|+1} \\
&= \Big(z^{2c(G_1)+|E(G_1)|} \sum_{A_1\subseteq E(G_1)} w^{|A_1|} z^{-|F_1(A_1)|-|F_1(A_1^c)|}\Big) \\
&\quad\times \Big(z^{2c(G_2)+|E(G_2)|} \sum_{A_2\subseteq E(G_2)} w^{|A_2|} z^{-|F_2(A_2)|-|F_2(A_2^c)|}\Big) \\
&= {}^{\partial}\tilde{\varepsilon}_{G_1}(w,z)\,{}^{\partial}\tilde{\varepsilon}_{G_2}(w,z),
\end{aligned}$$

where $Z_{\vee} = z^{2c(G_1\vee G_2)+|E(G_1\vee G_2)|} = z^{2c(G_1)+2c(G_2)-2+|E(G_1)|+|E(G_2)|}$ and $F_{\cup}$, $F_{\vee}$, F_1, F_2 count the numbers of face-disks of the subgraphs of $G_1\cup G_2, G_1\vee G_2, G_1$ and G_2, respectively. ■

To construct a *bar-amalgamation* of two disjoint ribbon graphs G_1 and G_2, denoted by $G_1 \mathbin{\square\!\!\!\!\!=} G_2$, the authors of [11] start the ribbon-join operation with selecting arcs p_1 and p_2, on the boundaries of vertex-disks v_1 and v_2 of G_1 and G_2, and then paste one end of a new ribbon to p_1 and the other end to p_2.

PROPOSITION 7.2. *Let G_1 and G_2 be disjoint ribbon graphs. Then*

$$^{\partial}\tilde{\varepsilon}_{G_1\mathbin{\square\!\!\!\!\!=}G_2}(w,z) = (1+w)^{\partial}\tilde{\varepsilon}_{G_1}(w,z)^{\partial}\tilde{\varepsilon}_{G_2}(w,z).$$

Proof. Similar to the proof of Proposition 7.1, let us use $F_{\square\!\!\!\!\!=}$ to denote the number of face-disks of the subgraphs of $G_1 \mathbin{\square\!\!\!\!\!=} G_2$. Then we have

$$\begin{aligned}
^{\partial}\tilde{\varepsilon}_{G_1\mathbin{\square\!\!\!\!\!=}G_2}(w,z) &= Z_{\square\!\!\!\!\!=} \sum_{A\subseteq E(G_1\mathbin{\square\!\!\!\!\!=}G_2)} w^{|A|}z^{-|F_{\square\!\!\!\!\!=}(A)|-|F_{\square\!\!\!\!\!=}(A^c)|}\\
&= Z_{\square\!\!\!\!\!=}\Bigg(\sum_{e\notin A\subseteq E(G_1\mathbin{\square\!\!\!\!\!=}G_2)} + \sum_{e\in A\subseteq E(G_1\mathbin{\square\!\!\!\!\!=}G_2)}\Bigg)w^{|A|}z^{-|F_{\square\!\!\!\!\!=}(A)|-|F_{\square\!\!\!\!\!=}(A^c)|}\\
&= Z_{\square\!\!\!\!\!=}\Bigg(\sum_{\substack{A_1\in E(G_1)\\ A_2\in E(G_2)}} w^{|A_1|+|A_2|}z^{-|F_1(A_1)|-|F_2(A_2)|-|F_1(A_1^c)|-|F_2(A_2^c)|+1}\\
&\quad + \sum_{\substack{A_1\in E(G_1)\\ A_2\in E(G_2)}} w^{|A_1|+|A_2|+1}z^{-|F_1(A_1)|-|F_2(A_2)|-|F_1(A_1^c)|-|F_2(A_2^c)|+1}\Bigg)\\
&= {}^{\partial}\tilde{\varepsilon}_{G_1}(w,z)^{\partial}\tilde{\varepsilon}_{G_2}(w,z) + w^{\partial}\tilde{\varepsilon}_{G_1}(w,z)^{\partial}\tilde{\varepsilon}_{G_2}(w,z)\\
&= (1+w)^{\partial}\tilde{\varepsilon}_{G_1}(w,z)^{\partial}\tilde{\varepsilon}_{G_2}(w,z),
\end{aligned}$$

where $Z_{\square\!\!\!\!\!=} = z^{2c(G_1\mathbin{\square\!\!\!\!\!=}G_2)+|E(G_1\mathbin{\square\!\!\!\!\!=}G_2)|} = z^{2c(G_1)+2c(G_2)+|E(G_1)|+|E(G_2)|-1}$ and e denotes the ribbon-edge added to connect G_1 and G_2. ■

By setting $w = 1$, we obtain

$$^{\partial}\varepsilon_{G_1\mathbin{\square\!\!\!\!\!=}G_2}(z) = 2^{\partial}\varepsilon_{G_1}(z)^{\partial}\varepsilon_{G_2}(z),$$

this recovers the result of [11, Proposition 3.2(b)].

