INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

DISSERTATIONES
MATHEMATICAE

Online First version

**JEAN MOULIN-OLLAGNIER, SASHO IVANOV POPOV and
JEAN-MARIE STRELCYN**

The Euler–Poisson equations; an elementary approach to partial integrability conditions, Goryachev–Chaplygin and beyond

WARSZAWA 2026

Jean Moulin-Ollagnier
Laboratoire LIX, École Polytechnique
F-91128 Palaiseau, France
E-mail: jean.moulin-ollagnier@polytechnique.edu

Sasho Ivanov Popov
Institute of Metal Science, Equipment and Technologies
with Hydro- and Aerodynamics Centre "Acad. A. Balevski"
Bulgarian Academy of Sciences
1574 Sofia, Bulgaria
E-mail: sasho_popov2003@yahoo.com

Jean-Marie Strelcyn
CNRS, UMR 7539, Laboratoire Analyse, Géométrie et Applications, LAGA
Université Sorbonne Paris Nord
F-93430 Villetaneuse, France
E-mail: strelcyn@math.univ-paris13.fr

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute

Abstracted/Indexed in: Mathematical Reviews, Zentralblatt MATH, Science Citation Index Expanded, Journal Citation Reports/Science Edition, Google Science, Scopus, EBSCO Discovery Service.

Available online at **http://journals.impan.pl**

doi: 10.4064/dm881-3-2025

ISSN 0012-3862

Contents

Abstract

We consider, in the complex domain, the Euler–Poisson equations describing the motion of a heavy rigid body about a fixed point. The real domain is a particular case of it. The Euler–Poisson equations admit three functionally independent first integrals H_1, H_2, H_3, i.e. the area, geometrical and energy first integrals. In four cases (Euler, Lagrange, Kovalevskaya, kinetic symmetry) a fourth functionally independent first integral appears. It can be found among polynomials that do not depend on all variables.

We study when, apart from the four cases above, the Euler–Poisson equations, restricted to the level manifolds of H_1, H_2 and H_3 and all their mutual intersections, admit a new first integral which does not depend on all variables. In this way we cover the partially integrable Goryachev–Chaplygin case and describe, in the complex domain, a new class of partially integrable cases on the level manifold $\{H_1 = 0, H_2 = 0\}$. We also deduce their uniqueness. We also cover the Sretenskii case of partial integrability of the gyrostat equations (which generalizes the Goryachev–Chaplygin case) and describe a new class of their integrable cases in the complex domain.

We provide a general quasi-algorithmic method to find all these cases and corresponding partial integrals.

The use of computer algebra is unavoidable to carry out our investigations. By following the link (https://sdrive.cnrs.fr/public.php/dav/files/bKmGokMQnM5Jo5f/?accept=zip), the reader can verify all reported computations using EPEPcomp.zip available at that link. Note that a public user can only download files without the edition option.

Acknowledgements. We warmly thank Islam Boussaada, Daniel Bennequin, Piotr Biler, Alain Chenciner, Jaques Fejoz and Piotr Mormul for interesting discussions and Marie-Claude Werquin for her help in checking our English. Moreover, we are grateful to Islam Boussaada for his help with the cloud repository EPEPcomp.zip. We thank the anonymous referee for important comments that enabled us to substantially improve our text.

2020 *Mathematics Subject Classification*: Primary 34C45; Secondary 34A05, 34A34, 34M04, 34M45, 58A30, 70E05, 70E17, 13P10, 13P15.

Key words and phrases: Euler–Poisson equations, rigid body, first integrals, partial integrals, polynomial systems, Gröbner bases, Lie bracket, Frobenius integrability theorem, gyrostat.

Received 13 February 2023; revised 8 October 2024.

Hâtez-vous lentement, et sans perdre courage,
Vingt fois sur le métier remettez votre ouvrage,
Polissez-le sans cesse, et le repolissez,
Ajoutez quelquefois, et souvent effacez.

Nicolas Boileau, *Art poétique* (1674) ([1])

In memory of our dear friend Andrzej Nowicki
who passed away while we were finishing our work

1. Introduction

This paper is one more contribution to the study of the classical problem of Euler–Poisson equations describing the motion of a heavy rigid body about a fixed point. It can be considered as a natural continuation of [67] but it can be read completely independently.

1.1. The problem. Let us briefly describe the content of [67] which is devoted to the search for the so called *fourth integral* of Euler–Poisson equations (see below), but only when this integral does not depend on all the variables. Let us recall some basic facts about the Euler–Poisson equations, which we consider without distinguishing the real case from the complex one. The Euler–Poisson equations are given by the system

$$\begin{aligned}
I_1\frac{d\omega_1}{dt} &= (I_2 - I_3)\omega_2\omega_3 + Mg(c_3\gamma_2 - c_2\gamma_3),\\
I_2\frac{d\omega_2}{dt} &= (I_3 - I_1)\omega_1\omega_3 + Mg(c_1\gamma_3 - c_3\gamma_1),\\
I_3\frac{d\omega_3}{dt} &= (I_1 - I_2)\omega_1\omega_2 + Mg(c_2\gamma_1 - c_1\gamma_2),\\
\frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \omega_2\gamma_3,\\
\frac{d\gamma_2}{dt} &= \omega_1\gamma_3 - \omega_3\gamma_1,\\
\frac{d\gamma_3}{dt} &= \omega_2\gamma_1 - \omega_1\gamma_2,
\end{aligned} \tag{1.1}$$

where $M \neq 0$, $g \neq 0$, $I_1 \neq 0$, $I_2 \neq 0$, $I_3 \neq 0$, c_1, c_2, c_3 are complex parameters.

([1]) In classical English translation of John Dryden (1683):

Gently make haste, of Labour not afraid;
A hundred times consider what you've said:
Polish, repolish, every Colour lay,
And sometimes add; but oft'ner take away.

Studying the Euler–Poisson equations (1.1) from a mechanical point of view, one considers only the real case with $H_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, together with the inequalities

$$M > 0, g > 0, I_1 > 0, I_2 > 0, I_3 > 0, I_1 + I_2 \geq I_3, I_2 + I_3 \geq I_1 \text{ and } I_3 + I_1 \geq I_2.$$

Let us note that the Euler–Poisson equations with non-zero real parameters I_1, I_2 and I_3 of different signs appear in the theory of equilibria of elastic rods [41, 47].

Equations (1.1) describe the motion of a heavy rigid body of mass $M > 0$ about a fixed point O. We consider a body fixed frame $Oxyz$ with origin at O and axes coinciding with the principal axes of inertia through O. Here I_1, I_2, I_3 are the corresponding principal moments of inertia, c_1, c_2, c_3 the coordinates of the body mass center, g is the acceleration of gravity, $g > 0$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the body and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vector directed upwards. For brevity we introduce the notation $\mathcal{I}c = (I_1, I_2, I_3, c_1, c_2, c_3)$.

As in [67], in the present paper we study these equations as a purely mathematical problem considering the general complex case $\mathcal{I}c \in \mathbb{C}^6$, without any restrictions on the parameters except $M \neq 0, g \neq 0, I_1 \neq 0$, $I_2 \neq 0$, $I_3 \neq 0$, which will always be assumed.

It is well known that without any loss of generality one can suppose that $Mg = 1$, which we will do from now on. Indeed, instead of system (1.1) with principal moments of inertia I_1, I_2, I_3, it suffices to consider such a system but with $I_1/(Mg)$, $I_2/(Mg)$, $I_3/(Mg)$ as new principal moments of inertia. As we study the totality of the Euler–Poisson equations (1.1), such a rescaling does not change anything.

Equations (1.1) always have three functionally independent first integrals:

$$\begin{aligned} H_1 &= I_1\omega_1\gamma_1 + I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3, \\ H_2 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \\ H_3 &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2(c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3). \end{aligned} \tag{1.2}$$

In the real case these are the *area* (2), *geometrical* and *conservation of energy* first integrals of system (1.1).

In the real case, in order to be integrable [5, Sec. 28], system (1.1) needs a supplementary fourth first integral H_4, functionally independent of H_1, H_2, H_3, called briefly a *fourth integral.* The only known cases when such a fourth integral exists (both in the real and complex cases) are the following four cases: the *Euler case*, defined by the condition

$$c_1 = c_2 = c_3 = 0, \tag{1.3}$$

as well as the following two cases, which up to appropriate numbering of principal moments of inertia are the *Lagrange case*, defined by the conditions

$$I_1 = I_2, \qquad c_1 = c_2 = 0,\ c_3 \neq 0, \tag{1.4}$$

and the *Kovalevskaya case*, defined by the conditions

$$I_1 = I_2 = 2I_3, \qquad (c_1, c_2) \neq (0,0),\ c_3 = 0. \tag{1.5}$$

---

(2) We use the terminology of the reference books [13] and [88]. In [67] “the area first integral” is by inadvertence called “kinetic moment first integral”.

Let us note that in the real case, in the Kovalevskaya case we can always take $c_2 = 0$, which is reached by an appropriate rotation of the frame of principal axes of inertia around the z axis. In this case we suppose $c_1 \neq 0$. Let us stress that in the complex case the reduction to $c_2 = 0$ by a linear change of variables is not always possible. The fourth case is the *kinetic symmetry case*, defined by

$$I_1 = I_2 = I_3. \tag{1.6}$$

We denote the sets of parameters satisfying cases (1.3) by $\mathcal{E}$, and (1.4) and (1.5) (up to appropriate numbering of principal moments of inertia) by $\mathcal{L}$ and $\mathcal{K}$, respectively.

The fourth integral in cases (1.3)–(1.6) is given as follows:

$$\begin{aligned}
&H_4 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \quad \text{when } \mathcal{I}c \in \mathcal{E},\\
&H_4 = \omega_3 \quad \text{when } \mathcal{I}c \in \mathcal{L},\\
&H_4 = \left(\omega_1^2 - \omega_2^2 - \frac{c_1\gamma_1 - c_2\gamma_2}{I_3}\right)^2 + \left(2\omega_1\omega_2 - \frac{c_2\gamma_1 + c_1\gamma_2}{I_3}\right)^2 \text{ when } \mathcal{I}c \in \mathcal{K},\\
&H_4 = c_1\omega_1 + c_2\omega_2 + c_3\omega_3 \quad \text{in the kinetic symmetry case.}
\end{aligned} \tag{1.7}$$

For $\mathcal{I}c \in \mathcal{K}$, when $c_2 = 0$, we recover

$$H_4 = \left(\omega_1^2 - \omega_2^2 - \frac{c_1}{I_3}\gamma_1\right)^2 + \left(2\omega_1\omega_2 - \frac{c_1}{I_3}\gamma_2\right)^2,$$

the standard form of the fourth integral in the real Kovalevskaya case with $c_2 = 0$ [3, 6, 11, 13, 24, 26, 28, 44, 45, 62, 88].

These four cases are called the *classical* cases of integrability of the Euler–Poisson equations.

One sees that in the above four cases the fourth integral does not depend on all variables, so that the question whether there is another case when the fourth integral does not depend on all the variables is natural. In [67] for $\mathcal{I}c \in \mathbb{C}^6$, $I_1 \neq 0$, $I_2 \neq 0$, $I_3 \neq 0$, we answered this question negatively.

Usually one also cites the so called *Goryachev–Chaplygin case* of *partial integrability*, which up to appropriate numbering of principal moments of inertia is the following. Let $I_1 = I_2 = 4I_3$, $(c_1, c_2) \neq (0, 0)$, $c_3 = 0$. In this case the restriction of the Euler–Poisson equations to the five-dimensional level manifold $\{H_1 = 0\}$ admits a supplementary first integral functionally independent of H_2 and H_3. It is given by

$$H_4 = I_3\omega_3(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3. \tag{1.8}$$

As in the above four cases, the first integral (1.8) depends on strictly fewer variables than is the dimension of the manifold $\{H_1 = 0\}$.

The first integral like H_4 in the Goryachev–Chaplygin case is an example of the so called *partial integral*, i.e. a function defined on the whole phase space that is constant along not all but a family of solutions [13, Sec. 2.4]. In [88] such functions are called *particular solutions.*

We shall use the following terminology. Let a dynamical system be defined on a space M and let $N \subsetneq M$ be an invariant subspace. Let φ be a partial integral defined on M, which is a first integral when restricted to N. We will then say that φ is a *partial integral with respect to* N. The cases where the system on M is integrable will be ignored.

