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Zero mean curvature submanifolds as generalizations of rotational surfaces in Minkowski space

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Abstract. Catenoids are rotationally symmetric hypersurfaces with zero mean curvature in Minkowski space. This paper considers three generalizations of catenoids. First, we construct a generalization by replacing the rotational orbits of catenoids with minimal submanifolds within these orbits. Second, we present another generalization: $O(m) \times O_1(n)$-invariant hypersurfaces in $\mathbb{L}^{m+n+2}$ with zero mean curvature, where $O_1(n)$ is the group of Lorentz transformations, and we classify all profile curves. Finally, we consider two types of birotationally symmetric functions. These functions are the sum of two functions, each depending on a radial variable, and their graphs have zero mean curvature. If the graph is not a hyperplane, one of the functions is linear, while the other represents a catenoid of the corresponding dimension under rotation.

1. Introduction. Maximal surfaces play a crucial role in differential geometry and mathematical physics, particularly in general relativity and the study of spacetime structures. As spacelike surfaces with zero mean curvature, they naturally model critical points of the area functional in Minkowski space. The study of maximal surfaces offers insights into the behavior of spacetime singularities and the initial value problem in general relativity. Moreover, minimal surface theory is closely connected to maximal surfaces as critical points of the area functional for all compactly supported variations. A rigorous understanding of the former is fundamental for investigating analogous properties of the latter.

Among minimal surfaces in $\mathbb{R}^3$, the catenoid is the only minimal surface of revolution other than the plane. An n-dimensional catenoid in $\mathbb{R}^{n+1}$ is invariant under the action of $O(n)$ and is foliated by coaxial $(n-1)$-dimensional spheres with varying radii. Several generalizations of a catenoid as a minimal surface of revolution in $\mathbb{R}^3$ have been considered: Bombieri, de Giorgi, and

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Giusti [4] considered the minimal graph of a function $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ for $m \geq 4$ where the domain is $O(m) \times O(m)$-invariant, and proved the existence of an entire minimal graph in $\mathbb{R}^{n+1}$ for $n \geq 8$ which is not a hyperplane. Moreover, they constructed $O(m) \times O(m)$-invariant minimal hypersurfaces in $\mathbb{R}^{2m}$. Alencar et al. [3] classified all $O(m) \times O(n)$-invariant minimal hypersurfaces in $\mathbb{R}^{m+n}$. Moreover, various $O(m) \times O(n)$-invariant hypersurfaces have been studied (see [7, 10, 12] and references therein). Choe and Hoppe [5] constructed another generalization of a catenoid in which they replace the $(n-1)$-dimensional spheres that form the orbits of the rotation with m-dimensional minimal submanifolds for $m < n-1$ within the orbit sphere.

In this paper, we aim to generalize catenoids in Minkowski space by extending the aforementioned generalizations in Euclidean space. A *catenoid* is a rotationally symmetric hypersurface with zero mean curvature everywhere, foliated by n-dimensional coaxial orbits whose radii vary along the rotation axis. In $\mathbb{L}^{n+1}$, various types of rotations exist, and hypersurfaces may be spacelike, timelike or contain mixed regions. Thus, seven types of catenoids exist in $\mathbb{L}^{n+1}$ (see [2, 9, 11] and references therein). In Section 2, we construct a generalization of catenoids in $\mathbb{L}^{n+1}$ as follows: Let $\mathbb{H}^n$ and $d\mathbb{S}^n$ denote the n-dimensional hyperbolic space and de Sitter space, respectively, immersed in $\mathbb{L}^{n+1}$ and defined by

$$\begin{aligned} d\mathbb{S}^n &= \{x \in \mathbb{L}^{n+1} \mid \langle x, x\rangle = 1\}, \\ \mathbb{H}^n &= \{x \in \mathbb{L}^{n+1} \mid \langle x, x\rangle = -1\}. \end{aligned}$$

We denote by $d\mathbb{S}^n(r)$ and $\mathbb{H}^n(r)$ the scalings of $d\mathbb{S}^n$ and $\mathbb{H}^n$ by r, respectively. Let P be a hyperplane in $\mathbb{L}^{n+1}$. Then the intersections $\mathbb{H}^n \cap P$ and $d\mathbb{S}^n \cap P$ form $(n-1)$-dimensional spheres in $P \subset \mathbb{L}^{n+1}$. Here, an $(n-1)$*-dimensional sphere* is defined as the set of points in P that are equidistant from a fixed point. Let $\mathbb{S}^{n-1}$, $\mathbb{H}^{n-1}$, $d\mathbb{S}^{n-1}$, and $\mathcal{P}^{n-1}$ denote the $(n-1)$-dimensional sphere, hyperbolic space, de Sitter space, and parabolic space in $\mathbb{L}^{n+1}$, respectively. Each of these spaces is considered an $(n-1)$-dimensional sphere in $\mathbb{L}^{n+1}$. More precisely, we define the *parabolic space* (also called parabolic slice) in $\mathbb{H}^n$ as the intersection of the hyperboloid with a lightlike hyperplane:

$$\mathcal{P}^{n-1} = \{x \in \mathbb{H}^n \subset \mathbb{L}^{n+1} \mid \langle x - e_{n+1}, v\rangle = 0\},$$

where $e_{n+1} = (0, \ldots, 0, 1)$ (which may be replaced by $-e_{n+1}$), and v is a fixed lightlike vector. In particular, a rotationally symmetric hypersurface in $\mathbb{L}^{n+1}$ is foliated by $(n-1)$-dimensional coaxial spheres, and the hyperplanes containing these spheres are perpendicular to the rotation axis. Then, we replace the orbits in the catenoid with k-dimensional minimal submanifolds immersed in these orbits, and this $(k+1)$-dimensional submanifold has zero mean curvature in $\mathbb{L}^{n+1}$. The following theorem is obtained by combining Theorems 2.1, 2.3, 2.4, and 2.6:

Theorem 1.1. *Let N be an n-dimensional catenoid in $\mathbb{L}^{n+1}$, foliated by rotational orbits $M(r)$ with radius r. Suppose that $A(r)$ is a k-dimensional compelete submanifold embedded in $M(r)$ with zero mean curvature, where $k < n-1$. If Σ is a $(k+1)$-dimensional submanifold of $\mathbb{L}^{n+1}$ obtained by replacing the foliation of N by $M(r)$ with $A(r)$, then Σ is a zero mean curvature submanifold in $\mathbb{L}^{n+1}$.*

Even if $A(r)$ is immersed in $M(r)$, it has zero mean curvature. The embeddedness condition prevents singularities other than those arising from the orbit radii. If $M(r)$ is either $\mathbb{S}^{n-1}(r)$ or $\mathbb{H}^{n-1}(r)$, then the causality of Σ is determined by that of N. If $M(r)$ coincides with $d\mathbb{S}^{n-1}(r)$, then the causality of Σ is determined by that of $A(r)$. In Section 3, let $O_1(n)$ be the group of Lorentz transformations, called the Lorentz group. We then fully classify $O(m) \times O_1(n)$-invariant hypersurfaces in $\mathbb{L}^{m+n+2}$ with zero mean curvature. There are two classes of hypersurfaces: one consists of m-dimensional spheres in $\mathbb{R}^{m+1}$ and n-dimensional hyperbolic spaces in $\mathbb{L}^{n+1}$, and the other consists of m-dimensional spheres in $\mathbb{R}^{m+1}$ and n-dimensional de Sitter spaces in $\mathbb{L}^{n+1}$.

