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Estimates of singular integrals and their commutators on weighted Hardy spaces

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Abstract. We prove that singular integrals T with standard Calderón–Zygmund kernel and $\mathcal{S}$ with variable kernel are bounded on appropriate weighted Hardy spaces. Similar results hold for the commutators T_b and $\mathcal{S}_b$ when b belongs to a suitable subspace of $\mathrm{BMO}(\mathbb{R}^n)$.

1. Introduction. Estimates on singular integrals and the associated commutators are certainly of interest. A driving force for such an interest lies in the observation that the estimates can lead to regularity of solutions to partial differential equations. Namely, the following facts (most relevant to us) are known in the literature:

- The L^p-boundedness of Calderón–Zygmund singular integrals can be used to derive local regularity estimates up to the second order for strong solutions to the elliptic equation
$$\sum_{i,j} a_{ij}\partial^2_{ij}u + \sum_i b_i\partial_i u + cu = f$$
with continuous leading coefficients a_{ij} and bounded lower-order coefficients b_i and c. See [GT83, Theorem 9.11].
- [CFL93, Theorems 4.1 and 4.2] considered the elliptic equation
$$\sum_{i,j} a_{ij}\partial^2_{ij}u = f \tag{1.1}$$
with leading VMO-coefficients a_{ij} and re-established [GT83, Theorem 9.11]. This is a significant improvement since VMO-functions may not

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be continuous. The proof relies on the L^p-boundedness of the associated commutators of Calderón–Zygmund singular integrals with coefficients in VMO-spaces.

- [SS06, Theorems 3.1 and 4.1] proved the boundedness of singular integrals with standard and variable kernels respectively as well as their commutators in the Hardy space $H^1_q(\mathbb{R}^n) := H^1(\mathbb{R}^n) + L^q(\mathbb{R}^n)$. Then those authors obtained an interior second-order estimate for the strong solution to (1.1) with a_{ij} being vanishing LMO-functions [SS06, Theorem 5.1]. This regularity estimate is interesting because [GT83, Theorem 9.11] and [CFL93, Theorems 4.1 and 4.2] may fail for L^1-spaces. With the focus on singular integrals and their commutators, our main results in fact extend [SS06, Theorems 3.1 and 4.1] to a weighted setting. See also [TT$^+$25b].
- Concerning weighted Hardy spaces, the authors of [LKY16] investigated δ-Calderón–Zygmund operators T and showed that their commutators are bounded from $H^1_w(\mathbb{R}^n)$ to $L^1_w(\mathbb{R}^n)$ as long as the commutator coefficient belongs to the space $\mathrm{BMO}_w(\mathbb{R}^n)$, where w is a Muckenhoupt weight of class $A_{1+\delta/n}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n}\,dx < \infty$. Observe that $\mathrm{BMO}_w(\mathbb{R}^n)$ is a proper subspace of $\mathrm{BMO}(\mathbb{R}^n)$. This boundedness contrasts with the well-known fact that such a result does not hold when $w \equiv 1$, unless the commutator coefficient is a constant, which also explains the significance of the result. Furthermore, the boundedness of the commutators on $H^1_w(\mathbb{R}^n)$ also holds under an extra assumption that $T^*1 = 0$. For more details, see [LKY16, Theorems 1.3 and 1.5].
- The authors of [SS05, Theorem 1.1] noted that the local estimate in [SS06, Theorem 5.1] could be made global if the leading coefficients a_{ij} are log-continuous with a vanishing property. In turn, the H^p-version appeared later in [TT$^+$25a, Theorem 1.1] under the assumption that a_{ij} are homogeneous Lipschitz continuous functions with a vanishing property. The proofs of [SS05, Theorem 1.1] and [TT$^+$25a, Theorem 1.1] are in the spirit of [GT83, Theorem 9.11], making use of the boundedness of Calderón–Zygmund singular integrals.

In another direction, estimates on commutators are a subject of independent interest, with various extensions and generalizations regarding both types of operators (generalized Calderón–Zygmund operators, operators with rough kernels, pseudo-differential operators, etc.) and functional settings (Hardy spaces, BMO-spaces, Herz-type spaces, etc.). See [S93, LWY02, SS05, CW14, CL99, DY03, HK15, D25] and the references therein.

Back to our considerations, this paper investigates the boundedness of singular integral operators T and $\mathcal{S}$ with standard Calderón–Zygmund kernel and variable kernel respectively as well as their commutators

$$T_b(\cdot) := T(b\,\cdot) - bT(\cdot) \quad \text{and} \quad \mathcal{S}_b(\cdot) := \mathcal{S}(b\,\cdot) - b\mathcal{S}(\cdot)$$

on the weighted Hardy spaces

$$H^{1,q}_\omega(\mathbb{R}^n) := H^1_\omega(\mathbb{R}^n) + L^q(\mathbb{R}^n) \quad \text{with } 1 < q < \infty \text{ and } n \geq 3,$$

where ω is a Muckenhoupt weight. The coefficients b belong to suitable function spaces. Our main results, Proposition 3.1, Theorems 3.3, 3.7 and 3.8, extend [SS05, Theorems 2.1 and 3.2] and [SS06, Theorems 3.1, 4.1 and 4.2] to a weighted setting. For comparison, we recall that [SS06, Theorems 3.1, 4.1 and 4.2] assert the boundedness of T_b and $\mathcal{S}_b$ on $H^{1,q}(\mathbb{R}^n)$ when b belongs to the space $\mathrm{LMO}(\mathbb{R}^n)$, which is a subset of $\mathrm{BMO}(\mathbb{R}^n)$ with the extra decaying property on balls of small radii. In our weighted setting, for the boundedness on $H^{1,q}_\omega(\mathbb{R}^n)$ we require b to be a member of the so-called $\mathrm{L^2MO}$-space, which is a subset of $\mathrm{LMO}(\mathbb{R}^n)$. The $\mathrm{L^2MO}$-space is uniquely tied to the weighted setting and does not appear in the unweighted setting of [SS06, Theorems 3.1, 4.1 and 4.2]. See Section 2.3 below.

As an application, it has been observed in [SS06, Theorem 5.1] that the boundedness of the commutators T_b and $\mathcal{S}_b$ can lead to local Hessian regularity estimates for second-order linear elliptic equations in the local Hardy space $h^1(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$. The global version of [SS06, Theorem 5.1] is available in [SS09, Theorems 1.1 and 1.2]. Under the same approach, the h^p-counterpart of [SS09, Theorems 1.1 and 1.2] with $\frac{n}{n+1} < p < 1$ was established in [TT⁺25a, Theorem 1.1]. See also [TTD21]. We anticipate that the boundedness results in the present paper will enable analogous regularity estimates in weighted Hardy spaces. We wish to follow this idea in a future work.

Our main results here include Proposition 3.1, Theorems 3.3, 3.7 and 3.8. Proposition 3.1 and Theorem 3.7 assert the boundedness of T and $\mathcal{S}$ on $H^{1,q}_\omega(\mathbb{R}^n)$. Theorems 3.3 and 3.8 present the boundedness of the commutators T_b and $\mathcal{S}_b$ on $H^{1,q}_\omega(\mathbb{R}^n)$. Before these, the essential background on Muckenhoupt weights and function spaces is collected in Section 2.

Notation. Set $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$. Constants C and c are always assumed to be positive, independent of the main parameters, and with values changing from line to line. For any two functions f and g, we write $f \lesssim g$ and $f \sim g$ to mean $f \leq Cg$ and $cg \leq f \leq Cg$ respectively. Given a $p \in [1, \infty)$, the Hölder-conjugate index of p is denoted by p'.

2. Preliminaries.

In this section, we introduce Muckenhoupt weights and essential function spaces together with their properties to be used later on. The essential function spaces include weighted Lebesgue spaces, BMO-spaces, $\mathrm{L^2MO}$-spaces and weighted Hardy spaces. Interactions between these spaces are also examined.

2.1. Muckenhoupt weights and weighted Lebesgue spaces

Definition 2.1. Let $0 \leq \omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $t \in [1, \infty)$.

If $t > 1$, we say that $\omega \in A_t$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-t'}\,dx\right)^{t-1} \leq C$$

for all balls $B \subset \mathbb{R}^n$.

If $t = 1$, we say that $\omega \in A_1$ if there exists a constant $C > 0$ such that

$$\int_B \omega\,dx \leq C|B| \inf_B \omega$$

for all balls $B \subset \mathbb{R}^n$.

The smallest constant C in the definition of ω is called the *A_t-constant* of ω and is denoted by $[\omega]_{A_t}$. We also write $A_\infty := \bigcup_{t\geq 1} A_t$.

We keep in mind the following important properties of Muckenhoupt weights (see [S93, Chapter 5] and [G09, Chapter 9]).

Lemma 2.2. *Let $t \in [1, \infty)$ and $\omega \in A_t$. Then the following statements hold:*

(i) $\omega(\lambda B) \leq [\omega]_{A_t}\lambda^{nt}\omega(B)$ *for all balls $B \subset \mathbb{R}^n$ and $\lambda \geq 1$.*

(ii) $A_t \subset A_q$ *whenever $t \leq q$.*

(iii) $\omega \in A_q$ *for some $q \in [1, t)$. Set*

$$t_\omega := \inf\{q \geq 1 : \omega \in A_q\}$$

which is called the critical index *of ω.*

(iv) *There exists a constant $\delta = \delta(n, [\omega]_{A_t}) \in (0, 1)$ such that*

$$C(n, [\omega]_{A_t})\left(\frac{|B|}{|E|}\right)^{\delta} \leq \frac{\omega(B)}{\omega(E)} \leq [\omega]_{A_t}\left(\frac{|B|}{|E|}\right)^{t}$$

for all balls $B \subset \mathbb{R}^n$ and measurable subsets $E \subset B$, where $|E|$ is the Lebesgue measure of E and $\omega(E) := \int_E \omega(x)\,dx$.

Next we define weighted Lebesgue spaces.

Definition 2.3. Let $0 \leq \omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $p \in (0, \infty)$. The *weighted Lebesgue space* $L^p_\omega(\mathbb{R}^n)$ is defined by

$$L^p_\omega(\mathbb{R}^n) := \Big\{f : \mathbb{R}^n \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^n} |f(x)|^p\omega(x)\,dx < \infty\Big\}.$$

We endow $L^p_\omega(\mathbb{R}^n)$ with the (quasi-)norm

$$\|f\|_{L^p_\omega} := \Big(\int_{\mathbb{R}^n} |f(x)|^p\omega(x)\,dx\Big)^{1/p} < \infty.$$

If $p = \infty$ then $L^\infty_\omega(\mathbb{R}^n) := L^\infty(\mathbb{R}^n)$.

2.2. BMO-space. We recall the definition of the BMO-space $\mathrm{BMO}(\mathbb{R}^n)$.

Definition 2.4. Define

$$\mathrm{BMO}(\mathbb{R}^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx$$

with the supremum taken over all balls $B \subset \mathbb{R}^n$, and

$$f_B := \frac{1}{|B|} \int_B f(x)\, dx.$$

Also recall two technical lemmas concerning BMO-functions.

Lemma 2.5 ([SS06, Lemma 2.4]). *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. For any $p \geq 1$ and $\alpha > 0$ there exists a constant $C = C(n, p, \alpha) > 0$ such that*

$$\int_{\mathbb{R}^n \setminus B} |f(y) - f_B|^p |x - y|^{-n-\alpha}\, dy \leq C r^{-\alpha} \|f\|^p_{\mathrm{BMO}(\mathbb{R}^n)},$$

where $B = B_r(x)$ is the ball of radius r centered at x.