Thus the following problems become natural. Let us consider the complex manifolds of complex dimension 5:

$$\{H_1 = U_1\}, \quad \{H_2 = U_2\}, \quad \{H_3 = U_3\},$$

where U_1, U_2, U_3 are arbitrary complex numbers. These level manifolds are always invariant manifolds for the Euler–Poisson equations. When do there exist partial integrals of the Euler–Poisson equations with respect to these level manifolds and all their mutual intersections, which do not depend on all variables?

In more detailed form these questions can be formulated as follows.

Let $1 \leq i, j \leq 3$.

(a) When does there exist a partial integral of the Euler–Poisson equations with respect to the five-dimensional complex level manifold $\{H_i = U_i\}$, which depends on at most four variables?

(b) When does there exist a partial integral of the Euler–Poisson equations with respect to the four-dimensional complex level manifold $\{H_i = U_i,\ H_j = U_j\}$, $i \neq j$, which depends on at most three variables?

(c) When does there exist a partial integral of the Euler–Poisson equations with respect to the three-dimensional complex level manifold $\{H_1 = U_1,\ H_2 = U_2,\ H_3 = U_3\}$, which depends on at most two variables?

In this paper, we give a complete answer to all these questions.

Indeed, in (a) we recover the Goryachev–Chaplygin case and in (b) we find a supplementary partial integral on the level manifold $\{H_1 = 0, H_2 = 0\}$. By a meticulous and detailed analysis, we show that these two cases are unique for (a), (b) and (c) when there exists a partial integral which does not depend on all the variables.

Let us consider a system of $k, 3 \leq k \leq 6$, differential equations in the complex domain. As in the real case of Euler–Poisson equations, we shall call it *integrable* when the number of its functionally independent first integrals is greater than $k - 3$.

By direct computations one verifies that in cases (a) and (b) when partial integrals exist, the restrictions of the Euler–Poisson equations (1.1) to the respective level manifolds are integrable.

Let us underline that in the paper [29] by D. N. Goryachev from 1900 where the case of Goryachev–Chaplygin appears for the first time, as well as in the paper by S. A. Chaplygin [15] from 1901, there is no explanation how this case was found. To the best of our knowledge no such explanation was published until 1983, when S. L. Ziglin [90] published it for the first time. See also the 2005 paper [51] where A. J. Maciejewski and M. Przybylska present such a deduction in a very clever and clear manner. Nevertheless these deductions are trying and in no way can be considered as simple or elementary.

On the contrary, the deduction of the Goryachev–Chaplygin case from the general principles described in Sec. 1.2 is presented in Sec. 5.2. It only uses facts that were already well known in 1900. It is natural and direct, although some tedious computations are required. They are now easy through the use of elementary computer algebra.

The Euler–Poisson equations have many modifications which describe different mathematical models related to the movements of rigid bodies with a fixed point [10–12, 31, 37, 38, 53, 54, 88]. One of the simplest of these is the system of equations describing the motion of the so-called gyrostat, the equations of which, in the simplest case, are only slightly modified Euler–Poisson equations (1.1). Indeed, the gyrostat equations differ from the Euler–Poisson equations only in the first three equations, where the linear terms $b_3\omega_2 - b_2\omega_3$, $b_1\omega_3 - b_3\omega_1$, $b_2\omega_1 - b_1\omega_2$, $b_1, b_2, b_3 \in \mathbb{C}$, are respectively added to the first three Euler–Poisson equations (1.1). When $b_1 = b_2 = b_3 = 0$ we recover the Euler–Poisson equations (1.1). The gyrostat equations are explicitly written in [25, 28] and in [71, 72] (see also [13, Sec. 2.7], [88, Sec. 5.1.1] and [44–46, 52, 81]). The four classical integrable cases of Euler–Poisson equations admit natural extensions to gyrostat equations. As proved by L. N. Sretenskii [71, 72], the same concerns the Goryachev–Chaplygin case of partial integrability. Its gyrostatic analogue is named the Sretenskii case. By applying the method of Sec. 5.2 which leads to the Goryachev–Chaplygin case, in Sec. 6.2 we recover the Sretenskii case. We also find a class of new cases of integrability of gyrostat equations in the complex domain. Indeed, as will be proved in Sec. 6.3, in the complex domain the gyrostat equations can have a fourth integral apart from the four classical cases.

Let us note that this kind of deduction of Goryachev–Chaplygin and Sretenskii cases appeared for the first time in [19], but our approach is more general.

In summary, our problem is to establish, having a multiparameter family of ordinary differential equations, how to find the values of the parameters for which there exists a supplementary first integral (i.e. non-obvious or not yet known) that does not depend on all variables.

What happens if we search for a fourth integral that does depend on all variables? As for the Euler–Poisson equations (1.1) in the real domain, there are no other cases of integrability than the four cases (Euler, Lagrange, Kovalevskaya, kinetic symmetry) listed previously. The study of the integrability conditions in algebraic functions for the Euler–Poisson equations (1.1) began with E. Husson [39, 40] and culminated in the epoch-making papers of S. L. Ziglin [89, 90] concerning a meromorphic fourth integral. See also [51]. Let us note that certain doubtful points of [39] have been corrected in [40]. See [8] for more historical details. See also [20, Chaps. 2 and 3].

Below, when we speak about *smooth functions*, we always mean C^1 functions in the real case and analytic functions in the complex case. Indeed, in the complex case any function having complex partial derivatives at any point of some open subset of $\mathbb{C}^n$ is analytic on it (see [59]).

Let us stress that we only require the C^1 differentiability of the first integral we are looking for. Although in the complex domain C^1 differentiability implies analyticity, we shall never explicitly use this fact. Moreover, all the considerations are *local.* We never use the fact that such a first integral is globally defined. We only require that it be defined on an open subset of phase space and not constant on any open subset of it (see Sec. 1.2.2). But the results obtained in all known examples are global because the explicit formulas that we obtain for them are globally defined. Let us note that in the complex case multivalued analytic functions can appear.

An important open question is whether, in the examples under study, there are cases with partial integral depending on all variables while there is no partial integral that does not depend on all variables.

It should be emphasized that there is a substantial difference between [50, 67] and the present paper. In both the cited papers, the use of computer algebra could in principle be avoided by tedious hand calculations. This is not the case here, where the huge systems of polynomial equations in several variables that appear cannot even be written down without the use of computer algebra. By following the link

(https://sdrive.cnrs.fr/public.php/dav/files/bKmGokMQnM5Jo5f/?accept=zip),

the reader can verify all reported computations using EPEPcomp.zip available at that link. Note that a public user can only download files without the edition option.

1.2. The method. Following [67], let us explain the approach used which is general and can be applied to many frequently encountered systems of ordinary differential equations. To avoid unnecessary repetitions, we describe it only in the real case but it also works in the complex case. Only in one place (Proposition 1.1) is the real case slightly more subtle than the complex one.

Let

$$\frac{dx}{dt} = G(x) \tag{1.9}$$

be an autonomous system of ordinary differential equations defined on an open connected subset $U \subset \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $G = (G_1, \dots, G_n)$, G is of class C^∞. Let us note that $G = \sum_{i=1}^n G_i \frac{\partial}{\partial x_i}$ is the vector field that defines the system (1.9) and for a function $f \in C^1(U)$, $G(f) = \sum_{i=1}^n G_i \frac{\partial f}{\partial x_i}$.

The vector field will be called *smooth* if all its components are of class C^∞. In the complex case this corresponds to analyticity.

A function $F \in C^1(U)$ is a *first integral* of system (1.9) if F is constant along the orbits of (1.9), that is,

$$G(F) = \sum_{i=1}^n G_i \frac{\partial F}{\partial x_i} = 0, \tag{1.10}$$

and F is not constant on any open subset of U.

Equation (1.10) is a linear equation, generally with variable coefficients, with respect to the partial derivatives $\left\{\frac{\partial F}{\partial x_i}\right\}_{1\le i\le n}$. Since the sought first integral F does not depend on all the variables, by the procedure described below we are able to write many other linear equations of this type. Then by the standard means of linear algebra, one can deduce the existence or non-existence of the first integral F, and if it exists, write it out explicitly.

1.2.1. Part one. If F is a twice differentiable first integral common for two smooth vector fields X and Y defined on some open subset U of $\mathbb{R}^n$, $n \ge 2$, then F is also a first integral of their Lie bracket (also known as the Jacobi–Lie bracket or the commutator of vector fields) $[X, Y]$, defined by $[X, Y](f) = X(Y(f)) - Y(X(f))$ for all twice differentiable functions f. Indeed, if $X(F) = Y(F) = 0$, then evidently $[X, Y](F) = X(Y(F)) -$

$Y(X(F)) = 0$. But what happens when the first integral is only of class C^1? In the complex case there is no problem because class C^1 implies analyticity. But what happens in the real case?

For smooth vector fields

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}, \tag{1.11}$$

simple computations give

$$[X,Y] = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right) \right] \frac{\partial}{\partial x_i}. \tag{1.12}$$

Consequently, $[X,Y]$ is also smooth.

From now on, the Lie bracket $[X,Y]$ is defined exclusively by (1.12) and $[X,Y](f)$ also has meaning for functions f of class C^1.

As in [67], the main tools used to decide if two smooth vector fields could have a common first integral of class C^1 are the simplest facts from linear algebra and the following proposition.

PROPOSITION 1.1. (^{3}) *Let $U \subset \mathbb{R}^n$ be an open set and let X and Y be two smooth vector fields defined by* (1.11). *Let $F \in C^1(U)$ be their common first integral. Then F is also a first integral of their Lie bracket $[X,Y]$:*

$$\textit{if } X(F) = Y(F) = 0 \textit{ then } [X,Y](F) = 0. \tag{1.13}$$

Proof. We can suppose that the vector field X does not vanish identically on U. Let $a \in U$ and $X(a) \neq 0$.

Vector fields, Lie bracket, first integrals and degree of differentiability are intrinsic notions that do not depend on the chosen coordinate system. Thus it is enough to prove (1.13) in one well chosen coordinate system.

By the flow-box (or rectification) theorem ([60, Proposition 1.29] for the real case and [42, Theorem 1.18] for the holomorphic case) there exists an open set V and $a \in V \subset U$ such that in a C^∞ coordinate system $y = (y_1, \ldots, y_n)$ the vector field X takes the form

$$\widetilde{X} = \sum_{i=1}^{n} \widetilde{X}_i \frac{\partial}{\partial y_i} = \frac{\partial}{\partial y_1}, \tag{1.14}$$

that is, $\widetilde{X}_1 = 1$, $\widetilde{X}_k = 0$, $2 \leq k \leq n$, and the vector field Y takes the form

$$\widetilde{Y} = \sum_{i=1}^{n} \widetilde{Y}_i \frac{\partial}{\partial y_i},$$

where $\widetilde{Y} \in C^\infty(V)$, $1 \leq i \leq n$. In this coordinate system the first integral F takes the form $\widetilde{F} = \widetilde{F}(y)$, where $\widetilde{F} \in C^1(V)$.

It remains to prove that

$$\text{if } \widetilde{X}(\widetilde{F}) = \widetilde{Y}(\widetilde{F}) = 0 \text{ then } [\widetilde{X}, \widetilde{Y}](\widetilde{F}) = 0. \tag{1.15}$$

(^{3}) Unfortunately, in [67] this proposition was taken as obvious but not proved.

As $\widetilde{X}(\widetilde{F}) = \frac{\partial \widetilde{F}}{\partial y_1} = 0$ on V, it follows that on V the first integral $\widetilde{F}$ does not depend on y_1: we have $\widetilde{F} = \widetilde{F}(y_2, \dots, y_n)$.

