STUDIA MATHEMATICA

Online First version

The Bergman kernel and projection on a class of bounded Hartogs domains

by

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Abstract. We study the Bergman kernel and projection on the class of bounded Hartogs domains defined by

$$H^{\mathbf{p}}_{k+m} := \{z = (\widetilde{z}, z_{k+1}, \dots, z_{k+m}) \in \mathbb{C}^{k+m} : \|\widetilde{z}\|^{p_1} < |z_{k+1}|^{p_{k+1}} < \dots < |z_{k+m}|^{p_{k+m}} < 1\},$$

where $k, m \in \mathbb{Z}^+$, $\mathbf{p} := (p_1, p_{k+1}, \dots, p_{k+m}) \in (\mathbb{R}^+)^{m+1}$ and $\widetilde{z} := (z_1, \dots, z_k) \in \mathbb{C}^k$. We obtain an explicit formula for the Bergman kernel of $H^{\mathbf{p}}_{k+m}$ and use it to establish an optimal estimate. We then study the L^p regularity and irregularity of the Bergman projection and further investigate the weak-type estimates at the endpoints of the L^p boundedness range.

1. Introduction. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. As usual, for $p > 0$, the space $L^p(\Omega)$ consists of all Lebesgue measurable functions f on Ω such that

$$\|f\|_{L^p} := \Big(\int_{\Omega} |f(z)|^p \, dV(z)\Big)^{1/p} < +\infty,$$

where dV denotes the standard Lebesgue measure. We set $A^p(\Omega) := L^p(\Omega) \cap H(\Omega)$, where $H(\Omega)$ denotes the space of holomorphic functions on Ω. The *Bergman space* $A^2(\Omega)$ is the Hilbert space of all square-integrable and holomorphic functions on Ω with the inner product $\langle f, g\rangle_{L^2} := \int_{\Omega} f(z)\overline{g(z)}\, dV(z)$. It is well known that $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$.

We consider the natural orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$, also known as the *Bergman projection*. The Bergman projection of Ω will be written as B_{Ω}. It is elementary that B_{Ω} is self-adjoint with respect to the above inner product. The Bergman projection can be expressed as an

2020 *Mathematics Subject Classification*: Primary 47G10; Secondary 32A36, 32A25.
Key words and phrases: Bergman kernel, Bergman projection, Hartogs domain.
Received 14 February 2025; revised 17 February 2026.
Published online 3 July 2026.

DOI: 10.4064/sm250214-27-2

integral operator

$$B_\Omega f(z) = \int_\Omega K_\Omega(z,w) f(w)\, dV(w), \qquad f \in L^2(\Omega), \tag{1.1}$$

where $K_\Omega : \Omega \times \Omega \to \mathbb{C}$ is the Bergman kernel of Ω. It should be remarked that when the integral in (1.1) converges, it is taken as the definition of $B_\Omega f$, even if $f \notin L^2(\Omega)$. If $\{\phi_\alpha\}$ is an orthonormal basis for the Bergman space $A^2(\Omega)$, then the Bergman kernel has the formula

$$K_\Omega(z,w) = \sum_\alpha \phi_\alpha(z)\overline{\phi_\alpha(w)}.$$

For a long time, it was difficult to obtain the explicit formula of the Bergman kernel function. Thus it is one of central problems in several complex variables to find the explicit formula of the Bergman kernel on various domains. We refer the readers to [24] for more information on Bergman theory.

Recently, the Bergman kernel and projection on a class of bounded Hartogs domains, that is, the Hartogs triangle and its variants, have attracted attention of many authors. We recall that the classical *Hartogs triangle* is the domain defined by

$$H := \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$

It is a pseudoconvex domain that is the source of pathological phenomena in several complex variables [27]. The Bergman kernel of H can be easily obtained from the fact that H is biholomorphic to the product domain $\mathbb{D}\times\mathbb{D}^*$, where $\mathbb{D}$ and $\mathbb{D}^*$ denote the unit disc and the punctured unit disc in $\mathbb{C}$, respectively. It was shown by Chakrabarti–Zeytuncu [6] that the Bergman projection on the Hartogs triangle H is L^p-regular if and only if $p \in \left(\frac{4}{3}, 4\right)$. Furthermore, Huo–Wick [21] proved that on the classical Hartogs triangle H, the Bergman projection is not of weak-type $\left(\frac{4}{3}, \frac{4}{3}\right)$, but is of weak-type $(4,4)$.

In [8], Chen investigated the Bergman kernel and projection on the class of generalized Hartogs triangles defined by

$$H_k^n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \|(z_1, \dots, z_k)\| < |z_{k+1}| < \cdots < |z_n| < 1\}. \tag{1.2}$$

In [14], Edholm obtained the explicit formula for the Bergman kernel of the 2-dimensional power-generalized Hartogs triangle

$$H_k := \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2| < 1\}, \qquad k \in \mathbb{Z}^+, \tag{1.3}$$

and the L^p mapping properties for the associated Bergman projection were investigated by Edholm–McNeal [15]. Moreover, Christopherson–Koenig [11] showed that the weak-type estimate for the Bergman projection of H_k holds at the upper endpoint, but fails at the lower endpoint of the L^p boundedness range. Zhang [29] studied Bergman theory on n-dimensional power-generalized Hartogs triangles. Applying the well-known Bell formula (see [3]), Chakrabarti et al. [5] derived an explicit formula for the Bergman kernel of the elementary Reinhardt domain

$$\mathcal{H}_{\mathbf{k}} := \Big\{ z = (z_1, \cdots, z_n) \in \mathbb{D}^n : |z_1|^{k_1} < \prod_{j=2}^{n} |z_j|^{k_j} \Big\} \tag{1.4}$$

with $\mathbf{k} := (k_1, \ldots, k_n) \in (\mathbb{Z}^+)^n$, and the L^p boundedness for the Bergman projection of $\mathcal{H}_{\mathbf{k}}$ was investigated by Zhang [30]. Furthermore, Bender et al. [4] completely characterized the L^p boundedness of the Bergman projection associated to so-called monomial polyhedra. We also refer the readers to [1, 17, 26, 28] for more works on this topic.

It is important to observe that the Hartogs triangle and its generalizations mentioned above share one crucial property: they are all proper holomorphic (or even biholomorphic) to some product domains of the unit ball $\mathbb{B}^n$ (or the unit disc $\mathbb{D}$) and the punctured unit disc $\mathbb{D}^*$. We call it the *product property*. For example, the classical Hartogs triangle H is biholomorphic to the product domain $\mathbb{D} \times \mathbb{D}^*$; the generalized Hartogs triangle defined by (1.2) is biholomorphic to $\mathbb{B}^k \times (\mathbb{D}^*)^{n-k}$; and H_k and $\mathcal{H}_{\mathbf{k}}$, which are defined by (1.3) and (1.4), are proper holomorphic to $\mathbb{D} \times \mathbb{D}^*$ and $\mathbb{D} \times (\mathbb{D}^*)^{n-1}$, respectively. Similarly, the monomial polyhedra studied in [4] are also proper holomorphic to the product of some copies of the unit disc and some copies of the punctured unit disc; see [4, Theorem 3.12]. The product property for the domains not only makes it possible to use Bell's formula to calculate the Bergman kernel, but also enables us to use Bell's transformation rule for the Bergman projection under proper holomorphic mappings (see Bell [2, 3]) to study the L^p boundedness of the Bergman projection. See [10, 9, 13].

Inspired by this, it is therefore reasonable and interesting to investigate the Bergman theory on some generalizations of the Hartogs triangles without the product property. In this paper, we study the Bergman kernel and projection on the generalized Hartogs triangle defined by

$$H^{\mathbf{p}}_{k+m} := \{ z = (\widetilde{z}, z_{k+1}, \ldots, z_{k+m}) \in \mathbb{C}^{k+m} : \|\widetilde{z}\|^{p_1} < |z_{k+1}|^{p_{k+1}} < \cdots < |z_{k+m}|^{p_{k+m}} < 1 \},$$

where $k, m \in \mathbb{Z}^+$, $\mathbf{p} := (p_1, p_{k+1}, \ldots, p_{k+m}) \in (\mathbb{R}^+)^{m+1}$, $\widetilde{z} := (z_1, \ldots, z_k) \in \mathbb{C}^k$ and $\|\widetilde{z}\| := (|z_1|^2 + \cdots + |z_k|^2)^{1/2}$ is the Euclidean norm in $\mathbb{C}^k$. Clearly, when $\mathbf{p} := (1, \ldots, 1)$, then $H^{\mathbf{p}}_{k+m}$ coincides with the domain considered by Chen [8]. If $k = 1$ or $m = 1$, then $H^{\mathbf{p}}_{k+m}$ reduces to the generalized Hartogs triangles studied in [29] or [18]. Also, $H^{\mathbf{p}}_{k+m}$ contains many other types of bounded Hartogs domains: see [1, 28]. Thus, our setting unifies and extends most of the previously considered domains. More importantly, from the definition of $H^{\mathbf{p}}_{k+m}$, we can find that $H^{\mathbf{p}}_{k+m}$ is not proper holomorphic to any product domain in general. We therefore expect that the analysis developed here will provide new insights and tools for the Bergman theory on generalized Hartogs triangles beyond the product property framework.

The paper is organized as follows. In Section 2, we study the Bergman kernel of $H^{\mathbf{p}}_{k+m}$. For simplicity, we write the Bergman kernel as $K_{\mathbf{p},k+m} := K_{H^{\mathbf{p}}_{k+m}}$. First, we obtain an explicit formula for the Bergman kernel in Section 2.1: see Theorem 2.4. As mentioned before, in general the domain $H^{\mathbf{p}}_{k+m}$ does not have the product property, so we cannot apply Bell's transformation formula to compute the Bergman kernel as in the previous works. Our strategy combines the method of sub-Bergman kernels and some new ideas. After giving the explicit formula for the Bergman kernel $K_{\mathbf{p},k+m}$, we find that the form of $K_{\mathbf{p},k+m}$ is too complicated to use, and it seems hard to get a closed form of it. Therefore, in Section 2.2, we use some technical methods to obtain the optimal estimate for the Bergman kernel – see Theorem 2.6, which is a crucial tool in studying the Bergman projection in the subsequent sections.

Section 3 is devoted to the L^p boundedness of the Bergman projection on $H^{\mathbf{p}}_{k+m}$. We first consider the rational case, that is, the domain $H^{\mathbf{p}}_{k+m}$ with $\mathbf{p} := (p_1, p_{k+1}, \dots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. The key tools are the Forelli–Rudin type estimate on $H^{\mathbf{p}}_{k+m}$ and a generalized version of Schur's test; see Proposition 3.2 and Lemma 3.3, respectively. Our main result is Theorem 3.7, which gives the L^p boundedness range for the Bergman projection. On the other hand, for the irrational case, we show that the Bergman projection $B_{\mathbf{p},k+m}$ completely degenerates as an L^p mapping; see Theorem 3.10.

Following this line, in Section 4, we further consider the weak-type regularity for the Bergman projection of $H^{\mathbf{p}}_{k+m}$ at the endpoints of the L^p boundedness range (I_1, I_2) obtained in Theorem 3.7. We prove that the weak-type estimate fails at the lower endpoint I_1, but holds at the upper endpoint I_2 (see Theorems 4.1 and 4.3). These weak-type estimates are new even in the special case of $m = 1$. They not only complement Theorem 3.7 but also extend the earlier work of Huo–Wick [21], Jing et al. [22] and Christopherson–Koenig [11, 12] to a more general setting.

Throughout this paper, we use the following notations. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we write as usual $\alpha! := \alpha_1! \cdots \alpha_n!$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. For given functions $A(z)$ and $B(z)$, we use $A(z) \lesssim B(z)$ to signify that $A(z) \le kB(z)$ for some constant $k > 0$, and we also write $A(z) \approx B(z)$ to mean that $A(z) \lesssim B(z) \lesssim A(z)$.

2. The Bergman kernel of $H^{\mathbf{p}}_{k+m}$

2.1. The explicit formula for the Bergman kernel. We begin with the following lemma, which characterizes when the holomorphic monomials z^α belong to $L^p(H^{\mathbf{p}}_{k+m})$ and gives their L^p norms.

LEMMA 2.1. *Let* $\alpha := (\widetilde{\alpha}, \alpha_{k+1}, \ldots, \alpha_{k+m}) \in \mathbb{Z}^{k+m}$ *be a multi-index, where* $\widetilde{\alpha} := (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k$. *For any generalized Hartogs triangle* $H^{\mathbf{p}}_{k+m}$ *with* $\mathbf{p} := (p_1, p_{k+1}, \ldots, p_{k+m}) \in (\mathbb{R}^+)^{m+1}$ *and* $1 < p < \infty$, *define the set of multi-indices*

$$\mathcal{A}^p(H^{\mathbf{p}}_{k+m}) := \Big\{ \alpha \in \mathbb{Z}^{k+m} : \alpha_i \geq 0,\ i = 1, \ldots, k, \frac{|\widetilde{\alpha}|p + 2k}{p_1} + \sum_{t=k+1}^{j} \frac{\alpha_t p + 2}{p_t} > 0,\ j = k+1, \ldots, k+m \Big\}.$$

Then $z^\alpha \in A^p(H^{\mathbf{p}}_{k+m})$ *if and only if* $\alpha \in \mathcal{A}^p(H^{\mathbf{p}}_{k+m})$. *Moreover, for any* $\alpha \in \mathcal{A}^p(H^{\mathbf{p}}_{k+m})$ *we have*

$$\|z^\alpha\|^p_{L^p} = 2^{m+1} \pi^{k+m} \frac{\prod_{i=1}^k \Gamma\big(\frac{\alpha_i p}{2} + 1\big)}{(|\widetilde{\alpha}|p + 2k)\Gamma\big(\frac{|\widetilde{\alpha}|p}{2} + k\big)} \prod_{j=k+1}^{k+m} \frac{1}{s_j(p) + 2},$$

where

$$s_j(p) := p_j \bigg(\frac{|\widetilde{\alpha}|p + 2k}{p_1} + \sum_{t=k+1}^{j} \frac{\alpha_t p + 2}{p_t} \bigg) - 2, \qquad j = k+1, \ldots, k+m.$$

Proof. We first prove the lemma for $m \geq 2$. Let $\alpha := (\widetilde{\alpha}, \alpha_{k+1}, \ldots, \alpha_{k+m}) \in \mathbb{Z}^{k+m}$ be a multi-index, and let $1 < p < \infty$. Assume that $z^\alpha \in A^p(H^{\mathbf{p}}_{k+m})$. Note that z_i can be zero for $i = 1, \ldots, k$, which implies $\alpha_i \geq 0$ for $i = 1, \ldots, k$.

Now we compute the L^p norm $\|z^\alpha\|_{L^p}$. By a direct calculation, we obtain

$$\begin{aligned}(2.1)\quad \|z^\alpha\|^p_{L^p} &= \int\limits_{H^{\mathbf{p}}_{k+m}} |\widetilde{z}^{\widetilde{\alpha}}|^p \prod_{j=k+1}^{k+m} |z_j|^{\alpha_j p} \, dV(z) \\ &= \int\limits_{\mathbb{D}^*} |z_{k+m}|^{\alpha_{k+m} p} \, dV(z_{k+m}) \cdots \int\limits_{0 < |z_{k+1}| < |z_{k+2}|^{\frac{p_{k+2}}{p_{k+1}}}} |z_{k+1}|^{\alpha_{k+1} p} \, dV(z_{k+1}) \\ &\quad \times \int\limits_{\|\widetilde{z}\| < |z_{k+1}|^{\frac{p_{k+1}}{p_1}}} |\widetilde{z}^{\widetilde{\alpha}}|^p \, dV(\widetilde{z}).\end{aligned}$$

Using the transformation

$$\widetilde{\xi} = |z_{k+1}|^{-p_{k+1}/p_1} \widetilde{z},$$

we have

$$(2.2)\qquad \int\limits_{\|\widetilde{z}\| < |z_{k+1}|^{\frac{p_{k+1}}{p_1}}} |\widetilde{z}^{\widetilde{\alpha}}|^p \, dV(\widetilde{z}) = |z_{k+1}|^{\frac{p_{k+1}}{p_1}(|\widetilde{\alpha}|p + 2k)} \int\limits_{\mathbb{B}^k} |\widetilde{\xi}^{\widetilde{\alpha}}|^p \, dV(\widetilde{\xi}).$$

Substituting (2.2) into (2.1) yields

$$\|z^\alpha\|_{L^p}^p = \int_{\mathbb{B}^k} |\widetilde{\xi}^{\widetilde{\alpha}}|^p \, dV(\widetilde{\xi}) \int_{\mathbb{D}^*} |z_{k+m}|^{\alpha_{k+m}p} \, dV(z_{k+m}) \cdots \times \int_{0<|z_{k+1}|<|z_{k+2}|^{\frac{p_{k+2}}{p_{k+1}}}} |z_{k+1}|^{s_{k+1}(p)} \, dV(z_{k+1}), \tag{2.3}$$

where $s_{k+1}(p) := \alpha_{k+1}p + \frac{p_{k+1}}{p_1}(|\widetilde{\alpha}|p + 2k)$.