REMARK 7.3. These notations hide the information of the arcs and vertex-disks we choose, because it can be seen from these two properties that the graded partial-dual genus polynomial does not depend on the choices of these arcs and vertex-disks. Actually, sliding an edge-ribbon of a ribbon graph along the boundary from one vertex-disk to another one preserves the partial-dual genus polynomial [9, Lemma 2.2]. One can similarly prove that the graded partial-dual genus polynomial is also preserved under edge sliding. Based on this fact, it is easy to observe that the graded partial-dual genus polynomial of the bar-amalgamation of two connected ribbon graphs does not depend on the choices of the arc and vertex-disks. As for categorification, it is routine to check that the S-, F-, V-, W-cubes also do not depend on these choices. In other words, different choices of arcs and vertex-disks give rise to isomorphic cohomology groups. In particular, by placing the sliding edge as the last one, it is not difficult to observe that the cohomology groups are also preserved under edge sliding.

As we mentioned before, the classical Euler–Poincaré dual of a ribbon graph G has the same genus as G. For the generating function ${}^{\partial}\varepsilon_G(z)$, it is obvious to conclude from Definition 2.1 that the partial-dual genus polynomial is preserved under the partial dual with respect to any subset $A \subseteq E(G)$. In other words, for arbitrary $A \subseteq E(G)$ we have ${}^{\partial}\varepsilon_{G^A}(z) = {}^{\partial}\varepsilon_G(z)$. Next we re-prove this result by using Lemma 2.2, since this method will be used again later. The following lemma can be proved directly by using the fact that $F(A) = V(G^A)$, but we still want to present a pure geometrical proof here.

LEMMA 7.4. *Let G be a ribbon graph and $e \in E(G)$. For arbitrary $B \subseteq E(G) \setminus \{e\}$, we have*

$$F_{G^{\{e\}}}(B) = F_G(B \cup \{e\}) \quad \textit{and} \quad F_{G^{\{e\}}}(B \cup \{e\}) = F_G(B),$$

where $F_{G^{\{e\}}}(B)$ denotes the set of the face disks of the ribbon graph obtained from $G^{\{e\}}$ by removing all the edges not in B.

Proof. Consider the edge e which connects two disks in Figure 2. Since $e \notin B$, one observes that the ribbon graph obtained from $G^{\{e\}}$ by removing all the edges not in B is homeomorphic to the ribbon graph obtained from G by removing all the edges not in $B \cup \{e\}$. It follows that $F_{G^{\{e\}}}(B) = F_G(B \cup \{e\})$. For the cases that e connects a disk to itself, either twisted or untwisted, one can also prove that there exists a homeomorphism between these two ribbon graphs.

By replacing G with $G^{\{e\}}$, one obtains the second equality. ∎

PROPOSITION 7.5. *For any ribbon graph G and any subset $A \subseteq E(G)$, we have*

$${}^{\partial}\varepsilon_G(z) = {}^{\partial}\varepsilon_{G^A}(z).$$

Proof. It suffices to verify the case where A contains exactly one edge. The general case can be proved inductively by using $G^{A\cup\{e\}} = (G^A)^{\{e\}}$.

Assume $A = \{e\}$, and first notice that

$$c(G^{\{e\}}) = c(G) \quad \text{and} \quad |E(G^{\{e\}})| = |E(G)|.$$

By using the natural one-to-one correspondence between the elements in $E(G^{\{e\}})$ and $E(G)$, we set $2^{E(G^{\{e\}})} = 2^{E(G)} = E_1 \cup E_2$, where $E_1 =$

$\{A \subseteq E(G) \mid e \in A\}$ and $E_2 = \{A \subseteq E(G) \mid e \notin A\}$. Then one computes

$$\begin{aligned}
{}^{\partial}\varepsilon_{G^{\{e\}}}(z) &= z^{2c(G^{\{e\}})+|E(G^{\{e\}})|} \sum_{B\subseteq E(G^{\{e\}})} z^{-|F_{G^{\{e\}}}(B)|-|F_{G^{\{e\}}}(B^c)|} \\
&= z^{2c(G)+|E(G)|}\Big(\sum_{B\in E_1} + \sum_{B\in E_2}\Big) z^{-|F_{G^{\{e\}}}(B)|-|F_{G^{\{e\}}}(B^c)|} \\
&= z^{2c(G)+|E(G)|}\Big(\sum_{B\in E_1} z^{-|F_G(B\backslash\{e\})|-|F_G(B^c\cup\{e\})|} \\
&\quad + \sum_{B\in E_2} z^{-|F_G(B\cup\{e\})|-|F_G(B^c\backslash\{e\})|}\Big) \\
&= z^{2c(G)+|E(G)|}\Big(\sum_{B\in E_2} z^{-|F_G(B)|-|F_G(B^c)|} + \sum_{B\in E_1} z^{-|F_G(B)|-|F_G(B^c)|}\Big) \\
&= {}^{\partial}\varepsilon_G(z). \blacksquare
\end{aligned}$$