Proposition 2.6 ([LKY16, Lemma 2.3]). *Let $\omega \in A_\infty$ and $q \in [1, \infty)$. Then there exists a positive constant C such that*

$$\left(\frac{1}{\omega(B)} \int_B |f(x) - f_B|^q \omega(x)\, dx \right)^{1/q} \leq C \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}$$

for all $f \in \mathrm{BMO}(\mathbb{R}^n)$ and all balls $B \subset \mathbb{R}^n$.

2.3. $\mathrm{L}^2\mathrm{MO}$-space and its properties. Here we discuss the function space $\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$, which will be of main interest in our analysis. Its various properties are also examined. The space $\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$ was first introduced in [BB05, p. 518]. More discussion of this function space and its generalizations can be found therein.

Definition 2.7. The space $\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$ is a subspace of $\mathrm{BMO}(\mathbb{R}^n)$ and is equipped with the norm

$$\begin{aligned} \|f\|_{\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)} := \sup_{0<r<1} & \frac{1 + |\ln r|^2}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|\, dx \\ & + \sup_{r \geq 1} \frac{1}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|\, dx < \infty \end{aligned}$$

for each $f \in \mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$, where B_r denotes an arbitrary ball with radius r.

Similarly, the space $\mathrm{L}^2\mathrm{MO}^p(\mathbb{R}^n)$ with $p \in [1,\infty)$ is defined to comprise all functions f satisfying

$$[f]_{\mathrm{L}^2\mathrm{MO}^p(\mathbb{R}^n)} := \sup_{0<r<1/2} (1+|\ln r|^2)\bigg(\frac{1}{|B_r|}\int_{B_r} |f(x)-f_{B_r}|^p\,dx\bigg)^{1/p} < \infty.$$

The space $\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$ resembles $\mathrm{LMO}(\mathbb{R}^n)$, the latter consisting of all functions f satisfying

$$\begin{aligned}\|f\|_{\mathrm{LMO}(\mathbb{R}^n)} := \sup_{0<r<1} \frac{1+|\ln r|}{|B_r|}&\int_{B_r} |f(x)-f_{B_r}|\,dx \\ &+ \sup_{r\geq 1}\frac{1}{|B_r|}\int_{B_r} |f(x)-f_{B_r}|\,dx < \infty.\end{aligned}$$

The appearance of $\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$ is due to a technical reason in the proof of Lemma 2.18. Specifically, in Case II in that proof, we derive the estimate

$$\int_{2r\leq|x-x_0|<1} |x-x_0|^{-n}\omega(x)\,dx \leq C(n)\frac{|\ln r|^2}{|B|}\omega(B)$$

using a dyadic decomposition, where $B := B(x_0,r)$. In turn, if $\omega = 1$, then

$$\int_{2r\leq|x-x_0|<1} |x-x_0|^{-n}\,dx = C(n)|\ln r|.$$

Likewise, $\mathrm{L}^2\mathrm{MO}^p(\mathbb{R}^n)$ resembles $\mathrm{LMO}^p(\mathbb{R}^n)$ (see [SS06]), the latter consisting of all functions f such that

$$[f]_{\mathrm{LMO}^p(\mathbb{R}^n)} := \sup_{r<1/2} (1+|\ln r|)\bigg(\frac{1}{|B_r|}\int_{B_r} |f(x)-f_{B_r}|^p\,dx\bigg)^{1/p} < \infty.$$

It is known that $\mathrm{LMO}(\mathbb{R}^n) \hookrightarrow \mathrm{LMO}^p(\mathbb{R}^n)$ as seminormed spaces (see [SS06, Lemma 2.5]). The next lemma tells us that a similar result holds for $\mathrm{L}^2\mathrm{MO}$-spaces.

LEMMA 2.8. *Let* $p \in [1,\infty)$ *and* $f \in \mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$. *Then there exists a constant* $C = C(n,p) > 0$ *such that*

$$[f]_{\mathrm{L}^2\mathrm{MO}^p(\mathbb{R}^n)} \leq C\|f\|_{\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)}.$$

Proof. Let $\Phi : [0,1] \to [0,\infty]$ be defined by

$$\Phi(r) = \frac{1}{1+(\ln r)^2}.$$

Then Φ is a continuous non-decreasing function which satisfies

$$\frac{1}{r(1+|\ln r|^2)} \geq \frac{1}{s(1+|\ln s|^2)} \quad \text{for all } 0\leq r<s\leq 1.$$

By [A92, Proposition 1.13], we obtain

$$\sup_{r<1/2}(1+|\ln r|^2)\bigg(\frac{1}{|B_r|}\int\limits_{B_r}|f(x)-f_{B_r}|^p\,dx\bigg)^{1/p}\le C(n,p)\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)}$$

as required. ∎

Another property of $\mathrm{L^2MO}$-spaces to keep in mind is the following.

Lemma 2.9. *Suppose* $f\in \mathrm{L^2MO}(\mathbb{R}^n)$. *Then there exists a constant* $C=C(n)>0$ *such that*

$$\int\limits_{\mathbb{R}^n\backslash B}|f(y)-f_B|\,|x-y|^{-n-1}\,dy\le Cr^{-1}(1+|\ln r|^2)^{-1}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)}$$

for all balls $B=B_r(x)$ *with* $r<1/2$.

Proof. Let $B:=B_r(x)$ with $r<1/2$. Set $B_j:=2^jB$ for each $j\in\mathbb{N}$. Choose $k\in\mathbb{N}^*$ such that $1/2^{k+1}\le r<1/2^k$, which implies $k\sim|\ln r|$. It follows that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n\backslash B}|f(y)-f_B|\,|x-y|^{-n-1}\,dy\\
&\quad\le\sum_{j\in\mathbb{N}}\int\limits_{B_{j+1}\backslash B_j}|f(y)-f_B|\,|x-y|^{-n-1}\,dy\\
&\quad\le C(n)\sum_{j\in\mathbb{N}}\frac{1}{2^jr|B_{j+1}|}\int\limits_{B_{j+1}}|f(y)-f_B|\,dy\\
&\quad\le C(n)\sum_{j\in\mathbb{N}}\frac{1}{2^jr}\bigg(\frac{1}{|B_{j+1}|}\int\limits_{B_{j+1}}|f(y)-f_{B_{j+1}}|\,dy+\sum_{i=0}^{j}|f_{B_{i+1}}-f_{B_i}|\bigg)\\
&\quad\le C(n)\sum_{j=0}^{k}\frac{1}{2^jr}\bigg(\frac{1}{|B_{j+1}|}\int\limits_{B_{j+1}}|f(y)-f_{B_{j+1}}|\,dy+\sum_{i=0}^{j}|f_{B_{i+1}}-f_{B_i}|\bigg)\\
&\qquad+C(n)\sum_{j=k+1}^{\infty}\frac{1}{2^jr}\bigg(\frac{1}{|B_{j+1}|}\int\limits_{B_{j+1}}|f(y)-f_{B_{j+1}}|\,dy+\sum_{i=0}^{j}|f_{B_{i+1}}-f_{B_i}|\bigg)\\
&\quad\le C(n)\sum_{j=0}^{k}\frac{1}{2^jr}\bigg(\frac{1+|\ln(2^{j+1}r)|^2}{(1+|\ln(2^{j+1}r)|^2)|B_{j+1}|}\int\limits_{B_{j+1}}|f-f_{B_{j+1}}|\,dy\\
&\qquad\qquad+\sum_{i=0}^{j}\frac{2^n}{|B_{i+1}|}\int\limits_{B_{i+1}}|f-f_{B_{i+1}}|\bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ C(n) \sum_{j=k+1}^{\infty} \frac{1}{2^j r}\Bigg(\frac{1}{|B_{j+1}|}\int\limits_{B_{j+1}} |f - f_{B_{j+1}}|\,dy \\
&\qquad\qquad + 2^n \sum_{i=0}^{j} \frac{1}{|B_{i+1}|}\int\limits_{B_{i+1}} |f - f_{B_{i+1}}|\,dy\Bigg) \\
&\le C(n)\sum_{j=0}^{k} \frac{1}{2^j r}\Bigg[\frac{1+|\ln(2^{j+1}r)|^2}{(1+|\ln(2^{j+1}r)|^2)|B_{j+1}|}\int\limits_{B_{j+1}} |f - f_{B_{j+1}}|\,dy \\
&\qquad\qquad + \sum_{i=0}^{j} \frac{1+|\ln(2^{i+1}r)|^2}{(1+|\ln(2^{i+1}r)|^2)|B_{i+1}|}\int\limits_{B_{i+1}} |f - f_{B_{i+1}}|\,dy\Bigg] \\
&+ C(n) \sum_{j=k+1}^{\infty} \frac{j+1}{2^j r}\|f\|_{\mathrm{BMO}} \\
&\le C(n)\sum_{j=0}^{k} \frac{1}{2^j r}\Bigg(\frac{1}{1+|\ln(2^{j+1}r)|^2}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&\qquad\qquad + \sum_{i=0}^{j} \frac{1}{1+|\ln 2^{i+1}r|^2}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)}\Bigg) \\
&+ C(n) \sum_{j=k+1}^{\infty} \frac{(j+1)(1+|\ln r|)^2}{2^j r(1+|\ln r|^2)}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&\le C(n)\frac{1}{r}\sum_{j=1}^{\infty}\sum_{i=0}^{j} \frac{1}{2^j(1+|\ln r+(i+1)\ln 2|^2)}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&+ C(n) \sum_{j=k+1}^{\infty} \frac{j^3+j^2}{2^j r(1+|\ln r|^2)}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&\le C(n)\frac{1}{r}\sum_{i=0}^{\infty}\sum_{j\ge i} \frac{1}{2^j(1+|\ln r+(i+1)\ln 2|^2)}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&+ C(n)r^{-1}(1+|\ln r|^2)^{-1}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&\le C(n)\frac{1}{r}\sum_{i=0}^{\infty} \frac{1}{2^i(1+|\ln r+(i+1)\ln 2|^2)}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&+ C(n)r^{-1}(1+|\ln r|^2)^{-1}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&\le C(n)\frac{1}{r}\sum_{i=0}^{\infty} \frac{(i+1)^2(\ln 2)^2}{2^i(1+|\ln r|^2)}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&+ C(n)r^{-1}(1+|\ln r|^2)^{-1}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \\
&\le C(n)r^{-1}(1+|\ln r|^2)^{-1}\|f\|_{\mathrm{L^2MO}(\mathbb{R}^n)},
\end{aligned}
$$

where we use the fact that

$$\frac{1}{1+|a+b|^2} \le \frac{10b^2}{1+|a|^2} \quad \text{for all } a<0 \text{ and } b>1/2$$

in the second-to-last step. ■

2.4. Weighted Hardy spaces. In this subsection, we introduce weighted Hardy spaces on $\mathbb{R}^n$ and their local versions.

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a *standard mollifier*, that is, φ is radial and satisfies

$$\operatorname{supp}\varphi \subset B(0,1), \quad \varphi \ge 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \varphi(x)\,dx = 1.$$

Denote

$$\varphi_t(x) := \frac{1}{t^n}\varphi\left(\frac{x}{t}\right)$$

for each $x \in \mathbb{R}^n$ and $t>0$.