After a straightforward computation, we conclude that

$$\int_{0<|z_{k+1}|<|z_{k+2}|^{\frac{p_{k+2}}{p_{k+1}}}} |z_{k+1}|^{s_{k+1}(p)} \, dV(z_{k+1}) < \infty$$

if and only if $s_{k+1}(p) > -2$. Substituting this into (2.3) and iterating the above process, we deduce that $\|z^\alpha\|_{L^p} < \infty$ if and only if

$$\alpha_i \geq 0, \quad i = 1, \ldots, k, \quad \text{and} \quad s_j(p) > -2, \quad j = k+1, \ldots, k+m, \tag{2.4}$$

where $s_{k+1}(p)$ is defined as above, and for $m \geq 2$, $s_j(p)$ $(k+2 \leq j \leq k+m)$ are given by the recurrence relation

$$s_j(p) := \alpha_j p + \frac{p_j}{p_{j-1}}(s_{j-1}(p) + 2), \qquad k+2 \leq j \leq k+m.$$

A direct calculation then shows that

$$s_j(p) = p_j\left(\frac{|\widetilde{\alpha}|p + 2k}{p_1} + \sum_{t=k+1}^{j} \frac{\alpha_t p + 2}{p_t}\right) - 2, \qquad j = k+1, \ldots, k+m.$$

Together with (2.4), this implies that $z^\alpha \in A^p(H^{\mathbf{p}}_{k+m})$ if and only if $\alpha \in \mathcal{A}^p(H^{\mathbf{p}}_{k+m})$, where $\mathcal{A}^p(H^{\mathbf{p}}_{k+m})$ is the set of multi-indices defined in the lemma.

For any $\alpha \in \mathcal{A}^p(H^{\mathbf{p}}_{k+m})$ we have

$$\|z^\alpha\|_{L^p}^p = \int_{\mathbb{B}^k} |\widetilde{\xi}^{\widetilde{\alpha}}|^p \, dV(\widetilde{\xi}) \cdot \prod_{j=k+1}^{k+m} \int_{\mathbb{D}^*} |\xi_j|^{s_j(p)} \, dV(\xi_j). \tag{2.5}$$

Applying polar integration gives

$$\int_{\mathbb{B}^k} |\widetilde{\xi}^{\widetilde{\alpha}}|^p \, dV(\widetilde{\xi}) = \int_0^1 r^{|\widetilde{\alpha}|p+2k-1} \, dr \int_{\partial\mathbb{B}^k} |\widetilde{\eta}^{\widetilde{\alpha}}|^p \, d\sigma(\widetilde{\eta}) = 2\pi^k \frac{\prod_{i=1}^k \Gamma\left(\frac{\alpha_i p}{2} + 1\right)}{(|\tilde{\alpha}|p + 2k)\Gamma\left(\frac{|\tilde{\alpha}|p}{2} + k\right)}.$$

Hence,

$$\|z^\alpha\|_{L^p}^p = 2^{m+1}\pi^{k+m} \frac{\prod_{i=1}^k \Gamma\left(\frac{\alpha_i p}{2} + 1\right)}{(|\tilde{\alpha}|p + 2k)\Gamma\left(\frac{|\tilde{\alpha}|p}{2} + k\right)} \prod_{j=k+1}^{k+m} \frac{1}{s_j(p) + 2}.$$

This proves the lemma for the case $m \geq 2$.

Now we assume $m=1$. In this case, (2.3) becomes

$$\|z^\alpha\|_{L^p}^p = \int\limits_{\mathbb{B}^k} |\widetilde{\xi}^{\widetilde{\alpha}}|^p \, dV(\widetilde{\xi}) \cdot \int\limits_{\mathbb{D}^*} |z_{k+1}|^{s_{k+1}(p)} \, dV(z_{k+1}).$$

Applying the same argument as in the calculation of (2.5), we obtain the desired result for $m=1$. ∎

Since $H^{\mathbf{p}}_{k+m}$ is a Reinhardt domain, the holomorphic monomials z^α belonging to $L^2(H^{\mathbf{p}}_{k+m})$ constitute an orthogonal basis for the Bergman space $A^2(H^{\mathbf{p}}_{k+m})$. Consequently, Lemma 2.1 yields the following result.

LEMMA 2.2. *The holomorphic monomials* $\{z^\alpha : \alpha \in \mathcal{A}^2(H^{\mathbf{p}}_{k+m})\}$ *form an orthogonal basis for* $A^2(H^{\mathbf{p}}_{k+m})$. *Moreover, for any* $\alpha \in \mathcal{A}^2(H^{\mathbf{p}}_{k+m})$,

$$\|z^\alpha\|_{L^2}^2 = 2^m \pi^{k+m} \frac{\widetilde{\alpha}!}{(|\widetilde{\alpha}|+k)!} \prod_{j=k+1}^{k+m} \frac{1}{s_j+2},$$

where

$$s_j := s_j(2) = 2p_j\biggl(\frac{|\widetilde{\alpha}|+k}{p_1} + \sum_{t=k+1}^{j} \frac{\alpha_t+1}{p_t}\biggr) - 2, \qquad j=k+1,\ldots,k+m.$$

The next two theorems constitute part of the main results of this paper. They provide the explicit formulas for the Bergman kernel of $H^{\mathbf{p}}_{k+m}$ with $\mathbf{p} := (p_1, p_{k+1}, \ldots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. We show that the Bergman kernel of $H^{\mathbf{p}}_{k+m}$ can be expressed as the sum of p_1 (for $m=1$) or $p_1 \prod_{j=k+1}^{k+m-1} p_j$ (for $m \geq 2$) sub-Bergman kernels. The notion of sub-Bergman kernel is introduced by Edholm and McNeal [16]. Roughly speaking, in some settings we can split the Bergman space into some orthogonal subspaces (called sub-Bergman spaces). The sub-Bergman kernels are then defined as the integral kernels of the orthogonal projections from the Bergman space onto these subspaces. We refer to [16] for further details. It should be noted, however, that in the proofs of Theorems 2.3 and 2.4 we handle the Bergman kernel directly, without decomposing the Bergman space or the Bergman projection into pieces as in [16].

The following theorem gives a formula for the Bergman kernel of $H^{\mathbf{p}}_{k+1}$.

THEOREM 2.3. *Let* $K_{\mathbf{p},k+1}(z,w)$ *be the Bergman kernel of* $H^{\mathbf{p}}_{k+1}$ *with* $\mathbf{p} := (p_1, p_{k+1}) \in (\mathbb{Z}^+)^2$. *Then* $K_{\mathbf{p},k+1}$ *is given by*

$$K_{\mathbf{p},k+1}(z,w) = \sum_{\beta_1=1}^{p_1} K_{\beta_1}(z,w), \tag{2.6}$$

where for each integer β_1 *with* $1 \leq \beta_1 \leq p_1$, *the* sub-Bergman kernel $K_{\beta_1}(z,w)$ *can be expressed as*

$$(2.7) \quad K_{\beta_1}(z,w) := \frac{1}{\pi^{k+1}}(z_{k+1}\overline{w}_{k+1})^{\widehat{\mu}_{k+1}(\beta_1)}\sum_{i=0}^{k}\binom{k}{i}\left(\frac{1}{1-u^{p_1}}\right)^{(i)}\Big|_{u=u_1}(u^{\beta_1+k-1})^{(k-i)}\Big|_{u=u_1} \times \frac{h(p_{k+1}E_1(\beta_1))(1-u_{k+1})+u_{k+1}}{(1-u_{k+1})^2}.$$

Here $h(x) := \{x\} - x$, and u_1, u_{k+1}, and $\widehat{\mu}_{k+1}(\beta_1)$ are defined by

$$u_1 := \frac{\langle \widetilde{z}, \widetilde{w}\rangle}{(z_{k+1}\overline{w}_{k+1})^{p_{k+1}/p_1}}, \qquad u_{k+1} := z_{k+1}\overline{w}_{k+1},$$

and

$$\widehat{\mu}_{k+1}(\beta_1) := \frac{p_{k+1}}{p_1}(\beta_1 - 1) + \{p_{k+1}E_1(\beta_1)\} - 1,$$

respectively. The symbol $\{x\}$ denotes the smallest integer strictly greater than $x \in \mathbb{R}$, and $E_1(\beta_1) := -(\beta_1 + k - 1)/p_1$.

Proof. We first calculate the Bergman kernel of $H^{\mathbf{p}}_{k+1}$ on the diagonal, i.e., $K_{\mathbf{p},k+1}(z,z)$. The off-diagonal Bergman kernel $K_{\mathbf{p},k+1}(z,w)$ will then be obtained via polarization.

Since $H^{\mathbf{p}}_{k+1}$ is a Reinhardt domain, we deduce from Lemma 2.2 that $\{z^\alpha/\|z^\alpha\|_{L^2}\}_{\alpha\in\mathcal{A}^2(H^{\mathbf{p}}_{k+1})}$ forms an orthonormal basis of the Bergman space $A^2(H^{\mathbf{p}}_{k+1})$. Applying Lemma 2.2, we obtain

$$(2.8) \quad K_{\mathbf{p},k+1}(z,z) = \sum_{\alpha\in\mathcal{A}^p(H^{\mathbf{p}}_{k+1})}\frac{|z^\alpha|^2}{\|z^\alpha\|^2_{L^2}} = \frac{1}{2\pi^{k+1}}\sum_{\tilde{\alpha}\in\mathbb{N}^k}\frac{(|\tilde{\alpha}|+k)!}{\tilde{\alpha}!}|\tilde{z}^{\tilde{\alpha}}|^2\sum_{\alpha_{k+1}>r_{k+1}}(s_{k+1}+2)|z_{k+1}|^{2\alpha_{k+1}},$$

where $r_{k+1} := -\frac{p_{k+1}}{p_k}(|\tilde{\alpha}|+k) - 1$, and s_{k+1} is defined as in Lemma 2.2.

Using the identity

$$\|x\|^{2l} = \sum_{|\alpha|=l}\frac{\Gamma(|\alpha|+1)}{\alpha!}x^{2\alpha}, \qquad x = (x_1,\dots,x_n),$$

we have

$$(2.9) \quad \sum_{i=0}^{\infty}\frac{(i+k)!}{i!}\|\tilde{z}\|^{2i} = \sum_{i=0}^{\infty}\sum_{|\tilde{\alpha}|=i}\frac{\Gamma(|\tilde{\alpha}|+1)}{\tilde{\alpha}!}\frac{(i+k)!}{i!}|\tilde{z}^{\tilde{\alpha}}|^2 = \sum_{i=0}^{\infty}\sum_{|\tilde{\alpha}|=i}\frac{\Gamma(i+1)}{\tilde{\alpha}!}\frac{(|\tilde{\alpha}|+k)!}{i!}|\tilde{z}^{\tilde{\alpha}}|^2 = \sum_{\tilde{\alpha}\in\mathbb{N}^k}\frac{(|\tilde{\alpha}|+k)!}{\tilde{\alpha}!}|\tilde{z}^{\tilde{\alpha}}|^2.$$

Observe that r_{k+1} and s_{k+1} depend only on $|\widetilde{\alpha}|$ (and not on the individual components $\alpha_1, \ldots, \alpha_k$). Then combining (2.8) and (2.9), we obtain

$$\begin{aligned}(2.10)\qquad & K_{\mathbf{p},k+1}(z,z) \\ &= \frac{1}{2\pi^{k+1}} \sum_{\alpha_1=0}^{\infty} \frac{(\alpha_1+k)!}{\alpha_1!} \|\widetilde{z}\|^{2\alpha_1} \sum_{\alpha_{k+1}>r'_{k+1}} (s_{k+1}+2)|z_{k+1}|^{2\alpha_{k+1}},\end{aligned}$$

where $r'_{k+1} := -\frac{p_{k+1}}{p_1}(\alpha_1+k)-1$.

We now decompose the summation index α_1 in (2.10) into residue classes modulo p_1. Namely, we write

$$\alpha_1 = n_1 p_1 + \beta_1 - 1, \qquad 1 \le \beta_1 \le p_1.$$

Note that

$$\alpha_{k+1} > r'_{k+1} \iff a_{k+1} > N_{k+1} := -p_{k+1} n_1 + p_{k+1} E_1(\beta_1) - 1,$$

where $E_1(\beta_1)$ is defined in the theorem. Consequently, the Bergman kernel admits the decomposition

$$K_{\mathbf{p},k+1}(z,z) = \sum_{\beta_1=1}^{p_1} K_{\beta_1}(z,z),$$

where for each integer β_1 with $1 \le \beta_1 \le p_1$,

$$\begin{aligned}(2.11)\qquad K_{\beta_1}(z,z) &= \frac{1}{2\pi^{k+1}} \sum_{n_1=0}^{\infty} \frac{(n_1p_1+\beta_1-1+k)!}{(n_1p_1+\beta_1-1)!} \|\widetilde{z}\|^{2(n_1p_1+\beta_1-1)} \\ &\quad \times \sum_{\alpha_{k+1}>N_{k+1}} (s_{k+1}+2)|z_{k+1}|^{2\alpha_{k+1}}.\end{aligned}$$

Set $\alpha'_{k+1} := \alpha_{k+1} + p_{k+1}n_1 + 1$. A direct computation yields

$$\begin{aligned}(2.12)\qquad & \sum_{\alpha_{k+1}>N_{k+1}} (s_{k+1}+2)|z_{k+1}|^{2\alpha_{k+1}} \\ &= 2 \sum_{\alpha_{k+1}>N_{k+1}} (p_{k+1}n_1 - p_{k+1}E_1(\beta_1) + \alpha_{k+1} + 1)|z_{k+1}|^{2\alpha_{k+1}} \\ &= 2|z_{k+1}|^{-2(p_{k+1}n_1+1)} \sum_{\alpha'_{k+1}>p_{k+1}E_1(\beta_1)} (\alpha'_{k+1} - p_{k+1}E_1(\beta_1))|z_{k+1}|^{2\alpha'_{k+1}} \\ &= 2|z_{k+1}|^{-2(p_{k+1}n_1+1)} \sum_{\alpha'_{k+1}=\{p_{k+1}E_1(\beta_1)\}}^{\infty} (\alpha'_{k+1} - p_{k+1}E_1(\beta_1))|z_{k+1}|^{2\alpha'_{k+1}} \\ &= 2|z_{k+1}|^{-2p_{k+1}n_1-2+2\{p_{k+1}E_1(\beta_1)\}} \cdot \frac{h(p_{k+1}E_1(\beta_1))(1-u_{k+1}) + u_{k+1}}{(1-u_{k+1})^2},\end{aligned}$$

where we have used the identity

$$\sum_{n=0}^{\infty}(An+B)z^n = \frac{(A-B)z+B}{(1-z)^2}, \qquad |z|<1,\ A,B\in\mathbb{C}. \tag{2.13}$$

Substituting (2.12) into (2.11) gives

$$\begin{aligned} K_{\beta_1}(z,z) &= \frac{1}{\pi^{k+1}}\cdot\frac{h(p_{k+1}E_1(\beta_1))(1-u_{k+1})+u_{k+1}}{(1-u_{k+1})^2}\cdot|z_{k+1}|^{2\{p_{k+1}E_1(\beta_1)\}-2}\\ &\quad\times\sum_{n_1=0}^{\infty}\frac{(n_1p_1+\beta_1-1+k)!}{(n_1p_1+\beta_1-1)!}\frac{\|\widetilde{z}\|^{2(n_1p_1+\beta_1-1)}}{|z_{k+1}|^{2p_{k+1}n_1}}. \end{aligned} \tag{2.14}$$

The summation in (2.14) can be calculated as

$$\begin{aligned} &\sum_{n_1=0}^{\infty}\frac{(n_1p_1+\beta_1-1+k)!}{(n_1p_1+\beta_1-1)!}\frac{\|\widetilde{z}\|^{2(n_1p_1+\beta_1-1)}}{|z_{k+1}|^{2p_{k+1}n_1}}\\ &= |z_{k+1}|^{\frac{2p_{k+1}}{p_1}(\beta_1-1)}\frac{d^k}{du^k}\frac{u^{\beta_1-1+k}}{1-u^{p_1}}\bigg|_{u=u_1}\\ &= |z_{k+1}|^{\frac{2p_{k+1}}{p_1}(\beta_1-1)}\sum_{i=0}^{k}\binom{k}{i}\left(\frac{1}{1-u^{p_1}}\right)^{(i)}\bigg|_{u=u_1}(u^{\beta_1+k-1})^{(k-i)}\big|_{u=u_1}. \end{aligned} \tag{2.15}$$

Combining (2.14) with (2.15), we obtain the desired expression for the diagonal sub-Bergman kernel $K_{\beta_1}(z,z)$.

Finally, applying polarization yields the required formula for the full Bergman kernel $K_{\mathbf{p},k+1}(z,w)$. ∎

The next theorem provides the Bergman kernel for the generalized Hartogs triangle $H^{\mathbf{p}}_{k+m}$ with $m\geq 2$.