Theorem 1.2. *Let $\mathbb{S}^m$ and $\mathbb{H}^n$ be the m-dimensional sphere and n-dimensional hyperbolic space, respectively, as submanifolds of $\mathbb{L}^{m+n+2}$. Suppose that a unit speed spacelike* (*resp. timelike*) *curve $\gamma(s) = (x(s), y(s))$ on $\mathbb{L}^2$ satisfies the following equation:*

$$\frac{x''y' - x'y''}{x'^2 - y'^2} - m\frac{y'}{x} - n\frac{x'}{y} = 0. \tag{1.1}$$

Then $\Sigma = \{(x(s)p, y(s)q) \in \mathbb{L}^{m+n+2} \mid p \in \mathbb{S}^m,\ q \in \mathbb{H}^n,\ s \in \mathbb{R}\}$ is a maximal (*resp. timelike minimal*) *hypersurface in $\mathbb{L}^{m+n+2}$.*

Theorem 1.3. *Let $\mathbb{S}^m$ and $d\mathbb{S}^n$ be the m-dimensional sphere and n-dimensional de Sitter space, respectively, as submanifolds of $\mathbb{L}^{m+n+2}$. Suppose that a unit speed curve $(x(s), y(s))$ on $\mathbb{R}^2$ satisfies the following equation:*

$$x'y'' - x''y' + m\frac{y'}{x} - n\frac{x'}{y} = 0. \tag{1.2}$$

Then $\Sigma = \{(x(s)p, y(s)q) \in \mathbb{L}^{m+n+2} \mid p \in \mathbb{S}^m, q \in d\mathbb{S}^n, s \in \mathbb{R}\}$ is a timelike minimal hypersurface in $\mathbb{L}^{m+n+2}$.

In particular, the orbits of each rotation of an $O(m) \times O_1(n)$-invariant hypersurface can be replaced by minimal submanifolds in the orbits. In Section 4, we consider zero mean curvature graphs of birotationally symmetric functions that take the form of the sum of two functions, each depending on a radial variable. To clarify terminology, we call invariance under $O(m)$ or $O_1(n)$ *rotation invariance*, and invariance under $O(m) \times O_1(n)$ *birotation* *invariance*, thereby emphasizing the simultaneous symmetry with respect to two orthogonal subspaces.

A graph in $\mathbb{L}^{m+n+1}$ can be represented in two ways: as the graph of a function on $\mathbb{R}^{m+n}$, and as the graph of a function on $\mathbb{L}^{m+n}$; both are embedded in $\mathbb{L}^{m+n+1}$. If the graph is not a hyperplane, then one of the functions is linear, and the other represents one of the catenoids of the corresponding dimension under the rotation. In Minkowski space, n-dimensional catenoids are classified into three types according to their symmetry group and the geometry of their orbits: elliptic catenoids, with circular orbits under $O(n)$ symmetry; hyperbolic catenoids, with hyperbolic orbits (type 1) or de Sitter orbits (type 2) under $O_1(n)$ symmetry; and parabolic catenoids, with parabolic orbits corresponding to parabolic symmetry. We henceforth denote by $\|\cdot\|_E$ the Euclidean norm. With this notation, we state the following theorems.

THEOREM 1.4. *Let $f : U \to \mathbb{R}$ be a birotationally symmetric function of the form $f(x, y) = \alpha(\|x\|_E) + \beta(\|y\|_E)$, where $U \subset \mathbb{R}^m \times \mathbb{R}^n$ and α and β are smooth functions on some interval of $\mathbb{R}$. Let $\Gamma_1(f)$ be the graph of f in $\mathbb{L}^{m+n+1}$ given by*

$$\Gamma_1(f) = \{(x, y, f(x, y)) \in \mathbb{L}^{m+n+1} \mid (x, y) \in U \subset \mathbb{R}^m \times \mathbb{R}^n\}.$$

If $\Gamma_1(f)$ has zero mean curvature, then it is either a hyperplane or a product of elliptic catenoids.

THEOREM 1.5. *Let $f : U \to \mathbb{R}$ be a birotationally symmetric function of the form $f(x, y) = \alpha(\|x\|_E) + \beta(\|y\|)$, where $U \subset \mathbb{R}^m \times \mathbb{R}^n$ and α and β are smooth functions on some interval of $\mathbb{R}$. Let $\Gamma_2(f)$ be the graph of f in $\mathbb{L}^{m+n+1}$ given by*

$$\Gamma_2(f) = \{(f(x, y), x, y) \in \mathbb{L}^{m+n+1} \mid (x, y) \in U \subset \mathbb{R}^m \times \mathbb{L}^n\}.$$

If $\Gamma_2(f)$ has zero mean curvature, then it is a hyperplane, a product of hyperbolic catenoids of type 1 or type 2, or a product of Euclidean catenoids.

2. Submanifolds generalizing a rotationally symmetric hypersurface in $\mathbb{L}^{n+2}$.

Let $\mathbb{L}^{n+1}$ denote the $(n+1)$-dimensional Minkowski space, which is the vector space $\mathbb{R}^{n+1}$ endowed with the bilinear form

$$\langle \cdot, \cdot \rangle = \sum_{k=1}^{n} dx_k^2 - dx_{n+1}^2.$$

A vector v in $\mathbb{L}^{n+1}$ is said to be *spacelike*, *timelike*, or *lightlike* (*null*) if $\langle v, v\rangle > 0$, $\langle v, v\rangle < 0$ or $\langle v, v\rangle = 0$, respectively. The norm induced by the above metric is defined as $\|v\| = \sqrt{|\langle v, v\rangle|}$. We also denote by $\|\cdot\|_E$ the Euclidean norm induced by the Euclidean inner product $\langle \cdot, \cdot \rangle_E$. A hypersurface in $\mathbb{L}^{n+1}$ is said to be *spacelike* if the induced metric is positive definite. Equivalently, the normal vector field ν of the spacelike hypersurface is timelike, i.e., $\langle \nu, \nu\rangle < 0$. A *maximal hypersurface* in $\mathbb{L}^{n+1}$ is a spacelike hypersurface whose mean curvature is zero. A *minimal timelike hypersurface* in $\mathbb{L}^{n+1}$ is a timelike hypersurface that has a spacelike unit normal vector, with zero mean curvature.

THEOREM 2.1. *Let M be a k-dimensional complete minimal submanifold of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Suppose that a function r defined on $\mathbb{R}$ satisfies*

$$\begin{aligned} & rr'' - k(r'^2 - 1) = 0, \\ & r'^2 - 1 > 0 \quad (\textit{resp. } r'^2 - 1 < 0). \end{aligned}$$

Then $\Sigma = \{(r(t)p, t) \in \mathbb{L}^{n+2} \mid p \in M \subset \mathbb{S}^n,\ t \in \mathbb{R}\}$ is a $(k+1)$-dimensional maximal (resp. timelike minimal) submanifold in $\mathbb{L}^{n+2}$.

Proof. Let $\mathcal{C}_p = \{(r(t)p, t) \mid t \in \mathbb{R}\}$ for a fixed point $p \in M$ be a profile curve of Σ. Let $r(t)M$ denote $\Sigma \cap \Pi_t$ where $\Pi_t = \{x \in \mathbb{R}^{n+2} \mid x_{n+2} = t\}$. Then we can easily find the principal vectors v_i for integers $1 \le i \le k+1$ of Σ such that v_j for $1 \le j \le k$ is tangent to $r(t)M$ and v_{k+1} is tangent to $\mathcal{C}_p$. In particular, the curvature of $\mathcal{C}_p$ is

$$\kappa_{k+1} = \frac{r''}{(r'^2 - 1)\sqrt{|1 - r'^2|}}.$$

Since M is a minimal submanifold of $\mathbb{S}^n$, $r(t)M$ is minimal in $\mathbb{S}^n(r(t))$. Thus, $\sum_{j=1}^n \overline{\nabla}_{v_j} v_j$ is normal to $\mathbb{S}^n(r(t))$ where $\overline{\nabla}$ denotes the Levi-Civita connection of $\mathbb{L}^{n+2}$. Let ν be the outward unit normal vector field to $\mathbb{S}^n(r(t))$, parallel to Π_p, and let $\overline{\nu}$ be the outward unit normal vector field to $\Sigma \subset \mathbb{L}^{n+2}$,

$$\overline{\nu} = \frac{1}{\sqrt{|1 - r'^2|}}(p + r' e_{n+1}),$$

where $e_{n+1} = (0, \ldots, 0, 1)$. By direct computation, we find that for any $1 \le j \le k$,

$$\begin{aligned} & \langle \overline{\nabla}_{v_j} v_j, \nu\rangle = -\langle v_j, \overline{\nabla}_{v_j} \nu\rangle = -\frac{1}{r(t)}, \\ & \sum_{j=1}^k \overline{\nabla}_{v_j} v_j = -\frac{k}{r(t)}\nu. \end{aligned}$$