It is worth mentioning that the graded partial-dual genus polynomial ${}^{\partial}\tilde{\varepsilon}_G(w, z)$ is not preserved under the partial dual in general. More precisely, consider a ribbon graph G and an edge $e \in E(G)$, and let us still denote $E_1 = \{A \subseteq E(G) \mid e \in A\}$ and $E_2 = \{A \subseteq E(G) \mid e \notin A\}$. Then we have

$$ {}^{\partial}\tilde{\varepsilon}_G(w, z) = \sum_{A\in E_1} w^{|A|} z^{\varepsilon(G^A)} + \sum_{A\in E_2} w^{|A|} z^{\varepsilon(G^A)}. $$

For convenience, we denote $\sum_{A\in E_1} w^{|A|} z^{\varepsilon(G^A)}$ and $\sum_{A\in E_2} w^{|A|} z^{\varepsilon(G^A)}$ by $u(w, z)$ and $v(w, z)$, respectively. In other words, ${}^{\partial}\tilde{\varepsilon}_G(w, z) = u(w, z) + v(w, z)$. Mimicking the proof of Proposition 7.5, one computes

$$\begin{aligned}
{}^{\partial}\tilde{\varepsilon}_{G^{\{e\}}}(w, z) &= z^{2c(G^{\{e\}})+|E(G^{\{e\}})|} \sum_{A\subseteq E(G^{\{e\}})} w^{|A|} z^{-|F_{G^{\{e\}}}(A)|-|F_{G^{\{e\}}}(A^c)|} \\
&= z^{2c(G)+|E(G)|}\Big(\sum_{A\in E_1} + \sum_{A\in E_2}\Big) w^{|A|} z^{-|F_{G^{\{e\}}}(A)|-|F_{G^{\{e\}}}(A^c)|} \\
&= z^{2c(G)+|E(G)|}\Big(\sum_{A\in E_1} w^{|A|} z^{-|F_G(A\backslash\{e\})|-|F_G(A^c\cup\{e\})|} \\
&\quad + \sum_{A\in E_2} w^{|A|} z^{-|F_G(A\cup\{e\})|-|F_G(A^c\backslash\{e\})|}\Big) \\
&= z^{2c(G)+|E(G)|}\Big(\sum_{A'\in E_2} w^{|A'|+1} z^{-|F_G(A')|-|F_G((A')^c)|} \\
&\quad + \sum_{A''\in E_1} w^{|A''|-1} z^{-|F_G(A'')|-|F_G((A'')^c)|}\Big) \\
&= w^{-1}u(w, z) + wv(w, z).
\end{aligned}$$

Here $A' = A \setminus \{e\}$ and $A'' = A \cup \{e\}$.

Example 7.6. Consider the ribbon graph G indexed by 11 in Figure 3. Direct calculation shows that

$$^{\partial}\tilde{\varepsilon}_{G^{\{e_1\}}}(w,z) = z + 2z^2w + zw^2.$$

Comparing with the result about $^{\partial}\tilde{\varepsilon}_G(w,z)$ in Example 2.5, we deduce that the partial dual operation does not preserve the graded partial-dual genus polynomial in general.

Although the graded partial-dual genus polynomial is not preserved under partial dual in general, the next proposition tells us that it is invariant under the classical Euler–Poincaré dual.

Proposition 7.7. *For any ribbon graph G, we have*

$$^{\partial}\tilde{\varepsilon}_G(w,z) = {}^{\partial}\tilde{\varepsilon}_{G^*}(w,z).$$

Proof. Assume $|E(G)| = n$ and $^{\partial}\tilde{\varepsilon}_G(w,z) = \sum_{i=0}^{n} f_i(z)w^i$. Then one computes

$$\begin{aligned} ^{\partial}\tilde{\varepsilon}_{G^*}(w,z) &= \sum_{A\subseteq E(G^*)} w^{|A|}z^{\varepsilon((G^*)^A)} = \sum_{A\subseteq E(G)} w^{|A|}z^{\varepsilon(G^{A^c})} \\ &= \sum_{A\subseteq E(G)} w^{n-|A|}z^{\varepsilon(G^A)} = \sum_{i=0}^{n} f_{n-i}(z)w^i \\ &= \sum_{i=0}^{n} f_i(z)w^i = {}^{\partial}\tilde{\varepsilon}_G(w,z) \end{aligned}$$

The penultimate equality follows from the fact that $f_i(z) = f_{n-i}(z)$, which was given at the end of Section 2. ■

Later in Section 7.4, we show that G and G^* actually have isomorphic cubes, as well as cochain complexes and cohomology groups, which illustrates that Proposition 7.7 can be lifted.