Theorem 2.4. *Let $H^{\mathbf{p}}_{k+m}$ be the generalized Hartogs triangle with $\mathbf{p}:=(p_1,p_{k+1},\ldots,p_{k+m})\in(\mathbb{Z}^+)^{m+1}$, where $m\geq 2$. Denote by $K_{\mathbf{p},k+m}(z,w)$ the Bergman kernel of $H^{\mathbf{p}}_{k+m}$. Then*

$$K_{\mathbf{p},k+m}(z,w) = \sum_{\beta\in J_{\mathbf{p}}} K_\beta(z,w), \tag{2.16}$$

where

$$J_{\mathbf{p}} := \{\beta=(\beta_1,\beta_{k+1},\ldots,\beta_{k+m-1})\in(\mathbb{Z}^+)^m : \\ 1\leq\beta_j\leq p_j,\ j=1,k+1,\ldots,k+m-1\},$$

and for each $\beta \in J_{\mathbf{p}}$, the sub-Bergman kernel $K_\beta(z,w)$ can be expressed as

$$(2.17)\qquad K_\beta(z,w) :=$$
$$\frac{\prod_{j=k+1}^{k+m-1} p_j}{\pi^{k+m}} \cdot \prod_{j=k+1}^{k+m} (z_j\overline{w}_j)^{\mu_j(\beta)} \cdot \sum_{i=0}^{k} \binom{k}{i} \left(\frac{1}{1-u^{p_1}}\right)^{(i)}\Big|_{u=u_1} (u^{\beta_1+k-1})^{(k-i)}\Big|_{u=u_1}$$
$$\times \prod_{j=k+1}^{k+m-1} \frac{h(E_j(\beta))(1-u_j)+u_j}{(1-u_j)^2} \cdot \frac{h(p_{k+m}E_{k+m-1}(\beta))(1-u_{k+m})+u_{k+m}}{(1-u_{k+m})^2}.$$

Here, $h(x)$ is defined as in Theorem 2.3 , and for $z,w \in H^{\mathbf{p}}_{k+m}$ we set

$$u_1 := \frac{\langle \widetilde{z}, \widetilde{w}\rangle}{(z_{k+1}\overline{w}_{k+1})^{\frac{p_{k+1}}{p_1}}},\ u_j := \frac{(z_j\overline{w}_j)^{p_j}}{(z_{j+1}\overline{w}_{j+1})^{p_{j+1}}}, \qquad j = k+1,\dots,k+m-1,$$
$$u_{k+m} := z_{k+m}\overline{w}_{k+m}.$$

The constants $E_j(\beta)$ and $\mu_j(\beta)$, which depend on the multi-index β, are defined by

$$E_j(\beta) := -\frac{\beta_1+k-1}{p_1} - \sum_{t=k+1}^{j} \frac{\beta_t}{p_t}, \qquad j = k+1,\dots,k+m-1,$$
$$\mu_{k+1}(\beta) := \frac{p_{k+1}}{p_1}(\beta_1-1) + p_{k+1}\{E_{k+1}(\beta)\} + \beta_{k+1} - 1,$$
$$\mu_j(\beta) := p_j(\{E_j(\beta)\} - \{E_{j-1}(\beta)\}) + \beta_j - 1,$$
$$j = k+2,\dots,k+m-1 \ (\textit{if } m \geq 3),$$
$$\mu_{k+m}(\beta) := \{p_{k+m}E_{k+m-1}(\beta)\} - p_{k+m}\{E_{k+m-1}(\beta)\} - 1.$$

Proof. Following the approach in the proof of Theorem 2.3, we first compute the Bergman kernel of $H^{\mathbf{p}}_{k+m}$ on the diagonal.

Using the same reasoning as in (2.8), we get

$$(2.18)\qquad K_{\mathbf{p},k+m}(z,z)$$
$$= C_1 \sum_{\widetilde{\alpha}\in\mathbb{N}^k} \frac{(|\widetilde{\alpha}|+k)!}{\widetilde{\alpha}!} |\widetilde{z}^{\widetilde{\alpha}}|^2$$
$$\times \sum_{\alpha_{k+1}>r_{k+1}} (s_{k+1}+2)|z_{k+1}|^{2\alpha_{k+1}} \cdots \sum_{\alpha_{k+m}>r_{k+m}} (s_{k+m}+2)|z_{k+m}|^{2\alpha_{k+m}},$$

where $C_1 := \frac{1}{2^m\pi^{k+m}}$, and

$$r_j := -\frac{p_j}{p_1}(|\widetilde{\alpha}|+k) - p_j \sum_{t=k+1}^{j-1} \frac{\alpha_t+1}{p_t} - 1, \qquad j = k+1,\dots,k+m.$$

Applying (2.9) to (2.18) yields

$$
\begin{aligned}
(2.19)\qquad & K_{\mathbf{p},k+m}(z,z) \\
&= C_1 \sum_{\alpha_1=0}^{\infty} \frac{(\alpha_1+k)!}{\alpha_1!} \|\widetilde{z}\|^{2\alpha_1} \sum_{\alpha_{k+1}>r'_{k+1}} (s_{k+1}+2)|z_{k+1}|^{2\alpha_{k+1}} \dots \\
&\quad \times \sum_{\alpha_{k+m}>r'_{k+m}} (s_{k+m}+2)|z_{k+m}|^{2\alpha_{k+m}},
\end{aligned}
$$

where

$$
r'_j := -\frac{p_j}{p_1}(\alpha_1+k) - p_j \sum_{t=k+1}^{j-1} \frac{\alpha_t+1}{p_t} - 1, \qquad j = k+1, \dots, k+m.
$$

In order to compute (2.19), we decompose the summation indices α_j into residue classes modulo p_j for $j = 1, k+1, \dots, k+m-1$. We write

$$
\alpha_j = n_j p_j + \beta_j - 1, \qquad n_j \in \mathbb{Z}^+,\ 1 \le \beta_j \le p_j,\ j = 1, k+1, \dots, k+m-1.
$$

It follows that

$$
\alpha_j > r'_j \iff n_j > N_j := -n_1 - \sum_{t=k+1}^{j-1} n_t + E_j(\beta),\ j = k+1, \dots, k+m-1,
$$

and

$$
\begin{aligned}
&\alpha_{k+m} > r'_{k+m} \\
&\iff \alpha_{k+m} > N_{k+m} := -p_{k+m}\Big(n_1 + \sum_{t=k+1}^{k+m-1} n_t\Big) + p_{k+m}E_{k+m-1}(\beta) - 1,
\end{aligned}
$$

and

$$
s_j + 2 := \begin{cases} 2p_j\big(n_1 + \sum_{t=k+1}^{j} n_t - E_j(\beta)\big), & j = k+1, \dots, k+m-1, \\ 2\big(p_{k+m}\big(n_1 + \sum_{t=k+1}^{k+m-1} n_t\big) - p_{k+m}E_{k+m-1}(\beta) + \alpha_{k+m} + 1\big), & \\ & j = k+m, \end{cases}
$$

where $E_j(\beta)$ is as defined in the theorem.

Consequently, the Bergman kernel $K_{\mathbf{p},k+m}(z,z)$ can be decomposed into the sum of $p_1 \prod_{j=k+1}^{k+m-1} p_j$ sub-Bergman kernels, that is,

$$
(2.20)\qquad K_{\mathbf{p},k+m}(z,z) = \sum_{\beta:=(\beta_1,\beta_{k+1},\dots,\beta_{k+m-1})\in J_{\mathbf{p}}} K_\beta(z,z),
$$

where

$$
J_{\mathbf{p}} := \{\beta \in (\mathbb{Z}^+)^m : 1 \le \beta_j \le p_j,\ j = 1, k+1, \dots, k+m-1\}.
$$

Moreover, for each multi-index $\beta \in J_{\mathbf{p}}$, we have

$$
\begin{aligned}
(2.21)\qquad & K_\beta(z,z) \\
&= C_1 \prod_{j=k+1}^{k+m-1} |z_j|^{2(\beta_j-1)} \cdot \sum_{n_1=0}^{\infty} \frac{(n_1p_1+\beta_1-1+k)!}{(n_1p_1+\beta_1-1)!} \|\widetilde{z}\|^{2(n_1p_1+\beta_1-1)} \\
&\quad \times \sum_{n_{k+1}>N_{k+1}} (s_{k+1}+2)|z_{k+1}|^{2p_{k+1}n_{k+1}} \dots \\
&\quad \times \sum_{n_{k+m-1}>N_{k+m-1}} (s_{k+m-1}+2)|z_{k+m-1}|^{2p_{k+m-1}n_{k+m-1}} \\
&\quad \times \sum_{\alpha_{k+m}>N_{k+m}} (s_{k+m}+2)|z_{k+m}|^{2\alpha_{k+m}}.
\end{aligned}
$$

Now we compute the sub-Bergman kernel $K_\beta(z,z)$. By a direct calculation, we have

$$
\begin{aligned}
(2.22)\qquad & \sum_{\alpha_{k+m}>N_{k+m}} (s_{k+m}+2)|z_{k+m}|^{2\alpha_{k+m}} \\
&= 2 \sum_{\alpha_{k+m}>N_{k+m}} \Big(p_{k+m}\Big(n_1 + \sum_{t=k+1}^{k+m-1} n_t\Big) - p_{k+m}E_{k+m-1}(\beta) + \alpha_{k+m} + 1\Big) \\
&\qquad\qquad \times |z_{k+m}|^{2\alpha_{k+m}} \\
&= 2|z_{k+m}|^{-2p_{k+m}(n_1+\sum_{t=k+1}^{k+m-1} n_t)-2} \\
&\quad \times \sum_{\alpha'_{k+m}>p_{k+m}E_{k+m-1}(\beta)} (\alpha'_{k+m} - p_{k+m}E_{k+m-1}(\beta))|z_{k+m}|^{2\alpha'_{k+m}} \\
&= 2|z_{k+m}|^{-2p_{k+m}(n_1+\sum_{t=k+1}^{k+m-1} n_t)-2} \\
&\quad \times \sum_{\alpha'_{k+m}=\{p_{k+m}E_{k+m-1}(\beta)\}}^{\infty} (\alpha'_{k+m} - p_{k+m}E_{k+m-1}(\beta))|z_{k+m}|^{2\alpha'_{k+m}} \\
&= 2|z_{k+m}|^{-2p_{k+m}(n_1+\sum_{t=k+1}^{k+m-1} n_t)-2+2\{p_{k+m}E_{k+m-1}(\beta)\}} \\
&\quad \times \sum_{\alpha''_{k+m}=0}^{\infty} (\alpha''_{k+m} + h(p_{k+m}E_{k+m-1}(\beta)))|z_{k+m}|^{2\alpha''_{k+m}} \\
&= 2|z_{k+m}|^{-2p_{k+m}(n_1+\sum_{t=k+1}^{k+m-1} n_t)-2+2\{p_{k+m}E_{k+m-1}(\beta)\}} \\
&\quad \times \frac{h(p_{k+m}E_{k+m-1}(\beta))(1-u_{k+m}) + u_{k+m}}{(1-u_{k+m})^2}.
\end{aligned}
$$

Here, $\{x\}$ and $h(x)$ are defined in Theorem 2.3. In the derivation above we introduced the new summation indices

$$\alpha'_{k+m} := \alpha_{k+m} + p_{k+m}\Big(n_1 + \sum_{t=k+1}^{k+m-1} n_t\Big) + 1,$$

$$\alpha''_{k+m} := \alpha'_{k+m} - \{p_{k+m}E_{k+m-1}(\beta)\},$$

and the last equality in (2.22) follows from the identity (2.13).

Next we claim that

$$K_\beta(z,z) = \frac{\prod_{j=k+1}^{k+m-1} p_j}{\pi^{k+m}} \cdot |z_{k+1}|^{2\mu_{k+1}(\beta)-\frac{2p_{k+1}}{p_1}(\beta_1-1)} \times \prod_{j=k+2}^{k+m} |z_j|^{2\mu_j(\beta)} \cdot \prod_{j=k+1}^{k+m-1} \frac{h(E_j(\beta))(1-u_j)+u_j}{(1-u_j)^2} \times \frac{h(p_{k+m}E_{k+m-1}(\beta))(1-u_{k+m})+u_{k+m}}{(1-u_{k+m})^2} \times \sum_{n_1=0}^{\infty} \frac{(n_1p_1+\beta_1-1+k)!}{(n_1p_1+\beta_1-1)!} \frac{\|\widetilde{z}\|^{2(n_1p_1+\beta_1-1)}}{|z_{k+1}|^{2p_{k+1}n_1}}. \tag{2.23}$$

Indeed, substituting (2.22) into (2.21) yields

$$\begin{aligned} K_\beta(z,z) &= 2C_1|z_{k+m}|^{2\{p_{k+m}E_{k+m-1}(\beta)\}-2} \\ &\times \prod_{j=k+1}^{k+m-1} |z_j|^{2(\beta_j-1)} \cdot \frac{h(p_{k+m}E_{k+m-1}(\beta))(1-u_{k+m})+u_{k+m}}{(1-u_{k+m})^2} \\ &\times \sum_{n_1=0}^{\infty} \frac{(n_1p_1+\beta_1-1+k)!}{(n_1p_1+\beta_1-1)!} \frac{\|\widetilde{z}\|^{2(n_1p_1+\beta_1-1)}}{|z_{k+m}|^{2n_1p_{k+m}}} \\ &\times \sum_{n_{k+1}>N_{k+1}} (s_{k+1}+2)\left(\frac{|z_{k+1}|^{2p_{k+1}}}{|z_{k+m}|^{2p_{k+m}}}\right)^{n_{k+1}} \cdots \\ &\times \sum_{n_{k+m-1}>N_{k+m-1}} (s_{k+m-1}+2)\left(\frac{|z_{k+m-1}|^{2p_{k+m-1}}}{|z_{k+m}|^{2p_{k+m}}}\right)^{n_{k+m-1}}. \end{aligned} \tag{2.24}$$

By a similar argument to that in (2.22), we have

$$\begin{aligned} &\sum_{n_{k+m-1}>N_{k+m-1}} (s_{k+m-1}+2)\left(\frac{|z_{k+m-1}|^{2p_{k+m-1}}}{|z_{k+m}|^{2p_{k+m}}}\right)^{n_{k+m-1}} \\ &= 2p_{k+m-1}|z_{k+m-1}|^{2p_{k+m-1}\{E_{k+m-1}(\beta)\}}|z_{k+m}|^{-2p_{k+m}\{E_{k+m-1}(\beta)\}} \\ &\times \left(\frac{|z_{k+m-1}|^{2p_{k+m-1}}}{|z_{k+m}|^{2p_{k+m}}}\right)^{-n_1-\sum_{t=k+1}^{k+m-2} n_t} \frac{h(E_{k+m-1}(\beta))(1-u_{k+m-1})+u_{k+m-1}}{(1-u_{k+m-1})^2}. \end{aligned} \tag{2.25}$$

If $m = 2$, then combining (2.25) with (2.24) immediately gives (2.23). If $m \geq 3$, then inserting (2.25) into (2.24) leads to

$$
\begin{aligned}
K_\beta(z,z) = {} & 4p_{k+m-1}C_1|z_{k+m-1}|^{2p_{k+m-1}\{E_{k+m-1}(\beta)\}} \\
& \times |z_{k+m}|^{2\{p_{k+m}E_{k+m-1}(\beta)\}-2p_{k+m}\{E_{k+m-1}(\beta)\}-2} \\
& \times \prod_{j=k+1}^{k+m-1}|z_j|^{2(\beta_j-1)}\frac{h(E_{k+m-1}(\beta))(1-u_{k+m-1})+u_{k+m-1}}{(1-u_{k+m-1})^2} \\
& \times \frac{h(p_{k+m}E_{k+m-1}(\beta))(1-u_{k+m})+u_{k+m}}{(1-u_{k+m})^2} \\
& \times \sum_{n_1=0}^{\infty}\frac{(n_1p_1+\beta_1-1+k)!}{(n_1p_1+\beta_1-1)!}\frac{\|\widetilde{z}\|^{2(n_1p_1+\beta_1-1)}}{|z_{k+m-1}|^{2n_1p_{k+m-1}}} \\
& \times \sum_{n_{k+1}>N_{k+1}}(s_{k+1}+2)\left(\frac{|z_{k+1}|^{2p_{k+1}}}{|z_{k+m-1}|^{2p_{k+m-1}}}\right)^{n_{k+1}}\cdots \\
& \times \sum_{n_{k+m-2}>N_{k+m-2}}(s_{k+m-2}+2)\left(\frac{|z_{k+m-2}|^{2p_{k+m-2}}}{|z_{k+m-1}|^{2p_{k+m-1}}}\right)^{n_{k+m-2}}.
\end{aligned}
$$

Repeating this process eventually yields (2.23). Thus the claim (2.23) is proved for all $m \geq 2$.

Finally, combining (2.23) and (2.15) gives the formula for $K_\beta(z,z)$. Applying polarization gives the desired formula for the Bergman kernel $K_{\mathbf{p},k+m}(z,w)$. ■

2.2. The optimal estimate for the Bergman kernel. Although the explicit formulas for the Bergman kernel of $H^{\mathbf{p}}_{k+m}$ have been established in Theorems 2.3 and 2.4, these expressions are too complicated to use, and a closed form seems difficult to obtain. Motivated by this, we now turn to studying optimal estimates for the Bergman kernel $K_{\mathbf{p},k+m}(z,w)$.

In what follows, we retain the notations of Theorems 2.3 and 2.4. From these theorems we now directly derive estimates for the sub-Bergman kernels.

PROPOSITION 2.5. *Let $H^{\mathbf{p}}_{k+m}$ be the generalized Hartogs triangle with* $\mathbf{p} := (p_1, p_{k+1}, \ldots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$.