Then

$$\sum_{j=1}^k \kappa_j = \sum_{j=1}^k \langle \overline{\nabla}_{v_j} v_j, \overline{\nu}\rangle = -\frac{k}{r(t)}\langle \nu, \overline{\nu}\rangle = -\frac{k}{r\sqrt{|1 - r'^2|}},$$

where $\nu = p$. Thus,

$$H = \sum_{i=1}^{k+1} \kappa_i = \frac{r''}{(r'^2 - 1)\sqrt{|1 - r'^2|}} - \frac{k}{r\sqrt{|1 - r'^2|}}$$
$$= \frac{1}{r(r'^2 - 1)\sqrt{|1 - r'^2|}}(rr'' - k(r'^2 - 1)). \ \blacksquare$$

Remark 2.2. In the construction of Theorem 2.1, if the submanifold M is immersed in $\mathbb{S}^n$, then it can still have zero mean curvature as desired. Note that singularities obviously arise due to the behavior of the orbit radii. Except for these expected singularities, we aim to exclude any other types of singularities. The embeddedness condition is necessary to avoid such undesired singularities and to ensure a well-behaved geometric structure globally. The embeddedness condition is also assumed in Theorems 2.3, 2.4 and 2.6.

Theorem 2.3. *Let M be a k-dimensional complete minimal submanifold of $\mathbb{H}^n$ in $\mathbb{L}^{n+1}$. Suppose that a function r defined on $\mathbb{R}$ satisfies*

$$rr'' + k(1 - r'^2) = 0,$$
$$1 - r'^2 > 0 \quad (\textit{resp. } 1 - r'^2 < 0).$$

Then $\Sigma = \{(t, r(t)p) \in \mathbb{L}^{n+2} \mid p \in M \subset \mathbb{H}^n,\ t \in \mathbb{R}\}$ is a $(k+1)$-dimensional maximal (resp. timelike minimal) submanifold in $\mathbb{L}^{n+2}$.

Proof. Let $\mathcal{C}_p = \{(t, r(t)p) \mid t \in \mathbb{R}\}$ for a fixed point $p \in M \subset \mathbb{H}^n$ be a profile curve of Σ. Let $r(t)M$ denote $\Sigma \cap \Pi_t$ where $\Pi_t = \{x \in \mathbb{R}^{n+2} \mid x_1 = t\}$. In particular, the curvature of $\mathcal{C}_p$ is

$$\kappa_{k+1} = -\frac{r''}{(1 - r'^2)\sqrt{|r'^2 - 1|}}.$$

Since M is a minimal submanifold of $\mathbb{H}^n$, $r(t)M$ is minimal in $\mathbb{H}^n(r(t))$, so $\sum_{j=1}^{n} \overline{\nabla}_{v_j} v_j$ is normal to $\mathbb{H}^n(r(t))$. Let ν be the outward unit normal vector field to $\mathbb{H}^n(r(t))$, parallel to Π_p, and let $\overline{\nu}$ be the outward unit normal vector field to $\Sigma \subset \mathbb{L}^{n+2}$,

$$\overline{\nu} = \frac{1}{\sqrt{|r'^2 - 1|}}(r' e_1 + p),$$

where $e_1 = (1, 0, \dots, 0)$. By direct computation we obtain, for any $1 \le j \le k$,

$$\langle \overline{\nabla}_{v_j} v_j, \nu \rangle = -\langle v_j, \overline{\nabla}_{v_j} \nu \rangle = -\frac{1}{r(t)},$$
$$\sum_{j=1}^{k} \overline{\nabla}_{v_j} v_j = -\frac{k}{r(t)}\nu,$$

where $\nu = p$. Then

$$\sum_{j=1}^{k} \kappa_j = \sum_{j=1}^{k} \langle \overline{\nabla}_{v_j} v_j, \overline{\nu} \rangle = -\frac{k}{r(t)} \langle \nu, \overline{\nu} \rangle = -\frac{k}{r\sqrt{|r'^2 - 1|}}.$$

Thus,

$$\begin{aligned} H = \sum_{i=1}^{k+1} \kappa_i &= -\frac{r''}{(1-r'^2)\sqrt{|r'^2-1|}} - \frac{k}{r\sqrt{|r'^2-1|}} \\ &= -\frac{r''r + k(1-r'^2)}{r(1-r'^2)\sqrt{|r'^2-1|}}. \ \blacksquare \end{aligned}$$

Theorem 2.4. *Let M be a k-dimensional complete minimal submanifold of $d\mathbb{S}^n$ in $\mathbb{L}^{n+1}$. Suppose that a function r defined on $\mathbb{R}$ satisfies*

$$rr'' - k(r'^2 + 1) = 0.$$

Then $\Sigma = \{(t, r(t)p) \in \mathbb{L}^{n+2} \mid p \in M \subset d\mathbb{S}^n,\ t \in \mathbb{R}\}$ is a $(k+1)$-dimensional timelike minimal submanifold in $\mathbb{L}^{n+2}$.

Proof. Let $\mathcal{C}_p = \{(t, r(t)p) \mid t \in \mathbb{R}\}$ for a fixed $p \in M \subset d\mathbb{S}^n$ be a profile curve of Σ. Let $r(t)M$ denote $\Sigma \cap \Pi_t$ where $\Pi_t = \{x \in \mathbb{R}^{n+2} \mid x_1 = t\}$. In particular, the curvature of $\mathcal{C}_p$ is

$$\kappa_{k+1} = \frac{r''}{(r'^2+1)^{3/2}}.$$

Since M is a minimal submanifold of $d\mathbb{S}^n$, $r(t)M$ is minimal in $d\mathbb{S}^n(r(t))$, so $\sum_{j=1}^{n} \overline{\nabla}_{v_j} v_j$ is normal to $d\mathbb{S}^n(r(t))$. Let ν be the outward unit normal vector field to $d\mathbb{S}^n(r(t))$, parallel to Π_p, and let $\overline{\nu}$ be the outward unit normal vector field to $\Sigma \subset \mathbb{L}^{n+2}$,

$$\overline{\nu} = \frac{1}{\sqrt{1+r'^2}}(-r'e_1 + p).$$

By direct computation, for any $1 \le j \le k$,

$$\begin{aligned} \langle \overline{\nabla}_{v_j} v_j, \nu \rangle &= -\langle v_j, \overline{\nabla}_{v_j} \nu \rangle = -\frac{1}{r(t)}, \\ \sum_{j=1}^{k} \overline{\nabla}_{v_j} v_j &= -\frac{k}{r(t)} \nu, \end{aligned}$$

where $\nu = p$. Then

$$\sum_{j=1}^{k} \kappa_j = \sum_{j=1}^{k} \langle \overline{\nabla}_{v_j} v_j, \overline{\nu} \rangle = -\frac{k}{r} \langle \nu, \overline{\nu} \rangle = -\frac{k}{r\sqrt{r'^2+1}}.$$

Thus,

$$H=\sum_{i=1}^{k+1}\kappa_i=-\frac{k}{r\sqrt{r'^2+1}}+\frac{r''}{(r'^2+1)^{3/2}}=\frac{rr''-k(r'^2+1)}{r(r'^2+1)^{3/2}}.\ \blacksquare$$

REMARK 2.5. The solutions of the ordinary differential equations in Theorems 2.1, 2.3, and 2.4 can be expressed in terms of the Gauss hypergeometric function. In the timelike case, when $k=1$, the solutions reduce to sinh, sinh, and cosh, respectively, whereas in the spacelike case, they reduce to sin and sin, but no cos-type solution appears. For $k=2$, the solutions can also be expressed in terms of Jacobi elliptic functions. In particular, the case $k=1$ is as described in [1, 9], while for $k\geq 2$, related solutions can be derived from the results in [6].