Let us compute $[\widetilde{X}, \widetilde{Y}](\widetilde{F})$. Taking into account (1.14) and (1.12), we get

$$\begin{aligned}[\widetilde{X}, \widetilde{Y}](\widetilde{F}) &= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left(\widetilde{X}_j \frac{\partial \widetilde{Y}_i}{\partial y_j} - \widetilde{Y}_j \frac{\partial \widetilde{X}_i}{\partial y_j} \right) \right] \frac{\partial \widetilde{F}}{\partial y_i} = \sum_{i=1}^{n} \widetilde{X}_1 \frac{\partial \widetilde{Y}_i}{\partial y_1} \frac{\partial \widetilde{F}}{\partial y_i} \\ &= \sum_{i=1}^{n} \frac{\partial \widetilde{Y}_i}{\partial y_1} \frac{\partial \widetilde{F}}{\partial y_i} = \frac{\partial}{\partial y_1} \left(\sum_{i=1}^{n} \widetilde{Y} i \frac{\partial \widetilde{F}}{\partial y_i} \right) = 0.\end{aligned}$$

Indeed, $\widetilde{F}$ does not depend on y_1 and $\widetilde{F}$ is a first integral of the vector field $\widetilde{Y}$. ∎

For $x \in U$, let us denote by $\mathcal{D}(x)$ the two-dimensional vector subspace of $\mathbb{R}^n$ spanned by the vectors $X(x)$ and $Y(x)$. Let us denote by $\mathcal{D}_0$ the set of all C^∞ vector fields Z defined on U such that for all $x \in U$, $Z(x) \in \mathcal{D}(x)$. $\mathcal{D}_0$ is a vector subspace of the vector space of all vector fields defined on U. Let us set $\mathcal{D}_1 = \mathcal{D}_0 + [\mathcal{D}_0, \mathcal{D}_0] = \mathcal{D}_0 + \{[A, B]; A, B \in \mathcal{D}_0\}$, where $[A, B] = AB - BA$ is the Lie bracket of the vector fields A and B. Let us set $\mathcal{D}_2 = \mathcal{D}_1 + [\mathcal{D}_1, \mathcal{D}_1] = \mathcal{D}_1 + \{[A, B]; A, B \in \mathcal{D}_1\}$, etc., where $A + B = \{a + b;\ a \in A,\ b \in B\}$. Thus $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots$. For some $k \leq n$, necessarily $[\mathcal{D}_k, \mathcal{D}_k] = \mathcal{D}_k$, so $\mathcal{D}_k$ is the smallest Lie algebra generated by the vector fields X and Y.

The equation $X(F)(x) = 0$, $x \in U$, can be considered as a linear homogeneous equations with unknowns $\left\{ \frac{\partial F}{\partial x_i}(x) \right\}_{1 \leq i \leq n}$. The same is true for the equations $Y(F)(x) = 0$ and $[X, Y](F)(x) = 0$ (see (1.13)). More generally, this is true for all vector fields from $\mathcal{D}_k$. Then if $a \in U$ and if $\dim \mathcal{D}_k(a) = n$, by continuity $\dim \mathcal{D}_k(x) = n$ for x belonging to some neighborhood $V \subset U$ of a. Choosing an arbitrary basis $v_1, \dots, v_n$ of the vector bundle $\mathcal{D}_k(V) = \bigcup_{x \in V} \mathcal{D}_k(x)$ and writing the corresponding linear homogeneous equations with (in general) variable coefficients and unknowns $\left\{ \frac{\partial F}{\partial x_i}(x) \right\}_{1 \leq i \leq n}$, as $\dim \mathcal{D}_k(x) = n$ for $x \in V$, one deduces that $\frac{\partial F}{\partial x_i}(x) = 0$ for $x \in V$, and finally $F|_V = \text{const}$. This contradicts the assumption that F is a first integral.

Thus the condition $\dim \mathcal{D}_k(a) < n$ is necessary for the existence of common first integrals of both X and Y.

The *Frobenius Integrability Theorem* implies that if on some neighborhood $W \subset U$ of a one has $\dim \mathcal{D}_k(a) = \text{const} = p < n$ for all $x \in W$, then there exist $n - p$ functionally independent first integrals of both X and Y, defined on some neighborhood of a. But we do not know if these first integrals are the restrictions of first integrals defined on the whole phase space U for system (1.9).

All this remains valid in the complex case with $G = (G_1, \dots, G_n)$ analytic in some open, connected subset of $\mathbb{C}^n$ (with obvious changes), because the Frobenius integrability theorem can also be formulated in the complex framework [42, 59].

The vector fields X and Y such that $\dim \mathcal{D}_k(a) < n$ will be said to be *compatible* at a. If this is the case on a dense open subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$), the vector fields X and Y will be said to be *compatible.*

Let us return to system (1.9) and let us suppose that $\dim \mathcal{D}_k(x) = n - 2$ for $x \in W$, where W is an open subset of U. Let us suppose that for some fixed i, $1 \leq i \leq n$, F does not depend on x_i, so $\frac{\partial F}{\partial x_i}(x) = 0$ for all $x \in W$. Without restricting generality we can choose $i = 1$.

In this case, for $x \in W$, $\{\frac{\partial F}{\partial x_i}(x)\}_{2\leq i\leq n}$ satisfy some system of $n-2$ linearly independent linear homogeneous equations. Let $\{\varphi_i(x)\}_{2\leq i\leq n}$ be a fixed non-zero solution of this system. Any other solution is of the form $\{\mu(x)\varphi_i(x)\}_{2\leq i\leq n}$ for some smooth function μ.

Thus if F is a first integral, then for some function μ, $\frac{\partial F}{\partial x_i}(x) = \mu(x)\varphi_i(x)$, which means that μ is an integrating factor of the differential form $\sum_{i=2}^{n}\varphi_i(x)dx_i$. Surprisingly, in this work in all cases when this situation arises, that is, when $\dim \mathcal{D}_k = n-2$ and $n \leq 6$, MAPLE is able to compute *explicitly* the first integral F, globally defined. This is precisely in this way we compute all the unknown first integrals.

Let us emphasize that although the Frobenius Integrability Theorem asserts that in the above framework, at least locally, the sought first integral exists, in our approach this theorem does not play any role. Nevertheless, it is our guiding force.

1.2.2. Part two. Let us search for a first integral F of system (1.9) that does not depend on x_1, so $F = F(\widehat{x})$, where $\widehat{x} = (x_2, \dots, x_n)$, or equivalently $\frac{\partial F}{\partial x_1} = 0$ identically. Here we have privileged x_1, but similar conditions can be written for every x_r, $1 \leq r \leq n$. Then for every x,

$$G(F(x)) = \sum_{i=2}^{n} G_i(x)\frac{\partial F}{\partial x_i}(\widehat{x}) = 0.$$

As F does not depend on x_1, for every $k \geq 1$ one has

$$\sum_{i=2}^{n}\frac{\partial^k G_i}{\partial x_1^k}(x)\frac{\partial F}{\partial x_i}(\widehat{x}) = 0.$$

In other words, if one denotes by Y_k the vector field

$$Y_k = \sum_{i=2}^{n}\frac{\partial^k G_i}{\partial x_1^k}(x)\frac{\partial}{\partial x_i}, \tag{1.16}$$

then for every $k \geq 0$ one has $Y_k(F) = 0$, where $Y_0 = G$, that is, F is a first integral of all these vector fields. All these vector fields are defined on an open connected subset $U \subset \mathbb{R}^n(x)$.

If among the vector fields $\{Y_k\}_{k\geq 0}$ one can find $n-1$ of them that are linearly independent at some point $a \in \mathbb{R}^n(x)$, then by continuity they are also linearly independent on some open neighborhood W of a. As $Y_k(F) = 0$ for all $k \geq 0$, one deduces that $\operatorname{grad} F$ vanishes identically on W and consequently $F|_W = \text{const}$. Then F is not a first integral of system (1.9) because by definition a first integral is non-constant on any open subset of its domain of definition. The same argument works also when arbitrary $n-1$ linearly independent vector fields $\{Z_i\}_{1\leq i\leq n-1}$ such that $Z_i(F) = 0$, $1 \leq i \leq n-1$, are given and $F = F(\widehat{x})$. Such a criterion of non-existence of a first integral will be frequently used in future.

Let us suppose now that the vector field G is of the form

$$G(x) = \sum_{i=0}^{p} x_1^i \widetilde{Y}_{p-i}(\widehat{x}) = x_1^p \widetilde{Y}_0(\widehat{x}) + \cdots + x_1\widetilde{Y}_{p-1}(\widehat{x}) + \widetilde{Y}_p(\widehat{x}) \tag{1.17}$$

for some smooth vector fields $\{\widetilde{Y}_i\}_{0\leq i\leq p}$ defined on some open subset V of $\mathbb{R}^{n-1}(\widehat{x})$. Then as the first integral $F = F(\widehat{x})$ does not depend on x_1, one has

$$0 = G(F)(x) = x_1^p \widetilde{Y}_0(F)(\widehat{x}) + \cdots + x_1 \widetilde{Y}_{p-1}(F)(\widehat{x}) + \widetilde{Y}_p(F)(\widehat{x}).$$

As $\widetilde{Y}_p(\widehat{x})$ does not depend on x_1, one deduces that $\widetilde{Y}_p(F) = 0$. Thus $G(x) = x_1 G_1(x) + \widetilde{Y}_p(\widehat{x})$ where the smooth vector field G_1 is

$$G_1(x) = x_1^{p-1} \widetilde{Y}_0(\widehat{x}) + \cdots + x_1 \widetilde{Y}_{p-2}(\widehat{x}) + \widetilde{Y}_{p-1}(\widehat{x}).$$

As above, one deduces that $\widetilde{Y}_{p-1}(F) = 0$, etc. Finally, one deduces that $\widetilde{Y}_i(F) = 0$ for all $0 \le i \le p$. Thus all vector fields $\widetilde{Y}_i$ defined on V have a common first integral F that does not depend on x_1.

1.2.3. Part three. Let us consider now systems of ordinary differential equations like (1.9) but depending on parameters $\lambda = (\lambda_1, \ldots, \lambda_k)$,

$$\frac{dx}{dt} = G(x, \lambda), \tag{1.18}$$

where $G \in C^\infty(U, \mathbb{R}^n)$ and for $f \in C^1(U)$,

$$G(f, \lambda)(x) = \sum_{i=1}^{n} G_i(x, \lambda) \frac{\partial f}{\partial x_i}(x).$$

All the content of Sec. 1.2.2 without parameters remains valid also with parameters. So, like (1.16) we have the vector fields $Y_k(x, \lambda)$, $k \ge 0$, etc.

As an example, let us consider the simple case when all functions $G_i = G_i(x, \lambda)$, $1 \le i \le n$, are of the form

$$G_i(x, \lambda) = x_1 g_i(\widehat{x}, \lambda) + h_i(\widehat{x}, \lambda), \quad 1 \le i \le n.$$

Let us search for a first integral $F = F(\widehat{x}, \lambda)$ of system (1.18) that does not depend on x_1. We repeat the whole Sec. 1.2.2 but now new data depending on λ appear.

This leads us to the identity

$$0 = G(F(\widehat{x}, \lambda), \lambda) = x_1 \widetilde{Y}_0(F(\widehat{x}, \lambda), \lambda) + \widetilde{Y}_1(F(\widehat{x}, \lambda), \lambda),$$

where

$$\widetilde{Y}_0(\widehat{x}, \lambda) = \sum_{i=2}^{n} g_i(\widehat{x}, \lambda) \frac{\partial}{\partial x_i}, \quad \widetilde{Y}_1(\widehat{x}, \lambda) = \sum_{i=2}^{n} h_i(\widehat{x}, \lambda) \frac{\partial}{\partial x_i}.$$

In particular, for every $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $x \in U$, starting from the vector fields $\widetilde{Y}_0(\widehat{x}, \lambda)$ and $\widetilde{Y}_1(\widehat{x}, \lambda)$, computing their Lie bracket and Lie brackets of higher orders we define a Lie algebra $\mathcal{D}_{k(\lambda)}(\widehat{x}, \lambda)$ like the Lie algebra $\mathcal{D}_k(x)$ in Sec. 1.2.1.

We search for a C^1 function $F(x, \lambda)$ such that for any fixed λ, F is a first integral of system (1.18). This leads to the necessary condition

$$\dim \mathcal{D}_{k(\lambda)}(\widehat{x}, \lambda) \le n - 2 \tag{1.19}$$

for the existence of such a first integral.