Definition 2.10. Let $0 \le \omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $p \in (0,1]$. The *weighted Hardy space* $H^p_\omega(\mathbb{R}^n)$ is defined by

$$H^p_\omega(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : M_\varphi(f)(x) := \sup_{t>0} |\varphi_t * f(x)| \in L^p_\omega(\mathbb{R}^n)\Big\}.$$

We endow $H^p_\omega(\mathbb{R}^n)$ with the quasi-norm

$$\|f\|_{H^p_\omega(\mathbb{R}^n)} := \|M_\varphi(f)\|_{L^p_\omega(\mathbb{R}^n)}.$$

Definition 2.11. Let $\omega \in A_1$ and $0<p\le 1$. Let $[n(1/p-1)] \le s \in \mathbb{Z}$, where $[\cdot]$ denotes the largest integer not exceeding the argument. A real-valued function a is called a *$(p,s)_\omega$-atom* if

(i) a is supported in a cube Q,
(ii) $\|a\|_{L^\infty(\mathbb{R}^n)} \le \omega(Q)^{-1/p}$,
(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for every multi-index α with $|\alpha| \le s$.

One of the most important properties of $H^p_\omega(\mathbb{R}^n)$ is atomic decomposition.

Proposition 2.12. *Let $\omega \in A_1$ and $0<p\le 1$. Let $f \in H^p_\omega(\mathbb{R}^n)$. Then there exist sequences $c_j \subset l^p(\mathbb{R})$ and a_j of $(p,s)_\omega$-atoms such that*

$$f = \sum_{j\in\mathbb{N}} c_j a_j \quad \text{in } H^p_\omega(\mathbb{R}^n). \tag{2.1}$$

For all $f \in H^p_\omega(\mathbb{R}^n)$, define the quasi-norm

$$\|f\|_{\mathrm{atom}} = \inf\Big\{\Big(\sum_{j\in\mathbb{N}} |c_j|^p\Big)^{1/p}\Big\},$$

where the infimum is taken over all decompositions in (2.1). *Then*

$$(H^p_\omega(\mathbb{R}^n), \|\cdot\|_{H^p_\omega(\mathbb{R}^n)}) = (H^p_\omega(\mathbb{R}^n), \|\cdot\|_{\mathrm{atom}}).$$

Proof. This follows from [LL02, Theorem F]. ■

Next we introduce weighted local Hardy spaces.

DEFINITION 2.13. Let $\omega \in A_1$ and $p \in (0,1]$. The *weighted local Hardy space* $h^p_\omega(\mathbb{R}^n)$ is defined by

$$h^p_\omega(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : m_\varphi(f)(x) := \sup_{0<t<1} |\varphi_t * f(x)| \in L^p_\omega(\mathbb{R}^n) \Big\}.$$

We endow $h^p_\omega(\mathbb{R}^n)$ with the quasi-norm

$$\|f\|_{h^p_\omega(\mathbb{R}^n)} := \|m_\varphi(f)\|_{L^p_\omega(\mathbb{R}^n)}.$$

DEFINITION 2.14. Let $0 < p \leq 1$ and $\omega \in A_1$. Let $s \in \mathbb{Z}$ satisfy $s \geq \max\{[n(1/p-1)], -1\}$. A real-valued function a is called a *type*-(a) $(p,s)_\omega$-*atom* if

(i) a is supported in a cube Q with side length $\ell(Q) < 1$,
(ii) $\|a\|_{L^\infty(\mathbb{R}^n)} \leq \omega(Q)^{-1/p}$,
(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha \, dx = 0$ for every multi-index α with $|\alpha| \leq s$.

A real-valued function a is called a *type*-(b) $(p,s)_\omega$-*atom* if it is supported in a cube Q with side length $\ell(Q) \geq 1$ and satisfies (ii).

Naturally, we define the space $h^{p,s}_{\omega,\mathrm{atom}}(\mathbb{R}^n)$.

DEFINITION 2.15. Let $\omega \in A_1$ and $0 < p \leq 1$. Let $s \in \mathbb{Z}$ satisfy $s \geq \max\{[n(1/p-1)], -1\}$. Then

$$h^{p,s}_{\omega,\mathrm{atom}}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : f = \sum_j \lambda_j a_j + \sum_j \mu_j b_j \Big\},$$

where each a_j is a type-(a) $(p,s)_\omega$-atom and each b_j is a type-(b) $(p,s)_\omega$-atom satisfying

$$\Big\| \sum_{j\in\mathbb{N}^*} \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{1/p}} \Big\|_{L^p_\omega(\mathbb{R}^n)} + \Big\| \sum_{j\in\mathbb{N}^*} \frac{\mu_j \chi_{P_j}}{\omega(P_j)^{1/p}} \Big\|_{L^p_\omega(\mathbb{R}^n)} < \infty.$$

We equip $h^{p,s}_{\omega,\mathrm{atom}}(\mathbb{R}^n)$ with the quasi-norm

$$\|f\|_{h^{p,s}_{\omega,\mathrm{atom}}(\mathbb{R}^n)} := \inf \Big\{ \Big\| \sum_{j\in\mathbb{N}^*} \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{1/p}} \Big\|_{L^p_\omega(\mathbb{R}^n)} + \Big\| \sum_{j\in\mathbb{N}^*} \frac{\mu_j \chi_{P_j}}{\omega(P_j)^{1/p}} \Big\|_{L^p_\omega(\mathbb{R}^n)} \Big\},$$

where the infimum is taken over all decompositions of f as above, Q_j is the supporting ball of the $(p,s)_\omega$-atom a_j and P_j is the supporting ball of the $(p,s)_\omega$-atom b_j.

Next we have atomic decomposition in $h^p_\omega(\mathbb{R}^n)$.

PROPOSITION 2.16. *Let* $\omega \in A_1$ *and* $0 < p \leq 1$. *Let* $f \in h^p_\omega(\mathbb{R}^n)$. *Then there exist sequences* $\{\lambda_j\}_j, \{\mu_j\}_j \subset l^p(\mathbb{R})$, $\{a_j\}$ *of type*-(a) $(p,s)_\omega$-*atoms,* *and* $\{b_j\}$ *of type-*(b) $(p,s)_\omega$*-atoms such that*

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j + \sum_{j\in\mathbb{N}} \mu_j b_j \quad \text{in } h^p_\omega(\mathbb{R}^n). \tag{2.2}$$

Moreover,

$$(h^p_\omega(\mathbb{R}^n), \|\cdot\|_{h^p_\omega(\mathbb{R}^n)}) = (h^p_\omega(\mathbb{R}^n), \|\cdot\|_{h^{p,s}_{\omega,\mathrm{atom}}(\mathbb{R}^n)}).$$

Proof. Apply [CT24, Theorem 1.6] and [AF03, Theorem 2.14]. ■

Let us write

$$H^{p,q}_\omega(\mathbb{R}^n) := H^p_\omega(\mathbb{R}^n) + L^q(\mathbb{R}^n) \quad \text{and} \quad h^{p,q}_\omega(\mathbb{R}^n) := h^p_\omega(\mathbb{R}^n) + L^q(\mathbb{R}^n)$$

for each $1 < q < \infty$ and $0 < p \le 1$.

Lemma 2.17. *Let* $\omega \in A_1$, $1 < q < \infty$ *and* $0 < p \le 1$. *Then*

$$H^{p,q}_\omega(\mathbb{R}^n) = h^{p,q}_\omega(\mathbb{R}^n)$$

as quasi-normed spaces.

Proof. In view of Proposition 2.16, it suffices to show that

$$H^{p,q}_\omega(\mathbb{R}^n) = h^{p,q}_{\omega,\mathrm{atom}}(\mathbb{R}^n)$$

as quasi-normed spaces. The inclusion $H^{p,q}_\omega(\mathbb{R}^n) \subset h^{p,q}_{\omega,\mathrm{atom}}(\mathbb{R}^n)$ is clear from Proposition 2.12. For the reverse inclusion, let $f \in h^{p,q}_{\omega,\mathrm{atom}}(\mathbb{R}^n)$. Then

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j + \sum_{j\in\mathbb{N}} \mu_j b_j := \lim_{k\to\infty} \Big(\sum_{j=1}^k \lambda_j a_j + \sum_{j=1}^k \mu_j b_j\Big) =: \lim_{k\to\infty} (g_k + h_k)$$

in $\mathcal{S}'$, where $\{\lambda_j\}_{j\in\mathbb{N}}$ and $\{\mu_j\}_{j\in\mathbb{N}}$ are sequences of non-negative numbers, and the sequences of pairs $\{(a_j, Q_j)\}_{j\in\mathbb{N}}$ and $\{(b_j, P_j)\}_{j\in\mathbb{N}}$ satisfy

$$\Big\|\sum_{j\in\mathbb{N}^*} \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{1/p}}\Big\|_{L^p_\omega(\mathbb{R}^n)} + \Big\|\sum_{j\in\mathbb{N}^*} \frac{\mu_j \chi_{P_j}}{\omega(P_j)^{1/p}}\Big\|_{L^p_\omega(\mathbb{R}^n)} < \infty.$$

Using the definitions of type-(a) and type-(b) $(p,q)_\omega$-atoms of $h^p_\omega(\mathbb{R}^n)$ as well as referring to [CT24, Theorem 1.7], we see that $\{g_k\}_{k\in\mathbb{N}}$ converges in $H^p_\omega(\mathbb{R}^n)$ and $\{h_k\}_{k\in\mathbb{N}}$ converges in $L^q(\mathbb{R}^n)$. Hence $f \in H^{p,q}_\omega(\mathbb{R}^n)$. ■

2.5. Interactions between function spaces. For boundedness on weighted H^1-spaces, the coefficients of the commutators are required to be elements of L^{2}MO-spaces. We present two results on the interactions between the L^{2}MO-spaces and the weighted H^1-spaces.

Lemma 2.18. *Let* $\psi \in \mathrm{L^2MO}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\omega \in A_1$ *and* $f \in h^1_\omega(\mathbb{R}^n)$. *Then* $\psi f \in h^1_\omega(\mathbb{R}^n)$. *Moreover, there exists a constant* $C = C(n, [\omega]_{A_1}) > 0$ *such that*

$$\|\psi h\|_{h^1_\omega(\mathbb{R}^n)} \le C(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{\mathrm{L^2MO}(\mathbb{R}^n)})\|f\|_{h^1_\omega(\mathbb{R}^n)}.$$

Proof. Taking advantage of the atomic decomposition of an $h^1_\omega(\mathbb{R}^n)$-function, we only need to show that

$$\|\psi a\|_{h^1_\omega(\mathbb{R}^n)} \le C(n,[\omega]_{A_1})(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{\mathrm{L^2MO}(\mathbb{R}^n)}) \tag{2.3}$$

for any $(1,s)_\omega$-atom of $h^1_\omega(\mathbb{R}^n)$.