THEOREM 2.6. *Let M be a k-dimensional complete minimal submanifold of $\mathcal{P}^n$ in $\mathbb{L}^{n+2}$. Suppose that a function h on some interval of $\mathbb{R}$ has the following form:*

$$h(t)=\frac{c_1}{1+2k}t^{2k+1}+c_2,$$

where $c_1>0$ (resp. $c_1<0$) and c_2 are constants. Then $\Sigma=\{(h(t)+t)(e_1+e_{n+2})-2tp\in\mathbb{L}^{n+2}\mid p\in M\subset\mathcal{P}^n,\ t\in\mathbb{R}\}$ is a $(k+1)$-dimensional maximal (resp. timelike minimal) submanifold in $\mathbb{L}^{n+2}$. In particular, singularities occur at $t=0$.

Proof. Let $\mathcal{C}_p=\{(h(t)+t)(e_1+e_{n+2})-2tp\mid t\in\mathbb{R}\}$ for a fixed $p\in M\subset\mathcal{P}^n$ be a profile curve of Σ. Let $2tM$ denote $\Sigma\cap\Pi_t$ where $\Pi_t=\{x\in\mathbb{R}^{n+2}\mid x_1-x_{n+2}=2t\}$. In particular, the curvature of $\mathcal{C}_p$ is

$$\kappa_{k+1}=-\frac{th''}{4|t|h'\sqrt{|h'|}},$$

and κ_{k+1} depends on the sign of t to maintain the direction of the unit normal vector of $\mathcal{P}^n$. Since M is a minimal submanifold of $\mathcal{P}^n$, $2tM$ is minimal in $\mathcal{P}^n(2t)$, so $\sum_{j=1}^n\overline{\nabla}_{v_j}v_j$ is normal to $\mathcal{P}^n(2t)$. Let ν be the outward unit normal vector field to $\mathcal{P}^n(2t)$, parallel to Π_p, and let $\overline{\nu}$ be the outward unit normal vector field to $\Sigma\subset\mathbb{L}^{n+2}$,

$$\overline{\nu}=\frac{t}{2|t|\sqrt{|h'|}}(2p+(h'-1)(e_1+e_{n+2})).$$

By direct computation, for any $1\leq j\leq k$,

$$\sum_{j=1}^{k}\overline{\nabla}_{v_j}v_j=\frac{k}{2|t|}\nu,$$

where $\nu = -\frac{t}{|t|}(e_1 + e_{n+2})$. Then

$$\sum_{j=1}^{k} \kappa_j = \sum_{j=1}^{k} \langle \overline{\nabla}_{v_j} v_j, \overline{\nu} \rangle = \frac{k}{2|t|} \langle \nu, \overline{\nu} \rangle = \frac{k}{2|t|\sqrt{|h'|}}.$$

Thus,

$$H = \sum_{i=1}^{k+1} \kappa_i = \frac{k}{2|t|\sqrt{|h'|}} - \frac{th''}{4|t|h'\sqrt{|h'|}} = \frac{2kh' - th''}{4|t|h'\sqrt{|h'|}}. \quad \blacksquare$$

3. A birotationally symmetric hypersurface in $\mathbb{L}^{m+n+2}$. We define a hypersurface in $\mathbb{L}^{m+n+2}$ as an immersion of $I \times \mathbb{S}^m \times \mathbb{H}^n$ into $\mathbb{L}^{m+n+2}$ parametrized by

$$X(s, \phi_1, \dots, \phi_m, \psi_1, \dots, \psi_n) = (x(s)\Phi(\phi_1, \dots, \phi_m), y(s)\Psi(\psi_1, \dots, \psi_n)),$$

where Φ and Ψ are orthogonal parametrizations of the m-dimensional unit spheres and the n-dimensional unit hyperboloid, respectively. The profile curve $\gamma(s) = (x(s), y(s))$ is defined on $\mathbb{L}^2$. Let ν be the unit normal vector field given by

$$\nu(s, \phi_1, \dots, \phi_m, \psi_1, \dots, \psi_n) = \frac{(y'(s)\Phi(\phi_1, \dots, \phi_m), x'(s)\Psi(\psi_1, \dots, \psi_n))}{\sqrt{|x'(s)^2 - y'(s)^2|}}.$$

The principal curvatures κ_i, κ_j and κ_{m+n+1} are

$$\begin{aligned} \kappa_1 &= \frac{x''y' - x'y''}{(x'^2 - y'^2)\sqrt{|x'^2 - y'^2|}}, \\ \kappa_i &= -\frac{y'}{x\sqrt{|x'^2 - y'^2|}}, \qquad 2 \le i \le m+1, \\ \kappa_j &= -\frac{x'}{y\sqrt{|x'^2 - y'^2|}}, \qquad m+2 \le j \le m+n+1. \end{aligned}$$

The mean curvature H of the hypersurface is

$$H = \frac{x''y' - x'y''}{(x'^2 - y'^2)\sqrt{|x'^2 - y'^2|}} - m\frac{y'}{x\sqrt{|x'^2 - y'^2|}} - n\frac{x'}{y\sqrt{|x'^2 - y'^2|}},$$

which is scaling invariant. The profile curve $(x(s), y(s))$ on $\mathbb{L}^2$, which has unit speed, of a birotationally symmetric maximal hypersurface satisfies the following ordinary differential equation:

$$\frac{x''y' - x'y''}{x'^2 - y'^2} - m\frac{y'}{x} - n\frac{x'}{y} = 0. \tag{3.1}$$

Similarly, the following equation is obtained for a hypersurface in $\mathbb{L}^{m+n+2}$ as an immersion of $I \times \mathbb{S}^m \times d\mathbb{S}^n$ into $\mathbb{L}^{m+n+2}$:

$$x'y'' - x''y' + m\frac{y'}{x} - n\frac{x'}{y} = 0. \tag{3.2}$$

Therefore, we obtain Theorems 1.2 and 1.3.

Remark 3.1. In Theorems 1.2 and 1.3, the spaces $\mathbb{S}^m$, $\mathbb{H}^n$, and $d\mathbb{S}^n$ appear as orbit spaces under group actions. By replacing each orbit with a minimal submanifold lying within the corresponding orbit space, we can construct new minimal submanifolds in the ambient space, as demonstrated in Section 2.

Remark 3.2. In [3], Alencar et al. constructed minimal hypersurfaces in the Euclidean space $\mathbb{R}^{m+n}$ invariant under $O(m) \times O(n)$. In contrast, we consider hypersurfaces in the Minkowski space $\mathbb{L}^{m+n}$ invariant under $O(m) \times O_1(n)$, where $O_1(n)$ is the Lorentz group. In particular, Theorem 1.3 treats the case where the orbits of $O_1(n)$ are de Sitter spaces. Although the resulting ODE is the same as in [3], the geometric setting and symmetry groups differ. We also note that Alencar et al. classified all solutions to (3.2).

Starting in the following section, we classify all solutions of (3.1) into two cases: when Σ is spacelike and when Σ is timelike.

3.1. Σ is spacelike.