As in all examples treated below, $n \le 6$, and without difficulty we compute explicitly a base of the vector space $\mathcal{D}_{k(\lambda)}(\widehat{x}, \lambda)$. Let $M(\widehat{x}, \lambda)$ be a matrix whose rows are the coordinates of the vectors of the above base. Then condition (1.19) is nothing other than

$$\operatorname{rank} M(\widehat{x}, \lambda) \le n - 2 \tag{1.20}$$

for all $x \in U$. Using MAPLE and the method described in Sec. 1.2.3, in all our examples we manage to determine all parameters λ such that (1.20) holds, and thus also that (1.19) is satisfied for all $x \in U$.

Having a concrete example to examine, we compute the associated vector fields and their Lie brackets. Afterwards, we determine the parameters λ that solve the problem posed: existence or nonexistence of a supplementary first integral. We stress that this is where the real difficulties arise. Practically every example requires individual treatment (see Sections 5–6, 8–9) with unavoidable intensive use of computer algebra, and there is no repetition among the very large number of individual cases studied in this paper.

1.3. History. Today, the standard approach for the detection of integrable versus non-integrable cases of ordinary differential equations of quite general nature follows mainly the ideas that begun with S. V. Kovalevskaya (1889) and A. M. Lyapunov (1894) on the one hand and J. Liouville (around 1840), E. Picard (1883–1896) and E. Vessiot (1892) on the other and culminate in the so-called Morales–Ramis theory. The history of this subject, as well as some of its applications, can be found in [2, 27, 55–57]. See also [7, 48, 75, 76, 78–80].

The method of compatible vector fields that is used in this paper was initiated independently by three people: Anatolii Dokshevich [19], Vladimir Bogaevskii [9] and Stefan Rauch-Wojciechowski (formerly Wojciechowski) [84].

Let us make a digression on the problem of priority between A. Dokshevich and V. Bogaevskii. The book including A. Dokshevich's paper went to print on June 30, 1964. The paper of V. Bogaevskii was received by the editors on October 20, 1964. In a footnote on page 93 of [9] he says that the paper was submitted before the publication of [19]. Let us also stress that [9] was published in a widely known mathematical journal published in Moscow while [19] was published in the proceedings of a conference in Tashkent, at the time the capital of Soviet Uzbekistan. There is no doubt that the two authors independently of each other discovered and applied the method of compatible vector fields.

Let us give a very short review of the two papers. Both are devoted to the study of real Euler–Poisson equations having a supplementary first integral that does not depend on all variables. In both papers the condition $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ is assumed.

In Dokshevich's paper [19], using the method of compatible vector fields it is proved that if the supplementary first integral is of the form $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2)$ then this occurs only in the Kovalevskaya case and when $c_2 = 0$, an explicit formula for the Kovalevskaya first integral is deduced. The other cases studied in [19] concern supplementary first integrals of the form $F = F(\omega_3)$, $F = F(\omega_1, \omega_2)$, $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2)$, $F = F(\omega_1, \omega_2, \omega_3, \gamma_3)$ which lead to the cases of integrability of Lagrange and Euler, to the invariant relation of Hess (if $I_1(I_2-I_3)c_1^2 = I_2(I_3-I_1)c_2^2$, $c_3 = 0$ then $I_1c_1\omega_1 + I_2c_2\omega_2 = 0$ is an invariant manifold for the Euler–Poisson equations) and to the Goryachev–Chaplygin partial integrability case respectively. The author notes also that Sretenskii's case of partial integrability of gyrostat equations can be found along the same lines as the Goryachev–Chaplygin case.

Let us mention the last paragraph of [19], where the principle of the method of compatible vector fields is clearly stated.

In Bogaevskii's paper [9] one considers supplementary first integrals of the form $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$ for the Euler–Poisson equations of motion of a rigid body with a fixed point in the potential force field $U = U(\gamma_1, \gamma_2, \gamma_3)$. When $U = c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3$, we recover the standard Euler–Poisson equations (1.1). Applying the method of compatible vector fields one identifies the classical cases of integrability: Euler, Lagrange and Kovalevskaya.

Afterwards, one considers the problem of finding the general form of the potential $U = U(\gamma_1, \gamma_2, \gamma_3)$ when there exists a supplementary first integral $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$. A new generalization of the Kovalevskaya case appears.

Unlike [67] and the present paper, where we sweep up all possible cases of first integrals that do not depend on all variables, in [19] and [9] the authors only focus on a few concrete cases that enable them to catch integrable cases.

This line of research was pursued by Yu. A. Arkhangelskii [3, 4] directly inspired by [19] and [9] and also by S. I. Popov [63–65]. For further developments see [50, 66, 67] and the present paper.

Around twenty years later, around 1986, Stefan Rauch-Wojciechowski motivated by [82, 83], where implicitly the Euler equations on the duals of Lie algebras appear (see [60, Ch. 6] and also [11, 13, 22, 44, 61, 74]) discovered independently the method of compatible vector fields (the name coined by him) advocating their application to three-dimensional systems [30, 73, 84–86]. For further developments see [32, 43, 58].

The method of compatible vector fields uses exclusively notions and facts already well known by Jacobi even if their formal settings were not perfect. Jacobi and some of his contemporaries already knew and understood vector fields, Jacobi–Lie bracket and the link between compatible vector fields and existence of common first integrals for them, i.e. Frobenius theorem ([14, Ch. Groupes de Lie et algèbres de Lie, p. 310], [33, Sec. 3], [34, Sec. 2.5], [35, Sec. 5], [36, Sec. 6.4.3], [69, p. 26], [79, 80]). On page 26 of [69] we even find the Deahna–Clebsch–Frobenius theorem instead of the traditional Frobenius theorem [16, 18, 23]. On page 417 of [35] and on page 188 of [36], the Jacobi–Clebsch–Deahna–Frobenius theorem appears. We cannot therefore exclude that the method of compatible vector fields appeared in certain works now forgotten, in the period from the second half of the nineteenth century, or even earlier, before the publication of [9, 19].

The present paper is organized as follows. In Sec. 2 an important technical tool, the so called *permutational symmetries*, is briefly described. Sec. 3 is devoted to the use of Gröbner bases to obtain by MAPLE the solutions of the enormous systems of polynomial equations which appear in this article. The direct approach used in [67] is totally inappropriate here. This is one of the pivots of the paper. The very short Sec. 4 gives some algebraic complements in a form that suits us.

Sec. 5 is devoted to the study of five-dimensional invariant manifolds $\{H_i = U_i\}$, $1 \leq i \leq 3$, that is, problem (a) formulated above. This leads us to recover in a natural way the Goryachev–Chaplygin case. This is the content of Sec. 5.2. In Secs. 5.3 and 5.4, without giving the tedious and long proofs, we briefly report on what happens on the manifolds $\{H_3 = U_3\}$ and $\{H_2 = 0\}$, respectively. The case of the manifolds $\{H_2 = U_2 \neq 0\}$ was completely elucidated in [67, Sec. 5]. In Sec. 6 we sketch the study of gyrostat equations and the derivation of the Sretenskii case of partial integrability which generalizes the Goryachev–Chaplygin case. Furthermore, in the complex domain, we have found new integrable cases of the gyrostat equations that were still unknown. In Sec. 7, somewhat isolated from the rest of the paper, we determine the so-called domain of the Sretenskii partial integral, which includes the Goryachev–Chaplygin partial integral as a special case. In Secs. 8 and 9 we study what happens on the four- and three-dimensional invariant manifolds

$$\begin{aligned}&\{H_i = U_i,\, H_j = U_j\}, \qquad 1 \le i, j \le 3,\, i \ne j, \text{ and}\\ &\{H_1 = U_1,\, H_2 = U_2,\, H_3 = U_3\},\end{aligned}$$

respectively. In Sec. 8.2.1, in the complex domain, a new class of partially integrable cases on the level manifold $\{H_1 = 0,\, H_2 = 0\}$ is described. By meticulous and detailed analysis we also deduce their uniqueness when there exists an additional partial integral that does not depend on all variables.

We refer to [67] for some supplementary details.

The method we use is of general interest and is probably the most interesting point of this paper. It can also be applied in many other circumstances (see for example [11–13]).

2. Permutational symmetries

The Euler–Poisson equations (1.1) have an invariance property which we call *permutational symmetry*. The permutational symmetries can be described in a general framework as follows. Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, and let $V(x, \lambda) = (V_1(x, \lambda), \ldots, V_n(x, \lambda))$ depend smoothly on x. Let us consider the system of differential equations

$$\frac{dx}{dt} = V(x, \lambda). \tag{2.1}$$

Let σ be an element of the symmetric group S_n, i.e., the group of all permutations of $\{1, \ldots, n\}$. For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ we will write $\sigma(a) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$.

A permutation $\sigma \in \mathrm{S}_n$ will be called a *permutational symmetry* of system (2.1) if for all $x \in \mathbb{C}^n$, $\lambda \in \mathbb{C}^n$, one has

$$V_k(\sigma(x), \sigma(\lambda)) = \varepsilon V_{\sigma(k)}(x, \lambda), \qquad 1 \le k \le n, \tag{2.2}$$

where $\varepsilon = \pm 1$ is a constant depending on k but independent of x and λ. It is obvious that all permutational symmetries of a given equation form a group.

Let us recall that a subset $M \subset \mathbb{C}^n$ is an *invariant subset* of system (2.1) if M is formed by the entire orbits of the system. That means that if for some $t_0 \in \mathbb{C}$, $x(t_0) \in M$, then $x(t) \in M$ for all $t \in \mathbb{C}$ such that $x(t)$ is well defined.

Let us formulate the following theorem, already proved in [50] and [67]. We formulate it in the complex setting, but it also remains valid in the real framework. For the sake of completeness, we also report its proof.

Theorem 2.1. *The permutational symmetries of the Euler–Poisson equations* (1.1) *are the following:*

$$\begin{aligned}
\sigma_1 &= \{(1,2,3),(1,2,3)\}, \quad \varepsilon = 1,\\
\sigma_2 &= \{(1,3,2),(1,3,2)\}, \quad \varepsilon = -1,\\
\sigma_3 &= \{(2,3,1),(2,3,1)\}, \quad \varepsilon = 1,\\
\sigma_4 &= \{(2,1,3),(2,1,3)\}, \quad \varepsilon = -1,\\
\sigma_5 &= \{(3,1,2),(3,1,2)\}, \quad \varepsilon = 1,\\
\sigma_6 &= \{(3,2,1),(3,2,1)\}, \quad \varepsilon = -1,
\end{aligned} \tag{2.3}$$

where $\sigma\{(i_1,i_2,i_3),(j_1,j_2,j_3)\}$, $1 \le i_r, j_r \le 3$, $1 \le r \le 3$, *is the permutation*

$$\sigma(s_1,s_2,s_3,t_1,t_2,t_3) = (s_{i_1},s_{i_2},s_{i_3},t_{j_1},t_{j_2},t_{j_3}).$$

Proof. The permutation σ_1 with $\varepsilon = 1$ is evidently a permutational symmetry for the Euler–Poisson equations. One can see from these equations that σ_2 with $\varepsilon = -1$ is a permutational symmetry too. The same is true for σ_3 with $\varepsilon = 1$.

Taking into account the equalities

$$\sigma_4 = \sigma_2 \circ \sigma_3, \quad \sigma_5 = \sigma_3^2, \quad \sigma_6 = \sigma_3 \circ \sigma_2$$

we deduce that σ_4 with $\varepsilon = -1$, σ_5 with $\varepsilon = 1$ and σ_6 with $\varepsilon = -1$ are permutational symmetries for the Euler–Poisson equations.

To complete the proof it remains to note that if the permutation

$$\sigma(1,2,3,4,5,6) = (l_1,l_2,l_3,l_4,l_5,l_6)$$

is a permutational symmetry for the Euler–Poisson equations then $l_1,l_2,l_3 \in \{1,2,3\}$ and $l_4,l_5,l_6 \in \{4,5,6\}$. Thus $\sigma = \{(l_{i_1},l_{i_2},l_{i_3}),(l_{j_1},l_{j_2},l_{j_3})\}$. Now, from the Euler–Poisson equations one deduces easily that $i_k = j_k$, $1 \le k \le 3$. ■

It is interesting to observe that the three sets $\mathcal{E}$, $\mathcal{L}$ and $\mathcal{K}$ are invariant with respect to the permutational symmetries. The same concerns the kinetic symmetry case.