(1) *If $m = 1$, then for the sub-Bergman kernels $K_{\beta_1}(z,w)$ $(1 \leq \beta_1 \leq p_1)$ given by* (2.7),

$$|K_{\beta_1}(z,w)| \lesssim \frac{|z_{k+1}\overline{w}_{k+1}|^{\widehat{\mu}_{k+1}(\beta_1)}}{|1-u_1^{p_1}|^{k+1}|1-u_{k+1}|^2}, \qquad z,w \in H^{\mathbf{p}}_{k+1}.$$

(2) *If $m \geq 2$, then for $\beta \in J_{\mathbf{p}}$,*

$$|K_\beta(z,w)| \lesssim \frac{\prod_{j=k+1}^{k+m} |z_j \overline{w}_j|^{\mu_j(\beta)}}{|1-u_1^{p_1}|^{k+1} \prod_{j=k+1}^{k+m} |1-u_j|^2}, \qquad z,w \in H_{k+m}^{\mathbf{p}}.$$

Proof. We first prove (1). Assume $m = 1$. For each $1 \leq \beta_1 \leq p_1$, it is easy to check that the sum in (2.7) can be calculated as

$$\sum_{i=0}^{k} \binom{k}{i} \left(\frac{1}{1-u^{p_1}}\right)^{(i)}\Big|_{u=u_1} (u^{\beta_1+k-1})^{(k-i)}\big|_{u=u_1} = \frac{u_1^{\beta_1-1} \cdot Q(u_1^{p_1})}{(1-u_1^{p_1})^{k+1}},$$

where $Q(u_1^{p_1})$ is a polynomial in $u_1^{p_1}$ of degree at most k. Observe that for $z,w \in H_{k+1}^{\mathbf{p}}$,

$$|u_1| = \frac{|\langle \widetilde{z}, \widetilde{w}\rangle|}{|z_{k+1}\overline{w}_{k+1}|^{\frac{p_{k+1}}{p_1}}} \leq \frac{\|\widetilde{z}\| \cdot \|\widetilde{w}\|}{|z_{k+1}\overline{w}_{k+1}|^{\frac{p_{k+1}}{p_1}}} < 1.$$

Hence,

$$\left|\frac{u_1^{\beta_1-1} \cdot Q(u_1^{p_1})}{(1-u_1^{p_1})^{k+1}}\right| \lesssim \frac{1}{|1-u_1^{p_1}|^{k+1}}. \tag{2.26}$$

Similarly, one readily checks that

$$\left|\frac{h(p_{k+1}E_1(\beta_1))(1-u_{k+1}) + u_{k+1}}{(1-u_{k+1})^2}\right| \lesssim \frac{1}{|1-u_{k+1}|^2}. \tag{2.27}$$

Combining (2.26), (2.27) and (2.7) yields (1).

We next prove (2). Assume $m \geq 2$. Direct calculations give

$$\left|\frac{h(E_j(\beta))(1-u_j) + u_j}{(1-u_j)^2}\right| \lesssim \frac{1}{|1-u_j|^2}, \qquad j = k+1, \dots, k+m-1, \tag{2.28}$$

and

$$\left|\frac{h(p_{k+m}E_{k+m-1}(\beta))(1-u_{k+m}) + u_{k+m}}{(1-u_{k+m})^2}\right| \lesssim \frac{1}{|1-u_{k+m}|^2}. \tag{2.29}$$

Substituting (2.28), (2.29) and (2.26) into (2.17) leads to the required estimate for $K_\beta(z,w)$. ■

We now present an optimal upper estimate for the Bergman kernel $K_{\mathbf{p},k+m}(z,w)$. This estimate will play a crucial role in our study of the L^p regularity of the Bergman projection on $H_{k+m}^{\mathbf{p}}$ in Section 3.

THEOREM 2.6. *Let $H_{k+m}^{\mathbf{p}}$ be the generalized Hartogs triangle with $\mathbf{p} := (p_1, p_{k+1}, \dots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. Then*

$$|K_{\mathbf{p},k+m}(z,w)| \lesssim \frac{\prod_{j=k+1}^{k+m} |z_j \overline{w}_j|^{A_j}}{|1-u_1^{p_1}|^{k+1} \prod_{j=k+1}^{k+m} |1-u_j|^2}, \qquad z,w \in H_{k+m}^{\mathbf{p}},$$

where the constants A_j *depend only on the domain* $H^{\mathbf{p}}_{k+m}$ *and are given by*

$$A_j := \begin{cases} \frac{p_{k+1}}{M_{k+1}} - \frac{p_{k+1}}{p_1}k - 1, & j = k+1, \\ \frac{p_j}{M_j} - \frac{p_j}{M_{j-1}} - 1, & j = k+2, \dots, k+m \ (\textit{if } m \geq 2), \end{cases}$$

and $M_j := \operatorname{lcm}(p_1, p_{k+1}, \dots, p_j)$ *for* $j = k+1, \dots, k+m$.

Proof. We divide the proof into two cases. First, assume $m = 1$. A basic fact from elementary number theory tells us that

$$\{p_{k+1}E_1(\beta_1)\} = \left\lceil \frac{p_{k+1}}{M_{k+1}} + p_{k+1}E_1(\beta_1) \right\rceil$$

for $1 \leq \beta_1 \leq p_1$, where $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to $x \in \mathbb{R}$. Consequently,

$$\{p_{k+1}E_1(\beta_1)\} \geq \frac{p_{k+1}}{M_{k+1}} + p_{k+1}E_1(\beta_1),$$

and therefore

$$\begin{aligned} \widehat{\mu}_{k+1}(\beta_1) &\geq \frac{p_{k+1}}{p_1}(\beta_1 - 1) + \frac{p_{k+1}}{M_{k+1}} + 2p_{k+1}E_1(\beta_1) - 1 \\ &= \frac{p_{k+1}}{M_{k+1}} - \frac{p_{k+1}}{p_1}k - 1 = A_{k+1}. \end{aligned}$$

This yields

$$\sum_{\beta_1=1}^{p_1} |z_{k+1}|^{\widehat{\mu}_{k+1}(\beta_1)} \lesssim |z_{k+1}|^{A_{k+1}}, \qquad z \in H^{\mathbf{p}}_{k+1}.$$

This, together with (2.6) and Proposition 2.5(1), completes the proof for $m = 1$.

Now assume $m \geq 2$. Again from elementary number theory we know that for every $\beta \in J_{\mathbf{p}}$,

$$\{E_j(\beta)\} = \left\lceil \frac{1}{M_j} + E_j(\beta) \right\rceil, \qquad j = k+1, \dots, k+m-1,$$

and

$$\{p_{k+m}E_{k+m-1}(\beta)\} = \left\lceil \frac{p_{k+m}}{M_{k+m}} + p_{k+m}E_{k+m-1}(\beta) \right\rceil.$$

It follows from the above two equations that

$$\{E_j(\beta)\} \geq \frac{1}{M_j} + E_j(\beta), \qquad j = k+1, \dots, k+m-1, \tag{2.30}$$

and

$$\{p_{k+m}E_{k+m-1}(\beta)\} \geq \frac{p_{k+m}}{M_{k+m}} + p_{k+m}E_{k+m-1}(\beta). \tag{2.31}$$

For each $\beta \in J_{\mathbf{p}}$, we define the functions

$$g_{k+2}(x) := (x - A_{k+1})\frac{p_{k+2}}{p_{k+1}} + \mu_{k+2}(\beta),$$
$$\vdots$$
$$g_{k+m}(x) := (x - A_{k+m-1})\frac{p_{k+m}}{p_{k+m-1}} + \mu_{k+m}(\beta),$$

and set $G_j(x) := g_j \circ \cdots \circ g_{k+2}(x)$, $j = k+2, \ldots, k+m$. A straightforward computation yields

$$\begin{aligned} G_j(\mu_{k+1}(\beta)) &= \mu_j(\beta) + p_j \sum_{i=k+1}^{j-1} \frac{\mu_i(\beta) - A_i}{p_i} \\ &= p_j\{E_j(\beta)\} - p_j E_j(\beta) - \frac{p_j}{M_{j-1}} - 1, \\ &\qquad\qquad j = k+2, \ldots, k+m-1 \text{ (if } m \geq 3) \end{aligned}$$

and

$$G_{k+m}(\mu_{k+1}(\beta)) = \{p_{k+m}E_{k+m-1}(\beta)\} - p_{k+m}E_{k+m-1}(\beta) - \frac{p_{k+m}}{M_{k+m-1}} - 1.$$

For each $\beta \in J_{\mathbf{p}}$, repeating the argument used for $m = 1$ gives

$$\mu_{k+1}(\beta) \geq A_{k+1}. \tag{2.32}$$

From (2.30) and (2.31), for $j = k+2, \ldots, k+m-1$ (if $m \geq 3$) we obtain

$$\begin{aligned} G_j(\mu_{k+1}(\beta)) &= p_j\{E_j(\beta)\} - p_j E_j(\beta) - \frac{p_j}{M_{j-1}} - 1 \\ &\geq p_j\left(\frac{1}{M_j} + E_j(\beta)\right) - p_j E_j(\beta) - \frac{p_j}{M_{j-1}} - 1 = A_j \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} &G_{k+m}(\mu_{k+1}(\beta)) \\ &\quad = \{p_{k+m}E_{k+m-1}(\beta)\} - p_{k+m}E_{k+m-1}(\beta) - \frac{p_{k+m}}{M_{k+m-1}} - 1 \\ &\quad \geq \frac{p_{k+m}}{M_{k+m}} - \frac{p_{k+m}}{M_{k+m-1}} - 1 = A_{k+m}. \end{aligned} \tag{2.34}$$

Consequently, $G_j(\mu_{k+1}(\beta)) \geq A_j$ for $j = k+1, \ldots, k+m$.

Now recall that for any $z \in H^{\mathbf{p}}_{k+m}$ we have

$$|z_{j-1}| < |z_j|^{\frac{p_j}{p_{j-1}}} \quad (k+2 \leq j \leq k+m) \quad \text{and} \quad |z_{k+m}| < 1.$$

Using these inequalities together with (2.32)–(2.34), we obtain

$$\sum_{\beta\in J_{\mathbf p}} |z_{k+1}|^{\mu_{k+1}(\beta)}\cdots|z_{k+m}|^{\mu_{k+m}(\beta)}$$
$$\le |z_{k+1}|^{A_{k+1}} \sum_{\beta\in J_{\mathbf p}} |z_{k+2}|^{G_{k+2}(\mu_{k+1}(\beta))}\cdots|z_{k+m}|^{\mu_{k+m}(\beta)}$$
$$= \prod_{j=k+1}^{k+2}|z_j|^{A_j}\cdot\sum_{\beta\in J_{\mathbf p}} |z_{k+2}|^{G_{k+2}(\mu_{k+1}(\beta))-A_{k+2}}\cdots|z_{k+m}|^{\mu_{k+m}(\beta)} \le \cdots$$
$$\le \prod_{j=k+1}^{k+m-1}|z_j|^{A_j}\cdot\sum_{\beta\in J_{\mathbf p}} |z_{k+m}|^{G_{k+m}(\mu_{k+1}(\beta))}$$
$$\lesssim \prod_{j=k+1}^{k+m}|z_j|^{A_j}, \qquad z\in H^{\mathbf p}_{k+m}.$$

Finally, combining this estimate with (2.16) and Proposition 2.5(2) yields the desired result. ∎

Remark 2.7. (1) For $k=1$, Theorem 2.6 reduces to [29, Proposition 3.2], which provides the optimal estimate for the Bergman kernel of $H^{\mathbf p}_{1+m}$. When $m=1$, Theorem 2.6 coincides with the optimal estimate for the Bergman kernel of $H^{\mathbf p}_{k+1}$ obtained in [18]. (2) In Section 3, the estimate established in Theorem 2.6 will be used to determine the range of p for which the Bergman projection is L^p bounded (see Propositions 3.2 and 3.4). Subsequently, this range will be shown to be sharp (see Theorem 3.7). Consequently, the kernel estimate given in Theorem 2.6 is optimal.

3. An L^p estimate for the Bergman projection on $H^{\mathbf p}_{k+m}$. In this section, using the main results on the Bergman kernel in Section 2, we consider another interesting and important topic in Bergman theory: the L^p regularity (also referred to as L^p boundedness) for the Bergman projection. A generalized Hartogs triangle $H^{\mathbf p}_{k+m}$ with $\mathbf p := (p_1, p_{k+1}, \dots, p_{k+m}) \in (\mathbb R^+)^{m+1}$ is said to be *rational* if $\frac{p_1}{p_{k+1}}, \dots, \frac{p_1}{p_{k+m}}$ are all rational numbers; otherwise, it is called *irrational*.

3.1. L^p regularity for the Bergman projection on $H^{\mathbf p}_{k+m}$ in rational cases. In this subsection, we investigate the L^p regularity for the Bergman projection on rational generalized Hartogs triangles $H^{\mathbf p}_{k+m}$. From the definition of $H^{\mathbf p}_{k+m}$, it is clear that we only need to study the problem for domains $H^{\mathbf p}_{k+m}$ with $\mathbf p := (p_1, p_{k+1}, \dots, p_{k+m}) \in (\mathbb Z^+)^{m+1}$.

The following lemma provides a Forelli–Rudin type estimate on the unit ball. As its proof is analogous to that of [19, Lemma 10], we omit it here.

LEMMA 3.1. *Let $s \in \mathbb{Z}^+$, $t > -1$, $r > -2$ and $c > 0$. Then*

$$\int_{\mathbb{B}^n} \frac{\|\eta\|^r (1-\|\eta\|^{2s})^t}{|1-\langle \xi, \eta\rangle^s|^{n+t+c+1}}\, dV(\eta) \lesssim (1-\|\xi\|^{2s})^{-c}, \qquad \xi \in \mathbb{B}^n.$$

We define

$$\psi(z) := (|z_{k+1}|^{2p_{k+1}} - \|\widetilde{z}\|^{2p_1})^{\frac{k+1}{2}} \prod_{j=k+1}^{k+m} (|z_{j+1}|^{2p_{j+1}} - |z_j|^{2p_j}), \qquad z \in H^{\mathbf{p}}_{k+m},$$

where we set $z_{k+m+1} := 1$. The function $\psi(z)$ measures the distance from $z \in H^{\mathbf{p}}_{k+m}$ to the boundary $\partial H^{\mathbf{p}}_{k+m}$.

The next proposition establishes a Forelli–Rudin type estimate on $H^{\mathbf{p}}_{k+m}$, which is a key tool in studying the L^p boundedness for the Bergman projection on this domain.

PROPOSITION 3.2. *Let $H^{\mathbf{p}}_{k+m}$ be a generalized Hartogs triangle with $\mathbf{p} := (p_1, p_{k+1}, \ldots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. Suppose a, b, and c_j $(j = k+1, \ldots, k+m)$ are real numbers satisfying $a \geq 1$ and $-\frac{2}{k+1} < b < 0$. Denote*

$$B_j := \begin{cases} A_{k+1}a + p_{k+1}(k+1)b + c_{k+1} + \dfrac{2p_{k+1}}{p_1}k, & j = k+1, \\ A_j a + 2p_j b + c_j + \dfrac{2p_j}{p_{j-1}}, & j = k+2, \ldots, k+m \ (\text{if } m \geq 2), \end{cases}$$

where A_j $(j = k+1, \ldots, k+m)$ are the constants defined in Theorem 2.6. If $B_{k+1} > -2$ and

$$B_j + p_j \sum_{t=k+1}^{j-1} \frac{B_t}{p_t} > -2, \qquad j = k+2, \ldots, k+m \ (\text{if } m \geq 2),$$

then

$$\int_{H^{\mathbf{p}}_{k+m}} |K_{\mathbf{p},k+m}(z,w)|^a \psi(w)^b \prod_{j=k+1}^{k+m} |w_j|^{c_j}\, dV(w) \lesssim \psi(z)^{-2a+b+2} \tag{3.1}$$

$$\times |z_{k+1}|^{(A_{k+1}+2(k+1)p_{k+1})a-(k+1)p_{k+1}b-2(k+1)p_{k+1}} \prod_{j=k+2}^{k+m} |z_j|^{(A_j+4p_j)a-2p_j b-4p_j}.$$

Here, we adopt the convention that if $m = 1$, then the product $\prod_{j=k+2}^{k+m}(\ldots)$ in (3.1) is interpreted as 1.

Proof. Without loss of generality, we prove the proposition for the case $m = 2$; the general cases follow by the same method.