Let a be a $(1,s)_\omega$-atom of $h^1_\omega(\mathbb{R}^n)$ which is supported in $B = B(x_0,r)$. If a is of type (b), let $b = \|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}\psi a$. Then clearly b is supported in B with $r \ge 1$. Moreover,

$$|b(x)| = \big|\|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}\psi a\big| \le \|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}\|\psi\|_{L^\infty(\mathbb{R}^n)}\|a\|_{L^\infty(\mathbb{R}^n)} \le \omega(B)^{-1}.$$

Thus, b is also a $(1,s)_\omega$-atom of type (b). Let $\mathcal{M}$ denote the Hardy–Littlewood operator. By the fact that $m_\varphi(f) \le \mathcal{M}f$ and by L^2-boundedness of $\mathcal{M}$, we obtain

$$\begin{aligned}
\|b\|_{h^1_\omega(\mathbb{R}^n)} &= \|m_\varphi(\|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}\psi a)\|_{L^1_\omega(\mathbb{R}^n)} = \|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}\|m_\varphi(\psi a)\|_{L^1_\omega(\mathbb{R}^n)}\\
&\le \|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}\|\mathcal{M}(\psi a)\|_{L^1_\omega(\mathbb{R}^n)}\\
&= \|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}\Big(\int_{2B}|\mathcal{M}(\psi a)|\omega(x)^{1/2}\omega(x)^{1/2}\,dx\Big)\\
&\le C(n)\|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}|\omega(B)|^{1/2}\|\mathcal{M}(\psi a)\|_{L^2_\omega(\mathbb{R}^n)}\\
&\le C(n)\|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}|\omega(B)|^{1/2}\|\psi a\|_{L^2_\omega(\mathbb{R}^n)}\\
&\le C(n)\|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}|\omega(B)|^{1/2}\|\psi\|_{L^\infty(\mathbb{R}^n)}\|a\|_{L^2_\omega(\mathbb{R}^n)}\\
&\le C(n)\|\psi\|^{-1}_{L^\infty(\mathbb{R}^n)}|\omega(B)|^{1/2}\|\psi\|_{L^\infty(\mathbb{R}^n)}\omega(B)^{1/2-1} \le C(n).
\end{aligned}$$

Next suppose a is a type-(a) $(1,s)_\omega$-atom. By definition, we only need to show

$$\|m_\varphi(\psi a)\|_{L^1_\omega(\mathbb{R}^n)} \le C(n,[\omega]_{A_1})(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{\mathrm{L^2MO}(\mathbb{R}^n)}),$$

where φ is a fixed mollifier.

CASE I. If $x \in 2B$, we obtain

$$\begin{aligned}
\int_{2B}|m_\varphi(\psi a)(x)|\omega(x)\,dx &= \int_{2B}|m_\varphi(\psi a)(x)|\omega(x)^{1/2}\omega(x)^{1/2}\,dx\\
&\le \Big(\int_{2B}|m_\varphi(\psi a)(x)|^2\omega(x)\,dx\Big)^{1/2}\Big(\int_{2B}\omega(x)\,dx\Big)^{1/2}\\
&\le \|m_\varphi(\psi a)\|_{L^2_\omega(2B)}\omega(2B)^{1/2}\\
&\le \|\mathcal{M}(\psi a)\|_{L^2_\omega(\mathbb{R}^n)}\omega(2B)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le \|\psi a\|_{L^2_\omega(\mathbb{R}^n)}\omega(2B)^{1/2}\\
&\le C(n,[\omega]_{A_1})\|\psi\|_{L^\infty(\mathbb{R}^n)}\|a\|_{L^2_\omega(\mathbb{R}^n)}\omega(B)^{1/2}\\
&\le C(n,[\omega]_{A_1})\|\psi\|_{L^\infty(\mathbb{R}^n)}\omega(B)^{1/2-1}\omega(B)^{1/2}\\
&\le C(n,[\omega]_{A_1})\|\psi\|_{L^\infty(\mathbb{R}^n)},
\end{aligned}
$$

where we use Hölder's inequality in the second step, Lemma 2.2(i) in the sixth step and the size condition for atoms in the seventh step.

Case II. If $x \notin 2B$, then for $y \in B$, $t < 1$ and by the mean value theorem,

$$
|\varphi_t(x-x_0)| \le C(n)|x-x_0|^{-n},
$$

$$
|\varphi_t(x-y)-\varphi_t(x-x_0)| \le C(n)|y-x_0|\,|x-x_0|^{-n-1}.
$$

By definition of type-(a) $(1,s)_\omega$-atom, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2B} |m_\varphi(\psi a)(x)|\omega(x)\,dx\\
&\quad= \int_{\mathbb{R}^n\setminus 2B}\Big|\sup_{0<t<1}\int_{\mathbb{R}^n}\varphi_t(x-y)\psi(y)a(y)\,dy\Big|\omega(x)\,dx\\
&\quad= \int_{\mathbb{R}^n\setminus 2B}\Big|\sup_{0<t<1}\int_{\mathbb{R}^n}\big((\varphi_t(x-y)-\varphi_t(x-x_0))\psi(y)a(y)\\
&\qquad+\varphi_t(x-x_0)(\psi(y)-\psi_B)a(y)-\varphi_t(x-x_0)\psi_B a(y)\big)\,dy\Big|\omega(x)\,dx\\
&\quad\le \int_{\mathbb{R}^n\setminus 2B}\Big|\sup_{0<t<1}\int_{\mathbb{R}^n}(\varphi_t(x-y)-\varphi_t(x-x_0))\psi(y)a(y)\,dy\Big|\omega(x)\,dx\\
&\qquad+\int_{\mathbb{R}^n\setminus 2B}\Big|\sup_{0<t<1}\int_{\mathbb{R}^n}\varphi_t(x-x_0)(\psi(y)-\psi_B)a(y)\,dy\Big|\omega(x)\,dx\\
&\quad\le \int_{\mathbb{R}^n\setminus 2B}\sup_{0<t<1}\int_B|\varphi_t(x-y)-\varphi_t(x-x_0)|\,|\psi(y)|\,|a(y)|\,dy\,\omega(x)\,dx\\
&\qquad+\int_{\mathbb{R}^n\setminus 2B}\sup_{0<t<1}\int_B|\varphi_t(x-x_0)|\,|\psi(y)-\psi_B|\,|a(y)|\,dy\,\omega(x)\,dx\\
&\quad\le C(n)\int_{\mathbb{R}^n\setminus 2B}\int_B|y-x_0|\,|x-x_0|^{-n-1}|\psi(y)|\,|a(y)|\,dy\,\omega(x)\,dx\\
&\qquad+C(n)\int_{\mathbb{R}^n\setminus 2B}\int_B|x-x_0|^{-n}|\psi(y)-\psi_B|\,|a(y)|\,dy\,\omega(x)\,dx\\
&\quad\le C(n)r\|\psi\|_{L^\infty(\mathbb{R}^n)}\Big(\int_B|a(y)|\,dy\Big)\Big(\int_{\mathbb{R}^n\setminus 2B}|x-x_0|^{-n-1}\omega(x)\,dx\Big)\\
&\qquad+C(n)\Big(\int_B|a(y)|\,|\psi(y)-\psi_B|\,dy\Big)\Big(\int_{2r\le|x-x_0|<1}|x-x_0|^{-n}\omega(x)\,dx\Big)
\end{aligned}
$$

$$\begin{aligned}
&\le C(n)r\|\psi\|_{L^\infty(\mathbb{R}^n)}\Big(\int_B |a(y)|\,dy\Big)\\
&\qquad\qquad\times\Big(\sum_{j\in\mathbb{N}^*}\ \int_{2^j r<|x-x_0|<2^{j+1}r} |x-x_0|^{-n-1}\omega(x)\,dx\Big)\\
&\quad + C(n)\Big(\|a\|_{L^\infty(\mathbb{R}^n)}\int_B |\psi(y)-\psi_B|\,dy\Big)\Big(\int_{2r\le|x-x_0|<1} |x-x_0|^{-n}\omega(x)\,dx\Big)\\
&\le C(n)r\|\psi\|_{L^\infty(\mathbb{R}^n)}\Big(\int_B |a(y)|\,dy\Big)\Big(\sum_{j\in\mathbb{N}^*}(2^j r)^{-n-1}\omega(2^{j+1}B)\Big)\\
&\quad + C(n)\Big(\omega(B)^{-1}\int_B |\psi(y)-\psi_B|\,dy\Big)\\
&\qquad\qquad\times\Big(\sum_{j=1}^{[\log_2\frac{1}{r}]-1}\ \int_{2^j r\le|x-x_0|<2^{j+1}r} |x-x_0|^{-n}\omega(x)\,dx\Big)\\
&\le C(n,[\omega]_{A_1})r\|\psi\|_{L^\infty(\mathbb{R}^n)}\Big(\int_B |a(y)|\,dy\Big)\Big(\omega(B)r^{-n-1}\sum_{j=1}^{\infty}(2^j)^{-n-1+n}\Big)\\
&\quad + C(n)\omega(B)^{-1}\Big(\int_B |\psi(y)-\psi_B|\,dy\Big)\\
&\qquad\qquad\qquad\times\Big(\sum_{j=1}^{[\log_2\frac{1}{r}]-1}\ \int_{2^j r\le|x-x_0|<2^{j+1}r} (2^j r)^{-n}\omega(x)\,dx\Big)\\
&\le C(n,[\omega]_{A_1})r\|\psi\|_{L^\infty(\mathbb{R}^n)}\Big(\int_B |a(y)|\,dy\Big)\omega(B)r^{-n-1}\\
&\quad + C(n)\omega(B)^{-1}\Big(\int_B |\psi(y)-\psi_B|\,dy\Big)\Big(\sum_{j=1}^{[\log_2\frac{1}{r}]-1}(2^j r)^{-n}(2^{j+1})^n\omega(B)\Big)\\
&\le C(n,[\omega]_{A_1})r\|\psi\|_{L^\infty(\mathbb{R}^n)}\Big(\int_B |a(y)|\,dy\Big)r^{-n-1}|B|\inf_B\omega(x)\\
&\quad + C(n,[\omega]_{A_1})\frac{|\ln r|^2}{|B|}\Big(\int_B |\psi(y)-\psi_B|\,dy\Big)\\
&\le C(n,[\omega]_{A_1})\Big(\|\psi\|_{L^\infty(\mathbb{R}^n)}\|a\|_{L^1_\omega(\mathbb{R}^n)}+\frac{|\ln r|^2}{|B|}\int_B |\psi(y)-\psi_B|\,dy\Big)\\
&\le C(n,[\omega]_{A_1})(\|\psi\|_{L^\infty(\mathbb{R}^n)}+\|\psi\|_{\mathrm{L^2MO}(\mathbb{R}^n)}).
\end{aligned}$$

Combining all the estimates, we see that (2.3) holds for every $(1,s)_\omega$-atom of $h^1_\omega(\mathbb{R}^n)$. Thus Lemma 2.18 follows. ∎

LEMMA 2.19. *Let* $\omega\in A_1$, $f\in H^{1,q}_\omega(\mathbb{R}^n)$ *with* $1<q<\infty$ *and* $\psi\in L^\infty(\mathbb{R}^n)\cap\mathrm{L^2MO}(\mathbb{R}^n)$. *Then* $\psi f\in H^{1,q}_\omega(\mathbb{R}^n)$. *Moreover, there exists a con-*

stant $C = C(n, [\omega]_{A_1}, q) > 0$ *such that*

$$\|\psi f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \le C(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{\mathrm{L^2MO}(\mathbb{R}^n)})\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)}.$$

Proof. Let $f \in H^{1,q}_\omega(\mathbb{R}^n)$. By Lemma 2.17, we have $f \in h^{1,q}_\omega(\mathbb{R}^n)$. Let $f = h + g$ be any decomposition for f with $h \in h^1_\omega(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. By Lemma 2.18,

$$\begin{aligned}
&\|\psi f\|_{h^{1,q}_\omega(\mathbb{R}^n)} \le \|\psi h\|_{h^1_\omega(\mathbb{R}^n)} + \|\psi g\|_{L^q(\mathbb{R}^n)}\\
&\quad\le C(n, [\omega]_{A_1})(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{\mathrm{L^2MO}(\mathbb{R}^n)})\|h\|_{h^1(\mathbb{R}^n)} + C\|\psi\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}\\
&\quad\le C(n, [\omega]_{A_1})(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{\mathrm{L^2MO}(\mathbb{R}^n)})(\|h\|_{h^1_\omega(\mathbb{R}^n)} + \|g\|_{L^q(\mathbb{R}^n)}).
\end{aligned}$$

Therefore, by Lemma 2.17,

$$\begin{aligned}
\|\psi f\|_{H^{1,q}_\omega(\mathbb{R}^n)} &\le C(n, [\omega]_{A_1}, q)\|\psi f\|_{h^{1,q}_\omega(\mathbb{R}^n)}\\
&\le C(n, q)(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{\mathrm{L^2MO}(\mathbb{R}^n)})\|f\|_{h^{1,q}_\omega(\mathbb{R}^n)}\\
&\le C(n, [\omega]_{A_1}, q)(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{\mathrm{L^2MO}(\mathbb{R}^n)})\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)}
\end{aligned}$$

as required. ∎

2.6. Spherical harmonic functions. Later on, we utilize spherical harmonic functions to prove the boundedness of singular integrals with variable kernels on weighted Hardy spaces. We devote this subsection to the essential background.