In other words, the set of permutational symmetries of the Euler–Poisson equations (1.1) coincides with S_3, where the same permutation is simultaneously applied to variables $\{\omega_1,\omega_2,\omega_3\}$ and $\{\gamma_1,\gamma_2,\gamma_3\}$ and to parameters $\{I_1,I_2,I_3\}$ and $\{c_1,c_2,c_3\}$.

It is also important to note that the first integrals H_1, H_2 and H_3 are invariant with respect to all permutational symmetries of the Euler–Poisson equations. This means that for any such permutational symmetry σ one has

$$H_i(I,c,\omega,\gamma) = H_i(\sigma(I),\sigma(c),\sigma(\omega),\sigma(\gamma)), \quad 1 \le i \le 3,$$

where for $\sigma \in S_3$, $\sigma(a_1,a_2,a_3) = (a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)})$.

This leads to the following general statement that will be frequently used in the future.

Let us define the function Φ_0, $\Phi_0(x,\lambda) = 1$ for all $x \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}^n$. Let $U_0 = 1$ and set $M(U_0,\lambda) = \{x \in \mathbb{C}^n;\ \Phi_0(x,\lambda) = 1\} = \mathbb{C}^n$.

Let $\lambda \in \mathbb{C}^n$ be fixed. Let $\Phi_i = \Phi_i(x,\lambda)$, $1 \le i \le k < n$, be a finite number of first integrals of system (2.1) which are all invariant with respect to all permutational symmetries σ of the system (2.1), that is,

$$\Phi_i(x, \lambda) = \Phi_i(\sigma(x), \sigma(\lambda)), \quad 1 \le i \le k. \tag{2.4}$$

For $U_1, \dots, U_k \in \mathbb{C}$ and $k \ge 0$ let us set

$$M(U_0, \dots, U_k, \lambda) = \{x \in \mathbb{C}^n;\ \Phi_i(x, \lambda) = U_i,\ 0 \le i \le k\}. \tag{2.5}$$

In the future, without repeating it each time, we will only consider the cases where $M(U_0, \dots, U_k, \lambda)$ is either $\mathbb{C}^n$ (when $k = 0$) or a non-empty submanifold (perhaps with singularities) of $\mathbb{C}^n$ (when $k \ge 1$).

All these submanifolds of $\mathbb{C}^n$ are invariant manifolds of the system (2.1) and from (2.4) it follows that they are all invariant with respect to all permutational symmetries of system (2.1).

Theorem 2.2. *Let $k \ge 0$. Let σ be some permutational symmetry of system* (2.1)*. Let us consider the system* (2.1) *restricted to the invariant manifold $M(U_0, \dots, U_k, \lambda)$ and its local first integral $F = F(x)$ defined on some open subset $W_F \subset M(U_0, \dots, U_k, \lambda)$. Then the function $G = F \circ \sigma^{-1}$, i.e. $G(x) = F(\sigma^{-1}(x))$, defined on the open subset $\sigma(W_F) = \{x \in M(U_0, \dots, U_k, \lambda);\ \sigma^{-1}(x) \in W_F\}$ of $M(U_0, \dots, U_k, \lambda)$ is a local first integral of the system*

$$\frac{dx}{dt} = V(x, \sigma(\lambda)) \tag{2.6}$$

restricted to $M(U_0, \dots, U_k, \lambda)$.

Proof. As F is a first integral of system (2.1) restricted to W_F, for every $x \in W_F$ we have

$$\sum_{k=1}^{n} V_k(x, \lambda) \left(\frac{\partial F}{\partial x_k} \right)(x) = 0.$$

As σ is a permutation of $\{1, \dots, n\}$, the last equality is equivalent to

$$\sum_{k=1}^{n} V_{\sigma(k)}(x, \lambda) \left(\frac{\partial F}{\partial x_{\sigma(k)}} \right)(x) = 0.$$

Taking into account (2.2), we can write this as

$$\sum_{k=1}^{n} V_k(\sigma(x), \sigma(\lambda)) \left(\frac{\partial F}{\partial x_{\sigma(k)}} \right)(x) = 0.$$

The last equality is satisfied for every $x \in W_F$. Then putting $\sigma^{-1}(x)$ instead of x, we find that for every $x \in \sigma(W_F)$,

$$\sum_{k=1}^{n} V_k(x, \sigma(\lambda)) \left(\frac{\partial F}{\partial x_{\sigma(k)}} \right)(\sigma^{-1}(x)) = 0.$$

On the other hand, a function $G = G(x)$ is a first integral of system (2.6) if

$$\sum_{k=1}^{n} V_k(x, \sigma(\lambda)) \left(\frac{\partial G}{\partial x_k} \right)(x) = 0.$$

Thus to finish the proof it remains to prove that for $G = F \circ \sigma^{-1}$ and $1 \leq k \leq n$ one has

$$\left(\frac{\partial F}{\partial x_{\sigma(k)}}\right)(\sigma^{-1}(x)) = \left(\frac{\partial G}{\partial x_k}\right)(x),$$

but this is obvious. ■

For $k = 0$, Theorem 2.2 coincides with Theorem 2.1 from [50] (see also [67]).

Theorem 2.2 shows that from the point of view of integrability/non-integrability, systems (2.1) and (2.6), both restricted to $M(U_0, \ldots, U_k, \lambda)$, are exactly of the same nature.

In the future, when considering local first integrals, the word "local" will frequently be omitted.

Hereafter, we will always have $\Phi_i = H_i$, $1 \leq i \leq 3$.

3. Solving some systems of polynomial equations

To solve all systems of polynomial equations encountered in this paper we use the theory of Gröbner bases of polynomial rings [17, 21, 68].

Let us recall some basic facts concerning them and their MAPLE implementations. For all computations we use exclusively the monomial order

$$\operatorname{tdeg}(U_1, U_2, U_3, I_1, I_2, I_3, c_1, c_2, c_3)$$

with ordering $U_1 > U_2 > U_3 > I_1 > I_2 > I_3 > c_1 > c_2 > c_3$ (1).

For a fixed monomial order a Gröbner basis of an ideal of the polynomial ring $\mathbb{Q}[U, I, c]$ is characterized by the property that the leading monomial of every polynomial in the ideal is divisible by the leading monomial of some polynomial in the Gröbner basis.

A *Maple reduced* Gröbner basis is a Gröbner basis such that if we remove a polynomial from it, the remaining polynomials no longer form a Gröbner basis and it has the additional property that no monomial of any polynomial in the basis is divisible by any of the leading monomials (other than itself). If all polynomials in a Maple reduced Gröbner basis have leading coefficient 1, then this basis is unique up to permutation of its elements and is called the *reduced* Gröbner basis. Let us stress that the reduced Gröbner basis always exists.

As proved by the following simple example, in general, a Maple reduced Gröbner basis is not the reduced Gröbner basis.

The MAPLE command `Groebner[Basis]` (2) computes Maple reduced Gröbner bases for ideals of polynomial rings.

Let us consider the polynomial ring $\mathbb{C}[x, y, z]$ where $x > y > z$ and its ideal L generated by the polynomials

$$\{3y^2 - 8z^3, xy^2 + yz^3, x^2 - 2xz + 5\}.$$

---

(1) https://www.maplesoft.com/support/help/Maple/view.aspx?path=Groebner/MonomialOrders.

(2) https://www.maplesoft.com/support/help/Maple/view.aspx?path=Groebner/Basis.

With monomial order $\mathrm{tdeg}(x, y, z)$, the command `Groebner[Basis]` gives the following Maple reduced Gröbner basis of L:

$$[x^2 - 2xz + 5, 8z^3 - 3y^2, 8xy^2 + 3y^3, 9y^4 + 48y^3z + 320y^2]$$

with leading coefficients $[1, 8, 8, 9]$. With monomial order $\mathrm{plex}(x, y, z)$ we obtain the Maple reduced basis

$$[1600z^3 - 96z^8 + 240z^6 + 9z^9, -40z^5 + 32z^7 - 3z^8 + 80yz^3,\\ 3y^2 - 8z^3, 120z^5 - 96z^7 + 9z^8 + 640xz^3, x^2 - 2xz + 5]$$

with leading coefficients $[9, 80, 3, 640, 1]$.

We observe that the resulting Maple reduced Gröbner bases consist of polynomials, each with integer coprime coefficients and positive leading coefficient.

The reason that Maple in its definition of the reduced Gröbner bases does not require that the leading coefficients be 1 is due to avoiding the use of rational non-integer numbers.

All factorizations are over $\mathbb{Q}$, that is, in the polynomial ring

$$\mathbb{Q}[U, I, c] = \mathbb{Q}[U_1, U_2, U_3, I_1, I_2, I_3, c_1, c_2, c_3].$$

Let us consider polynomials $P_i = P_i(U, I, c) = P_i(U_1, U_2, U_3, I_1, I_2, I_3, c_1, c_2, c_3) \in \mathbb{Q}[U, I, c]$, $1 \le i \le n$. We want to find all complex solutions of the system $P_i(U, I, c) = 0$, $1 \le i \le n$, such that $I_j \neq 0$, $1 \le j \le 3$. Such solutions will be called *good solutions*. To find them we proceed as follows (steps A.1–A.3) and in all cases encountered we achieve a success.

Let us set $\mathbb{C}^9_g = \{(U, I, c) \in \mathbb{C}^9;\ I_i \neq 0,\ 1 \le i \le 3\}$. The good solutions are in $\mathbb{C}^9_g$.

A.1. With the Maple command `factor`, we factorize over $\mathbb{Q}$ all polynomials $P_i(U, I, c)$, $1 \le i \le n$,

$$P_i = I_1^{\alpha_{i1}} I_2^{\alpha_{i2}} I_3^{\alpha_{i3}} \prod_{k=1}^{r_i} D_{ik}^{\beta_{ik}},$$

where $\beta_{ik} \in \mathbb{N} = \{1, 2, \ldots\}$, $\alpha_{i1}, \alpha_{i2}, \alpha_{i3} \in \mathbb{N} \cup \{0\}$, $D_{ik} \in \mathbb{Q}[U, I, c]$. Moreover, for $k \neq l$, the polynomials D_{ik} and D_{il} are relatively prime and irreducible in $\mathbb{Q}[U, I, c]$, $1 \le k, l \le r_i$, $1 \le i \le n$.

Then

$$\{(U, I, c) \in \mathbb{C}^9_g;\ P_i(U, I, c) = 0,\ 1 \le i \le n\}\\ = \{(U, I, c) \in \mathbb{C}^9_g;\ \widehat{P}_i(U, I, c) = 0,\ 1 \le i \le n\}, \qquad (3.1)$$

where $\widehat{P}_i = \prod_{k=1}^{r_i} D_{ik}$ is a square-free factorization of $\prod_{k=1}^{r_i} D_{ik}^{\beta_{ik}}$. Let us stress that in (3.1) we have identity of zeros but perhaps not of their multiplicities.

It is clear that the following inclusion of ideals in the ring $\mathbb{Q}[U, I, c]$ holds:

$$\{P_1, \ldots, P_n\} \subset \{\widehat{P}_1, \ldots, \widehat{P}_n\}, \qquad (3.2)$$

where $\{R_1, \ldots, R_q\}$ denotes the ideal in $\mathbb{Q}[U, I, c]$ generated by the polynomials

$$R_1, \ldots, R_q \in \mathbb{Q}[U, I, c].$$

A.2. Using the MAPLE command `Groebner[Basis]` we compute in the ring $\mathbb{Q}[U, I, c]$ a Maple reduced Gröbner basis $\{Q_1, \ldots, Q_m\}$ of the ideal $\{\widehat{P}_1, \ldots, \widehat{P}_n\} \subset \mathbb{Q}[U, I, c]$. The polynomials $Q_1, \ldots, Q_m$ can have multiple factors in $\mathbb{Q}[U, I, c]$.