For given real numbers a, b, and c_j satisfying the conditions in the proposition, set

$$F(z) := \int\limits_{H^{\mathbf{p}}_{k+2}} |K_{\mathbf{p},k+2}(z,w)|^a \psi(w)^b \prod_{j=k+1}^{k+2} |w_j|^{c_j}\, dV(w), \qquad z \in H^{\mathbf{p}}_{k+2}.$$

By Theorem 2.6 we obtain

$$\begin{aligned}(3.2)\qquad F(z) \lesssim |z_{k+1}|^{A_{k+1}a} |z_{k+2}|^{A_{k+2}a} \int\limits_{\mathbb{D}^*} \frac{(1-|w_{k+2}|^2)^b |w_{k+2}|^{B_{k+2}-\frac{2p_{k+2}}{p_{k+1}}}}{|1-z_{k+2}\overline{w}_{k+2}|^{2a}} \\ \times \Bigg[\int\limits_{0<|w_{k+1}|<|w_{k+2}|^{\frac{p_{k+2}}{p_{k+1}}}} \frac{\big(1-\frac{|w_{k+1}|^{2p_{k+1}}}{|w_{k+2}|^{2p_{k+2}}}\big)^b |w_{k+1}|^{B_{k+1}-\frac{2p_{k+1}}{p_1}k}}{\big|1-\frac{(z_{k+1}\overline{w}_{k+1})^{p_{k+1}}}{(z_{k+2}\overline{w}_{k+2})^{p_{k+2}}}\big|^{2a}} \cdot M\, dV(w_{k+1}) \Bigg] \\ \times dV(w_{k+2}),\end{aligned}$$

where

$$M := \int\limits_{\|\widetilde{w}\|<|w_{k+1}|^{\frac{p_{k+1}}{p_1}}} \frac{\big(1-\frac{\|\widetilde{w}\|^{2p_1}}{|w_{k+1}|^{2p_{k+1}}}\big)^{\frac{k+1}{2}b}}{\big|1-\frac{\langle \widetilde{z},\widetilde{w}\rangle^{p_1}}{(z_{k+1}\overline{w}_{k+1})^{p_{k+1}}}\big|^{(k+1)a}}\, dV(\widetilde{w}),$$

and B_j ($j = k+1,\dots,k+m$) are defined in the proposition. Changing variables, we have

$$\begin{aligned} M &= |w_{k+1}|^{\frac{2p_{k+1}}{p_1}k} \int\limits_{\mathbb{B}^k} \frac{\big(1-\|\widetilde{\eta}\|^{2p_1}\big)^{\frac{k+1}{2}b}}{\big|1-\frac{\langle\widetilde{z},\widetilde{\eta}\rangle^{p_1}}{z_{k+1}^{p_{k+1}}}\big|^{(k+1)a}}\, dV(\widetilde{\eta}) \\ &\approx |w_{k+1}|^{\frac{2p_{k+1}}{p_1}k} \left(1-\frac{\|\widetilde{z}\|^{2p_1}}{|z_{k+1}|^{2p_{k+1}}}\right)^{\frac{k+1}{2}(-2a+b+2)}, \end{aligned}$$

where the last line follows from Lemma 3.1. This, together with the lemma, implies that the bracketed integral $[\cdots]$ in (3.2) can be estimated as

$$\begin{aligned}(3.3)\qquad &= \int\limits_{0<|w_{k+1}|<|w_{k+2}|^{\frac{p_{k+2}}{p_{k+1}}}} \frac{|w_{k+1}|^{B_{k+1}}\big(1-\frac{|w_{k+1}|^{2p_{k+1}}}{|w_{k+2}|^{2p_{k+2}}}\big)^b}{\big|1-\frac{(z_{k+1}\overline{w}_{k+1})^{p_{k+1}}}{(z_{k+2}\overline{w}_{k+2})^{p_{k+2}}}\big|^{2a}}\, dV(w_{k+1}) \\ &= |w_{k+2}|^{(B_{k+1}+2)\frac{p_{k+2}}{p_{k+1}}} \int\limits_{\mathbb{D}^*} \frac{|\eta_{k+1}|^{\frac{B_{k+1}+2}{p_{k+1}}-2}(1-|\eta_{k+1}|^2)^b}{\big|1-\frac{z_{k+1}^{p_{k+1}}}{z_{k+2}^{p_{k+2}}}\overline{\eta}_{k+1}\big|^{2a}}\, dV(\eta_{k+1}) \\ &\lesssim |w_{k+2}|^{(B_{k+1}+2)\frac{p_{k+2}}{p_{k+1}}} \left(1-\frac{|z_{k+1}|^{2p_{k+1}}}{|z_{k+2}|^{2p_{k+2}}}\right)^{-2a+b+2}. \end{aligned}$$

Here, from the first line to the second line, we used the transformation $\eta_{k+1} := w_{k+1}^{p_{k+1}}/w_{k+2}^{p_{k+2}}$.

Substituting (3.3) into (3.2) yields

$$\begin{aligned}F(z) &\lesssim |z_{k+1}|^{A_{k+1}a}|z_{k+2}|^{A_{k+2}a}\left(1-\frac{\|\widetilde{z}\|^{2p_1}}{|z_{k+1}|^{2p_{k+1}}}\right)^{\frac{k+1}{2}(-2a+b+2)} \\ &\quad\times\left(1-\frac{|z_{k+1}|^{2p_{k+1}}}{|z_{k+2}|^{2p_{k+2}}}\right)^{-2a+b+2} \\ &\quad\times\int_{\mathbb{D}^*}\frac{|w_{k+2}|^{B_{k+2}+B_{k+1}\frac{p_{k+2}}{p_{k+1}}}(1-|w_{k+2}|^2)^b}{|1-z_{k+2}\overline{w}_{k+2}|^{2a}}\,dV(w_{k+2}), \qquad z\in H_{k+m}^{\mathbf{p}}.\end{aligned}$$

Since $a\geq 1$, $-1<b<0$, and $B_{k+2}+B_{k+1}\frac{p_{k+2}}{p_{k+1}}>-2$, we may apply Lemma 3.1 to obtain

$$\begin{aligned}F(z) &\lesssim |z_{k+1}|^{A_{k+1}a}|z_{k+2}|^{A_{k+2}a}\left(1-\frac{\|\widetilde{z}\|^{2p_1}}{|z_{k+1}|^{2p_{k+1}}}\right)^{\frac{k+1}{2}(-2a+b+2)} \\ &\quad\times\left(1-\frac{|z_{k+1}|^{2p_{k+1}}}{|z_{k+2}|^{2p_{k+2}}}\right)^{-2a+b+2}(1-|z_{k+2}|^2)^{-2a+b+2} \\ &=\psi(z)^{-2a+b+2}|z_{k+1}|^{[A_{k+1}+2(k+1)p_{k+1}]a-(k+1)p_{k+1}b-2(k+1)p_{k+1}} \\ &\quad\times|z_{k+2}|^{(A_{k+2}+4p_{k+2})a-2p_{k+2}b-4p_{k+2}}, \qquad z\in H_{k+m}^{\mathbf{p}}. \ \blacksquare\end{aligned}$$

The well-known Schur's test is a standard tool for establishing the L^p boundedness of integral operators. For the classical version, we refer to [31, Theorem 2.9]. We now present a generalized variant of Schur's test, introduced by Khanh, Liu, and Thuc [23, Theorem 5.1] for studying the L^p-L^q regularity of Toeplitz operators.

LEMMA 3.3 (see [23]). *Let (X,μ) and (Y,υ) be measure spaces with σ-finite, positive measures; let $1<p\leq q<\infty$ and $r\in\mathbb{R}$. Let $K: X\times Y\to\mathbb{C}$ and $\psi: Y\to\mathbb{C}$ be measurable functions. Assume that there exist positive measurable functions h_1,h_2 on Y and g on X such that*

$$h_1^{-1}h_2\in L^\infty(Y,d\upsilon)$$

and the inequalities

$$\int_Y |K(x,y)|^{rp'}h_1(y)^{p'}\,d\upsilon(y)\leq C_1 g(x)^{p'}, \tag{3.4}$$

$$\int_X |K(x,y)|^{(1-r)q}g(x)^q\,d\mu(x)\leq C_2 h_2(y)^q \tag{3.5}$$

hold for almost all $x\in(X,\mu)$ and $y\in(Y,\upsilon)$, where $1/p+1/p'=1$ and C_1,C_2 are positive constants. Then the integral operator T associated to the *kernel* $K(x,y)$ *and defined by*

$$T(f)(x) := \int_Y K(x,y)f(y)\,d\upsilon(y), \qquad f \in L^p(Y,\upsilon),$$

is bounded from $L^p(Y,\upsilon)$ *to* $L^q(X,\mu)$.

Denote by $B_{\mathbf{p},k+m}$ the Bergman projection of $H^{\mathbf{p}}_{k+m}$, and write

$$M_j := \operatorname{lcm}(p_1, p_{k+1}, \ldots, p_j), \qquad j = k+1, \ldots, k+m.$$

We now present a sufficient condition for the L^p boundedness of $B_{\mathbf{p},k+m}$.

PROPOSITION 3.4. *Let* $H^{\mathbf{p}}_{k+m}$ *be a generalized Hartogs triangle with* $\mathbf{p} := (p_1, p_{k+1}, \ldots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. *Then the associated Bergman projection* $B_{\mathbf{p},k+m}$ *is bounded on* $L^p(H^{\mathbf{p}}_{k+m})$ *for*

$$\frac{2\left(\frac{k}{p_1} + \sum_{j=k+1}^{k+m} \frac{1}{p_j}\right)}{\frac{k}{p_1} + \sum_{j=k+1}^{k+m} \frac{1}{p_j} + \frac{1}{M_{k+m}}} < p < \frac{2\left(\frac{k}{p_1} + \sum_{j=k+1}^{k+m} \frac{1}{p_j}\right)}{\frac{k}{p_1} + \sum_{j=k+1}^{k+m} \frac{1}{p_j} - \frac{1}{M_{k+m}}}. \tag{3.6}$$

Proof. Let $p' := \frac{p}{p-1}$ be the conjugate exponent of p. Choose $0 < \beta < \min\left\{\frac{1}{p}, \frac{1}{p'}\right\}$, and let γ_j ($j = k+1, \ldots, k+m$) be parameters to be determined that satisfy

(3.7)

$$\gamma_j + p_j \sum_{t=k+1}^{j-1} \frac{\gamma_t}{p_t} < p_j\left(1 - \frac{1}{p}\right)\left(\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t} + \frac{1}{M_j}\right), \qquad j = k+1, \ldots, k+m.$$

Here and in what follows, we adopt the convention that for $j = k+1$, the empty sum $\sum_{t=k+1}^{k}(\ldots)$ in (3.7) is taken to be 0.

The rest of the proof is divided into two steps.

STEP 1. We first verify that both of the following choices of parameters satisfy the hypotheses of Proposition 3.2:

$$(a, b, c_j) = (1, -\beta p', (t_j p_j \beta - \gamma_j)p'), \tag{3.8}$$

$$(a, b, c_j) = \left(1, -\beta p, \frac{(A_j + t_j p_j \beta p')p}{p'}\right). \tag{3.9}$$

Here, the constants A_j are as in Theorem 2.6, and $t_{k+1} := k+1$ and $t_j := 2$ for $j = k+2, \ldots, k+m$ (if $m \geq 2$).

For the choice (3.8), we clearly have $a \geq 1$ and $-1 < b < 0$. From the definition of B_j in Proposition 3.2, we compute

$$B_j + p_j \sum_{t=k+1}^{j-1} \frac{B_t}{p_t} = p_j\left(\frac{k}{p_1} + \sum_{t=k+1}^{j-1} \frac{1}{p_t} + \frac{1}{M_j}\right) - p'\left(\gamma_j + p_j \sum_{t=k+1}^{j-1} \frac{\gamma_t}{p_t}\right) - 1$$

for $j = k+1, \dots, k+m$. Therefore, straightforward calculations show that

$$\begin{aligned} B_j + p_j \sum_{t=k+1}^{j-1} \frac{B_t}{p_t} > -2 \\ \iff \gamma_j + p_j \sum_{t=k+1}^{j-1} \frac{\gamma_t}{p_t} < p_j\Big(1 - \frac{1}{p}\Big)\Big(\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t} + \frac{1}{M_j}\Big) \end{aligned}$$

for $j = k+1, \dots, k+m$. This implies that the conditions in Proposition 3.2 are satisfied.

For the choice (3.9), it is also obvious that $a \geq 1$ and $-1 < b < 0$. Since

$$p < \frac{2\big(\frac{k}{p_1} + \sum_{j=k+1}^{k+m} \frac{1}{p_j}\big)}{\frac{k}{p_1} + \sum_{j=k+1}^{k+m} \frac{1}{p_j} - \frac{1}{M_{k+m}}} = \frac{2M_{k+m}\big(\frac{k}{p_1} + \sum_{j=k+1}^{k+m} \frac{1}{p_j}\big)}{M_{k+m}\big(\frac{k}{p_1} + \sum_{j=k+1}^{k+m} \frac{1}{p_j}\big) - 1},$$

by combining this with the facts that the function $x \mapsto \frac{x}{x-1}$ is decreasing on $(1, +\infty)$ and the term $M_j\big(\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t}\big)$ is increasing with respect to j, we find that

$$p < \frac{2\big(\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t}\big)}{\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t} - \frac{1}{M_j}}, \qquad j = k+1, \dots, k+m. \tag{3.10}$$

Note that $A_{k+1} < 0$. Then (3.9) implies that

$$\begin{aligned} B_j + p_j \sum_{t=k+1}^{j-1} \frac{B_t}{p_t} > -2 \\ \iff p_j p\Big(\frac{1}{M_j} - \sum_{t=k+1}^{j} \frac{1}{p_t} - \frac{k}{p_1}\Big) + 2p_j\Big(\frac{k}{p_1} + \sum_{t=k+1}^{j-1} \frac{1}{p_t}\Big) > -2 \\ \iff p < \frac{2\big(\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t}\big)}{\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t} - \frac{1}{M_j}}, \qquad j = k+1, \dots, k+m. \end{aligned}$$

This, together with (3.10), implies that the conditions in Proposition 3.2 are satisfied.

STEP 2. Now we prove that the Bergman projection $B_{\mathbf{p},k+m}$ is bounded on $L^p(H^{\mathbf{p}}_{k+m})$ for p in the range (3.6).

From the analysis in Step 1, we may apply Proposition 3.2 to obtain the estimates

$$\begin{aligned} \int_{H^{\mathbf{p}}_{k+m}} |K_{\mathbf{p},k+m}(z,w)| \psi(w)^{-\beta p'} \prod_{j=k+1}^{k+m} |w_j|^{(t_j p_j \beta - \gamma_j)p'} \, dV(w) \\ \lesssim \psi(z)^{-\beta p'} \prod_{j=k+1}^{k+m} |z_j|^{A_j + t_j p_j \beta p'}, \end{aligned}$$

$$\int_{H^{\mathbf{p}}_{k+m}} |K_{\mathbf{p},k+m}(z,w)|\psi(z)^{-\beta p} \prod_{j=k+1}^{k+m} |z_j|^{\frac{(A_j+t_jp_j\beta p')p}{p'}}\, dV(z)$$

$$\lesssim \psi(w)^{-\beta p} \prod_{j=k+1}^{k+m} |w_j|^{A_j+t_jp_j\beta p}.$$

For $z\in H^{\mathbf{p}}_{k+m}$ define

$$g(z) := \psi(z)^{-\beta} \prod_{j=k+1}^{k+m} |z_j|^{\frac{A_j+t_jp_j\beta p'}{p'}},$$

$$h_1(z) := \psi(z)^{-\beta} \prod_{j=k+1}^{k+m} |z_j|^{t_jp_j\beta-\gamma_j},$$

$$h_2(z) := \psi(z)^{-\beta} \prod_{j=k+1}^{k+m} |z_j|^{\frac{A_j+t_jp_j\beta p}{p}}.$$

Then inequalities (3.4) and (3.5) of Lemma 3.3 are satisfied.

To apply Lemma 3.3 and establish the L^p boundedness of the Bergman projection, it remains to choose the parameters γ_j so that

$$\frac{h_2(z)}{h_1(z)} = \prod_{j=k+1}^{k+m} |z_j|^{A_j/p+\gamma_j} \in L^\infty(H^{\mathbf{p}}_{k+m}). \tag{3.11}$$

A natural choice is $\gamma_j = -\frac{A_j}{p}$, which makes the quotient identically 1 and therefore trivially bounded. We must verify that this choice also satisfies condition (3.7). Substituting $\gamma_j=-\frac{A_j}{p}$ into (3.7) gives, for $j=k+1,\dots,k+m$,

$$(3.7) \iff -\frac{p_j}{p}\left(\frac{1}{M_j}-\frac{k}{p_1}-\sum_{t=k+1}^{j}\frac{1}{p_t}\right) < p_j\left(1-\frac{1}{p}\right)\left(\frac{k}{p_1}+\sum_{t=k+1}^{j}\frac{1}{p_t}+\frac{1}{M_j}\right)$$

$$\iff p > \frac{2\left(\frac{k}{p_1}+\sum_{t=k+1}^{j}\frac{1}{p_t}\right)}{\frac{k}{p_1}+\sum_{t=k+1}^{j}\frac{1}{p_t}+\frac{1}{M_j}}, \qquad j=k+1,\dots,k+m.$$

The lower bound for p in Proposition 3.4 can be rewritten as

$$p > \frac{2\left(\frac{k}{p_1}+\sum_{j=k+1}^{k+m}\frac{1}{p_j}\right)}{\frac{k}{p_1}+\sum_{j=k+1}^{k+m}\frac{1}{p_j}+\frac{1}{M_{k+m}}} = \frac{2M_{k+m}\left(\frac{k}{p_1}+\sum_{j=k+1}^{k+m}\frac{1}{p_j}\right)}{M_{k+m}\left(\frac{k}{p_1}+\sum_{j=k+1}^{k+m}\frac{1}{p_j}\right)+1}.$$

Then by combining this with the facts that the function $x\mapsto\frac{x}{x+1}$ is increasing on $(0,+\infty)$ and the term $M_j\left(\frac{k}{p_1}+\sum_{t=k+1}^{j}\frac{1}{p_t}\right)$ is increasing with respect to j, we obtain

$$p > \frac{2\left(\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t}\right)}{\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t} + \frac{1}{M_j}}, \qquad j = k+1, \dots, k+m,$$

which implies that (3.7) is satisfied for $\gamma_j = -\frac{A_j}{p}$. By Lemma 3.3, we conclude that the Bergman projection $B_{\mathbf{p},k+m}$ is bounded on $L^p(H^{\mathbf{p}}_{k+m})$ for every p in the interval stated in Proposition 3.4. ■

Before establishing a necessary condition for the L^p boundedness of the Bergman projection, we prove the following basic fact in elementary number theory, which will be used later.