7.2. The categorification of Proposition 7.1. Let G_1 and G_2 be two disjoint ribbon graphs. Without loss of generality, we can order (2) the face-disks connected in the ribbon-join operation of $G_1 \cup G_2$ to be the first two, and the face-disk obtained in $G_1 \vee G_2$ to be the first one. Then

$$f_S = \mathrm{id}, f_F = \mathrm{id}, \quad \text{and} \quad f_V = m\otimes \mathrm{id}, f_W = m\otimes \mathrm{id}$$

are S-, F-, V-, W-cubes maps from $G_1 \cup G_2$ to $G_1 \vee G_2$. And $f = f_S \otimes f_F \otimes f_V \otimes f_W$ is a cube map from $\mathrm{Cube}(G_1 \cup G_2)$ to $\mathrm{Cube}(G_1 \vee G_2)$. Moreover, it induces a surjective chain map from $C(G_1 \cup G_2)$ to $C(G_1 \vee G_2)$, as well

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(2) Here we want to remind the readers that to construct the homology, the set of face-disks need to be ordered, but this order does not affect the homology up to isomorphism.

as a homomorphism from $H(G_1 \cup G_2)$ to $H(G_1 \vee G_2)$, denoted by f and f^* respectively. We have the following short exact sequence:

$$0 \to \ker f \overset{i}{\hookrightarrow} C(G_1 \cup G_2) \xrightarrow{f} C(G_1 \vee G_2)\{-2\} \to 0.$$

The degree shift makes this exact sequence graded. And because f is a cochain map, $\ker f$ is a subcochain of $C(G_1 \cup G_2)$. Additionally, this short exact sequence gives a long exact sequence

$$\cdots \to H^*(\ker f) \xrightarrow{i^*} H^*(G_1 \cup G_2) \xrightarrow{f^*} H^*(G_1 \vee G_2)\{-2\} \xrightarrow{\delta^*} H^{*+1}(\ker f) \to \cdots, \tag{7.1}$$

where the map δ^* is the classical partial operator given by choosing a representative element, finding a preimage of f acted on by $\partial_{G_1 \cup G_2}$, finding a preimage of i, and selecting the corresponding cohomology class.

On the other hand, since $(\ker f)^i$ is isomorphic to

$$\bigoplus_{\substack{A \subseteq E(G_1 \cup G_2) \\ |A| = i}} N^{\otimes i} \otimes \ker\,(M^{\otimes 4} \xrightarrow{m \otimes m} M^{\otimes 2}\{-2\}) \otimes M^{\otimes |F(A)| + |F(A^c)| - 4}\{-2i\},$$

we obtain the graded Euler characteristic of $\ker f$, $\chi_q(\ker f)$, which is

$$\sum_{A \subseteq E(G_1 \cup G_2)} [-q^2(1+p)]^{|A|}(q^{-1}+1+q)^{|F(A)|+|F(A^c)|}[1-(q^{-1}+1+q)^{-2}q^2].$$

Note that the long exact sequence (7.1) shows that

$$\chi_q(G_1 \vee G_2) \cdot q^2 = \chi_q(G_1 \cup G_2) - \chi_q(\ker f),$$

which gives

$$\chi_q(G_1 \vee G_2) = (q^{-1}+1+q)^{-2}\chi_q(G_1 \cup G_2).$$

This equality can be used to re-prove the first equality of Proposition 7.1, since

$$\begin{aligned} {}^{\partial}\tilde{\varepsilon}_{G_1 \vee G_2}(w,z) &= z^{2c(G_1 \vee G_2) + |E(G_1 \vee G_2)|}\chi \\ &= z^{2c(G_1 \cup G_2) - 2 + |E(G_1 \cup G_2)|}(q^{-1}+1+q)^{-2}\chi \\ &= z^{2c(G_1 \cup G_2) + |E(G_1 \cup G_2)|}\chi \\ &= {}^{\partial}\tilde{\varepsilon}_{G_1 \cup G_2}(w,z), \end{aligned}$$

where $\chi = \chi_q(G_1 \cup G_2)\big|_{-q^2-pq^2=w,\,q^{-1}+1+q=z^{-1}}$. That is, the long exact sequence (7.1) categorifies the result of Proposition 7.1.

7.3. The categorification of Proposition 7.2. Suppose ribbon graphs G_1 and G_2 are disjoint again and we order the face-disks of $G_1 \cup G_2$ and $G_1 \mathrel{=\!\!\!\!\!\flat} G_2$ similarly to the former subsection. In other words, we order the face-disks connected in the bar-amalgamation operation of $G_1 \cup G_2$ to be the first two and the face-disk obtained in $G_1 \eqcirc G_2$ to be the first one. Then we can split the cochain group of $G_1 \eqcirc G_2$ into two parts as follows:

$$\begin{aligned}C^i(G_1 \eqcirc G_2) &= \bigoplus_{\substack{A\subseteq E(G_1\eqcirc G_2)\\ |A|=i}} N^i \otimes M^{F_{\eqcirc}(A)} \otimes M^{F_{\eqcirc}(A^c)}\{-2i\}\\ &= \Big(\bigoplus_{\substack{e\in A\subseteq E(G_1\eqcirc G_2)\\ |A|=i}} N^i \otimes M^{F_{\eqcirc}(A)} \otimes M^{F_{\eqcirc}(A^c)}\{-2i\}\Big)\\ &\quad \oplus \Big(\bigoplus_{\substack{e\notin A\subseteq E(G_1\eqcirc G_2)\\ |A|=i}} N^i \otimes M^{F_{\eqcirc}(A)} \otimes M^{F_{\eqcirc}(A^c)}\{-2i\}\Big),\end{aligned}$$