Suppose that the profile curve $\gamma(s) = (x(s), y(s))$ on $\mathcal{Q} = \{(x, y) \in \mathbb{L}^2 \mid x, y \geq 0\}$ satisfies the equation

$$\frac{x''y' - x'y''}{x'^2 - y'^2} - m\frac{y'}{x} - n\frac{x'}{y} = 0. \tag{3.3}$$

We consider a coordinate transformation by taking a suitable parameter $\tilde{s} \in (-\infty, \infty)$ and use the same notation s for convenience as follows:

$$\sin(a) = \frac{y'}{x'}, \qquad \tan(b) = \frac{y}{x},$$

satisfying

$$b' = \frac{\sin(a)\cos(b) - \sin(b)}{\cos(a)\sqrt{x^2 + y^2}}.$$

Changing coordinates in (3.3), we have

$$-a'\sec(a) + \frac{\csc(b)\sec(b)(m\tan(a)\sec(b) + n\sec(a)\csc(b))}{\sec(a)\sec(b) - \tan(a)\csc(b)}b' = 0.$$

Multiplying the above equation by $\cos(a)\sin(b)\cos(b)(\sin(b) - \sin(a)\cos(b))$, we can rewrite it as follows:

$$Aa' - Bb' = 0, \tag{3.4}$$

where

$$A = \sin(b)\cos(b)(\sin(a)\cos(b) - \sin(b)),$$
$$B = -\cos(a)(m\sin(a)\sin(b) + n\cos(b)).$$

Equation (3.4) yields the following system of first order ordinary differential equations:

$$a' = B, \qquad b' = A.$$

We define an associated vector field $V(a,b) = (V_1(a,b), V_2(a,b))$ by

$$V_1(a,b) = a' = B, \qquad V_2(a,b) = b' = A.$$

Each integral curve is a solution of (3.4) and after applying the coordinate transformation, the solution corresponds to a solution of (3.3). Thus, analyzing and classifying integral curves of V leads to the classification of the profile curves of (3.3). In particular, V satisfies $V_1 = 0$ on the lines $a = -\pi/2, \pi/2$, and $V_2 = 0$ on the lines $b = 0, \pi/2$. By the periodicity of V, we only consider the region $\Omega = [-\pi/2, \pi/2] \times [0, \pi/2]$, which is divided into two regions at $a = 0$, namely, $\Omega_1 = [-\pi/2, 0] \times [0, \pi/2]$ and $\Omega_2 = [0, \pi/2] \times [0, \pi/2]$. We can find the singular points of the vector field V on Ω from the following lemma:

Lemma 3.3. *The vector field V has the following properties on Ω:*

(1) *V_1 has the following zero set: the lines $a = -\pi/2, \pi/2$, and the graph of the function*
$$b(a) = -\arctan\left(\frac{n}{m\sin(a)}\right).$$
(2) *V_2 has the following zero set: the lines $b = 0, \pi/2$ and the graph of the function*
$$b(a) = \arctan(\sin(a)).$$

Lemma 3.4. *There are no periodic integral curves of V on Ω.*

Proof. Since V_1 is negative on Ω_2, each integral curve of V on Ω_2 passes through $a = 0$. Since V_2 is negative on Ω_1, each integral curve of V on Ω_2 goes downward. Thus, there are no periodic integral curves on Ω. ∎

Proposition 3.5. *The singular points of the vector field V on Ω are as follows:*

(1) *$p_1 = (-\pi/2, 0)$ is a stable node.*
(2) *$p_2 = (\pi/2, 0)$, $p_4 = (\pi/2, \pi/2)$ and $p_6 = (-\pi/2, \pi/2)$ are unstable nodes.*
(3) *$p_3 = (\pi/2, \pi/4)$ and $p_5 = (0, \pi/2)$ are saddle points.*

Proof. The Jacobian matrix of V is

$$DV = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned}
a_{11} &= n\sin(a)\cos(b) - m\cos(2a)\sin(b),\\
a_{12} &= \cos(a)(n\sin(b) - m\sin(a)\cos(b)),\\
a_{21} &= \cos(a)\sin(b)\cos^2(b),\\
a_{22} &= \tfrac{1}{4}(\sin(a)(\cos(b) + 3\cos(3b)) + \sin(b) - 3\sin(3b)).
\end{aligned}$$

The following table represents the eigenvalues and eigenvectors of p_i for $1 \le i \le 6$:

	p_1	p_2	p_3	p_4	p_5	p_6
λ_1	-1	n	$\frac{m+n}{\sqrt{2}}$	m	1	m
λ_2	$-n$	1	$-\frac{1}{\sqrt{2}}$	1	$-m$	1
v_1	$(0,1)$	$(1,0)$	$(1,0)$	$(1,0)$	$(n,m+1)$	$(1,0)$
v_2	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$

Hence the results follow. ∎

By the Poincaré–Bendixson theorem, there are three types of solutions on a bounded region: singular points as trivial solutions, connected integral curves, and periodic integral curves. Lemma 3.4 shows that periodic solutions do not exist, and since trivial solutions do not correspond to profile curves in the phase plane, we only consider connected integral curves. In particular, the profile curves of solutions to (3.3) consist of singular points and integral curves connecting these points.

Lemma 3.6. *Every integral curve of V on Ω is of one of the following types (see Fig. 1):*

(1) *There is a family of integral curves from p_2 to p_1.*

(2) *There exists a unique integral curve from p_3 to p_1.*

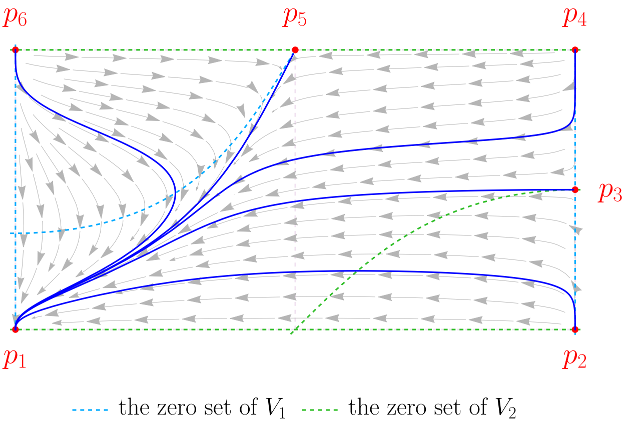

Fig. 1. The vector field V and its integral curves

(3) *There is a family of integral curves from* p_4 *to* p_1*.*
(4) *There exists a unique integral curve from* p_5 *to* p_1*.*
(5) *There is a family of integral curves from* p_6 *to* p_1*.*

Proof. By the Poincaré–Bendixson theorem, we only have integral curves between two singular points on Ω. The point p_1 is a stable node and there are no stable manifolds at p_i for $i = 2, \dots, 6$. Thus, integral curves in Ω remain in Ω, and every integral curve goes to p_1. Since p_2, p_4 and p_6 are unstable nodes and p_3 and p_5 are saddle points, integral curves starting at p_i for each $2 \leq i \leq 6$ tend to p_1. In particular, by the stable manifold theorem, there is an integral curve from p_3 and p_5. ■

In the following proposition, the order of items corredponds to the order of items in Lemma 3.6.