Lemma 3.5. *Let $y_1, \dots, y_n \in \mathbb{Z}^+$. Then*

$$\prod_{i=1}^{n} y_i = \operatorname{lcm}(y_1, \dots, y_n) \cdot \gcd\left(\frac{\prod_{i=1}^{n} y_i}{y_1}, \dots, \frac{\prod_{i=1}^{n} y_i}{y_n}\right).$$

Proof. We proceed by induction on n. The case $n = 2$ is straightforward. Assume the statement holds for some $n = l$, i.e.,

$$\prod_{i=1}^{l} y_i = \operatorname{lcm}(y_1, \dots, y_l) \cdot \gcd\left(\frac{\prod_{i=1}^{l} y_i}{y_1}, \dots, \frac{\prod_{i=1}^{l} y_i}{y_l}\right). \tag{3.12}$$

We now prove it for $n = l+1$. Set $c := \operatorname{lcm}(y_1, \dots, y_l)$. Then from (3.12) we have

$$\begin{aligned} \gcd(c, y_{l+1}) &= \frac{c y_{l+1}}{\operatorname{lcm}(y_1, \dots, y_{l+1})} \\ &= \frac{\prod_{i=1}^{l+1} y_i}{\gcd\left(\frac{\prod_{i=1}^{l} y_i}{y_1}, \dots, \frac{\prod_{i=1}^{l} y_i}{y_l}\right) \cdot \operatorname{lcm}(y_1, \dots, y_{l+1})}. \end{aligned} \tag{3.13}$$

Again using (3.12), we obtain

$$\begin{aligned} \gcd(c, y_{l+1}) &= \gcd\left(\frac{\prod_{i=1}^{l} y_i}{\gcd\left(\frac{\prod_{i=1}^{l} y_i}{y_1}, \dots, \frac{\prod_{i=1}^{l} y_i}{y_l}\right)}, y_{l+1}\right) \\ &= \frac{\gcd\left(\prod_{i=1}^{l} y_i, \gcd\left(\frac{\prod_{i=1}^{l} y_i}{y_1}, \dots, \frac{\prod_{i=1}^{l} y_i}{y_l}\right) y_{l+1}\right)}{\gcd\left(\frac{\prod_{i=1}^{l} y_i}{y_1}, \dots, \frac{\prod_{i=1}^{l} y_i}{y_l}\right)} \\ &= \frac{\gcd\left(\frac{\prod_{i=1}^{l+1} y_i}{y_1}, \dots, \frac{\prod_{i=1}^{l+1} y_i}{y_{1+1}}\right)}{\gcd\left(\frac{\prod_{i=1}^{l} y_i}{y_1}, \dots, \frac{\prod_{i=1}^{l} y_i}{y_l}\right)}. \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14) yields the desired identity for $n = l+1$. This completes the induction. ■

PROPOSITION 3.6. *Let $H^{\mathbf{p}}_{k+m}$ be a generalized Hartogs triangle with $\mathbf{p} := (p_1, p_{k+1}, \ldots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. If the associated Bergman projection $B_{\mathbf{p},k+m}$ is bounded on $L^p(H^{\mathbf{p}}_{k+m})$, then p satisfies* (3.6).

Proof. We denote

$$\lambda_j := p_1 \prod_{t=k+1}^{j} p_t, \quad \Lambda_j := \gcd\left(\frac{\lambda_j}{p_1}, \frac{\lambda_j}{p_{k+1}}, \ldots, \frac{\lambda_j}{p_j}\right), \qquad j = k+1, \ldots, k+m.$$

Lemma 2.2 shows that $z^\alpha \in A^2(H^{\mathbf{p}}_{k+m})$ if and only if

$$\frac{|\widetilde{\alpha}| + k}{p_1} + \sum_{t=k+1}^{j} \frac{\alpha_t + 1}{p_t} > 0, \qquad j = k+1, \ldots, k+m,$$

which is equivalent to

$$\frac{\lambda_j(|\widetilde{\alpha}| + k)}{p_1} + \sum_{t=k+1}^{j} \frac{\lambda_j(\alpha_t + 1)}{p_t} \geq \Lambda_j, \qquad j = k+1, \ldots, k+m.$$

Since $\Lambda_{k+m} = \gcd\left(\frac{\lambda_{k+m}}{p_1}, \frac{\lambda_{k+m}}{p_{k+1}}, \ldots, \frac{\lambda_{k+m}}{p_{k+m}}\right)$, there exist $\theta_1, \theta_{k+1}, \ldots, \theta_{k+m} \in \mathbb{Z}$ such that

$$\theta_1 \frac{\lambda_{k+m}}{p_1} + \theta_{k+1} \frac{\lambda_{k+m}}{p_{k+1}} + \cdots + \theta_{k+m} \frac{\lambda_{k+m}}{p_{k+m}} = \Lambda_{k+m}.$$

Clearly, for any $\sigma \in \mathbb{Z}^+$, we have

$$\begin{aligned}\text{(3.15)} \qquad \left(\theta_1 + \sigma \sum_{j=k+1}^{k+m} \frac{\lambda_{k+m}}{p_j}\right) \frac{\lambda_{k+m}}{p_1} + \left(\theta_{k+1} - \sigma \frac{\lambda_{k+m}}{p_1}\right) \frac{\lambda_{k+m}}{p_{k+1}} + \cdots \\ + \left(\theta_{k+m} - \sigma \frac{\lambda_{k+m}}{p_1}\right) \frac{\lambda_{k+m}}{p_{k+m}} = \Lambda_{k+m}.\end{aligned}$$

We choose $\sigma \in \mathbb{Z}^+$ sufficiently large so that

$$\begin{aligned} h_1 &:= \theta_1 + \sigma \sum_{j=k+1}^{k+m} \frac{\lambda_{k+m}}{p_j} \geq k, \\ h_{k+1} &:= \theta_{k+1} - \sigma \frac{\lambda_{k+m}}{p_1} \leq 0, \\ &\vdots \\ h_{k+m} &:= \theta_{k+m} - \sigma \frac{\lambda_{k+m}}{p_1} \leq 0, \end{aligned}$$

and fix such a σ.

Define

$$\delta_j := \begin{cases} h_1 - k, & j = 1, \\ h_j - 1, & j = k+1, \dots, k+m, \end{cases}$$

and consider

$$f(w) := w_1^{\delta_1} \prod_{j=k+1}^{k+m} \overline{w}_j^{-\delta_j}, \qquad w \in H^{\mathbf{p}}_{k+m}.$$

Because $\delta_1 \geq 0$ and $\delta_j < 0$ for $j = k+1, \dots, k+m$, the function f is bounded on $H^{\mathbf{p}}_{k+m}$; consequently, $f \in L^p(H^{\mathbf{p}}_{k+m})$ for every $p \geq 1$.

From (3.15) we obtain

$$(3.16)\quad (\delta_1 + k)\frac{\lambda_{k+m}}{p_1} + (\delta_{k+1} + 1)\frac{\lambda_{k+m}}{p_{k+1}} + \cdots + (\delta_{k+m} + 1)\frac{\lambda_{k+m}}{p_{k+m}} = \Lambda_{k+m}.$$

Lemma 2.2 together with (3.16) implies that $\delta := (\widetilde{\delta}, \delta_{k+1}, \dots, \delta_{k+m}) \in \mathcal{A}^2(H^{\mathbf{p}}_{k+m})$, where $\widetilde{\delta} := (\delta_1, 0, \dots, 0) \in \mathbb{Z}^k$. Recall that $H^{\mathbf{p}}_{k+m}$ is a Reinhardt domain. We deduce that

$$\begin{aligned} B_{\mathbf{p},k+m} f(z) &= \int_{H^{\mathbf{p}}_{k+m}} K_{\mathbf{p},k+m}(z,w) f(w)\, dV(w) \\ &= \sum_{\alpha \in \mathcal{A}^2(H^{\mathbf{p}}_{k+m})} \frac{z^\alpha}{\|z^\alpha\|^2_{L^2}} \int_{H^{\mathbf{p}}_{k+m}} \overline{w}^\alpha \cdot w_1^{\delta_1} \prod_{j=k+1}^{k+m} \overline{w}_j^{-\delta_j}\, dV(w) \\ &= \sum_{\alpha \in \mathcal{A}^2(H^{\mathbf{p}}_{k+m})} \frac{z^\alpha}{\|z^\alpha\|^2_{L^2}} \int_{H^{\mathbf{p}}_{k+m}} (\overline{w}_1^{\alpha_1} w_1^{\delta_1}) \prod_{i=2}^{k} w_i^{\alpha_i} \prod_{j=k+1}^{k+m} \overline{w}_j^{\alpha_j - \delta_j}\, dV(w) \\ &\approx z_1^{\delta_1} \prod_{j=k+1}^{k+m} z_j^{\delta_j}, \qquad z \in H^{\mathbf{p}}_{k+m}. \end{aligned}$$

Since the Bergman projection is bounded on $L^p(H^{\mathbf{p}}_{k+m})$, we get

$$B_{\mathbf{p},k+m} f(z) \approx z_1^{\delta_1} \prod_{j=k+1}^{k+m} z_j^{\delta_j} \in L^p(H^{\mathbf{p}}_{k+m}),$$

or equivalently

$$\int_{H^{\mathbf{p}}_{k+m}} |z_1|^{\delta_1 p} \prod_{j=k+1}^{k+m} |z_j|^{\delta_j p}\, dV(z) < +\infty.$$

Then Lemma 2.1 leads to

$$(3.17)\quad p\left(\frac{\delta_1}{p_1} + \sum_{t=k+1}^{j} \frac{\delta_t}{p_t}\right) + 2\left(\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t}\right) > 0, \qquad j = k+1, \dots, k+m.$$

Note that from (3.16) and Lemma 3.5 we obtain

$$\frac{\delta_1}{p_1} + \sum_{j=k+1}^{k+m} \frac{\delta_j}{p_j} = \frac{1}{M_{k+m}} - \sum_{j=k+1}^{k+m} \frac{1}{p_j} - \frac{k}{p_1}.$$

Inserting this into (3.17) yields the upper bound on p in (3.6).

Finally, because the Bergman projection is self-adjoint, the lower bound on p in (3.6) follows by duality. ∎

Combining Propositions 3.4 and 3.6, we obtain the complete characterization of L^p boundedness for the Bergman projection on rational $H^{\mathbf{p}}_{k+m}$.

THEOREM 3.7. *Let $H^{\mathbf{p}}_{k+m}$ be a generalized Hartogs triangle with $\mathbf{p} := (p_1, p_{k+1}, \dots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. Then the associated Bergman projection $B_{\mathbf{p},k+m}$ is bounded on $L^p(H^{\mathbf{p}}_{k+m})$ if and only if p satisfies* (3.6).

REMARK 3.8. Theorem 3.7 generalizes some previous results to more general cases. For example, when $k = 1$, then Theorem 3.7 reduces to Zhang's [29, Theorem 1.1]. When $m = 1$, then Theorem 3.7 coincides with the result of Fu–Deng [18, Corollary 3.8]. If we take $\mathbf{p} = (1, \dots, 1)$, then Theorem 3.7 becomes Chen's [8, Theorem 1.2].

3.2. L^p irregularity for the Bergman projection on $H^{\mathbf{p}}_{k+m}$ in irrational cases. The following proposition, usually called the theorem of Dirichlet, gives a method of approximating rational numbers by irrational numbers quantitatively.

PROPOSITION 3.9 (see [20, Theorem 187]). *If γ is irrational, then there exists a sequence of rational numbers $\{a_j/b_j\}$ with $a_j/b_j \to \gamma$ as $j \to \infty$ such that*

$$\left| \frac{b_j}{a_j} - \frac{1}{\gamma} \right| < \frac{1}{a_j^2}.$$

We now show that for irrational generalized Hartogs triangles, the L^p regularity of the Bergman projection $B_{\mathbf{p},k+m}$ completely degenerates. This extends earlier results of Edholm–McNeal [16, Theorem 1.2] and Fu–Deng [18, Theorem 2.2].

THEOREM 3.10. *Let $H^{\mathbf{p}}_{k+m}$ be a generalized Hartogs triangle with $\mathbf{p} := (p_1, p_{k+1}, \dots, p_{k+m}) \in (\mathbb{R}^+)^{m+1}$. If at least one of the ratios $p_1/p_{k+1}, \dots,$ p_1/p_{k+m} is irrational, then the Bergman projection $B_{\mathbf{p},k+m}$ is bounded on $L^p(H^{\mathbf{p}}_{k+m})$ if and only if $p = 2$.*

Proof. Fix any $p > 2$. By hypothesis, we may assume without loss of generality that $p_1/p_{k+1} \notin \mathbb{Q}$. Then Proposition 3.9 shows that there exists a sequence $\{p_{1,j}/p_{k+1,j}\}$ with $p_{1,j}, p_{k+1,j} \in \mathbb{Z}^+$ relatively prime such that

$p_{1,j}/p_{k+1,j} \to p_1/p_{k+1}$ as $j \to \infty$ and

$$\left|\frac{p_{k+1,j}}{p_{1,j}} - \frac{p_{k+1}}{p_1}\right| < \frac{1}{p_{1,j}^2}. \tag{3.18}$$

The proof of Proposition 3.9 indicates that we can require $p_{1,j} \to \infty$ as $j \to \infty$. Thus we may assume $p_{1,j} \geq k$.

For each $j \in \mathbb{Z}^+$, set $\mathbf{p}_j := (p_{1,j}, p_{k+1,j}, \ldots, p_{k+m})$. Since $p_{1,j}$ and $p_{k+1,j}$ are relatively prime, there exist integers $\beta_{1,j}$, $\beta_{k+1,j}$ with $0 \leq \beta_{1,j} \leq p_{1,j} - k$ and $\beta_{k+1,j} < 0$ satisfying

$$(\beta_{1,j} + k)p_{k+1,j} + (\beta_{k+1,j} + 1)p_{1,j} = 1.$$

Denote $\mathbf{b}_j := (\beta_{1,j}, \beta_{k+1,j}, 0, \ldots, 0)$. Then by Lemma 2.1 we know that $\mathbf{b}_j \in \mathcal{A}^2(H_{k+m}^{\mathbf{p}_j})$.

Now we show that $\mathbf{b}_j \in \mathcal{A}^2(H_{k+m}^{\mathbf{p}})$. If $p_1/p_{k+1} \leq p_{1,j}/p_{k+1,j}$, then since $\beta_{k+1,j} \leq -1$, we have

$$\beta_{1,j} + k + \frac{p_1}{p_{k+1}}(\beta_{k+1,j} + 1) \geq \beta_{1,j} + k + \frac{p_{1,j}}{p_{k+1,j}}(\beta_{k+1,j} + 1) = \frac{1}{p_{k+1,j}} > 0,$$

which gives $\mathbf{b}_j \in \mathcal{A}^2(H_{k+m}^{\mathbf{p}})$. If $p_1/p_{k+1} > p_{1,j}/p_{k+1,j}$, then (3.18) leads to

$$\begin{aligned} p_{k+1,j} &= \frac{1}{p_{1,j}} - (\beta_{1,j} + k)\left(\frac{p_{k+1,j}}{p_{1,j}} - \frac{p_{k+1}}{p_1}\right) - \frac{p_{k+1}}{p_1}(\beta_{1,j} + k) - 1 \\ &> -\frac{p_{k+1}}{p_1}(\beta_{1,j} + k) - 1. \end{aligned}$$

This, together with Lemma 2.1, implies that $\mathbf{b}_j \in \mathcal{A}^2(H_{k+m}^{\mathbf{p}})$.

Put $f_j(z) := z_1^{\beta_{1,j}} \overline{z}_{k+1}^{-\beta_{k+1,j}}$. Since $\beta_{1,j} \geq 0$ and $\beta_{k+1,j} \leq -1$, the function f_j belongs to $L^\infty(H_{k+m}^{\mathbf{p}})$. As we have already shown $\mathbf{b}_j \in \mathcal{A}^2(H_{k+m}^{\mathbf{p}})$, a computation analogous to that in Proposition 3.6 yields

$$B_{\mathbf{p},k+m}(f_j)(z) \approx z_1^{\beta_{1,j}} z_{k+1}^{\beta_{k+1,j}}.$$

From Lemma 2.1 we know that $B_{\mathbf{p},k+m}(f_j) \notin L^p(H_{k+m}^{\mathbf{p}})$ when

$$\frac{\beta_{1,j}p + 2k}{p_1} + \frac{\beta_{k+1,j}p + 2}{p_{k+1}} \leq 0,$$

or equivalently

$$p\left[1 + \frac{p_{k+1,j}k - 1}{p_{1,j}} + \beta_{1,j}\left(\frac{p_{k+1,j}}{p_{1,j}} - \frac{p_{k+1}}{p_1}\right)\right] \geq 2\left(\frac{p_{k+1}}{p_1}k + 1\right). \tag{3.19}$$

Because $0 \leq \beta_{1,j} \leq p_{1,j} - k < p_{1,j}$, it follows from (3.18) that

$$\beta_{1,j}\left|\frac{p_{k+1,j}}{p_{1,j}} - \frac{p_{k+1}}{p_1}\right| < \frac{1}{p_{1,j}}.$$

This implies

$$\text{LHS of (3.19)} > p\left(\frac{p_{k+1,j}}{p_{1,j}}k+1\right) - \frac{2p}{p_{1,j}}. \tag{3.20}$$

Since $p > 2$ and $p_{1,j} \to \infty$ as $j \to \infty$, we can choose j sufficiently large so that

$$p\left(\frac{p_{k+1,j}}{p_{1,j}}k+1\right) - \frac{2p}{p_{1,j}} > 2\left(\frac{p_{k+1}}{p_1}k+1\right).$$

This, together with (3.19) and (3.20), shows that $B_{\mathbf{p},k+m}(f_j) \notin L^p(H^{\mathbf{p}}_{k+m})$ for sufficiently large j. Consequently, the Bergman projection $B_{\mathbf{p},k+m}$ cannot be bounded on $L^p(H^{\mathbf{p}}_{k+m})$ for any $p > 2$. Note that the Bergman projection operator is self-adjoint. It follows that $B_{\mathbf{p},k+m}$ cannot be bounded on L^p for any $1 < p < 2$. ∎

4. A weak-type estimate of the Bergman projection on $H^{\mathbf{p}}_{k+m}$. Weak-type estimates for the Bergman projection are a classical subject in Bergman theory. Recall that a linear operator T acting on $L^p(X)$ is said to be *of weak-type* (p,p) if

$$V(\{x \in X : |Tf(x)| > \lambda\}) \lesssim \frac{\|f\|^p_{L^p(X)}}{\lambda^p},$$

where $V(\cdot)$ denotes the volume under the standard Lebesgue measure. Showing that the Bergman projection satisfies a certain weak-type estimate is an important method for obtaining L^p regularity. For a broad class of smooth bounded pseudoconvex domains, including finite type domains in $\mathbb{C}^2$, decoupled domains of finite type, and finite type convex domains in $\mathbb{C}^n$, McNeal [25] proved that the Bergman projection is of weak-type $(1,1)$. Charpentier–Dupain [7] also followed this approach in the case of smooth bounded pseudoconvex domains with locally diagonalizable Levi form. For weak-type estimates for the Bergman projection on non-smooth domains, we refer the readers to [11, 21].