where e is the edge added in the bar-amalgamation operation and $F_{\eqcirc}$ counts the number of face-disks of the corresponding subribbon graphs of $G_1 \eqcirc G_2$. Note that if $e \in A$, then $|F_{\eqcirc}(A)| = |F_{\vee}(A-e)| = |F_{\cup}(A-e)|-1$, $|F_{\eqcirc}(A^c)| = |F_{\vee}(A^c)|+1 = |F_{\cup}(A^c)|$, and if $e \notin A$, then $|F_{\eqcirc}(A)| = |F_{\vee}(A)|+1 = |F_{\cup}(A)|$, $|F_{\eqcirc}(A^c)| = |F_{\vee}(A^c)| = |F_{\cup}(A^c)| - 1$. We can construct the following cube maps:

$$\begin{array}{llll} g_S = \mathrm{id}, & g_F = \mathrm{id}, & g_V = \mathrm{id}, & g_W = m \otimes \mathrm{id},\\ h_S = \mathrm{id}, & h_F = \mathrm{id}, & h_V = m \otimes \mathrm{id}, & h_W = \mathrm{id},\\ r_S = \mathrm{id}, & r_F = \mathrm{id}, & r_V = m \otimes \mathrm{id}, & r_W = \mathrm{id},\\ t_S = -\mathrm{id}, & t_F = \mathrm{id}, & t_V = \mathrm{id}, & t_W = m \otimes \mathrm{id},\end{array}$$

where g is the cube map from the subcube of $G_1 \eqcirc G_2$ with the edge set containing e to the cube of $G_1 \vee G_2$, h is the cube map from the subcube of $G_1 \eqcirc G_2$ with the edge set without e to the cube of $G_1 \vee G_2$, r is the cube map from the cube of $G_1 \cup G_2$ to the subcube of $G_1 \eqcirc G_2$ with the edge set containing e, and t is the cube map from the cube of $G_1 \cup G_2$ to the subcube of $G_1 \eqcirc G_2$ with the edge set without e.

Based on the properties of TQFT, it can be easily verified that $g+h$ and $r+t$ induce two graded cochain maps

$$\begin{aligned} g+h &: C(G_1 \eqcirc G_2) \to C(G_1 \vee G_2)\{-1\},\\ r+t &: C(G_1 \cup G_2) \to C(G_1 \eqcirc G_2)\{-1\}.\end{aligned}$$

Furthermore, if we extend one of these two maps to a short exact sequence, we will obtain a long exact sequence of cohomology groups which categorifies Proposition 7.2. The process is quite the same as the one in the previous subsection.

REMARK 7.8. Note that $(g + h) \circ (r + t) = g \circ r + h \circ t = 0$, $g + h$ is surjective and $r + t$ is injective, and we obtain a graded short sequence *not* exact only at the middle term as follows:

$$0 \to C(G_1 \cup G_2) \xrightarrow{r+t} C(G_1 \mathrel{=\!\!\!\!\circ} G_2)\{-1\} \xrightarrow{g+h} C(G_1 \vee G_2)\{-2\} \to 0.$$

In a sense, this sequence is not satisfactory. We guess there is no short exact sequence utilizing these three cochain groups.

7.4. The categorification of Proposition 7.7. For a given ribbon graph G and a subset $A \subseteq E(G)$, the corresponding vertex space is $\mathrm{Cube}(G)_A = N^{\otimes|A|} \otimes M^{F_G(A)} \otimes M^{F_G(A^c)}$, and for its dual graph G^*, it is $\mathrm{Cube}(G^*)_A = N^{\otimes|A|} \otimes M^{\otimes F_{G^*}(A)} \otimes M^{\otimes F_{G^*}(A^c)}$. Notice that

$$F_G(A) = V(G^A) = V((G^*)^{A^c}) = F_{G^*}(A^c),$$
$$F_G(A^c) = V(G^{A^c}) = V((G^*)^A) = F_{G^*}(A).$$

Now we define a linear map between these two vertex spaces by

$$\begin{aligned} &P_A : \mathrm{Cube}(G)_A \to \mathrm{Cube}(G^*)_A, \\ &n_1 \otimes \cdots \otimes n_{|A|} \otimes m_1 \otimes \cdots \otimes m_{|F_G(A)|} \otimes m_1^c \otimes \cdots \otimes m^c_{|F_G(A^c)|} \\ &\qquad \mapsto n_1 \otimes \cdots \otimes n_{|A|} \otimes m_1^c \otimes \cdots \otimes m^c_{|F_G(A^c)|} \otimes m_1 \otimes \cdots \otimes m_{|F_G(A)|}. \end{aligned}$$

It is routine to verify that P_A is an isomorphism and commutes with the edge maps of these two cubes, which means that it induces a cube isomorphism. This leads to the following proposition:

PROPOSITION 7.9. *For any ribbon graph G,*

$$\mathrm{Cube}(G) \cong \mathrm{Cube}(G^*), \quad C(G) \cong C(G^*) \quad \textit{and} \quad H(G) \cong H(G^*).$$

The third isomorphism can be seen as a categorification of the equality in Proposition 7.7. However, Example 7.6 shows that this result does not hold for general partial dual graphs, because the partial dual operation may change the graded partial dual polynomial, not to mention the cohomology groups. We do not know whether a similar conclusion holds if the graded partial-dual polynomial is fixed by the partial dual operation. The following example shows that such a partial dual operation does exist, but unfortunately these two ribbon graphs have isomorphic cohomology groups.