Proposition 3.7. *Let* $\gamma(s)$ *be the profile curve corresponding to an integral curve of* V *on* $\mathcal{Q}$*. Then* γ *is one of the following types* (*see Fig.* 2)*:*

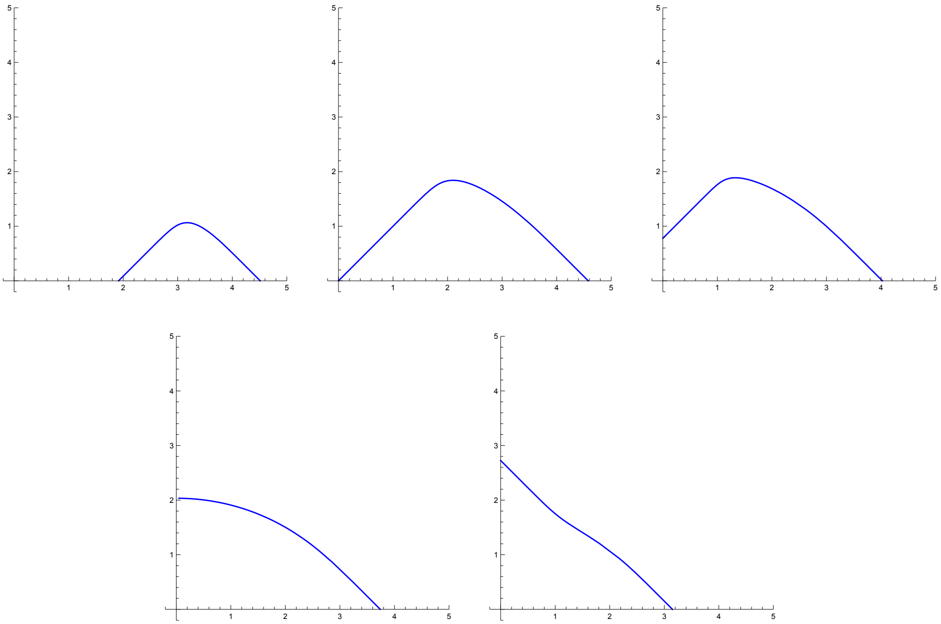

Fig. 2. Profile curves corresponding to integral curves in Fig. 1

(1) γ *is a concave curve with two limit points on the* x*-axis, where the curve approaches the axis at angles* $-\pi/4$ *and* $\pi/4$*, respectively.*
(2) γ *is a concave curve with two limit points, where the curve approaches the origin and the* x*-axis at angles* $\pi/4$ *and* $-\pi/4$*, respectively.*

(3) *γ is a concave curve with two limit points, where the curve approaches the y-axis and the x-axis at angles $\pi/4$ and $-\pi/4$, respectively.*
(4) *γ is a concave curve with two limit points, where the curve approaches the y-axis perpendicularly and the x-axis at an angle of $-\pi/4$, respectively.*
(5) *γ is a curve with two limit points, where the curve approaches the y-axis and the x-axis at angles $-\pi/4$ and $-\pi/4$, respectively.*

Proof. By Lemma 3.6, we can obtain the behavior of the limit points of profile curves. Consider an integral curve in the family of Lemma 3.6(1), which goes from p_2 to p_1. Then we have

$$\begin{aligned}\lim_{s\to-\infty}\frac{y'(s)}{x'(s)} &= \lim_{s\to-\infty}\sin(a(s)) = -1,\\ \lim_{s\to-\infty}\frac{y(s)}{x(s)} &= \lim_{s\to-\infty}\tan(b(s)) = 0,\\ \lim_{s\to\infty}\frac{y'(s)}{x'(s)} &= \lim_{s\to\infty}\sin(a(s)) = 1,\\ \lim_{s\to\infty}\frac{y(s)}{x(s)} &= \lim_{s\to\infty}\tan(b(s)) = 0.\end{aligned}$$

Both limit points of the profile curve corresponding to the integral curve are on the x-axis, and the angles of the tangent lines at the limit points are $-\pi/4$ and $\pi/4$, as determined by the above limit values. Since the integral curves of the family are in the region with $V_1 < 0$, $a(s)$ is monotone, which means that the corresponding profile curve is convex. In the same way, we can describe the behavior of all profile curves. ■

Provided that the radii of the orbits corresponding to each rotation group are allowed to take negative values, the solutions to (3.3) can be interpreted as integral curves of V. According to the Poincaré–Bendixson theorem, the qualitative behavior of this dynamical system is characterized by singular points and the trajectories connecting them. Consequently, each profile curve consists of smoothly joined components, each corresponding to different types classified in Proposition 3.7.

Remark 3.8. A discussion related to the construction of connected profile curves is presented in [8, Remark 3.1]. This approach is similarly used in the next section to construct complete profile curves.

3.2. Σ is timelike.

We follow the argument used in Section 3.1. Suppose that the profile curve $\gamma(s) = (x(s), y(s))$ on $\mathcal{Q} = \{(x,y) \in \mathbb{L}^2 \mid x \geq 0\}$ satisfies the equation

$$\frac{x''y' - x'y''}{x'^2 - y'^2} - m\frac{y'}{x} - n\frac{x'}{y} = 0. \tag{3.5}$$

We consider a coordinate transformation by taking a suitable parameter $\tilde{s} \in (-\infty, \infty)$ and use the same notation s for convenience as follows:

$$\csc(a) = \frac{y'}{x'}, \qquad \tan(b) = \frac{y}{x},$$

satisfying

$$b' = \frac{\cos(b) - \sin(a)\sin(b)}{\cos(a)\sqrt{x^2 + y^2}}.$$

Making the coordinate transformation in (3.5), we have

$$-a'\sec(a) + \frac{2\csc(2b)(m\sec(a)\sec(b) + n\tan(a)\csc(b))}{\sec(a)\csc(b) - \tan(a)\sec(b)} b' = 0.$$

Multiplying the above equation by $\cos(a)\sin(b)\cos(b)(\sin(a)\sin(b) - \cos(b))$ yields

$$Aa' - Bb' = 0,$$

where

$$A = \sin(b)\cos(b)(\cos(b) - \sin(a)\sin(b)),$$
$$B = -\cos(a)(n\sin(a)\cos(b) + m\sin(b)).$$

We define an associated vector field $V(a, b) = (V_1(a, b), V_2(a, b))$ by

$$V_1(a, b) = a' = B, \qquad V_2(a, b) = b' = A.$$

We consider $\Omega_1 = [-\pi/2, 0] \times [0, \pi/2]$ and $\Omega_2 = [0, \pi/2] \times [0, \pi/2]$. We can find the singular points of the vector field V on Ω from the following lemma:

Lemma 3.9. *The vector field V has the following properties on Ω:*

(1) *V_1 has the following zero set: the lines $a = -\pi/2, \pi/2$, and the graph of the function*

$$b(a) = -\arctan\left(\frac{n\sin(a)}{m}\right).$$

(2) *V_2 has the following zero set: the lines $b = 0, \pi/2$ and the graph of the function*

$$b(a) = \arctan(\csc(a)).$$

Lemma 3.10. *There are no periodic integral curves of V on Ω.*

Proposition 3.11. *The singular points of the vector field V on Ω are as follows:*

(1) *$p_1 = (-\pi/2, 0)$, $p_3 = (\pi/2, 0)$ and $p_5 = (\pi/2, \pi/2)$ are unstable nodes.*
(2) *$p_2 = (0, \pi/2)$ and $p_4 = (\pi/2, \pi/4)$ are saddle points.*
(3) *$p_6 = (-\pi/2, \pi/2)$ is a stable node.*

Lemma 3.12. *Every integral curve of V on Ω defined for all $s \in \mathbb{R}$ is of one of the following types* (*see Fig.* 3)*:*

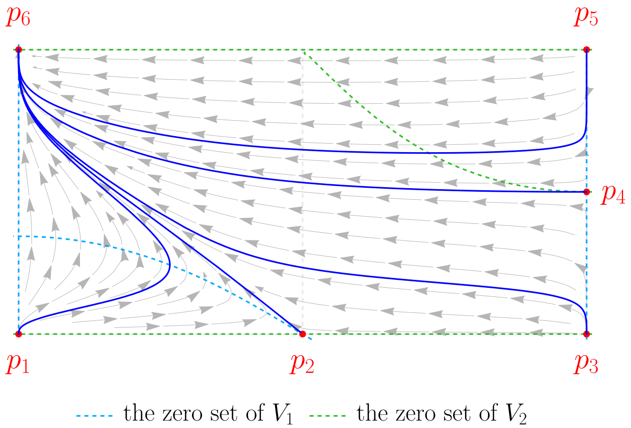

Fig. 3. The vector field V and its integral curves

(1) *There is a family of integral curves from* p_1 *to* p_6*.*
(2) *There exists a unique integral curve from* p_2 *to* p_6*.*
(3) *There is a family of integral curves from* p_3 *to* p_6*.*
(4) *There exists a unique integral curve from* p_4 *to* p_6*.*
(5) *There is a family of integral curves from* p_5 *to* p_6*.*