This section studies the weak-type estimate for the Bergman projection $B_{\mathbf{p}.k+m}$ on the generalized Hartogs triangle $H^{\mathbf{p}}_{k+m}$ with $\mathbf{p} := (p_1, p_{k+1}, \ldots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. Theorem 3.7 established that $B_{\mathbf{p},k+m}$ is bounded on $L^p(H^{\mathbf{p}}_{k+m})$ if and only if $p \in (I_1, I_2)$, where

$$I_1 := \frac{2\left(\frac{k}{p_1} + \sum_{j=k+1}^{k+m}\frac{1}{p_j}\right)}{\frac{k}{p_1} + \sum_{j=k+1}^{k+m}\frac{1}{p_j} + \frac{1}{M_{k+m}}}, \qquad I_2 := \frac{2\left(\frac{k}{p_1} + \sum_{j=k+1}^{k+m}\frac{1}{p_j}\right)}{\frac{k}{p_1} + \sum_{j=k+1}^{k+m}\frac{1}{p_j} - \frac{1}{M_{k+m}}}$$

are the endpoints of the L^p boundedness range. It should be mentioned that $I_1^{-1} + I_2^{-1} = 1$, that is, I_1 and I_2 are Hölder conjugate. Now we further show that the weak-type estimate for $B_{\mathbf{p},k+m}$ fails at the lower endpoint I_1, but holds at the upper endpoint I_2. Our results generalize the analogous ones obtained by Huo–Wick [21] and Christopherson–Koenig [11] on 2-dimensional Hartogs triangles, and also serve as examples that weak-type regularity may break down on non-smooth bounded pseudoconvex domains.

4.1. Failure of weak-type estimate at the lower endpoint I_1

THEOREM 4.1. *Let $H^{\mathbf{p}}_{k+m}$ be a generalized Hartogs triangle with $\mathbf{p} := (p_1, p_{k+1}, \dots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. Then the Bergman projection $B_{\mathbf{p},k+m}$ is not of weak-type (I_1, I_1).*

Proof. We will find functions $f_\lambda \in L^p(H^{\mathbf{p}}_{k+m})$ and constants c_λ with $c_\lambda \to \infty$ as $\lambda \to \infty$ such that

$$V(\{z \in H^{\mathbf{p}}_{k+m} : |B_{\mathbf{p},k+m} f_\lambda(z)| > \lambda\}) \geq c_\lambda \frac{\|f_\lambda\|^{I_1}_{L^{I_1}}}{\lambda^{I_1}},$$

which shows that $B_{\mathbf{p},k+m}$ is not of weak-type (I_1, I_1), as claimed.

For $\lambda > 0$, define

$$f_\lambda(z) := \overline{z}_1^{-l} |z_1|^l \prod_{j=k+1}^{k+m} \overline{z}_j^{a_j} |z_j|^{b_j},$$

where $l \in \mathbb{N}$, $a_j \in \mathbb{Z}^+$, and $b_j \in \mathbb{R}$ are parameters to be determined with l and a_j depending only on the domain $H^{\mathbf{p}}_{k+m}$, and b_j depending on $H^{\mathbf{p}}_{k+m}$ and λ. By Lemma 2.1 we have

$$\|f_\lambda\|^{I_1}_{L^{I_1}} \approx \prod_{j=k+1}^{k+m} \frac{1}{(p_j \sum_{t=k+1}^{j} \frac{a_t+b_t}{p_t}) I_1 + 2p_j (\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t})},$$

provided a_j and b_j satisfy

$$\frac{2k}{p_1} + \sum_{t=k+1}^{j} \frac{(a_t + b_t) I_1 + 2}{p_t} > 0, \qquad j = k+1, \dots, k+m. \tag{4.1}$$

A straightforward computation yields

$$\begin{aligned} B_{\mathbf{p},k+m} f_\lambda(z) &= \int_{H^{\mathbf{p}}_{k+m}} K_{\mathbf{p},\mathrm{k+m}}(z,w) f_\lambda(w)\, dV(w) \\ &= \sum_{\alpha \in \mathcal{A}^2} \int_{H^{\mathbf{p}}_{k+m}} \frac{z^\alpha \overline{w}^\alpha \overline{w}_1^{-l} |w_1|^l \prod_{j=k+1}^{k+m} \overline{w}_j^{a_j} |w_j|^{b_j}}{\|w^\alpha\|^2_{L^2}}\, dV(w) \\ &= \frac{1}{\|w_1^l \prod_{j=k+1}^{k+m} w_j^{-a_j}\|^2_{L^2}} \int_{H^{\mathbf{p}}_{k+m}} |w_1|^l \prod_{j=k+1}^{k+m} |w_j|^{b_j}\, dV(w) \cdot z_1^l \prod_{j=k+1}^{k+m} z_j^{-a_j} \end{aligned}$$

under the condition that $(l,0,\dots,0,-a_{k+1},\dots,-a_{k+m})\in\mathcal{A}^2(H^{\mathbf{p}}_{k+m})$, i.e.,

$$\frac{l+k}{p_1}+\sum_{t=k+1}^{j}\frac{1-a_t}{p_t}>0,\qquad j=k+1,\dots,k+m. \tag{4.2}$$

By Lemma 2.1,

$$\left\|w_1^l\prod_{j=k+1}^{k+m}w_j^{-a_j}\right\|_{L^2}^2=\frac{l!}{(l+k)!}\prod_{j=k+1}^{k+m}\frac{1}{p_j\left(\frac{l+k}{p_1}+\sum_{t=k+1}^{j}\frac{1-a_t}{p_t}\right)}.$$

Applying Lemma 2.1 again yields

$$\int_{H^{\mathbf{p}}_{k+m}}|w_1|^l\prod_{j=k+1}^{k+m}|w_j|^{b_j}\,dV(w)=\frac{\Gamma\left(\frac{l}{2}+1\right)}{(l+2k)\Gamma\left(\frac{l}{2}+k\right)}\prod_{j=k+1}^{k+m}\frac{1}{p_j\left(\frac{l+2k}{p_1}+\sum_{t=k+1}^{j}\frac{b_t+2}{p_t}\right)}$$

under the condition

$$\frac{l+2k}{p_1}+\sum_{t=k+1}^{j}\frac{b_t+2}{p_t}>0,\qquad j=k+1,\dots,k+m. \tag{4.3}$$

Combining the above equations, we obtain

(4.4)

$$B_{\mathbf{p},k+m}f_\lambda(z)=\frac{(l+k)!\,\Gamma\left(\frac{l}{2}+1\right)}{l!(l+2k)\Gamma\left(\frac{l}{2}+k\right)}\prod_{j=k+1}^{k+m}\frac{\frac{l+k}{p_1}+\sum_{t=k+1}^{j}\frac{1-a_t}{p_t}}{\frac{l+2k}{p_1}+\sum_{t=k+1}^{j}\frac{b_t+2}{p_t}}\cdot z_1^l\prod_{j=k+1}^{k+m}z_j^{-a_j}$$

under the conditions (4.1)–(4.3).

Through a similar process to that in the proof of Proposition 3.6, we can take $l,a_{k+1},\dots,a_{k+m}\in\mathbb{Z}^+$ such that

$$\frac{l+k}{p_1}+\sum_{t=k+1}^{k+m}\frac{1-a_t}{p_t}=\frac{1}{M_{k+m}}. \tag{4.5}$$

With this choice, condition (4.2) clearly holds. Next, if $m\geq 2$, choose real numbers $b_{k+1},\dots,b_{k+m-1}$ such that

$$\frac{l+2k}{p_1}+\sum_{t=k+1}^{j}\frac{b_t+2}{p_t}>0,\qquad j=k+1,\dots,k+m-1, \tag{4.6}$$

and fix them. Finally, determine b_{k+m} from the equation

$$p_{k+m}\left(\frac{l+2k}{p_1}+\sum_{t=k+1}^{k+m}\frac{b_t+2}{p_t}\right)=\lambda^{-\delta},$$

where $\delta\in(0,1)$ will be chosen shortly. It is obvious that (4.3) and (4.1) are also satisfied.

With the parameters chosen above, (4.4) becomes

$$B_{\mathbf{p},k+m}f_\lambda(z) \approx \lambda^\delta \cdot z_1^l \prod_{j=k+1}^{k+m} z_j^{-a_j}. \tag{4.7}$$

Using (4.5) and (4.6), we compute

$$\left(p_{k+m}\sum_{t=k+1}^{k+m}\frac{a_t+b_t}{p_t}\right)I_1 + 2p_{k+m}\left(\frac{k}{p_1}+\sum_{t=k+1}^{k+m}\frac{1}{p_t}\right) = I_1\lambda^{-\delta}.$$

Consequently, the L^{I_1} norm of f_λ satisfies

$$\|f_\lambda\|_{L^{I_1}}^{I_1} \approx \prod_{j=k+1}^{k+m}\frac{1}{(p_j\sum_{t=k+1}^{j}\frac{a_t+b_t}{p_t})I_1 + 2p_j(\frac{k}{p_1}+\sum_{t=k+1}^{j}\frac{1}{p_t})} \approx \lambda^\delta. \tag{4.8}$$

Set

$$\widetilde{H}_{k+m}^{\mathbf{p}} := \left\{z\in\mathbb{C}^{k+m} : \tfrac{1}{2}|z_{k+1}|^{p_{k+1}} < \|\tilde{z}\|^{p_1} < |z_{k+1}|^{p_{k+1}} < \cdots < |z_{k+m}|^{p_{k+m}} < 1\right\}.$$

Clearly, $\widetilde{H}_{k+m}^{\mathbf{p}} \subset H_{k+m}^{\mathbf{p}}$. For any $z\in\widetilde{H}_{k+m}^{\mathbf{p}}$, by (4.5) one gets

$$\begin{aligned}|z_1|^l\prod_{j=k+1}^{k+m}|z_j|^{-a_j} &\gtrsim |z_{k+1}|^{\frac{p_{k+1}}{p_1}l-a_{k+1}}|z_{k+2}|^{-a_{k+2}}\cdots|z_{k+m}|^{-a_{k+m}}\\ &\geq \cdots \geq |z_{k+m}|^{p_{k+m}(\frac{l}{p_1}-\sum_{j=k+1}^{k+m}\frac{a_j}{p_j})} = |z_{k+m}|^{p_{k+m}(\frac{1}{M_{k+m}}-\frac{k}{p_1}-\sum_{j=k+1}^{k+m}\frac{1}{p_j})}.\end{aligned}$$

Combining this with (4.7), we obtain

$$\begin{aligned}&V(\{z\in H_{k+m}^{\mathbf{p}} : |B_{\mathbf{p},k+m}f_\lambda(z)|>\lambda\}) \qquad (4.9)\\ &\geq V\left(\left\{z\in\widetilde{H}_{k+m}^{\mathbf{p}} : \lambda^\delta|z_1|^l\prod_{j=k+1}^{k+m}|z_j|^{-a_j}\gtrsim\lambda\right\}\right)\\ &\geq V\left(\left\{z\in\widetilde{H}_{k+m}^{\mathbf{p}} : |z_{k+m}|\lesssim\lambda^{(\delta-1)p_{k+m}^{-1}\left(\frac{k}{p_1}+\sum_{j=k+1}^{k+m}\frac{1}{p_j}-\frac{1}{M_{k+m}}\right)^{-1}}\right\}\right) =: V(T).\end{aligned}$$

A straightforward computation, together with (4.8), gives

$$V(T) = \int_T dV(z) \approx \lambda^{(\delta-1)I_2} \approx \frac{\|f_\lambda\|_{L^{I_1}}^{I_1}}{\lambda^{I_1}}\cdot\lambda^s, \tag{4.10}$$

where $s := (\delta-1)I_2 - \delta + I_1$.

Combining (4.9) and (4.10), we obtain

$$V(\{z\in H_{k+m}^{\mathbf{p}} : |B_{\mathbf{p},k+m}f_\lambda(z)|>\lambda\}) \gtrsim \frac{\|f_\lambda\|_{L^{I_1}}^{I_1}}{\lambda^{I_1}}\cdot\lambda^s. \tag{4.11}$$

Observe that $\frac{I_2-I_1}{I_2-1}<1$, and the condition $s>0$ is equivalent to $\delta>\frac{I_2-I_1}{I_2-1}$. Hence we may choose $\delta\in(0,1)$ satisfying $\delta>\frac{I_2-I_1}{I_2-1}$, which guarantees $s>0$. Taking $\lambda\to\infty$ on both sides of (4.11), we eventually deduce that the weak-type estimate for the Bergman projection at I_1 fails. ∎

4.2. Weak-type estimate at the upper endpoint I_2. Before proving the weak-type estimate at the upper endpoint, we give the L^p boundedness for some integral operators on $\mathbb{B}^k$ and $\mathbb{D}$.

PROPOSITION 4.2.

(i) *Let B_r be the operator on the unit ball $\mathbb{B}^k$ defined by*

$$B_r f(z) := \int_{\mathbb{B}^k} \frac{f(\xi)}{|1-\langle z,\xi\rangle^r|^{k+1}}\, dV(\xi), \qquad z \in \mathbb{B}^k.$$

Then for every $r > 1$ and $1 < p < \infty$, B_r is a bounded operator on $L^p(\mathbb{B}^k)$.

(ii) *Let C_t be the operator on the unit disc $\mathbb{D}$ defined by*

$$C_t f(z) := \int_{\mathbb{D}} \frac{|\xi|^t f(\xi)}{|1-z\bar{\xi}|^2}\, dV(\xi), \qquad z \in \mathbb{D}.$$

Then for every $t > -2$ and $1 < p < \infty$, C_t is a bounded operator on $L^p(\mathbb{D})$.

Proof. We prove (i); statement (ii) follows by a similar argument. Define $h(z) := 1 - \|z\|^{2r}$, $z \in \mathbb{B}^k$, and let p' be the Hölder conjugate of p. Applying Lemma 3.1, we obtain

$$B_r h^{p'}(z) = \int_{\mathbb{B}^k} \frac{(1-\|\xi\|^{2r})^{p'}}{|1-\langle z,\xi\rangle^r|^{k+1}}\, dV(\xi) \lesssim (1-\|z\|^{2r})^{p'} = h(z)^{p'},$$

$$B_r h^{p}(\xi) = \int_{\mathbb{B}^k} \frac{(1-\|z\|^{2r})^{p}}{|1-\langle z,\xi\rangle^r|^{k+1}}\, dV(z) \lesssim (1-\|\xi\|^{2r})^{p} = h(\xi)^{p}.$$

These two inequalities, together with Lemma 3.3, imply that B_r is bounded on $L^p(\mathbb{B}^k)$. ∎

THEOREM 4.3. *Let $H^{\mathbf{p}}_{k+m}$ be a generalized Hartogs triangle with $\boldsymbol{p} := (p_1, p_{k+1}, \ldots, p_{k+m}) \in (\mathbb{Z}^+)^{m+1}$. Then the Bergman projection $B_{\mathbf{p},k+m}$ is of weak-type (I_2, I_2).*

Proof. Let $f \in L^{I_2}(H^{\mathbf{p}}_{k+m})$. A direct calculation shows that

$$(4.12)\qquad \|f\|^{I_2}_{L^{I_2}(H^{\mathbf{p}}_{k+m})} \approx$$

$$\sum_{j_1=1}^{p_1} \cdots \sum_{j_{k+m}=1}^{p_{k+m}} \int_{\Omega} |f(e^{\frac{2\pi\sqrt{-1}(j_1-1)}{p_1}}(\xi_{k+1}\cdots\xi_n)^{\frac{1}{p_1}}\tilde{\xi}, \ldots, e^{\frac{2\pi\sqrt{-1}(j_{k+m}-1)}{p_{k+m}}}\xi_{k+m}^{\frac{1}{p_{k+m}}})|^{I_2}$$

$$\cdot \prod_{j=k+1}^{k+m} |\xi_j|^{2c_j}\, dV(\xi)$$

$$\approx \int_{\Omega} |g(\tilde{\xi}, \xi_{k+1}, \ldots, \xi_{k+m})|^{I_2}\, dV(\xi) = \|g\|^{I_2}_{L^{I_2}(\Omega)},$$

where $\Omega := \mathbb{B}^k \times \mathbb{D}^m$ is the product domain, $c_j := k/p_1 + \sum_{t=k+1}^{j} 1/p_t - 1$ for $j = k+1, \ldots, k+m$, and g is the function on Ω defined by

$$g(\xi) := \sum_{j_1=1}^{p_1} \cdots \sum_{j_{k+m}=1}^{p_{k+m}} \left|f\left(e^{\frac{2\pi\sqrt{-1}(j_1-1)}{p_1}}(\xi_{k+1}\cdots\xi_n)^{\frac{1}{p_1}}\tilde{\xi}, \ldots, e^{\frac{2\pi\sqrt{-1}(j_{k+m}-1)}{p_{k+m}}}\xi_{k+m}^{\frac{1}{p_{k+m}}}\right)\right| \cdot \prod_{j=k+1}^{k+m} |\xi_j|^{\frac{2c_j}{I_2}}.$$

In (4.12), we used the elementary fact that $(\sum a_i)^p \approx \sum a_i^p$ for all $a_i > 0$ and $p \geq 1$.