Formulas (3.1) and (3.2) imply respectively that

$$\{(U, I, c) \in \mathbb{C}^9_g;\ P_i(U, I, c) = 0,\ 1 \le i \le n\} \\ = \{(U, I, c) \in \mathbb{C}^9_g;\ Q_j(U, I, c) = 0,\ 1 \le j \le m\}$$

and

$$\{P_1, \ldots, P_n\} \subset \{Q_1, \ldots, Q_m\}.$$

As $\{R_1, \ldots, R_u\} \subset \{\widehat{R}_1, \ldots, \widehat{R}_u\}$ we have

$$\{P_1, \ldots, P_n\} \subset \{\widehat{Q}_1, \ldots, \widehat{Q}_m\}. \tag{3.3}$$

and

$$\{(U, I, c) \in \mathbb{C}^9_g;\ P_i(U, I, c) = 0,\ 1 \le i \le n\} \\ = \{(U, I, c) \in \mathbb{C}^9_g;\ \widehat{Q}_j(U, I, c) = 0,\ 1 \le j \le m\}. \tag{3.4}$$

The passage from the system $P_1 = 0, \ldots, P_n = 0$ to $\widehat{Q}_1 = 0, \ldots, \widehat{Q}_m = 0$ will be called *simplification.*

According to (3.4) the system obtained by simplification has the same good solutions as the source system and in all encountered cases the final system of equations is simpler than the original one.

As the ring $\mathbb{Q}[U, I, c]$ is Noetherian, after a finite number of consecutive simplifications we will arrive (see (3.3)) at the system $S_1 = 0, \ldots, S_t = 0$, that will not be modified by another simplification, that is, every polynomial S_i, $1 \le i \le t$, is square-free, without factors of the form $I_1^{\alpha_{i1}} I_2^{\alpha_{i2}} I_3^{\alpha_{i3}}$, and the polynomials $\{S_i\}_{1 \le i \le t}$ form a Maple reduced Gröbner basis of the ideal $\{S_1, \ldots, S_t\}$.

We call the system of equations $S_1 = 0, \ldots, S_t = 0$ the *reduced system* or the *reduction* of the source system $P_1 = 0, \ldots, P_n = 0$. The reduced system $\{S_j = 0\}$ has the same set of good solutions as the source system $\{P_i = 0\}$. The simplest MAPLE computational criterion for the system $S_1 = 0, \ldots, S_t = 0$ to be the reduction of the source system is that its simplification coincides with it. This criterion will be constantly used.

A.3. The final step is then to describe the set of all complex solutions of the reduced system $\{S_j = 0\}$, $1 \le j \le t$.

It is clear that if $t = 1$ and $S_1 = 1$, then the source system does not admit any good solution.

Fortunately, in an unexplained and unexpected way, in all other cases encountered below, the reduced systems are simple, of low degrees and all $\{S_j\}_{1 \le j \le t}$ factorize into products of factors that depend on only one kind of unknowns, $\{U_1, U_2, U_3\}$, $\{c_1, c_2, c_3\}$ or $\{I_1, I_2, I_3\}$. Moreover, every factor belongs to the following short list of possibilities:

$$U_1, \quad U_2, \quad U_3, \quad a_1 I_1 + a_2 I_2 + a_3 I_3, \quad c_1, \quad c_2, \quad c_3 \quad \text{and} \quad b_1 c_1^2 + b_2 c_2^2 + b_3 c_3^2, \tag{3.5}$$

where a_i and b_i, $1 \le i \le 3$, are some integers. There is only one exception in Sec. 8.2.1 where in one of the equations of the reduced system a factor appears that depends simultaneously on I_i and c_i, $1 \le i \le 3$, and it is $(I_2 - I_3)c_1^2 + (I_1 - I_3)c_2^2 + (I_2 - I_1)c_3^2$.

In many cases the situation is even simpler because some of the polynomials S_j, $1 \leq j \leq t$, merely coincide with some of the possibilities from the list (3.5). For example, in Sec. 7 (see formula (7.8)) we have $S_1 = c_3$.

Thus, without any difficulty, all the good solutions can be found either by hand or by applications of elementary computer algebra, MAPLE for example.

4. Some algebra

The following two simple propositions will be used repeatedly until the end of the article. The first one is well known and follows from the well known elementary properties of resultant ([17, Chap. 3, §6] and [21]).

Let $\mathbb{K}$ be a field of characteristic 0 and $\mathbb{K}[x]$ be as usual the ring of polynomials in one variable x with coefficients in $\mathbb{K}$.

PROPOSITION 4.1. *Let $g \in \mathbb{K}[x]$ and $h(x) = \frac{dg}{dx}$. Let ρ be the resultant of g and h and suppose $\rho \neq 0$. Let $\overline{x}$ be some root of g. Then*

(i) *$h(\overline{x}) \neq 0$,*
(ii) *g has no multiple roots.*

Proof. (i) follows immediately from the well known fact that if $f, g \in \mathbb{K}[x]$ then f and g have a common factor in $\mathbb{K}[x]$ if and only if their resultant is 0 or equivalently if f and g have a common root (perhaps in the algebraic closure of $\mathbb{K}$).

(ii) follows from the evident fact that if g has a multiple root, then $h(x) = \frac{dg}{dx}$ has the same root and thus g and h have a common factor. ∎

The second proposition is completely evident. Let $\mathbb{K}$ be a field of characteristic 0. Let $f, g \in \mathbb{K}[x]$ and $g \neq 0$. By Euclidean division we know that for some $q, r \in \mathbb{K}[x]$ one has

$$f(x) = q(x)g(x) + r(x), \quad \deg r < \deg g \text{ or } r = 0.$$

PROPOSITION 4.2. *Suppose that*

(i) *all roots of g are simple and are in $\mathbb{K}$,*
(ii) *all roots of g are also roots of f.*

Then g divides f in $\mathbb{K}[x]$, so the remainder r (which is in $\mathbb{K}[x]$) vanishes identically.

In the following, for fixed $n \geq 1$, let $\mathbb{K}_n = \mathbb{A}\mathrm{lg}(s_1, \ldots, s_n)$ be the field of algebraic functions of complex variables $(s_1, \ldots, s_n) \in \mathbb{C}^n$ [1, 70, 77]. The field $\mathbb{K}_n$ is of characteristic 0.

Let us explain this in more detail. Following [1], let $P_0, \ldots, P_k \in \mathbb{C}[x_1, \ldots, x_n]$ with $P_k(x_1, \ldots, x_n) \not\equiv 0$. A function $y = y(x_1, \ldots, x_n)$ is called a *complex algebraic function* if

$$P_k(x_1, \ldots, x_n)y^k + P_{k-1}(x_1, \ldots, x_n)y^{k-1} + \cdots + P_0(x_1, \ldots, x_n) = 0 \tag{4.1}$$

for all $(x_1, \ldots, x_n) \in \mathbb{C}^n$ and if the above polynomial in y is irreducible in $\mathbb{C}[x_1, \ldots, x_n]$. The number k is called the *degree* of the algebraic function y. If $k = 1$, an algebraic function is a rational function $y = -P_0(x_1, \ldots, x_n)/P_1(x_1, \ldots, x_n)$. For $k = 2, 3, 4$, an algebraic function can be expressed by square and cube roots of rational functions in $x_1, \ldots, x_n$. If $k \geq 5$, this is impossible in general [21].

If $k \geq 2$ an algebraic function is multivalued (for example $y = \sqrt{x}$) and in an open dense subset of $\mathbb{C}^n$, it locally admits holomorphic (analytic) branches. This follows from the complex implicit function theorem [42, 59].

Let us also note that any non-zero complex polynomial (4.1) can be factorized into irreducible factors [21]. Thus, the equation (4.1) defines algebraic functions even if the polynomial (4.1) is not irreducible.

Let us write for short $x = (x_1, \ldots, x_n)$. Let us compute the partial derivatives $\frac{\partial y}{\partial x_i}(x)$, $1 \leq i \leq n$, for an algebraic function of degree k. By differentiating the formula (4.1) with respect to x_i, $1 \leq i \leq n$, one easily deduces that

$$\frac{\partial y}{\partial x_i}(x) = -\frac{\frac{\partial P_k}{\partial x_i}(x)y^k + \frac{\partial P_{k-1}}{\partial x_i}(x)y^{k-1} + \cdots + \frac{\partial P_0}{\partial x_i}(x)}{kP_k(x)y^{k-1} + (k-1)P_{k-1}(x)y^{k-2} + \cdots + P_1(x)}. \tag{4.2}$$

The partial derivatives of higher order of the algebraic function $y = y(x)$ can be computed by consecutive differentiations of the formula (4.2).

As the degree of the algebraic function $y = y(x)$ is k, the denominator of (4.2) which is a non-vanishing algebraic function is non-zero on an open dense subset of $\mathbb{C}^n$, where the formula (4.2) gives the derivative, which is also an algebraic function of $x = (x_1, \ldots, x_n)$.

Now, let us consider, on some open subset U of $\mathbb{C}^n$, a holomorphic branch of a multivalued algebraic function $y = y(x)$, which we shall denote $f = f(x)$. Then, if in (4.2) instead of y one takes the function f, the formula remains valid.

Consequently, instead of analyzing separately all holomorphic branches of an algebraic function $y = y(x)$, it suffices to consider the multivalued algebraic function $y = y(x)$ as a whole, the derivatives of which are given by (4.2).

We shall also apply the following well known and easy proposition.

PROPOSITION 4.3. *Let $n \geq 2$ and let $V \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial that is not a square of another polynomial. Then $\sqrt{V} \notin \mathbb{C}(x_1, \ldots, x_n)$, that is, $\sqrt{V}$ is not a rational function of $x_1, \ldots, x_n$.*

5. Five-dimensional invariant manifolds $\{H_i = U_i\}$, $1 \leq i \leq 3$. Goryachev–Chaplygin case

5.1. Extraction procedure. In this section we study the existence of a partial integral of the Euler–Poisson equations (1.1) with respect to the invariant complex five-dimensional manifolds $\{H_1 = U_1\}$ and $\{H_3 = U_3\}$. We study when on each of them there exists a partial integral that depends on at most four variables. The same problem can be stated also for the manifold $\{H_2 = U_2\}$. For $U_2 = 1$ this case has been considered in [67, Sec. 5] and the general case of $U_2 \neq 0$ can be easily reduced to the case $U_2 = 1$. Thus it remains to study the case $\{H_2 = 0\}$.

Let us fix i, $1 \le i \le 3$. According to (2.5),

$$M(U_0, U_i, \mathcal{I}c) = \{x \in \mathbb{C}^6;\ H_i((\omega,\gamma), \mathcal{I}c) = U_i\},$$

where $(\omega,\gamma) = (\omega_1,\omega_2,\omega_3,\gamma_1,\gamma_2,\gamma_3)$ and $\dim M(U_0,U_i,\mathcal{I}c) = 5$.

We search for functions F of four variables, $F = F(s_1,s_2,s_3,s_4)$, where $(s_1,s_2,s_3,s_4) \in (\omega,\gamma)$, of class C^1, such that $\operatorname{grad} F$ does not vanish identically on any open subset of $M(U_0,U_i,\mathcal{I}c)$, which are partial integrals of the Euler–Poisson equations (1.1) with respect to $M(U_0,U_i,\mathcal{I}c)$.

Let $i = 1$. The unique intrinsic property of a C^1 function F that is a local first integral is that $\operatorname{grad} F$ does not vanish identically on any open subset of its domain of definition, which in this case is equal to $M(U_0,U_1,\mathcal{I}c)$. This implies that some of the partial derivatives of F may be identically zero. Thus the results of Sec. 5.2 also remain valid for functions of at most four variables.

As $\frac{\partial F}{\partial s_1}\frac{ds_1}{dt} + \frac{\partial F}{\partial s_2}\frac{ds_2}{dt} + \frac{\partial F}{\partial s_3}\frac{ds_3}{dt} + \frac{\partial F}{\partial s_4}\frac{ds_4}{dt} = 0$, where $\frac{ds_i}{dt}$, $1 \le i \le 4$, are given by the right-hand sides of Euler–Poisson equations (1.1), the order of the variables s_i, $1 \le i \le 4$, in $F(s_1,s_2,s_3,s_4)$ is irrelevant for F to be a first integral.

We have exactly 15 different four-element subsets of (ω,γ) and thus 15 cases of functions of four elements to examine. We will now describe an extraction procedure based on permutational symmetries which reduces the above 15 cases to only four.

These 15 functions of four variables (up to the order of variables) are shown in the table below.