Proposition 3.13. *Let* $\gamma(s)$ *be the profile curve corresponding to an integral curve of* V *on* $\mathcal{Q}$*. Then* γ *is of one of the following types* (*see Fig.* 4)*:*

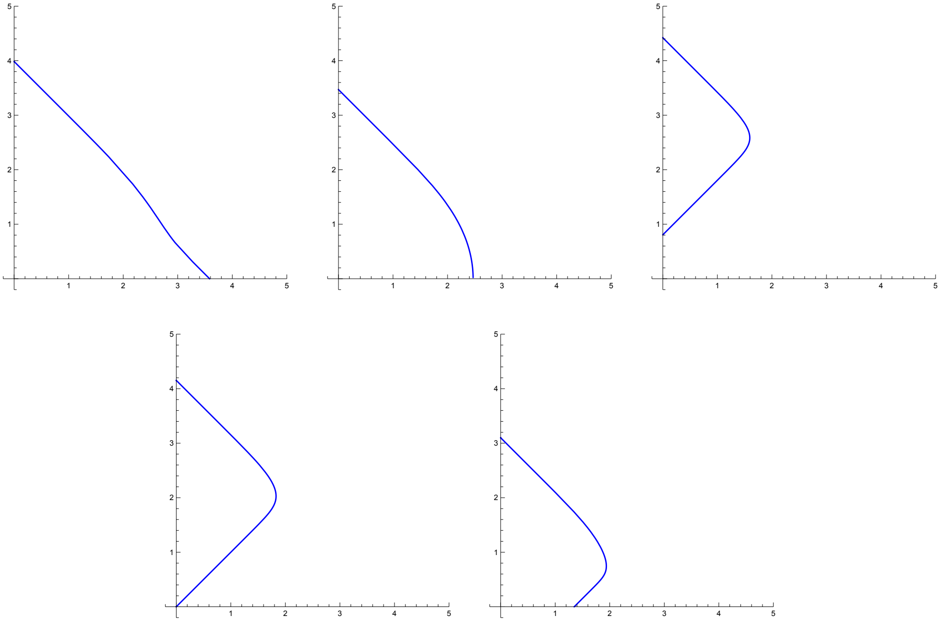

Fig. 4. Profile curves corresponding to Fig. 3.

(1) *γ is a curve with two limit points, where the curve approaches the y-axis and the x-axis at angles $-\pi/4$ and $-\pi/4$, respectively.*
(2) *γ is a concave curve with two limit points, where the curve approaches the x-axis perpendicularly and the y-axis at an angle of $-\pi/4$, respectively.*
(3) *γ is a concave curve with two limit points on the y-axis, where the curve approaches the y-axis at angles $-\pi/4$ and $\pi/4$, respectively.*
(4) *γ is a concave curve with two limit points, where the curve approaches the origin and the y-axis at angles $\pi/4$ and $-\pi/4$, respectively.*
(5) *γ is a concave curve with two limit points, where the curve approaches the x-axis and the y-axis at angles $\pi/4$ and $-\pi/4$, respectively.*

4. Graphs of birotationally symmetric functions. Let ϕ and ψ be functions defined on the first quadrant of $\mathbb{R}^2$, i.e., $\{(x,y)\in\mathbb{R}^2 \mid x,y>0\}$. A function $f:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}$ is said to be *birotationally symmetric* if

$$f(x,y)=\varphi(\|x\|_E,\|y\|_E)$$

for $x\in\mathbb{R}^m$ and $y\in\mathbb{R}^n$. On the other hand, there is another type of birotationally symmetric functions: $f:\mathbb{R}^m\times\mathbb{L}^n\to\mathbb{R}$ defined by

$$f(x,y)=\psi(\|x\|_E,\|y\|)$$

for $x\in\mathbb{R}^m$ and $y\in\mathbb{L}^n$. We distinguish two cases for ψ according to the sign of $\langle y,y\rangle$.

4.1. A function f defined on $\mathbb{R}^m\times\mathbb{R}^n$. Let f be a birotationally symmetric function on $\mathbb{R}^m\times\mathbb{R}^n$. The graph $\mathit{\Gamma}_1(f)$ of f is then defined by

$$\mathit{\Gamma}_1(f)=\{(x,y,f(x,y))\in\mathbb{L}^{m+n+1}\mid x\in\mathbb{R}^m, y\in\mathbb{R}^n\}.$$

In particular, $\mathit{\Gamma}_1(f)$ is spacelike if $1-\|\nabla f\|_E^2>0$, and timelike if $1-\|\nabla f\|_E^2<0$. We consider $f(x,y)=\varphi(u(x),v(y))$ where u and v are defined by $u(x)=\|x\|_E$ and $v(y)=\|y\|_E$. The following equation characterizes $\mathit{\Gamma}_1(\varphi)$ as a hypersurface with zero mean curvature:

$$\begin{aligned}(1-\varphi_v^2)\varphi_{uu}+(1-\varphi_u^2)\varphi_{vv}+2\varphi_u\varphi_v\varphi_{uv}\\ +(1-\varphi_u^2-\varphi_v^2)\biggl((m-1)\frac{\varphi_u}{u}+(n-1)\frac{\varphi_v}{v}\biggr)=0.\end{aligned}\tag{4.1}$$

One of the most famous examples of a maximal hypersurface in $\mathbb{L}^{n+1}$ is the maximal catenoid. The product of this hypersurface with $\mathbb{R}^m$ gives a typical example of a maximal birotationally symmetric graph over $\mathbb{R}^m\times\mathbb{R}^n$. Then we have Theorem 1.4, whose proof is given below.

Proof of Theorem 1.4. We define new variables $u = \|x\|_E$ and $v = \|y\|_E$ and consider the function $\varphi(u, v) = \alpha(u) + \beta(v)$. Equation (4.1) yields

$$\begin{aligned}(1-\alpha'^2)\Big(\beta'' + (n-1)\frac{\beta'}{v}\Big) + (1-\beta'^2)\Big(\alpha'' + (m-1)\frac{\alpha'}{u}\Big)& \\ - (m-1)\frac{\alpha'^3}{u} - (n-1)\frac{\beta'^3}{v} &= 0.\end{aligned} \tag{4.2}$$

Differentiating this with respect to u and v, we have

$$(\alpha'^2)'\Big(\beta'' + (n-1)\frac{\beta'}{v}\Big)' + (\beta'^2)'\Big(\alpha'' + (m-1)\frac{\alpha'}{u}\Big)' = 0.$$

We assume $\alpha'\alpha''\beta'\beta'' \neq 0$ and then

$$\frac{\big(\alpha'' + (m-1)\frac{\alpha'}{u}\big)'}{(\alpha'^2)'} = -\frac{\big(\beta'' + (n-1)\frac{\beta'}{v}\big)'}{(\beta'^2)'} = c_1,$$

where c_1 is a constant. We obtain the following two equations:

$$\begin{aligned}\Big(\alpha'' + (m-1)\frac{\alpha'}{u}\Big)' &= c_1(\alpha'^2)', \\ \Big(\beta'' + (n-1)\frac{\beta'}{v}\Big)' &= -c_1(\beta'^2)'\end{aligned}$$

Integrating both sides of each equation yields

$$\begin{aligned}\alpha'' &= c_1\alpha'^2 - (m-1)\frac{\alpha'}{u} + c_2, \\ \beta'' &= -c_1\beta'^2 - (n-1)\frac{\beta'}{v} + c_3,\end{aligned} \tag{4.3}$$

where c_2 and c_3 are constants.