Define

$$d_j := \sum_{t=k+1}^{j} \frac{A_t}{p_t} = \frac{1}{M_j} - \left(\frac{k}{p_1} + \sum_{t=k+1}^{j} \frac{1}{p_t}\right), \qquad j = k+1, \ldots, k+m,$$

where A_j $(j = k+1, \ldots, k+m)$ are the constants defined in Theorem 2.6. According to that theorem, we obtain

$$\begin{aligned}
(4.13)\qquad & |B_{\mathbf{p},k+m} f(z)| \\
&\lesssim \prod_{j=k+1}^{k+m} |z_j|^{A_j} \cdot \int_{H^{\mathbf{p}}_{k+m}} \frac{\prod_{j=k+1}^{k+m} |w_j|^{A_j} \cdot |f(w)|}{\left|1 - \frac{\langle \tilde{z}, \tilde{w}\rangle^{p_1}}{(z_{k+1}\overline{w}_{k+1})^{p_{k+1}}}\right|^{k+1} \prod_{j=k+1}^{k+m} \left|1 - \frac{(z_j \overline{w}_j)^{p_j}}{(z_{j+1}\overline{w}_{j+1})^{p_{j+1}}}\right|^2} \, dV(z) \\
&\approx \prod_{j=k+1}^{k+m} |z_j|^{A_j} \cdot \int_{\Omega} \frac{\prod_{j=k+1}^{k+m} |\xi_j|^{\gamma_j} \cdot g(\xi)}{\left|1 - \langle z_{k+1}^{-\frac{p_{k+1}}{p_1}} \tilde{z}, \tilde{\xi}\rangle^{p_1}\right|^{k+1} \prod_{j=k+1}^{k+m} \left|1 - z_j^{p_j} z_{j+1}^{-p_{j+1}} \overline{\xi}_j\right|^2} \, dV(\xi) \\
&= \prod_{j=k+1}^{k+m} |z_j|^{A_j} \cdot T(g)\left(z_{k+1}^{-\frac{p_{k+1}}{p_1}} \tilde{z}, z_{k+1}^{p_{k+1}} z_{k+2}^{-p_{k+2}}, \ldots, z_{k+m}^{p_{k+m}}\right),
\end{aligned}$$

where γ_j is given by

$$(4.14)\qquad \gamma_j := d_j + 2c_j - \frac{2c_j}{I_2} = d_j + \frac{2c_j}{I_1}, \qquad j = k+1, \ldots, k+m,$$

and $T(g)(z)$ is defined by

$$\begin{aligned}
T(g)(z) &:= T(g)(\tilde{z}, z_{k+1}, \ldots, z_{k+m}) \\
&= \int_{\Omega} \frac{\prod_{j=k+1}^{k+m} |\xi_j|^{\gamma_j} \cdot g(\xi)}{|1 - \langle \tilde{z}, \tilde{\xi}\rangle^{p_1}|^{k+1} \prod_{j=k+1}^{k+m} |1 - z_j \overline{\xi}_j|^2} \, dV(\xi).
\end{aligned}$$

From (4.14) and the fact that $1 < I_1 < 2$, it follows that

$$I_1\gamma_j = I_1 d_j + 2c_j > 2(c_j + d_j) = 2\left(\frac{1}{M_j} - 1\right) > -2,$$

and consequently $\gamma_j > -2$ for each $j = k+1, \dots, k+m$. The definition of $T(g)$ shows that

$$T(g) = B_{p_1} \circ C_{\gamma_{k+1}} \circ \cdots \circ C_{\gamma_{k+m}}(g),$$

where B_{p_1} and C_{γ_j} are the operators on $\mathbb{B}^k$ and $\mathbb{D}$, respectively, introduced in Proposition 4.2.

Therefore, by (4.13) we have

$$\begin{aligned}(4.15)\quad & V(\{z \in H^{\mathbf{p}}_{k+m} : |B_{\mathbf{p},k+m} f(z)| > \lambda\}) \\ &= \int\limits_{\{z \in H^{\mathbf{p}}_{k+m} : |B_{\mathbf{p},k+m} f(z)| > \lambda\}} dV(z) \\ &\lesssim \int\limits_{\{\xi \in \Omega : \prod_{j=k+1}^{k+m} |\xi_j|^{d_j} \cdot T(g)(\xi) > \lambda\}} \prod_{j=k+1}^{k+m} |\xi_j|^{2c_j} \, dV(\xi).\end{aligned}$$

We denote

$$\begin{aligned} \widetilde{Q}_1 &:= \{\tilde{\xi} \in \mathbb{B}^k : \|\tilde{\xi}\| > 1/2\}, \quad & \widetilde{Q}_2 &:= \{\tilde{\xi} \in \mathbb{B}^k : \|\tilde{\xi}\| \le 1/2\}, \\ Q_1 &:= \{\xi \in \mathbb{D} : |\xi| > 1/2\}, \quad & Q_2 &:= \{\xi \in \mathbb{D} : |\xi| \le 1/2\}. \end{aligned}$$

For any indices $i_1, \dots, i_{k+m} \in \{1, 2\}$, set

$$\mathcal{Q}_{i_1,\dots,i_{k+m}} := \{\xi \in \Omega : \tilde{\xi} \in \widetilde{Q}_{i_1},\, \xi_{k+1} \in Q_{i_{k+1}},\, \dots,\, \xi_{k+m} \in Q_{i_{k+m}}\}.$$

It is clear that

$$\begin{aligned}(4.16)\quad & \text{RHS of (4.15)} \\ &= \sum_{i_1,\dots,i_{k+m}} \int\limits_{\{\xi \in \Omega : \prod_{j=k+1}^{k+m} |\xi_j|^{d_j} \cdot T(g)(\xi) > \lambda\} \cap \mathcal{Q}_{i_1,\dots,i_{k+m}}} \prod_{j=k+1}^{k+m} |\xi_j|^{2c_j} \, dV(\xi).\end{aligned}$$

Now we consider the following three cases.

Case 1. When $\xi \in \mathcal{Q}_{2,\dots,2}$, that is,

$$\|\tilde{\xi}\| \le 1/2, \quad |\xi_j| \le 1/2, \quad j = k+1, \dots, k+m,$$

then

$$|1 - \langle \tilde{z}, \tilde{\xi} \rangle^{p_1}|^{k+1} \approx 1, \quad |1 - z_j \overline{\xi}_j|^2 \approx 1$$

for any $z \in \Omega$. This implies that $T(g)(\xi) \approx C(g)$, where $C(g)$ depends only on g (and is independent of ξ). Polar integration with respect to z_{k+m} gives

$$\begin{aligned}(4.17)\quad & \int\limits_{\{\xi \in \Omega : \prod_{j=k+1}^{k+m} |\xi_j|^{d_j} \cdot T(g)(\xi) > \lambda\} \cap \mathcal{Q}_{2,\dots,2}} \prod_{j=k+1}^{k+m} |\xi_j|^{2c_j} \, dV(\xi) \\ &= \int\limits_{\left\{\xi \in \Omega : |\xi_{k+m}| < \prod_{j=k+1}^{k+m-1} |\xi_j|^{-d_j d_{k+m}^{-1}} \cdot \left(\frac{T(g)(\xi)}{\lambda}\right)^{-d_{k+m}^{-1}}\right\} \cap \mathcal{Q}_{2,\dots,2}} \prod_{j=k+1}^{k+m} |\xi_j|^{2c_j} \, dV(\xi)\end{aligned}$$

$$\lesssim \int_{\mathbb{B}^k\times\mathbb{D}^{m-1}} \prod_{j=k+1}^{k+m-1} |\xi_j|^{2c_j}\, dV(\tilde{\xi},\dots,\xi_{k+m-1}) \times \int_0^{\prod_{j=k+1}^{k+m-1}|\xi_j|^{-d_j d_{k+m}^{-1}}\cdot(\frac{T(g)(\xi)}{\lambda})^{-d_{k+m}^{-1}}} r^{2c_{k+m}+1}\, dr$$

$$\approx \int_{\mathbb{D}^{m-1}} \prod_{j=k+1}^{k+m-1} |\xi_j|^{2c_j-(2c_j+2)\frac{d_j}{d_{k+m}}}\, dV(\xi_{k+1},\dots,\xi_{k+m-1})\cdot\left(\frac{T(g)(\xi)}{\lambda}\right)^{-\frac{2c_{k+m}+2}{d_{k+m}}}$$

$$\approx \left(\frac{T(g)(\xi)}{\lambda}\right)^{I_2},$$

where we have used the fact that $2c_j-(2c_j+2)\frac{d_j}{d_{k+m}} > -2$ and $-\frac{2c_{k+m}+2}{d_{k+m}} = I_2$. Note that $I_1\gamma_j > -2$. By Hölder's inequality, we obtain

$$\text{RHS of (4.17)} \approx \frac{(\int_\Omega \prod_{j=k+1}^{k+m}|\xi_j|^{\gamma_j} g(\xi)\, dV(\xi))^{I_2}}{\lambda^{I_2}}$$

$$\lesssim \frac{\int_\Omega |g(\xi)|^{I_2}\, dV(\xi)}{\lambda^{I_2}} = \frac{\|g\|^{I_2}_{L^{I_2}(\Omega)}}{\lambda^{I_2}} \approx \frac{\|f\|^{I_2}_{L^{I_2}(H^{\mathbf{p}}_{k+m})}}{\lambda^{I_2}},$$

which implies that

$$\int_{\{\xi\in\Omega:\, \prod_{j=k+1}^{k+m}|\xi_j|^{d_j}\cdot T(g)(\xi)>\lambda\}\cap\mathcal{Q}_{2,\dots,2}} \prod_{j=k+1}^{k+m} |\xi_j|^{2c_j}\, dV(\xi) \lesssim \frac{\|f\|^{I_2}_{L^{I_2}(H^{\mathbf{p}}_{k+m})}}{\lambda^{I_2}}. \tag{4.18}$$

Case 2. When $\xi\in\mathcal{Q}_{1,\dots,1}$, that is,

$$\|\tilde{\xi}\| > 1/2, \quad |\xi_j| > 1/2, \quad j = k+1,\dots,k+m,$$

then for any $\xi\in\Omega$ with $\prod_{j=k+1}^{k+m}|\xi_j|^{d_j}\cdot T(g)(\xi) > \lambda$, we have

$$\frac{T(g)(\xi)}{\lambda} > \prod_{j=k+1}^{k+m} |\xi_j|^{-d_j} \gtrsim 1.$$

Proposition 4.2 shows that C_{γ_j} and B_{p_1} are L^p bounded for all $1 < p < \infty$, which implies

$$\int_{\{\xi\in\Omega:\, \prod_{j=k+1}^{k+m}|\xi_j|^{d_j}\cdot T(g)(\xi)>\lambda\}\cap\mathcal{Q}_{1,\dots,1}} \prod_{j=k+1}^{k+m} |\xi_j|^{2c_j}\, dV(\xi) \tag{4.19}$$

$$\lesssim \int_\Omega \left|\frac{T(g)(\xi)}{\lambda}\right|^{I_2} dV(\xi) = \int_\Omega \left|\frac{B_{p_1}\circ\cdots\circ C_{\gamma_{k+m}}(g)(\xi)}{\lambda}\right|^{I_2} dV(\xi) \lesssim \frac{\|f\|^{I_2}_{L^{I_2}(H^{\mathbf{p}}_{k+m})}}{\lambda^{I_2}}.$$

CASE 3. If $\xi \in \mathcal{Q}_{i_1,\dots,i_{k+m}}$ with $(i_1,\dots,i_{k+m})$ different from $(1,\dots,1)$ and $(2,\dots,2)$, then without loss of generality we may assume $\xi \in \mathcal{Q}_{1,\dots,1,2,\dots,2}$, that is,

$$\|\tilde{\xi}\| > \frac{1}{2}, \quad |\xi_{k+1}| > \frac{1}{2}, \dots, |\xi_{k+j_0}| > \frac{1}{2}, |\xi_{k+j_0+1}| \le \frac{1}{2}, \dots, |\xi_{k+m}| \le \frac{1}{2}$$

for some $1 \le j_0 < m$ (if $m = 1$, the pattern reduces to $\xi \in \mathcal{Q}_{1,2}$, meaning $\|\tilde{\xi}\| > 1/2$ and $|\xi_{k+1}| \le 1/2$). Here we argue for $m \ge 2$, and the case $m = 1$ can be proved similarly.

As in Case 1, we have

$$T(g)(\xi) \approx \widetilde{C}(g)(\tilde{\xi}, \xi_{k+1}, \dots, \xi_{k+j_0}),$$

where $\widetilde{C}(g)$ is a function that depends only on the variables $\tilde{\xi}, \xi_{k+1}, \dots, \xi_{k+j_0}$. By a similar calculation to that in Case 1, we obtain

$$\begin{aligned}
(4.20)\qquad & \int_{\{\xi\in\Omega:\, \prod_{j=k+1}^{k+m} |\xi_j|^{d_j}\cdot T(g)(\xi)>\lambda\}\cap\mathcal{Q}_{1,\dots,1,2,\dots,2}} \prod_{j=k+1}^{k+m} |\xi_j|^{2c_j}\, dV(\xi) \\
&\approx \int_{(\mathbb{B}^k\times\mathbb{D}^{j_0})\cap\mathcal{Q}_{1,\dots,1,2,\dots,2}} \prod_{j=k+1}^{k+j_0} |\xi_j|^{2c_j-(2c_j+2)\frac{d_j}{d_{k+m}}} \\
&\quad\times \left(\frac{T(g)(\xi)}{\lambda}\right)^{I_2} dV(\tilde{\xi}, \xi_{k+1}, \dots, \xi_{k+j_0}) \\
&\lesssim \int_{\mathbb{B}^k\times\mathbb{D}^{j_0}} \left(\frac{T(g)(\xi)}{\lambda}\right)^{I_2} dV(\tilde{\xi}, \xi_{k+1}, \dots, \xi_{k+j_0}).
\end{aligned}$$

Here, to obtain the last line, we have used the fact that

$$\prod_{j=k+1}^{k+j_0} |\xi_j|^{2c_j-(2c_j+2)d_j/d_{k+m}} \approx 1, \qquad \xi \in \mathcal{Q}_{1,\dots,1,2,\dots,2}.$$

Under the present assumption, one easily verifies that

$$T(g)(\xi) \approx B_{p_1} \circ \dots \circ C_{\gamma_{j_0}}(G(g))(\tilde{\xi}, \dots, \xi_{k+j_0}),$$

where

$$G(g)(\tilde{\xi}, \dots, \xi_{k+j_0}) := \int_{\mathbb{D}^{m-j_0}} \prod_{j=k+j_0+1}^{k+m} |\xi_j|^{\gamma_j} \cdot g(\xi)\, dV(\xi_{k+j_0+1}, \dots, \xi_{k+m}).$$

Using the L^p boundedness of the operators $B_{p_1}, \dots, C_{\gamma_{j_0}}$ (Proposition 4.2), we get

$$\text{RHS of (4.20)} \approx \frac{\|T(g)(\xi)\|_{L^{I_2}(\Omega)}^{I_2}}{\lambda^{I_2}} \lesssim \frac{\left(\int_\Omega \prod_{j=k+j_0+1}^{k+m} |\xi_j|^{\gamma_j} \cdot g(\xi)\, dV(\xi)\right)^{I_2}}{\lambda^{I_2}} \lesssim \frac{\|g\|_{L^{I_2}(\Omega)}^{I_2}}{\lambda^{I_2}} \approx \frac{\|f\|_{L^{I_2}(H_{k+m}^{\mathbf{p}})}^{I_2}}{\lambda^{I_2}}, \tag{4.21}$$

where we have used Hölder's inequality and the fact that $I_1\gamma_j > -2$.

Now we return to (4.16). Combining it with (4.18), (4.19), and (4.21), we eventually obtain

$$V(\{z \in H_{k+m}^{\mathbf{p}} : |B_{\mathbf{p},k+m} f(z)| > \lambda\}) \lesssim \frac{\|f\|_{L^{I_2}(H_{k+m}^{\mathbf{p}})}^{I_2}}{\lambda^{I_2}},$$

which means that $B_{\mathbf{p},k+m}$ is of weak-type (I_2, I_2). ∎

Remark 4.4. When $\mathbf{p} = (1, \ldots, 1)$, Theorems 4.1 and 4.3 reduce to the main results of [12]. If $k = 1$, they reduce to the main results of [22]. Consequently, our weak-type theorems extend these earlier works to a substantially more general setting.

Acknowledgements. The authors are grateful to the referee for a careful reading of the manuscript and for valuable corrections and suggestions, which have greatly improved the paper. The authors also wish to thank Professor Zhenhan Tu and Professor Feng Rong for helpful comments and advice on this work.

Funding. Yanyan Tang is supported by the National Natural Science Foundation of China (No. 12101185). Shichao Yang is supported by the National Natural Science Foundation of China (No. 12271350), Guangdong Basic and Applied Basic Research Foundation (No. 2022A1515110887) and Professorial and Doctoral Scientific Research Foundation of Huizhou University (No. 2022JB037). Shuo Zhang is supported by the National Natural Science Foundation of China (Nos. 12401102 and 12271350) and the Tianjin Natural Science Foundation Project (No. 24JCQNJC00410).

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