Table 5.1

Functions	Case
$F(\omega_1,\omega_2,\omega_3,\gamma_i)$, $1 \le i \le 3$	(i)
$F(\omega_1,\omega_3,\gamma_1,\gamma_3)$, $F(\omega_1,\omega_2,\gamma_1,\gamma_2)$, $F(\omega_2,\omega_3,\gamma_2,\gamma_3)$	(ii)
$F(\omega_1,\omega_2,\gamma_1,\gamma_3)$, $F(\omega_1,\omega_3,\gamma_1,\gamma_2)$, $F(\omega_2,\omega_3,\gamma_1,\gamma_2)$, $F(\omega_1,\omega_2,\gamma_2,\gamma_3)$, $F(\omega_1,\omega_3,\gamma_2,\gamma_3)$, $F(\omega_2,\omega_3,\gamma_1,\gamma_3)$	(iii)
$F(\omega_i,\gamma_1,\gamma_2,\gamma_3)$, $1 \le i \le 3$	(iv)

It is easy to see that under the group of permutational symmetries (2.3) of the Euler–Poisson equations for every case (i)–(iv) from Table 5.1 each function from any case can be transformed into any other function from the same case.

Thus in virtue of Theorem 2.2 we can restrict ourselves to the study of only four functions, each belonging to a different case from Table 5.1 and chosen arbitrarily from the functions of this case.

We will call such four functions F_i, $1 \le i \le 4$, (up to the order of variables) a *basis*.

As Table 5.1 shows, the functions

$$F(\omega_1,\omega_2,\omega_3,\gamma_3),\ F(\omega_1,\omega_3,\gamma_1,\gamma_3),\ F(\omega_1,\omega_2,\gamma_1,\gamma_3),\ F(\omega_1,\gamma_1,\gamma_2,\gamma_3) \tag{5.1}$$

form a basis.

Being a local first integral of some vector field, defined in some open subset of some manifold, is an intrinsic property, independent of the system of coordinates used. Thus in $M(U_0,U_i,\mathcal{I}c)$ instead of the coordinates $(\omega_1,\omega_2,\omega_3,\gamma_1,\gamma_2,\gamma_3)$ inherited from the Euler–Poisson equations, we can consider for example the system of coordinates

$(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_3)$, where the coordinate (variable) γ_2 can be eliminated thanks to the identity $H_1 = U_1$. The same concerns all the remaining coordinates.

Using the coordinates $(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_3)$ on $M(U_0, U_i, \mathcal{I}c)$ we can verify if the first three functions of the basis (5.1) can be partial integrals or not. For the last function of (5.1) we will use the coordinates $(\omega_1, \omega_3, \gamma_1, \gamma_2, \gamma_3)$.

The following general remark also concerns Secs. 8 and 9. If we are interested in partial integrals that depend on at most three variables, for instance $F(\omega_2, \omega_3, \gamma_3)$, we can consider it as a particular case of $F(\omega_1, \omega_2, \omega_3, \gamma_3)$ (case (i)), of $F(\omega_2, \omega_3, \gamma_2, \gamma_3)$ (case (ii)) and of $F(\omega_2, \omega_3, \gamma_1, \gamma_3)$ (case (iii)). From the study of each of these functions, we can deduce the existence of the sought partial integral $F(\omega_2, \omega_3, \gamma_3)$.

5.2. Invariant manifold $\{H_1 = U_1\}$. Determination of the Goryachev–Chaplygin case. Here we present our method on the example $\{H_1 = U_1\}$. This invariant manifold not only gives results on non-existence of partial integrals on $U_1 \neq 0$, but when $U_1 = 0$ it also gives a nice derivation of the Goryachev–Chaplygin case.

5.2.1. Elimination of γ_2. Let us express γ_2 from the equation $H_1 = U_1$. We have

$$\gamma_2 = \frac{U_1 - I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3}{I_2\omega_2}. \tag{5.2}$$

We put this in the Euler–Poisson equations (1.1) and remove the fifth equation. In this way we obtain

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{c_3U_1 + I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1c_3\omega_1\gamma_1 - I_2c_2\omega_2\gamma_3 - I_3c_3\omega_3\gamma_3}{I_1I_2\omega_2},\\
\frac{d\omega_2}{dt} &= \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\gamma_3 - c_3\gamma_1}{I_2},\\
\frac{d\omega_3}{dt} &= \frac{-c_1U_1 + I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1c_1\omega_1\gamma_1 + I_2c_2\omega_2\gamma_1 + I_3c_1\omega_3\gamma_3}{I_2I_3\omega_2},\\
\frac{d\gamma_1}{dt} &= \frac{-I_1\omega_1\omega_3\gamma_1 - I_2\omega_2^2\gamma_3 - I_3\omega_3^2\gamma_3 + \omega_3U_1}{I_2\omega_2},\\
\frac{d\gamma_3}{dt} &= \frac{I_1\omega_1^2\gamma_1 + I_2\omega_2^2\gamma_1 + I_3\omega_1\omega_3\gamma_3 - \omega_1U_1}{I_2\omega_2}.
\end{aligned} \tag{5.3}$$

Looking for a first integral of system (5.3) which depends on four variables indicated in brackets, we come to the following five possible cases:

1. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$ (case (i)).
2. $F(\omega_1, \omega_2, \omega_3, \gamma_3)$ (case (i)).
3. $F(\omega_1, \omega_2, \gamma_1, \gamma_3)$ (case (iii)).
4. $F(\omega_1, \omega_3, \gamma_1, \gamma_3)$ (case (ii)).
5. $F(\omega_2, \omega_3, \gamma_1, \gamma_3)$ (case (iii)).

Here “case $(*)$” indicates in which case of Table 5.1 the corresponding partial integral appears.

Functions of types 2, 3 and 4 belong to the basis (5.1). We should study all of them. We start with partial integrals of type 2.

Type 2. Let us look for a first integral of system (5.3) that does not depend on γ_1, i.e. of type 2. Let us suppose that the function

$$F(\omega_1, \omega_2, \omega_3, \gamma_3) \tag{5.4}$$

is such a first integral of (5.3). It satisfies the identity

$$\begin{aligned}
\frac{dF}{dt} &= \frac{c_3U_1 + I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1c_3\omega_1\gamma_1 - I_2c_2\omega_2\gamma_3 - I_3c_3\omega_3\gamma_3}{I_1I_2\omega_2}\,\frac{\partial F}{\partial \omega_1}\\
&\quad + \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\gamma_3 - c_3\gamma_1}{I_2}\,\frac{\partial F}{\partial \omega_2}\\
&\quad + \frac{-c_1U_1 + I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1c_1\omega_1\gamma_1 + I_2c_2\omega_2\gamma_1 + I_3c_1\omega_3\gamma_3}{I_2I_3\omega_2}\,\frac{\partial F}{\partial \omega_3}\\
&\quad + \frac{I_1\omega_1^2\gamma_1 + I_2\omega_2^2\gamma_1 + I_3\omega_1\omega_3\gamma_3 - \omega_1U_1}{I_2\omega_2}\,\frac{\partial F}{\partial \gamma_3} = 0,
\end{aligned}$$

or equivalently

$$I_1I_2I_3\omega_2\frac{dF}{dt} = I_1\gamma_1Y_1(F) + Y_2(F) = 0, \tag{5.5}$$

where Y_1 and Y_2 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_3)$:

$$\begin{aligned}
Y_1 &= -I_3c_3\omega_1\frac{\partial}{\partial \omega_1} - I_3c_3\omega_2\frac{\partial}{\partial \omega_2} + (I_1c_1\omega_1 + I_2c_2\omega_2)\frac{\partial}{\partial \omega_3} + I_3(I_1\omega_1^2 + I_2\omega_2^2)\frac{\partial}{\partial \gamma_3},\\
Y_2 &= I_3[I_2(I_2 - I_3)\omega_2^2\omega_3 - I_2c_2\omega_2\gamma_3 - I_3c_3\omega_3\gamma_3 + c_3U_1]\frac{\partial}{\partial \omega_1}\\
&\quad - I_1I_3\omega_2[(I_1 - I_3)\omega_1\omega_3 - c_1\gamma_3]\frac{\partial}{\partial \omega_2}\\
&\quad - I_1[I_2(I_2 - I_1)\omega_1\omega_2^2 - I_3c_1\omega_3\gamma_3 + c_1U_1]\frac{\partial}{\partial \omega_3} + I_1I_3\omega_1(I_3\omega_3\gamma_3 - U_1)\frac{\partial}{\partial \gamma_3}.
\end{aligned}$$

As (5.5) is an identity with respect to all the variables and as $Y_1(F)$ and $Y_2(F)$ do not depend on γ_1 we have

$$Y_1(F) = Y_2(F) = 0. \tag{5.6}$$

We compute the Lie brackets $Y_3 = [Y_1, Y_2]/I_3$ and $Y_4 = [Y_1, Y_3]$ and obtain

$$\begin{aligned}
Y_3 &= m_{31}\frac{\partial}{\partial \omega_1} + m_{32}\frac{\partial}{\partial \omega_2} + m_{33}\frac{\partial}{\partial \omega_3} + m_{34}\frac{\partial}{\partial \gamma_3},\\
Y_4 &= m_{41}\frac{\partial}{\partial \omega_1} + m_{42}\frac{\partial}{\partial \omega_2} + m_{43}\frac{\partial}{\partial \omega_3} + m_{44}\frac{\partial}{\partial \gamma_3},
\end{aligned}$$

where

$$\begin{aligned}
m_{31} &= -I_1I_2I_3c_2\omega_1^2\omega_2 - I_1I_3^2c_3\omega_1^2\omega_3 - I_2^2I_3c_3\omega_2^2\omega_3 + I_1I_2(I_2 - I_3)c_1\omega_1\omega_2^2\\
&\quad - I_1I_3c_1c_3\omega_1\gamma_3 + I_2^2(I_2 - 2I_3)c_2\omega_2^3 - I_2I_3c_2c_3\omega_2\gamma_3 - I_3^2c_3^2\omega_3\gamma_3 + I_3c_3^2U_1,\\
m_{32} &= I_1\omega_2[I_2I_3c_1\omega_2^2 - I_1(I_1 - 2I_3)c_1\omega_1^2 - I_2(I_1 - I_3)c_2\omega_1\omega_2 + I_3(I_1 - I_3)c_3\omega_1\omega_3]\\
m_{33} &= I_1[I_1I_3c_1\omega_1^2\omega_3 + I_1c_1^2\omega_1\gamma_3 + I_2c_1c_2\omega_2\gamma_3 - I_2(I_2 - 2I_3)c_1\omega_2^2\omega_3\\
&\quad + I_2(I_1 - I_3)c_2\omega_1\omega_2\omega_3 - 3I_2(I_1 - I_2)c_3\omega_1\omega_2^2 + I_3c_1c_3\omega_3\gamma_3 - c_1c_3U_1],\\
m_{34} &= I_1I_3[I_1I_3\omega_1^3\omega_3 + I_1c_1\omega_1^2\gamma_3 - 2I_2c_1\omega_2^2\gamma_3 + 3I_2c_2\omega_1\omega_2\gamma_3\\
&\quad + I_2(2I_1 - 2I_2 + I_3)\omega_1\omega_2^2\omega_3 + I_3c_3\omega_1\omega_3\gamma_3 - c_3U_1\omega_1],
\end{aligned}$$

$$
\begin{aligned}
m_{41} &= I_3c_2[-2I_1^2I_3c_1\omega_1^3 - I_1I_2(3I_2 - I_3)c_1\omega_1\omega_2^2 - I_1I_3c_1c_2\omega_1\gamma_3 - 3I_2^2(I_2 - I_3)c_2\omega_2^3\\
&\qquad - I_2I_3c_2c_2\omega_2\gamma_3 + I_2I_3(I_2 - I_3)c_2\omega_2^2\omega_3 - I_3^2c_2^2\omega_3\gamma_3 + U_1I_3c_2^2],\\
m_{42} &= I_1\omega_2I_3c_3[-2I_2I_3c_1\omega_2^2 + (3I_1 - 5I_3)I_1c_1\omega_1^2\\
&\qquad + 3(I_1 - I_3)I_2c_2\omega_1\omega_2 - (I_1 - I_3)I_3c_3\omega_1\omega_3],\\
m_{43} &= I_1[2I_1^2I_2c_1c_2\omega_1^2\omega_2 + 2I_1^2I_3c_1^2\omega_1^3 + I_1I_3c_1^2c_3\omega_1\gamma_3 - 2I_2^2(I_2 - 2I_3)c_1c_2\omega_2^3\\
&\qquad + I_2I_3c_1c_2c_3\omega_2\gamma_3 + 3I_2I_3(I_2 - I_3)c_1c_3\omega_2^2\omega_3 - 3I_2I_3(I_1 - I_3)c_2c_3\omega_1\omega_2\omega_3\\
&\qquad + I_3^2c_1c_3^2\omega_3\gamma_3 - I_2\omega_1\omega_2^2(2I_1I_2c_1^2 - 2I_1I_2c_2^2 - 4I_1I_3c_1^2 - 9I_1I_3c_3^2 + 2I_2I_3c_2^2\\
&\qquad + 9I_2I_3c_3^2) - I_3c_1c_3^2U_1],\\
m_{44} &= I_1I_3[2I_1^2I_3c_1\omega_1^4 + 6I_1I_2I_3c_2\omega_1^3\omega_2 - 4I_2^2I_3c_1\omega_2^4 + 2I_1I_2(2I_1 - 2I_2 - I_3)c_1\omega_1^2\omega_2^2\\
&\qquad + I_1I_3c_1c_3\omega_1^2\gamma_3 + 2I_2^2(2I_1 - 2I_2 + 3I_3)c_2\omega_1\omega_2^3 + 4I_2I_3c_1c_3\omega_2^2\gamma_3\\
&\qquad - 3I_2I_3c_2c_3\omega_1\omega_2\gamma_3 - 8I_2I_3(I_1 - I_2)c_3\omega_1\omega_2^2\omega_3 + I_3^2c_3^2\omega_1\omega_3\gamma_3 - I_3c_3^2U_1\omega_1].
\end{aligned}
$$

Equations (5.6) imply that

$$
Y_3(F) = Y_4(F) = 0. \tag{5.7}
$$

Equations (5.6) and (5.7) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically on any open subset of the domain of definition of F, because F is non-constant on any such open subset.