Let $\mathcal{H}_m$ be the space of spherical harmonic functions of degree m. Then

$$g_m := \dim \mathcal{H}_m < \infty.$$

Define $\mathcal{H} := \bigcup_{m=0}^\infty \mathcal{H}_m$. It is well known that the set of all finite linear combinations of elements in $\mathcal{H}$ is dense in $L^2(\Sigma)$, where

$$\Sigma := \{x \in \mathbb{R}^n : |x| = 1\}. \tag{2.4}$$

Let $\{Y_{km}\}$ with $k \in \{1, \dots, g_m\}$ and $m \in \mathbb{N}$ be an orthonormal system of spherical harmonic functions in $L^2(\Sigma)$,

$$\int_\Sigma Y_{km}^2\, d\sigma = 1 \quad \text{and} \quad \int_\Sigma Y_{km}Y_{k'm'}\, d\sigma = 0$$

for any $(k, m) \neq (k', m')$. In what follows, we always identify a spherical harmonic function $p : \Sigma \to \mathbb{R}$, $x \mapsto p(x)$, with its extension $p : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, $x \mapsto p(x/|x|)$, which is a homogeneous function of degree zero. Finally, we introduce the operator

$$\Lambda u := |x|^2 \Delta u$$

for each homogeneous function u of degree zero. The ℓth power of Λ is denoted by Λ^ℓ for each $\ell \in \mathbb{N}^*$. The following properties of spherical harmonic functions will be useful (see [SS06, Section 4]). See also [CZ57, S93, SW71].

Lemma 2.20.

(a) $g_m = C_m^{n+m-1} - C_{m-2}^{n+m-3} \leq C(n) m^{n-2}$ *for all* $m \in \mathbb{N}$,

(b) $|Y_{km}(x)| \leq C(n) m^{\frac{n-2}{2}}$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$, $k \in \{1, \ldots, g_m\}$ *and* $m \in \mathbb{N}$,

(c) $Y_{km} = \dfrac{\Lambda^\ell Y_{km}}{(-m)^\ell (m+n-2)^\ell}$ *for any* $k \in \{1, \ldots, g_m\}$ *and* $\ell, m \in \mathbb{N}^*$.

Lemma 2.21. *Let* $f, g \in C^\infty(\mathbb{R}^n \backslash \{0\})$ *be homogeneous functions of degree zero. Then*

$$\int_\Sigma f \Lambda^l g \, d\sigma = \int_\Sigma g \Lambda^l f \, d\sigma$$

for all $\ell \in \mathbb{N}^*$.

3. Estimates for singular integrals and their commutators in weighted H^1-spaces. This section contains the main results of the paper. The first two results, Proposition 3.1 and Theorem 3.3, assert the boundedness of singular integrals with standard (non-variable) kernels and their commutators in weighted H^1-spaces. The next two results, Theorems 3.7 and 3.8, generalize Proposition 3.1 and Theorem 3.3 to the case of singular integrals with variable kernels. For the precise statements, the following two assumptions on the weight ω will be needed:

(W1) $\omega \in A_1$.

(W2) There exists a constant $\kappa_1 > 0$ such that

$$\omega(B(x, 1)) \geq \kappa_1$$

for all $x \in \mathbb{R}^n$.

3.1. Commutator of singular integrals with standard kernels. We first introduce the definition of singular integrals. Let $f \in C_c^\infty(\mathbb{R}^n)$ and let

$$Tf(x) := \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) \, dy \tag{3.1}$$

be the singular integral with kernel $K \in C(\mathbb{R}^n \setminus \{0\})$ satisfying

(i) T is a bounded operator on $L^2(\mathbb{R}^n)$,

(ii) $|K(x)| \leq C|x|^{-n}$ for all $x \neq 0$,

(iii) $|K(x-y) - K(x)| \leq C|y| \, |x|^{-n-1}$ for all $|x| > 2|y|$.

For each $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $f \in C_c^\infty(\mathbb{R}^n)$, define the commutator

$$T_b f := T(bf) - bT(f).$$

We call T_b the commutator *generated* by the singular integral operator T and the function b.

It is well known that T can be extended to bounded operators on L^q-spaces for all $q \in (1,\infty)$ (see [S70, Chapter 2]). A generalization of this result in the direction of weighted Hardy spaces is the following.

PROPOSITION 3.1. *Assume that* ω *satisfies* (W1). *Let* $q \in (1,\infty)$. *Then* T *is bounded on* $H^{1,q}_\omega(\mathbb{R}^n)$.

Moreover, there exists a constant $C = C(n,[\omega]_{A_1},q,\delta,K) > 0$ *such that*

$$\|Tf\|_{H^{1,q}_\omega(\mathbb{R}^n)} \le C\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)}$$

for all $f \in H^{1,q}_\omega(\mathbb{R}^n)$.

Proof. This follows since T is bounded on $L^q(\mathbb{R}^n)$ and at the same time bounded on $H^1_w(\mathbb{R}^n)$ (see [LL02, Theorem 4]). ∎

We aim to prove the boundedness of T_b on $H^{1,q}_w(\mathbb{R}^n)$. At this point, it is appropriate to emphasize the fact that T_b is bounded in $L^q(\mathbb{R}^n)$ for all $q \in (1,\infty)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Moreover, there exists a constant $C = C(n,q,K) > 0$ such that

$$\|T_b f\|_{L^q(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^q(\mathbb{R}^n)}.$$

The next estimate is known as the *molecular estimate* for singular integrals.

LEMMA 3.2. *Assume that* ω *satisfies* (W1). *Let* a *be a* $(1,s)_\omega$*-atom of* $H^1_\omega(\mathbb{R}^n)$ *supported in* $B(x_0,r)$. *Then there exists a constant* $C = C(n) > 0$ *such that*

$$|Ta(x)| \le Cr^{n+1}|x-x_0|^{-n-1}\omega(B)^{-1}$$

for all $x \in \mathbb{R}^n$ *with* $|x-x_0| \ge 2r$.

Proof. Let $x \in \mathbb{R}^n$ be such that $|x-x_0| \ge 2r$. We will write $B := B(x_0,r)$ for short. Then

$$\begin{aligned}
|Ta(x)| &= \Big|\int_{\mathbb{R}^n} K(x-y)a(y)\,dy\Big| = \Big|\int_B K(x-y)a(y)\,dy\Big| \\
&= \Big|\int_B \Big(\big(K(x-x_0-(y-x_0)) - K(x-x_0)\big)a(y) + K(x-x_0)a(y)\Big)\,dy\Big| \\
&\le \Big|\int_B \big(K(x-x_0-(y-x_0)) - K(x-x_0)\big)a(y)\,dy\Big| \\
&\quad + \Big|K(x-x_0)\int_{B(x_0,r)} a(y)\,dy\Big|
\end{aligned}$$

$$\begin{aligned}
&\le \int_B |K(x-x_0-(y-x_0))-K(x-x_0)|\,|a(y)|\,dy \\
&\le C(n)\int_B |y-x_0|\,|x-x_0|^{-n-1}|a(y)|\,dy \\
&\le C(n)r|x-x_0|^{-n-1}\int_{B(x_0,r)} |a(y)|\,dy \\
&\le C(n)r^{n+1}|x-x_0|^{-n-1}\|a\|_{L^\infty(\mathbb{R}^n)} \\
&\le C(n)r^{n+1}|x-x_0|^{-n-1}\omega(B)^{-1},
\end{aligned}$$

where we use the size condition for K and a in the fifth and last step respectively. ■

Our second result is as follows.

THEOREM 3.3. *Assume that* $b \in \mathrm{L^2MO}(\mathbb{R}^n)$ *and that* ω *satisfies* (W1) *and* (W2). *Let* $q \in (1,\infty)$. *Then* T_b *is bounded on* $H^{1,q}_\omega(\mathbb{R}^n)$. *Moreover, there exists a constant* $C = C(n,[\omega]_{A_1},q,\delta,K,\kappa_1) > 0$ *such that*

$$\|T_b f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \le C\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)}$$

for all $f \in H^{1,q}_\omega(\mathbb{R}^n)$.

Proof. By the L^q-boundedness of T_b, Lemma 2.17 and the atomic decomposition of $H^1_\omega(\mathbb{R}^n)$, we only need to show that for a $(1,s)$-atom a of $H^1_\omega(\mathbb{R}^n)$ we have $T_b a \in h^{1,q}_\omega(\mathbb{R}^n)$ and

$$\|T_b a\|_{h^{1,q}_\omega(\mathbb{R}^n)} \le C(n,[\omega]_{A_1},q,\delta,K,\kappa_1)\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}.$$

Suppose a is supported in $B = B(x_0,r)$. If $r \ge 1/8$, then by the L^q-boundedness of T_b,

$$\begin{aligned}
\|T_b a\|_{L^q(\mathbb{R}^n)} &\le C(n,q,K)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|a\|_{L^q(\mathbb{R}^n)} \\
&\le C(n,q,K)|B|^{1/q}\omega(B)^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\le C(n,q,K)|B|^{1/q}\Big(\frac{\omega(B(x_0,1/8))}{\omega(B)}\Big)\omega(B(x_0,1/8))^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\le (n,q,K,[\omega]_{A_1})\bigg(\frac{|B(x_0,1/8)|}{|B|}\bigg)^{\delta}|B|^{1/q}\omega(B(x_0,1/8))^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\le C(n,q,K,[\omega]_{A_1})|B|^{1/q-\delta}\omega(B(x_0,1))^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\le C(n,q,\delta,K,[\omega]_{A_1},\kappa_1)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\le C(n,q,\delta,K,[\omega]_{A_1},\kappa_1)\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)},
\end{aligned}\tag{3.2}$$

where we use Lemma 2.2(i, iv) in the fourth and fifth step respectively, as well as (W2) in the sixth step.

Now assume $r < 1/8$. For each $y \in \mathbb{R}^n$, decompose $T_b a$ as

$$\begin{aligned} T_b a(y) &= -b(y)Ta(y) + T(ba)(y) \\ &= -(b(y) - b_B)Ta(y) + T((b - b_B)a)(y) \\ &=: -I(y) + II(y). \end{aligned}$$

Furthermore,

$$\begin{aligned} I(y) &= (b(y) - b_B)Ta(y)\chi_{4B}(y) + (b(y) - b_B)Ta(y)\chi_{B_0\backslash 4B}(y) \\ &\quad + (b(y) - b_B)Ta(y)\chi_{B_0^C}(y) \\ &=: I_1(y) + I_2(y) + I_3(y) \end{aligned}$$

for each $y \in \mathbb{R}^n$, where $B_0 := B_1(x_0)$.