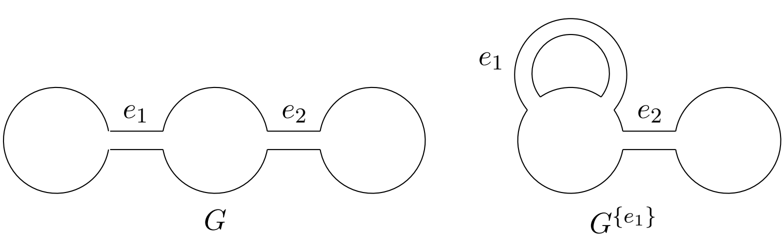

Fig. 11. A ribbon graph and one of its partial dual graphs

Example 7.10. Let G be the ribbon graph on the left side of Figure 11, with its two ribbons denoted by e_1 and e_2. Then one of its partial dual graphs $G^{\{e_1\}}$ is the right one. Their cubes are illustrated in Figures 12 and 13.

$$\begin{tikzcd}
 & N \otimes M^{\otimes 2} \otimes M^{\otimes 2} \arrow[dr, "u\otimes \mathrm{Id}_N \otimes m \otimes \Delta \otimes \mathrm{Id}_M"] & \\
M^{\otimes 3} \otimes M \arrow[ur, "u\otimes \mathrm{Id}_M \otimes m \otimes \Delta"] \arrow[dr, "u\otimes m \otimes \mathrm{Id}_M \otimes \Delta"'] & & N^{\otimes 2} \otimes M \otimes M^{\otimes 3} \\
 & N \otimes M^{\otimes 2} \otimes M^{\otimes 2} \arrow[ur, "-u\otimes \mathrm{Id}_N \otimes m \otimes \mathrm{Id}_M \otimes \Delta"'] &
\end{tikzcd}$$

Fig. 12. Cube(G)

$$\begin{tikzcd}
 & N \otimes M \otimes M^{\otimes 3} \arrow[dr, "u\otimes \mathrm{Id}_N \otimes \Delta \otimes m \otimes \mathrm{Id}_M"] & \\
M^{\otimes 2} \otimes M^{\otimes 2} \arrow[ur, "u\otimes m \otimes \mathrm{Id}_M \otimes \Delta"] \arrow[dr, "u\otimes \Delta \otimes \mathrm{Id}_M \otimes m"'] & & N^{\otimes 2} \otimes M^{\otimes 2} \otimes M^{\otimes 2} \\
 & N \otimes M^{\otimes 3} \otimes M \arrow[ur, "-u\otimes \mathrm{Id}_N \otimes \mathrm{Id}_M \otimes m \otimes \Delta"'] &
\end{tikzcd}$$

Fig. 13. Cube($G^{\{e_1\}}$)

Note that the vertex spaces on each vertex of their cubes are isomorphic, hence the ribbon graphs G and $G^{\{e_1\}}$ share the same graded partial-dual genus polynomial, $z^4(z^{-4} + 2wz^{-4} + w^2 z^{-4}) = (1 + w)^2$. At first sight, the readers may notice that they have different edge maps. However, it turns out that these two ribbon graphs have isomorphic cohomology groups as listed in Table 1. The integers in this table are the ranks of cohomology groups at corresponding degrees, and the empty cells and degrees out of the table mean that cochain groups vanish there. Additionally, the cohomology groups of these two ribbon graphs are torsion-free.

This example indicates that the knight move theorem for the chromatic graph cohomologies with rational coefficients proved in [7] does not hold. It motivates us to pose the following question.

Question 7.11. Do there exist two ribbon graphs that have the same graded partial-dual genus polynomial but different cohomology groups?

8. Some remarks on algebraic structures. There are plenty of algebraic structures that can be used to construct a categorification of the partial-dual genus polynomial. Fixing the Frobenius algebra as before, we

Table 1. Homology groups of G and $G^{\{e_1\}}$

$j=0$	H^0	H^1	H^2	$j=1$	H^1	H^2	$j=2$	H^2
$k=-4$	1			$k=-4$			$k=-4$	
$k=-3$	4			$k=-3$			$k=-3$	
$k=-2$	8	0		$k=-2$	2		$k=-2$	
$k=-1$	10	2		$k=-1$	8		$k=-1$	
$k=0$	8	8	0	$k=0$	19	1	$k=0$	1
$k=1$	4	16	0	$k=1$	28	4	$k=1$	4
$k=2$	1	20	1	$k=2$	29	11	$k=2$	10
$k=3$	0	16	4	$k=3$	20	20	$k=3$	16
$k=4$	0	8	8	$k=4$	9	27	$k=4$	19
$k=5$		2	10	$k=5$	2	26	$k=5$	16
$k=6$		0	8	$k=6$	0	18	$k=6$	10
$k=7$			4	$k=7$		4	$k=7$	4
$k=8$			1	$k=8$		1	$k=8$	1

may attempt different half-genus maps. It turns out that essentially the half-genus map defined in Section 3 is the unique one that can realize this categorification. And actually, we can construct a Frobenius algebra with half-genus map for any odd dimension greater than three.