Dividing (4.2) by $1 - \beta'^2$, we have

$$\begin{aligned}(1-\alpha'^2)\frac{\beta'' + (n-1)\frac{\beta'}{v}}{(1-\beta'^2)} + \alpha'' + (m-1)\frac{\alpha'}{u}& \\ - (m-1)\frac{\alpha'^3}{u}\frac{1}{1-\beta'^2} - (n-1)\frac{\beta'^3}{v(1-\beta'^2)} &= 0.\end{aligned}$$

Differentiating this with respect to u and v implies that there exists a constant c_4 such that

$$\left(\frac{\beta'' + (n-1)\frac{\beta'}{v}}{(1-\beta'^2)}\right)' \Big/ \left(\frac{1}{1-\beta'^2}\right)' = (m-1)\frac{\big(\frac{\alpha'^3}{u}\big)'}{(1-\alpha'^2)'} = c_4.$$

In particular, if $c_4 = 0$, then α' is a constant, which is a contradiction. We assume $c_4 \neq 0$ and obtain

$$(m-1)\left(\frac{\alpha'^3}{u}\right)' = c_4(1-\alpha'^2)',$$

$$\left(\frac{(\beta'' + (n-1)\frac{\beta'}{v})}{(1-\beta'^2)}\right)' = c_4\left(\frac{1}{1-\beta'^2}\right)'.$$

From the second equation, we have

$$\beta'' + (n-1)\frac{\beta'}{v} = c_4 + c_5(1-\beta'^2). \tag{4.4}$$

Using (4.3) and (4.4), we obtain

$$-c_1\beta'^2 + c_3 = c_4 + c_5(1-\beta'^2).$$

Then β' is a constant, which is a contradiction. Therefore, $\alpha'\alpha''\beta'\beta'' = 0$, which implies that $\mathit{\Gamma}_\varphi$ is a cylindrical hypersurface.

Without loss of generality, we consider the cases of $\alpha' = 0$ and $\alpha'' = 0$. We assume $\alpha(u) = c_1u + c_2$. Equation (4.2) can be rewritten as follows:

$$\beta''(1-c_1^2) + (1-c_1^2-\beta'^2)\left((m-1)\frac{c_1}{u} + (n-1)\frac{\beta'}{v}\right) = 0, \tag{4.5}$$

or equivalently

$$\left(\beta''(1-c_1^2) + (n-1)\frac{\beta'}{v}(1-c_1^2-\beta'^2)\right)u + (m-1)c_1 = 0.$$

Because the above equation consists of independent terms, the coefficients of each term must vanish identically. Thus, $c_1 = 0$, which means $\alpha' = 0$, and

$$\beta'' + (n-1)\frac{\beta'}{v}(1-\beta'^2) = 0.$$

Taking $\tilde{u} = (h \circ r)(\tilde{u})$, we have an equivalent ordinary differential equation:

$$r''r - (n-1)(r'^2 - 1) = 0.$$

From this equation, spacelike and timelike elliptic catenoids are obtained. If $\alpha' = 0$ and $\beta'\beta'' = 0$, then $\mathit{\Gamma}_\varphi$ is a hyperplane. ■

4.2. A function f defined on $\mathbb{R}^m \times \mathbb{L}^n$. Let f be a birotationally symmetric function on $\mathbb{R}^m \times \mathbb{L}^n$. Then the graph $\mathit{\Gamma}_2(f)$ of f is defined by

$$\mathit{\Gamma}_2(f) = \{(f(x,y), x, y) \in \mathbb{L}^{1+m+n} \mid (x,y) \in \mathbb{R}^m \times \mathbb{L}^n\}.$$

We classify the cases based on the sign of $\langle y, y\rangle$, which is represented by δ. In particular, $\mathit{\Gamma}_2(f)$ is spacelike if $1 + f_v^2 + \delta f_w^2 < 0$, and timelike if $1 + f_v^2 + \delta f_w^2 > 0$. We suppose that $f(x,y) = \psi(v,w)$ where v and w are new parameters as follows: $v = \|x\|_E$ and $w = \|y\|$. Then the graph of ψ satisfies the following equation:

$$(4.6)\qquad (1+\delta\psi_w^2)\psi_{vv}+\delta(1+\psi_v^2)\psi_{ww}-2\delta\psi_v\psi_w\psi_{vw} \\ +(1+\psi_v^2+\delta\psi_w^2)\bigg((m-1)\frac{\psi_v}{v}+\delta(n-1)\frac{\psi_w}{w}\bigg)=0.$$

Then we have Theorem 1.5, whose proof is given below.

Proof of Theorem 1.5. Let v and w be new variables satisfying $v=\|x\|_E$ and $w=\|y\|$ for $x\in\mathbb{R}^m$ and $y\in\mathbb{L}^n$. We first assume that $\varGamma_2(f)$ is spacelike, i.e., $1+f_v^2+\delta f_w^2<0$. Substituting $\psi(v,w)=\alpha(v)+\beta(w)$ into the equation (4.6) gives

$$\delta(1+\alpha'^2)\bigg(\beta''+(n-1)\frac{\beta'}{w}\bigg)+(1+\delta\beta'^2)\bigg(\alpha''+(m-1)\frac{\alpha'}{v}\bigg) \\ +(m-1)\frac{\alpha'^3}{v}+(n-1)\frac{\beta'^3}{w}=0,$$

and

$$1+\alpha'^2+\delta\beta'^2<0.$$

We follow the proof of Theorem 1.4 and assume that $\alpha'\alpha''\beta'\beta''\neq 0$. Then we have two equations corresponding to (4.3) and (4.4):

$$\beta''+(n-1)\frac{\beta'}{w}=-c_1\beta'^2+c_3,$$
$$\beta''+(n-1)\frac{\beta'}{w}=c_4+c_5(1+\delta\beta'^2).$$

Combining these equations, β' is a constant, which leads to a contradiction.

We first consider $\alpha(v)=c_1v+c_2$ and

$$1+c_1^2-\beta'^2<0.$$

Then we have the equivalent equation

$$\delta\bigg((1+c_1^2)\beta''+(n-1)\frac{\beta'}{w}(1+c_1^2+\delta\beta'^2)\bigg)v+c_1(m-1)(1+c_1^2+\delta\beta'^2)=0.$$

Since the above equation consists of independent terms, the coefficient of v must vanish identically. Thus, we have $c_1=0$ and

$$\beta''+(n-1)\frac{\beta'}{w}(1+\delta\beta'^2)=0.$$

If $\alpha'=0$ and $\beta'\beta''=0$, then $\varSigma$ is a hyperplane. Considering $\tilde{w}=(\beta\circ r)(\tilde{w})$, the above equation yields

$$\frac{(n-1)(\delta+r'^2)-rr''}{rr'^3}=0.$$

Thus, the hyperbolic catenoids of type 1 and 2 are obtained from this ordinary differential equation.

Secondly, we assume $\beta(w) = c_3 w + c_4$ and

$$1 + \alpha'^2 + \delta c_3^2 > 0.$$

Then we have the equivalent equation

$$\Big((1 + \delta c_3^2)\alpha'' + (m-1)\frac{\alpha'}{v}(1 + \alpha'^2 + \delta c_3^2)\Big)w \\ + \delta c_3 (n-1)(1 + \alpha'^2 + \delta c_3^2) = 0.$$

Since $c_3 = 0$, the above equation implies

$$\alpha'' + (m-1)\frac{\alpha'}{v}(1 + \alpha'^2) = 0.$$

Thus, this induces a product of a Euclidean catenoid. If $\beta' = 0$ and $\alpha'\alpha'' = 0$, then Σ is a hyperplane. ■

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