We will prove that $I_1, I_2 \in h^1_\omega(\mathbb{R}^n)$, $I_3 \in L^q(\mathbb{R}^n)$, while $II \in h^1_\omega(\mathbb{R}^n) + L^q(\mathbb{R}^n)$. Let φ be the mollifier given in Section 2.4.

Estimate on I_1. We have

$$\begin{aligned} \text{(3.3)} \qquad & \int_{8B} m_\varphi I_1(x)\omega(x)\,dx = \int_{8B} m_\varphi I_1(x)\omega(x)^{1/2}\omega(x)^{1/2}\,dx \\ &\le \omega(8B)^{1/2}\|m_\varphi I_1\|_{L^2_\omega(\mathbb{R}^n)} \le \omega(8B)^{1/2}\|\mathcal{M}(I_1)\|_{L^2_\omega(\mathbb{R}^n)} \le \omega(8B)^{1/2}\|I_1\|_{L^2_\omega(\mathbb{R}^n)} \\ &\le C(n,[\omega]_{A_1})\omega(B)^{1/2}\Big(\int_{4B} |b(y) - b_B|^2 |Ta(y)|^2 \omega(y)\,dy\Big)^{1/2} \\ &\le C(n,[\omega]_{A_1})\omega(B)^{1/2}\Big(\int_{4B} |b(y) - b_B|^4 \omega(y)\,dy\Big)^{1/4}\Big(\int_{4B} |Ta(y)|^4 \omega(y)\,dy\Big)^{1/4} \\ &\le C(n,[\omega]_{A_1})\omega(B)^{1/2}\Big(\int_{4B} |b(y) - b_B|^4 \omega(y)\,dy\Big)^{1/4}\Big(\int_{4B} |a(y)|^4 \omega(y)\,dy\Big)^{1/4} \\ &\le C(n,[\omega]_{A_1})\omega(B)^{1/2}\Big(\int_{4B} |b(y) - b_B|^4 \omega(y)\,dy\Big)^{1/4}\omega(B)^{-1}\omega(4B)^{1/4} \\ &\le C(n,[\omega]_{A_1})\omega(B)^{-1/4}\Big(\int_{4B} |b(y) - b_B|^4 \omega(y)\,dy\Big)^{1/4} \\ &\le C(n,[\omega]_{A_1})\Big(\frac{1}{\omega(4B)}\int_{4B} |b(y) - b_{4B}|^4 \omega(y)\,dy\Big)^{1/4} \\ &\quad + C(n)|b_{4B} - b_B| \\ &\le C(n,[\omega]_{A_1})\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C(n,[\omega]_{A_1})\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}, \end{aligned}$$

where we use Lemma 2.2(i) in the fifth and tenth steps, the boundedness of singular integrals in weighted Lebesgue spaces (see [H16] for details) in the seventh step and Proposition 2.6 in the eleventh step.

If $x \notin 8B$, for $y \in 4B$, $t < 1$, one obtains

$$|\varphi_t(x-y)| \le C(n)|x-x_0|^{-n}$$

and

$$\begin{aligned}
|\varphi * I_1(x)| &= \Big|\int_{\mathbb{R}^n} \varphi_t(x-y) I_1(y)\,dy\Big| = \Big|\int_{4B} \varphi_t(x-y)(b(y)-b_B)Ta(y)\,dy\Big| \\
&\le \int_{4B} |\varphi_t(x-y)(b(y)-b_B)Ta(y)|\,dy \\
&\le C(n)|x-x_0|^{-n} \int_{4B} |(b(y)-b_B)Ta(y)|\,dy.
\end{aligned}$$

Note that $\varphi_t * I_1(x) = 0$ for all $x \notin 2B_0$. Hence,

$$\begin{aligned}
(3.4)\qquad & \int_{\mathbb{R}^n\setminus 8B} |\varphi_t * I_1(x)|\omega(x)\,dx = \int_{2B_0\setminus 8B} |\varphi_t * I_1(x)|\omega(x)\,dx \\
&\le C(n) \int_{2B_0\setminus 8B} |x-x_0|^{-n}\omega(x)\,dx \int_{4B} |(b(y)-b_B)Ta(y)|\,dy \\
&\le C(n)\Big(\sum_{j=1}^{[\log_2 \frac{2}{r}]+1} \int_{2^j r \le |x-x_0| < 2^{j+1} r} |x-x_0|^{-n}\omega(x)\,dx\Big) \\
&\quad \times \int_{4B} |(b(y)-b_B)Ta(y)|\,dy \\
&\le C(n)\Big(\sum_{j=1}^{[\log_2 \frac{2}{r}]+1} (2^j r)^{-n}\omega(2^{j+1}B)\Big)\Big(\int_{4B} |b(y)-b_B|^2\,dy\Big)^{1/2} \|Ta\|_{L^2(\mathbb{R}^n)} \\
&\le C(n,[\omega]_{A_1}) r^{-n}\omega(B)(1+|\ln 4r|^2)\Big(\int_{4B} |b(y)-b_B|^2\,dy\Big)^{1/2} \|a\|_{L^2(\mathbb{R}^n)} \\
&\le C(n,[\omega]_{A_1}) r^{-n}\omega(B)(1+|\ln 4r|^2)\Big(\int_{4B} |b(y)-b_B|^2\,dy\Big)^{1/2} |B|^{1/2}\omega(B)^{-1} \\
&\le C(n,[\omega]_{A_1})(1+|\ln 4r|^2)\Big(\frac{1}{|4B|}\int_{4B} |b(y)-b_B|^2\,dy\Big)^{1/2} \\
&\le C(n,[\omega]_{A_1})[b]_{\mathrm{L^2MO}^2(\mathbb{R}^n)} \le C(n,[\omega]_{A_1})\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)},
\end{aligned}$$

where we use Lemma 2.2(i) together with the L^2-boundedness of T in the fifth step, the size condition for a in the seventh step and Lemma 2.8 in the last step.

Combining (3.3) and (3.4), we obtain

$$\|I_1\|_{h^1_\omega(\mathbb{R}^n)} \le C(n,[\omega]_{A_1})\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}. \tag{3.5}$$

Estimate on I_2. Note that if $x \in 2B$ and $y \notin 4B$, then $|y-x| \ge 2r$ and $|y-x_0| \ge 4r$. By Lemmas 3.2 and 2.5, we obtain

$$\begin{aligned}|\varphi_t * I_2(x)| &\le C(n) \int\limits_{1>|y-x_0|\ge 4r} |x-y|^{-n}|b(y)-b_B|\,|Ta(y)|\,dy\\ &\le C(n) r^{-n} r^{n+1}\omega(B)^{-1} \int\limits_{|y-x_0|\ge 4r} |b(y)-b_B|\,|y-x_0|^{-n-1}\,dy\\ &\le C(n)\omega(B)^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C(n)\omega(B)^{-1}\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}.\end{aligned}$$

It follows that

$$\begin{aligned}\int\limits_{2B} |\varphi_t * I_2(x)|\omega(x)\,dx &\le C(n)\omega(B)^{-1}\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \int\limits_{2B}\omega(x)\,dx\\ &\le C(n)\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}.\end{aligned} \tag{3.6}$$

On the other hand, if $x \notin 2B$ then we make use of the estimate

$$\begin{aligned}|\varphi_t * I_2(x)| &= \Big|\int\limits_{B_0\setminus 4B} \varphi_t(x-y)(b(y)-b_B)Ta(y)\,dy\Big|\\ &\le \Big|\int\limits_{(B_0\setminus 4B)\cap B_r(x)} \varphi_t(x-y)(b(y)-b_B)Ta(y)\,dy\Big|\\ &\quad + \Big|\int\limits_{(B_0\setminus 4B)\cap B_r^C(x)} \varphi_t(x-y)(b(y)-b_B)Ta(y)\,dy\Big|\\ &=: J_{1t}(x) + J_{2t}(x).\end{aligned}$$

We will estimate each term separately.

For J_{1t}, observe that

$$\tfrac{3}{4}|y-x_0| \le |x-x_0| \le \tfrac{5}{4}|y-x_0|$$

as $x \notin 2B$, $y \notin 4B$ and $y \in B_r(x)$. Set $B_j := 2^j B$. Then $x \in B_{j+1}\setminus B_j$ implies $y \in B_{j+2}$. By Lemma 3.2,

$$\begin{aligned}J_{1t}(x) &\le \int\limits_{(B_0\setminus 4B)\cap B_r(x)} \varphi_t(x-y)|b(y)-b_B|\,|Ta(y)|\,dy\\ &\le C(n) r^{n+1}\omega(B)^{-1} \int\limits_{B_r(x)} \varphi_t(x-y)|b(y)-b_B|\,|y-x_0|^{-n-1}\,dy\\ &\le C(n) r^{n+1}\omega(B)^{-1}|x-x_0|^{-n-1} \int\limits_{\mathbb{R}^n} \varphi_t(x-y)|b(y)-b_B|\chi_{B_{j+2}}\,dy\\ &\le C(n) r^{n+1}\omega(B)^{-1}|x-x_0|^{-n-1} m_\varphi(|b-b_B|\chi_{B_{j+2}})(x),\end{aligned}$$

whence

$$
\begin{aligned}
&(3.7)\qquad \int\limits_{\mathbb{R}^n\setminus 2B} J_{1t}(x)\omega(x)\,dx = \sum_{j\in\mathbb{N}^*}\ \int\limits_{B_{j+1}\setminus B_j} J_{1t}(x)\omega(x)\,dx\\
&\le C(n)r^{n+1}\omega(B)^{-1}\sum_{j\in\mathbb{N}^*}\ \int\limits_{B_{j+1}\setminus B_j} |x-x_0|^{-n-1}m_\varphi(|b-b_B|\chi_{B_{j+2}})(x)\omega(x)\,dx\\
&\le C(n)\omega(B)^{-1}\sum_{j\in\mathbb{N}^*}(2^j)^{-n-1}\int\limits_{B_{j+1}} m_\varphi(|b-b_B|\chi_{B_{j+2}})(x)\omega(x)\,dx\\
&\le C(n)\omega(B)^{-1}\sum_{j\in\mathbb{N}^*}(2^j)^{-n-1}\Big(\int\limits_{B_{j+1}} m_\varphi(|b-b_B|\chi_{B_{j+2}})^2(x)\omega(x)\,dx\Big)^{1/2}\\
&\quad\times\Big(\int\limits_{B_{j+1}}\omega(x)\,dx\Big)^{1/2}\\
&\le C(n)\omega(B)^{-1}\sum_{j\in\mathbb{N}^*}(2^j)^{-n-1}(2^{j+1})^{n/2}\omega(B)^{1/2}\|m_\varphi(|b-b_B|\chi_{B_{j+2}})\|_{L^2_\omega(B_{j+1})}\\
&\le C(n)\omega(B)^{-1/2}\sum_{j\in\mathbb{N}^*}(2^j)^{-n/2-1}\|\mathcal{M}(|b-b_B|\chi_{B_{j+2}})\|_{L^2_\omega(\mathbb{R}^n)}\\
&\le C(n)\omega(B)^{-1/2}\sum_{j\in\mathbb{N}^*}(2^j)^{-n/2-1}\||b-b_B|\chi_{B_{j+2}}\|_{L^2_\omega(\mathbb{R}^n)}\\
&\le C(n)\omega(B)^{-1/2}\sum_{j\in\mathbb{N}^*}(2^j)^{-n/2-1}\Big(\int\limits_{B_{j+2}}|b(y)-b_B|^2\omega(y)\,dy\Big)^{1/2}\\
&\le C(n)\omega(B)^{-1/2}\\
&\quad\times\sum_{j\in\mathbb{N}^*}(2^j)^{-n/2-1}(2^{j+2})^{n/2}\omega(B)^{1/2}\Big(\frac{1}{\omega(B_{j+2})}\int\limits_{B_{j+2}}|b(y)-b_B|^2\omega(y)\,dy\Big)^{1/2}\\
&\le C(n)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\le C(n)\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)},
\end{aligned}
$$

where we use Lemma 3.2 in the second step and Proposition 2.6 in the ninth step.