In addition, we may use the algebra $M' = \mathbb{C}[x]/\langle x^{2n+1} - 1\rangle$ to give a complex which is not graded. Similarly to what happens to the construction of Baldridge and McCarty's homology [4], this homology at degree 0 depends on the number of colorings on the graph.

In the last subsection, we deal with a 2-dimensional algebra, which is realized by an unoriented TQFT.

8.1. Other half-genus maps.

Recall that the half-genus map h is an endomorphism of the Frobenius algebra M with condition $h^2 = m \circ \Delta$. In order to give a TQFT corresponding to a punctured annulus, it has to satisfy

$$m \circ (h \otimes \mathrm{Id}_M) = h \circ m = m \circ (\mathrm{Id}_M \otimes h),$$
$$(h \otimes \mathrm{Id}_M) \circ \Delta = \Delta \circ h = (\mathrm{Id}_M \otimes h) \circ \Delta.$$

The first equation shows that $h \in \mathrm{End}_M(M)$, which implies that it is multiplication with a fixed element that belongs to M. Denote this element by $h = h_0 + h_1 x^1 + h_2 x^2$; then it is easy to check that these two equations are satisfied for any $h_1, h_2, h_3 \in \mathbb{Z}[\sqrt{3}]$. To get the half-genus relation $h^2 = m \circ \Delta$, after some elementary calculations, we find that $h_0 = 0, h_1 = \pm\sqrt{3}$ and h_2 can be any element in $\mathbb{Z}[\sqrt{3}]$.

Then for a general h_2, we can construct the punctured TQFT as before and the cochain complex whose graded Euler characteristic is the partial-dual genus polynomial, with a little modification. For a nonzero h_2, the half-genus map is no longer a graded map. Moreover, it has the same degree as m and Δ, hence it cannot be deduced that the graded Euler characteristic of the cohomology groups equals that of the cochain groups.

However, just as another differential in Khovanov homology theory brings out Lee's endomorphism [22], a new differential may give us more information. If we still use ∂ to denote the differential of the cochain complex given by $h = \sqrt{3}x$ while ∂' by $h' = \sqrt{3}x + x^2$, then a new differential $\Phi := \partial' - \partial$ is given by zero multiplication, unit, comultiplication, trace while only the half-genus map is multiplication with x^2. By applying this differential in a V-cube or W-cube, we can construct a bicomplex $(C(G), \partial, \Phi)$ whose spectral sequence is given by horizontal (or vertical, resp.) filtration, say E_{hor} (or E_{vert}, resp.). And both of them converge to $H(C(G), \partial')$, the cohomology induced by h'.

Remark 8.1. If a ribbon graph G is orientable, then both E_{hor} and E_{vert} converge at the second page, since $\Phi = 0$ in this case. We define the *orientability number* of G to be the minimal number of edges such that removing these edges yields an orientable surface. We guess that the orientability number of a ribbon graph G might be read off from some information of E_{hor} and E_{vert}.

8.2. Other Frobenius algebras. We can also use other algebraic structures to categorify the partial-dual genus polynomial. This is different from Khovanov's categorification of the Jones polynomial [16] and Helme-Guizon and Rong's categorification of the chromatic polynomial [14], because the former one lifts the skein relation of the Jones polynomial and the latter one lifts the deletion-contraction rule of the chromatic polynomial.

In fact, we can choose different M, such as $\mathbb{Z}[\sqrt{2n+1}, x]/x^{2n+1}$, with Frobenius trace ϵ mapping each polynomial to the coefficient of x^{2n}, the half-genus map being the multiplication with $\sqrt{2n+1}x^n$, and Φ being the map induced by the multiplication with x^{n+i} for any $i > 0$ to construct the bicomplex as well as the spectral sequence. This is exactly the extended Frobenius algebra from [4]. However, this will make the calculation much more complicated.

Remark 8.2. Unlike Khovanov homology [16] needing the existence of the unit map to preserve the homology groups under three kinds of Reidemeister moves, since we do not need the unit and trace map of the Frobenius algebra, we can select an algebra without such structure, which is called *nearly Frobenius algebra* in [1]. Maybe the simplest example is the ideal generated by the half-genus element, the one multiplied by the half-genus map.

The homology groups given in this way are much easier to calculate, but they may lose some properties as the TQFT does not work anymore.