To estimate $J_{2t}(x)$ we note that $\varphi_t(x-y)=0$ for all $|x-y|>1$ since $t<1$. Consequently,

$$
\begin{aligned}
&(3.8)\qquad \int\limits_{\mathbb{R}^n\setminus 2B} J_{2t}(x)\omega(x)\,dx\\
&\qquad=\int\limits_{\mathbb{R}^n\setminus 2B}\Big|\int\limits_{(B_0\setminus 4B)\cap B_r^C(x)}\varphi_t(x-y)(b(y)-b_B)Ta(y)\,dy\Big|\omega(x)\,dx\\
&\qquad\le C(n)\int\limits_{|y-x_0|\ge 4r}\ \int\limits_{r\le|y-x|<1}|x-y|^{-n}\omega(x)\,dx\,|b(y)-b_B|\,|Ta(y)|\,dy
\end{aligned}
$$

$$
\begin{aligned}
&\le C(n) \int\limits_{|y-x_0|\ge 4r} \Big(\sum_{i=0}^{[\log_2 \frac{1}{r}]-1} \int\limits_{2^j r<|x-y|<2^{j+1}r} |x-y|^{-n}\omega(x)\,dx\Big) \\
&\qquad\qquad \times |b(y)-b_B|\,|Ta(y)|\,dy \\
&\le C(n) r^{-n}\omega(B)(1+|\ln r|^2) \int\limits_{|y-x_0|\ge 4r} |b(y)-b_B|\,|Ta(y)|\,dy \\
&\le C(n) r^{-n}\omega(B)(1+|\ln r|^2) r^{n+1}\omega(B)^{-1} \\
&\quad \times \int\limits_{|y-x_0|\ge 4r} |y-x_0|^{-n-1}|b(y)-b_B|\,dy \\
&\le C(n) r(1+|\ln r|^2) \int\limits_{|y-x_0|\ge r} |y-x_0|^{-n-1}|b(y)-b_B|\,dy \\
&\le C(n)\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)},
\end{aligned}
$$

where we use Lemma 3.2 in the fifth step and Lemma 2.9 in the final step.

Combining (3.7) and (3.8) yields

$$
\int\limits_{\mathbb{R}^n\setminus 2B} m_\varphi I_2(x)\omega(x)\,dx \le C(n)\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}. \tag{3.9}
$$

With (3.6) and (3.9) in mind, we arrive at

$$
\|I_2\|_{h^1_\omega(\mathbb{R}^n)} = \|m_\varphi I_2\|_{L^1_\omega(\mathbb{R}^n)} \le C(n,[\omega]_{A_1})\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}. \tag{3.10}
$$

Estimate on I_3. In view of the molecular estimate in Lemma 3.2 as well as of Lemma 2.5 and (W2), we have

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n} |I_3(y)|^q\,dy &= \int\limits_{\mathbb{R}^n\setminus B_0} |(b(y)-b_B)Ta(y)|^q\,dy \\
&\le C(n) r^{(n+1)q}\omega(B)^{-q} \int\limits_{|y-x_0|\ge 1} |y-x_0|^{-nq-q}|b(y)-b_B|^q\,dy \\
&\le C(n,q,\kappa_1) r^{(n+1)q}\Big(\frac{\omega(B(x_0,1))}{\omega(B)}\Big)^q \int\limits_{|y-x_0|\ge 1} |y-x_0|^{-nq-q}|b(y)-b_B|^q\,dy \\
&\le C(n,q,\kappa_1,[\omega]_{A_1}) r^{(n+1)q}\frac{1}{|B|^q} r^{-q}\|b\|^q_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\le C(n,q,\kappa_1,[\omega]_{A_1})\|b\|^q_{\mathrm{L^2MO}(\mathbb{R}^n)}.
\end{aligned} \tag{3.11}
$$

Estimate on II. By replacing I_1 with $(b-b_B)a$ in "Estimate on I_1", similar arguments show that

$$
\|(b-b_B)a\|_{h^1_\omega(\mathbb{R}^n)} \le C(n,[\omega]_{A_1})\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}.
$$

Recall from Proposition 3.1 that T is bounded from $H^{1,q}_\omega(\mathbb{R}^n)$ to $H^{1,q}_\omega(\mathbb{R}^n)$.

Together with Lemma 2.17, we obtain

$$\begin{aligned}\|T(b-b_B)a\|_{h^{1,q}_\omega(\mathbb{R}^n)} &\le C(n,q,[\omega]_{A_1})\|T((b-b_B)a)\|_{H^{1,q}_\omega(\mathbb{R}^n)}\\ &\le C(n,q,[\omega]_{A_1})\|(b-b_B)a\|_{H^{1,q}_\omega(\mathbb{R}^n)}\\ &\le C(n,q,[\omega]_{A_1})\|(b-b_B)a\|_{h^{1}_\omega(\mathbb{R}^n)}\\ &\le C(n,q,[\omega]_{A_1})\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}.\end{aligned}\tag{3.12}$$

To complete the proof, put (3.2), (3.5) and (3.10)–(3.12) together. ∎

3.2. Commutators of singular integrals with variable kernels. We start with the definition of singular integrals with variable kernels.

Definition 3.4. A function $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is called a *Calderón–Zygmund kernel* if

(i) $k(tx) = t^{-n}k(x)$ for all $t > 0$,
(ii) $\int_\Sigma k(x)\,d\sigma(x) = 0$, where Σ is given by (2.4).

Let K be a real-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ such that $K(x,\cdot)$ is a Calderón–Zygmund kernel for a.e. $x \in \mathbb{R}^n$. For each $f \in C_c^\infty(\mathbb{R}^n)$ and for a.e. $x \in \mathbb{R}^n$, we define the *singular integral with variable kernel*

$$\mathcal{S}f(x) := \mathrm{p.v.}\int_{\mathbb{R}^n} K(x,x-y)f(y)\,dy.$$

Its *commutator* is defined by

$$\begin{aligned}\mathcal{S}_b f(x) &:= b(x)\mathcal{S}f(x) - \mathcal{S}(bf)(x)\\ &= \mathrm{p.v.}\int_{\mathbb{R}^n} K(x,x-y)[b(x)-b(y)]f(y)\,dy,\end{aligned}$$

where $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The boundedness of $\mathcal{S}$ on $L^q(\mathbb{R}^n)$ with $q \in (1,\infty)$ is well known.

Proposition 3.5 ([CFL93, Theorem 2.3]). *Let $1 < q < \infty$. Let $\mathcal{S}$ be a singular integral with variable kernel K satisfying*

$$\max_{|\beta|\le 2n}\left\|\frac{\partial^\beta}{\partial y^\beta}K(x,y)\right\|_{L^\infty(\mathbb{R}^n\times\Sigma)} \le C_1.\tag{3.13}$$

Then $\mathcal{S}$ can be extended to a bounded operator on $L^q(\mathbb{R}^n)$. Moreover, there exists a constant $C = C(n,q,C_1) > 0$ such that

$$\|\mathcal{S}f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^q(\mathbb{R}^n)}$$

for all $f \in L^q(\mathbb{R}^n)$.

We also need a technical lemma.

LEMMA 3.6. *Let K be a variable kernel satisfying* (3.13) *for each $y \in \Sigma$. Suppose further that $K(\cdot, y) \in \mathrm{L^2MO}(\mathbb{R}^n)$ and*

$$\max_{|\beta|\le 2n} \left\| \frac{\partial^\beta}{\partial y^\beta} K(x,y) \right\|_{L^\infty(\mathrm{L^2MO}(\mathbb{R}^n),\Sigma)} \le C_2, \tag{3.14}$$

where

$$\|h(x,y)\|_{L^\infty(\mathrm{L^2MO}(\mathbb{R}^n),\Sigma)} := \sup_{y\in\Sigma} \|h(\cdot,y)\|_{\mathrm{L^2MO}(\mathbb{R}^n)}$$

for each $y \in \Sigma$. Let $m \in \mathbb{N}^*$, $k \in \{1,\dots,g_m\}$ and define

$$a_{km}(x) := \int_\Sigma K(x,y) Y_{km}(y)\, d\sigma(y)$$

for a.e. $x \in \mathbb{R}^n$. Then $a_{km} \in L^\infty(\mathbb{R}^n) \cap \mathrm{L^2MO}(\mathbb{R}^n)$ and

$$\|a_{km}\|_{L^\infty(\mathbb{R}^n)} + \|a_{km}\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \le C(n)(C_1+C_2)m^{-2n}.$$

Proof. Observe that $|\cdot|^n K(x,\cdot)$ and Y_{km} are homogeneous functions of degree zero. Therefore, it follows from Lemmas 2.20 and 2.21 that

$$\begin{aligned} a_{km}(x) &= \frac{1}{(-m)^n(m+n-2)^n} \int_\Sigma |y|^n K(x,y) \Lambda^n(Y_{km}(y))\, d\sigma(y) \\ &= \frac{1}{(-m)^n(m+n-2)^n} \int_\Sigma \Lambda^n(|y|^n K(x,y)) Y_{km}(y)\, d\sigma(y) \\ &= \frac{1}{(-m)^n(m+n-2)^n} \int_\Sigma \sum_{|\beta|\le 2n} \rho_\beta(y) \frac{\partial^\beta}{\partial y^\beta} K(x,y) Y_{km}(y)\, d\sigma(y), \end{aligned}$$

where $\rho_\beta \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is a homogeneous function of degree $n+\beta$. Consequently,

$$\begin{aligned} \|a_{km}\|_{L^\infty(\mathbb{R}^n)} &\le m^{-2n} \int_\Sigma \sum_{|\beta|\le 2n} \left| \rho_\beta(y) \frac{\partial^\beta}{\partial y^\beta} K(x,y) Y_{km}(y) \right| d\sigma(y) \\ &\le C(n) m^{-2n} \int_\Sigma \sum_{|\beta|\le 2n} \left\| \frac{\partial^\beta}{\partial y^\beta} K(x,y) \right\|_{L^\infty(\mathbb{R}^n\times\Sigma)} d\sigma(y) \\ &\le C(n) C_1 m^{-2n}, \end{aligned}$$

where the last step makes use of (3.13).

Next, let B_r be a ball of radius $r > 0$ in $\mathbb{R}^n$. Then

$$\begin{aligned}&\int\limits_{B_r} |a_{km}(x) - (a_{km})_{B_r}|\,dx\\ &\le m^{-2n} \int\limits_{\Sigma}\int\limits_{B_r} \sum_{|\beta|\le 2n} \left|\left(\frac{\partial^\beta K(x,y)}{\partial y^\beta} - \left(\frac{\partial^\beta K(x,y)}{\partial y^\beta}\right)_{B_r}\right)\rho_\beta(y) Y_{km}(y)\right| dx\,d\sigma(y)\\ &\le C(n) m^{-2n} \int\limits_{\Sigma} \sum_{|\beta|\le 2n} \int\limits_{B_r} \left|\left(\frac{\partial^\beta K(x,y)}{\partial y^\beta} - \left(\frac{\partial^\beta K(x,y)}{\partial y^\beta}\right)_{B_r}\right)\right| dx\,d\sigma(y).\end{aligned}$$

In view of (3.14), we obtain

$$\|a_{km}\|_{\mathrm{L^2MO}(\mathbb{R}^n)} \le C(n) C_2 m^{-2n}.$$

This finishes the proof. ∎

The next result tells us that $\mathcal{S}$ is bounded on weighted Hardy spaces under some further conditions on the kernel K. This extends Proposition 3.1 to the setting of singular integrals with variable kernels.

THEOREM 3.7. *Assume ω satisfies* (W1). *Let $q \in (1,\infty)$. Let $\mathcal{S}$ be a singular integral with variable kernel K satisfying $K(\cdot,y) \in \mathrm{L^2MO}(\mathbb{R}^n)$,* (3.13) *and* (3.14) *for each $y \in \Sigma$. Let $f \in H^{1,q}_\omega(\mathbb{R}^n)$. Then $\mathcal{S} f \in H^{1,q}_\omega(\mathbb{R}^n)$. Moreover, there exists a constant $C = C(n, [\omega]_{A_1}, q, C_1, C_2) > 0$ such that*

$$\|\mathcal{S} f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \le C \|f\|_{H^{1,q}_\omega(\mathbb{R}^n)}.$$

Proof. For a.e. $x \in \mathbb{R}^n$, observe that

$$|\cdot|^n K(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\}) \quad \text{and} \quad \int\limits_{\Sigma} |y|^n K(x,y)\,d\sigma(y) = 0.$$

Hence

$$|y|^n K(x,y) = \sum_{m\in\mathbb{N}^*} \sum_{k=1}^{g_m} a_{km}(x) Y_{km}(y)$$

for all $y \in \mathbb{R}^n \setminus \{0\}$ and for a.e. $x \in \mathbb{R}^n$. In turn, the arguments in [C74] give

$$\mathcal{S} f(x) = \sum_{m\in\mathbb{N}^*} \sum_{k=1}^{g_m} a_{km}(x) R_{km} f(x),$$

where

$$a_{km}(x) := \int\limits_{\Sigma} K(x,y) Y_{km}(y)\,d\sigma(y),$$

$$R_{km} f(x) := \text{p.v.} \int\limits_{\mathbb{R}^n} \frac{Y_{km}(x-y)}{|x-y|^n} f(y)\,dy$$

for a.e. $x \in \mathbb{R}^n$.

By Lemma 3.6, $a_{km} \in \mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$ for all $m \in \mathbb{N}^*$ and $k \in \{1, \dots, g_m\}$. Moreover, R_{km} are Calderón–Zygmund singular integrals.

It follows that

$$\begin{aligned}
\|\mathcal{S}f\|_{H^{1,q}_\omega(\mathbb{R}^n)} &\le \sum_{m\in\mathbb{N}^*}\sum_{k=1}^{g_m} \|a_{km}R_{km}f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \\
&\le C(n,[\omega]_{A_1},q)\sum_{m\in\mathbb{N}^*}\sum_{k=1}^{g_m}(\|a_{km}\|_{L^\infty}+\|a_{km}\|_{\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)})\|R_{km}f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \\
&\le C(n,[\omega]_{A_1},q,C_1,C_2)\sum_{m\in\mathbb{N}^*}\sum_{k=1}^{g_m} m^{-2n}\|R_{km}f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \\
&\le C(n,[\omega]_{A_1},q,C_1,C_2)\sum_{m\in\mathbb{N}^*} m^{n-2}m^{-2n}\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \\
&\le C(n,[\omega]_{A_1},q,C_1,C_2)\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)},
\end{aligned}$$

where we use Lemma 2.19 in the second step, Lemma 3.6 in the third step and Lemma 2.20 and Proposition 3.1 in the fourth step. ∎

The boundedness of the commutator $\mathcal{S}_b$ also follows, which is our fourth main result. This extends Theorem 3.3 to the case of singular integrals with variable kernels.

THEOREM 3.8. *Assume ω satisfies* (W1) *and* (W2). *Let $q \in (1,\infty)$. Let $\mathcal{S}$ be a singular integral with variable kernel K satisfying $K(\cdot,y) \in \mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$,* (3.13) *and* (3.14) *for each $y \in \Sigma$. Let $b \in \mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)$. Then $\mathcal{S}_b f \in H^{1,q}_\omega(\mathbb{R}^n)$. Moreover, there exists a constant $C = C(n,[\omega]_{A_1},q,\delta,K,\kappa_1,C_1,C_2) > 0$ such that*

$$\|\mathcal{S}_b f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \le C\|b\|_{\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)}\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)}.$$

Proof. As in the proof of Theorem 3.7,

$$\mathcal{S}_b f(x) = \sum_{m\in\mathbb{N}^*}\sum_{k=1}^{g_m} a_{km}(x)R_{km,b}f(x)$$

for a.e. $x \in \mathbb{R}^n$, where $R_{km,b}$ is the commutator of R_{km} with b. Then

$$\begin{aligned}
&\|\mathcal{S}_b f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \\
&\quad\le \sum_{m\in\mathbb{N}^*}\sum_{k=1}^{g_m}\|a_{km}R_{km,b}f\|_{H^{1,q}_\omega(\mathbb{R}^n)} \\
&\quad\le C(n,[\omega]_{A_1},q)\sum_{m\in\mathbb{N}^*}\sum_{k=1}^{g_m}(\|a_{km}\|_{L^\infty}+\|a_{km}\|_{\mathrm{L}^2\mathrm{MO}(\mathbb{R}^n)})\|R_{km,b}f\|_{H^{1,q}_\omega(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&\le C(n,[\omega]_{A_1},q,\delta,K,\kappa_1)\\
&\quad\times \sum_{m\in\mathbb{N}^*}\sum_{k=1}^{g_m}(\|a_{km}\|_{L^\infty}+\|a_{km}\|_{\mathrm{L^2MO}(\mathbb{R}^n)})\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)}\\
&\le C(n,[\omega]_{A_1},q,\delta,K,\kappa_1,C_1,C_2)\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}\sum_{m\in\mathbb{N}^*} m^{n-2}m^{-2n}\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)}\\
&\le C(n,[\omega]_{A_1},q,\delta,K,\kappa_1,C_1,C_2)\|b\|_{\mathrm{L^2MO}(\mathbb{R}^n)}\|f\|_{H^{1,q}_\omega(\mathbb{R}^n)},
\end{aligned}
$$

where we use Lemma 2.19 in the second step, Lemma 3.2 in the third step, and Lemmas 2.20 and 3.6 in the fourth step. ∎

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References

[A92] P. Acquistapace, *On BMO regularity for linear elliptic systems*, Ann. Mat. Pura Appl. 161 (1992), 231–269.

[AF03] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd ed., Academic Press, Amsterdam, 2003.

[BB05] M. Bramanti and L. Brandolini, *Estimates of BMO type for singular integrals on spaces of homogeneous type and applications to hypoelliptic PDEs*, Rev. Mat. Iberoamer. 21 (2005), 511–556.

[CZ57] A. P. Calderón and A. Zygmund, *Singular integral operators and differential equations*, Amer. J. Math. 79 (1957), 901–921.

[CL99] D. C. Chang and Y. S. Li, *On the boundedness of multipliers, commutators and the second derivatives of Green's operators on H^1 and BMO*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 28 (1999), 341–356.

[CT24] X. Chen and J. Tan, *The atomic characterization of weighted local Hardy spaces and its applications*, Filomat 38 (2024), 5925–5949.

[CFL93] F. Chiarenza, M. Frasca, and P. Longo, *$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. 336 (1993), 841–853.

[C74] R. R. Coifman, *A real-variable characterization of H^p*, Studia Math. 51 (1974), 269–274.

[CW14] D. Cruz-Uribe and L.-A. D. Wang, *Variable Hardy spaces*, Indiana Univ. Math. J. 63 (2014), 447–493.

[D25] T. D. Do, *Boundedness of pseudo-differential operators on weighted Hardy spaces*, Vietnam J. Math. 53 (2025), 427–439.

[DY03] X. T. Duong and L. X. Yan, *Commutators of BMO functions and singular integral operators with non-smooth kernels*, Bull. Austral. Math. Soc. 67 (2003), 187–200.

[GT83] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grundlehren Math. Wiss. 224, Springer, Berlin, 1983.

[G09] L. Grafakos, *Modern Fourier Analysis*, 2nd ed., Grad. Texts in Math. 250, Springer, New York, 2009.

[H16] K.-P. Ho, *Singular integral operators, John–Nirenberg inequality and Triebel–Lizorkin type spaces on weighted Lebesgue space with variable exponents*, Rev. Un. Mat. Argentina 57 (2016), no. 1, 85–101.

[HK15] H. D. Hung and L. D. Ky, *An Hardy estimate for commutators of pseudo-differential operators*, Taiwanese J. Math. 19 (2015), 1097–1109.

[LL02] M.-Y. Lee and C.-C. Lin, *The molecular characterization of weighted Hardy spaces*, J. Funct. Anal. 188 (2002), 442–460.

[LKY16] Y. Liang, L. D. Ky, and D. Yang, *Weighted endpoint estimates for commutators of Calderón–Zygmund operators*, Proc. Amer. Math. Soc. 144 (2016), 5171–5181.

[LWY02] S. Z. Lu, Q. Wu, and D. C. Yang, *Boundedness of commutators on Hardy type spaces*, Sci. China Ser. A 45 (2002), 984–997.

[S70] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton Univ. Press, Princeton, NJ, 1970.

[S93] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. 43, Monogr. Harmon. Anal. III, Princeton Univ. Press, Princeton, NJ, 1993.

[SW71] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Math. Ser. 32, Princeton Univ. Press, Princeton, NJ, 1971.

[SS05] Y. Sun and W. Su, *Hardy type estimates for commutators of singular integrals*, Sci. China Ser. A 48 (2005), 563–575.

[SS06] Y. Sun and W. Su, *Interior h^1-estimates for second order elliptic equations with vanishing LMO coefficients*, J. Funct. Anal. 234 (2006), 235–260.

[SS09] Y. Sun and W. Su, *Solvability in Hardy space for second order elliptic equations*, Anal. Theory Appl. 25 (2009), 355–368.

[TTD21] N. N. Trong, L. X. Truong, and T. D. Do, *Boundedness of second-order Riesz transforms on weighted Hardy and BMO spaces associated with Schrödinger operators*, C. R. Math. Acad. Sci. Paris 359 (2021), 687–717.

[TT$^+$25a] L. X. Truong, N. D. Trung, N. N. Trong, and T. D. Do, *Global Hessian estimate for second-order elliptic equation in Hardy spaces*, Ric. Mat. 74 (2025), 1177–1198.

[TT$^+$25b] N. D. Trung, N. N. Trong, L. X. Truong and T. D. Do, *Commutators of singular integrals in variable exponent Hardy spaces*, Complex Anal. Operator Theory 19 (2025), no. 6, art. 124, 35 pp.

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