8.3. Lee-type homology and relation to the coloring number. It is also feasible to utilize the Frobenius algebra $M = \mathbb{C}[x]/\langle x^{2n+1} - 1\rangle$, with trace $\epsilon : 1, \dots, x^{2n-1} \mapsto 0$, $x^{2n} \mapsto 1$ and half-genus map $h : x^k \mapsto \sqrt{2n+1}x^{n+k}$, $k = 0, \dots, 2n$. The comultiplication induced by trace is $\varDelta : x^k \mapsto x^k \otimes x^{2n} + x^{k+1} \otimes x^{2n-1} + \dots + x^{2n+k} \otimes 1$, $k = 0, \dots, 2n$. Similarly to the one in [22], the structure of this algebra is simple under a specific basis, called the color basis, given by Baldridge and McCarty [4].

This basis can be given by

$$c_k = \frac{1}{2n+1}(1 + \lambda^k x + \lambda^{2k} x^2 + \dots + \lambda^{2nk} x^{2n}),$$

where $k = 0, 1, \dots, 2n$ and $\lambda = e^{\frac{2\pi i}{2n+1}}$, such that

$$c_j \cdot c_k = \delta^{jk} c_j,\ h(c_k) = \sqrt{2n+1}\lambda^{-nk} c_k, \qquad \varDelta(c_k) = (2n+1)\lambda^{-2nk} c_k \otimes c_k.$$

By applying the same strategy as in [4], we could conceive the relation between this homology and coloring numbers. Concretely speaking, if we denote by a the number of colorings on $F(\emptyset)$ such that adding any ribbon edge connects two different colored circles into one, and by b the number of colorings on $F(G)$ such that the deletion of any ribbon edge translates two different colored circles into one, then the dimension of the first homology group is

$$\dim H^0(G) = (2n+1)^{|F(G)|} a + (2n+1)^{|F(\emptyset)|} b - ab.$$

Moreover, although this homology cannot categorify the polynomial directly, it is related to the original homology by a spectral sequence similar to that in [22].

8.4. Another categorification using unoriented TQFT. To investigate link homology theories for stable equivalence classes of link diagrams on orientable surfaces, Turaev and Turner [28] introduced the notion of unoriented topology quantum field theory and extended Frobenius algebras, which are in one-to-one correspondence to each other. We omit the definition, but give an example of an extended Frobenius algebra.

Example 8.3. Let $\mathcal{A} = \mathbb{Z}[x]/x^2$, with natural unit map, multiplication, and Frobenius trace $\epsilon : 1 \mapsto 0$, $x \mapsto 0$. It becomes an extended Frobenius algebra after we equip it with an involution $\phi : 1 \mapsto 1$, $x \mapsto -x$ and an element $\theta = 0 \in \mathcal{A}$.

We can construct a V'-cube by using this rank two algebra. For each subgraph given by $A \subseteq E(G)$, define $V'_A = \mathcal{A}^{\otimes F(A)}$ and fix an orientation on each component of the face disks $F(A)$, where different choice does not change the homology. For all $A, B \subseteq E(G)$ satisfying the edge condition, if

- $|F(B)| = |F(A)| - 1$, then the addition of the new ribbon edge can be seen as a saddle giving a cobordism from two oriented circles to one. The orientation of the third circle induces the orientation of this cobordism, and if any orientation of the first two circles is incompatible with this cobordism, we apply an involution on $\mathcal{A}$ corresponding to this circle. Then we apply multiplication on $\mathcal{A} \otimes \mathcal{A}$ corresponding to these two circles.
- $|F(B)| = |F(A)| + 1$, then the addition of the new ribbon edge can be seen as a saddle giving a cobordism from one oriented circle to two. The orientation of the first circle induces an orientation of this cobordism, and if any orientation of the last two circles is incompatible with this cobordism, we apply an involution on $\mathcal{A}$ corresponding to this circle. Then we apply a comultiplication on $\mathcal{A}$ corresponding to the first circle.
- $|F(B)| = |F(A)|$, then the addition of the new ribbon edge can be seen as a saddle giving a cobordism from one oriented circle to another one. We apply a zero map on $\mathcal{A}$ corresponding to the first circle.

For other circles we apply identity maps and ultimately we tensor those maps together to obtain the edge map. The properties of the unoriented TQFT preserve the commutativity of this cube. The readers can reprove it by checking $m \circ (\phi \otimes \mathrm{Id}) \circ \Delta(1) = \theta^2$. Additionally, if we keep the degree of 1 and x as 1 and 0, this algebra is graded and each edge map of this cube is of degree -1 as well.

Replacing the W-cube by the dual of the V'-cube gives an anticommutative cube by tensoring it with S, F, and the V'-cube. Then everything happens as before and we obtain the homology which can be considered as a categorification of the graded partial-dual genus polynomial. As the rank of the algebra that we use is smaller, the calculation will be simpler but the cost is the addition of the step of analyzing the orientations.

Remark 8.4. There are many extended Frobenius algebras if we do not assume any restriction. As long as an extended Frobenius algebra is graded, it can be applied to categorify the graded partial-dual genus polynomial in the same way. For ungraded ones, it can be studied by using the spectral sequence from them to the graded ones.

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