If a new integral F exists, system (5.6)–(5.7) has at least one non-zero solution. Let us consider the 4×4 matrix A whose rows are the coefficients of the vector fields Y_1, Y_2, Y_3 and Y_4. The condition under which system (5.6)–(5.7) has at least one non-zero solution is

$$
\operatorname{rank} A \leq 3.
$$

We equate to zero the determinant D of the 4×4 matrix A of the coefficients of system (5.6)–(5.7) and study when the identity

$$
D = \det(A) \equiv 0 \tag{5.8}
$$

is fulfilled. We compute

$$
D = I_1^2I_2^2I_3^2\omega_2^3\widehat{D},
$$

where $\widehat{D}$ is a polynomial in ω_1, ω_2, ω_3 and γ_3 having 72 monomials and thus with 72 coefficients depending on $\mathcal{I}c$ and U_1. It is clear that to solve (5.8) is equivalent to finding all values of the parameters $\mathcal{I}c$ and U_1 for which the 72 coefficients of $\widehat{D}$ are zero. The expression for $\widehat{D}$ is too long to write it here. To solve this system of 72 equations we proceed as described in Sec. 3.

After four consecutive simplifications of the source system of 72 equations we obtain the reduced system having only nine equations:

$$
\begin{aligned}
&(I_2 - I_3)c_2c_3 = 0, \quad (I_1 - I_3)c_2c_3 = 0,\\
&(I_2 - I_3)c_1c_3 = 0, \quad (I_1 - I_3)c_1c_3 = 0, \quad (I_1 - I_2)c_1c_2 = 0,\\
&(I_2 - 4I_3)(I_1 - I_3)c_2 = 0, \quad (I_1 - I_3)(I_1 - 4I_3)c_2 = 0,\\
&(I_2 - I_3)(I_2 - 4I_3)c_1 = 0, \quad (I_2 - I_3)(I_1 - 4I_3)c_1 = 0.
\end{aligned}
$$

We solve these nine equations by the MAPLE command `solve` and obtain five solutions. Two of them lead to the Lagrange case, and one to the kinetic symmetry case. In this way we come to the following two cases that should be studied separately:

1. $I_1 = I_2 = 4I_3$, $c_3 = 0$, and $(c_1, c_2) \neq (0, 0)$ and U_1 are arbitrary,
2. $c_1 = c_2 = 0$, and $c_3 \neq 0$, $I_1 \neq 0$, $I_2 \neq 0$, $I_3 \neq 0$ and U_1 are arbitrary.

Let us study these cases.

Case 1: $I_1 = I_2 = 4I_3$, $c_3 = 0$, $(c_1, c_2) \neq (0, 0)$ and U_1 are arbitrary. Under this condition we have $\widehat{D} = 0$ and therefore the vector fields Y_i, $1 \leq i \leq 4$, are linearly dependent.

Let us denote by D_{ab} the determinant of the 3×3 matrix obtained from the 4×4 matrix A by removing row a and column b. Elementary MAPLE computations show that

$$D_{43} = 768 I_3^7 \omega_2^2 (\omega_1^2 + \omega_2^2)(-c_2\omega_1 + c_1\omega_2)(I_3\omega_1^2\omega_3 + I_3\omega_2^2\omega_3 - c_1\omega_1\gamma_3 - c_2\omega_2\gamma_3)$$

never vanishes identically unless $c_1 = c_2 = 0$, i.e. in the Euler case. Thus the vector fields Y_i, $1 \leq i \leq 3$, are linearly independent on an open dense subset of the space $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_3)$ for every $U_1 \in \mathbb{C}$, in particular for $U_1 = 0$.

We compute the Lie bracket $Y_5 = [Y_2, Y_3]$ and obtain

$$Y_5 = m_{51}\frac{\partial}{\partial \omega_1} + m_{52}\frac{\partial}{\partial \omega_2} + m_{53}\frac{\partial}{\partial \omega_3} + m_{54}\frac{\partial}{\partial \gamma_3},$$

where

$$\begin{aligned}
m_{51} = {} & I_3\omega_2[9I_3c_1\omega_1^2\omega_2\omega_3 - 9I_3c_1\omega_2^3\omega_3 - 4I_3c_2\omega_1^3\omega_3 + 14I_3c_2\omega_1\omega_2^2\omega_3 \\
& + 2c_1c_2\omega_1^2\gamma_3 + c_1c_2\omega_2^2\gamma_3 - (3c_1^2 + 2c_2^2)\omega_1\omega_2\gamma_3], \\
m_{52} = {} & I_3\omega_2[-2I_3c_1\omega_1^3\omega_3 + 16I_3c_1\omega_1\omega_2^2\omega_3 - 15I_3c_2\omega_1^2\omega_2\omega_3 + 3I_3c_2\omega_2^3\omega_3 \\
& - 2c_1^2\omega_1^2\gamma_3 - c_1c_2\omega_1\omega_2\gamma_3 + (-4c_1^2 - 3c_2^2)\omega_2^2\gamma_3], \\
m_{53} = {} & I_3^2c_1\omega_1^3\omega_3^2 - 17I_3^2c_1\omega_1\omega_2^2\omega_3^2 + 9I_3^2c_2\omega_1^2\omega_2\omega_3^2 - 9I_3^2c_2\omega_2^3\omega_3^2 \\
& + 4I_3c_1c_2\omega_1\omega_2\omega_3\gamma_3 + c_1^3\omega_1\gamma_3^2 + c_1^2c_2\omega_2\gamma_3^2 - (c_1^2 - 3c_2^2)I_3\omega_2^2\omega_3\gamma_3 \\
& + 2c_1^2U_1\omega_1^2 - 2c_1^2U_1\omega_2^2 + 4c_1c_2U_1\omega_1\omega_2, \\
m_{54} = {} & I_3[I_3^2\omega_1^4\omega_3^2 - 2I_3^2\omega_1^2\omega_2^2\omega_3^2 - 3I_3^2\omega_2^4\omega_3^2 - 20I_3c_1\omega_1\omega_2^2\omega_3\gamma_3 \\
& + 14I_3c_2\omega_1^2\omega_2\omega_3\gamma_3 - 6I_3c_2\omega_2^3\omega_3\gamma_3 + c_1^2\omega_1^2\gamma_3^2 + 2c_1U_1\omega_1^3 \\
& - 4c_1U_1\omega_1\omega_2^2 + 4c_2U_1\omega_1^2\omega_2 - 2c_2U_1\omega_2^3 + (4c_1^2 + 3c_2^2)\omega_2^2\gamma_3^2].
\end{aligned}$$

Equations (5.6)–(5.7) imply that $Y_5(F) = 0$. In this way we obtain the four equations

$$Y_1(F) = Y_2(F) = Y_3(F) = Y_5(F) = 0. \tag{5.9}$$

If a partial integral F exists, system (5.9) has at least one non-zero solution. We consider the 4×4 matrix B of the coefficients of this system and look for values of the parameters for which

$$\operatorname{rank} B \leq 3. \tag{5.10}$$

We have

$$\det(B) = -3840 I_3^7 \omega_2^4 U_1 (c_2\omega_1 - c_1\omega_2)^3 (I_3\omega_1^2\omega_3 - c_1\omega_1\gamma_3 + I_3\omega_2^2\omega_3 - c_2\omega_2\gamma_3).$$

Thus (5.10) will be fulfilled if and only if $U_1 = 0$, because $(c_1, c_2) \neq (0, 0)$.

Let $U_1 = 0$. Thus (5.10) is fulfilled. As Y_1, Y_2, Y_3 are linearly independent, Y_5 is linearly dependent on them. Moreover, as already mentioned, Y_4 is also linearly dependent on Y_i, $1 \le i \le 3$ (see (5.8)). Thus the equations

$$Y_i(F) = 0, \quad 1 \le i \le 3, \tag{5.11}$$

are in involution. They give a system of three first order linear homogeneous partial differential equations for determining the function F. We note here that the local solvability of system (5.11) around any point $(\omega_1, \omega_2, \omega_3, \gamma_3)$ where vector fields Y_1, Y_2 and Y_3 are linearly independent follows from the Frobenius Integrability Theorem (see [59, 60]). Hence equations (5.11) have, at least locally, a non-trivial solution. We shall now present two ways, (a) and (b), to identify F.

(a) We solve system (5.11) by the MAPLE command `pdsolve`. In this way we obtain the solution

$$F = G[I_3\omega_3(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3], \tag{5.12}$$

where G is an arbitrary smooth function. By direct computations one can verify that the function $L = I_3\omega_3(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3$ that corresponds to $G(x) = x$ is indeed a first integral of system (5.3) under the conditions $I_1 = I_2 = 4I_3$, $c_3 = 0$, $U_1 = 0$. Computation by hand or by MAPLE shows that the time derivative $\frac{\partial L}{\partial t}$ is not identically 0 for the Euler–Poisson equations (1.1). Thus L is a partial integral. H_2, H_3 and L are functionally independent. The Goryachev–Chaplygin case is thus integrable in the sense of Sec. 1.1.

(b) Although the use of the MAPLE command `pdsolve` immediately gives a solution of system (5.11), it is not difficult to solve it by hand starting from the following simple remark.

Let us consider the following linear partial differential equation with constant coefficients:

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = 0, \tag{5.13}$$

where $p \ne 0$, q are constants and $f = f(x, y)$ is a smooth function defined on some open subset of $\mathbb{C}^2$.

The linear change of variables $u = qx - py$, $v = x$ transforms (5.13) into $\frac{\partial\varphi(u,v)}{\partial v} = 0$, where $f(x, y) = f(v, \frac{qv-u}{p}) = \varphi(u, v)$. Equation (5.13) is then transformed into $\frac{\partial\varphi(u,v)}{\partial v} = 0$. The general solution of this equation is $\varphi(u, v) = \Phi(u)$, where Φ is an arbitrary smooth function. Consequently, the general solution of (5.13) is

$$f(x, y) = \Phi(qx - py). \tag{5.14}$$

Let us return to system (5.11). When $I_1 = I_2 = 4I_3$, $c_3 = 0$ and $U_1 = 0$, one has $I_3\omega_2Y_1 = Z$, where

$$Z = (c_1\omega_1 + c_2\omega_2)\frac{\partial}{\partial\omega_3} + I_3(\omega_1^2 + \omega_2^2)\frac{\partial}{\partial\gamma_3}.$$

We have $Y_1(F) = 0$ if and only if $Z(F) = 0$. The equation $Z(F) = 0$ is of type (5.13), with $x = \omega_3$, $y = \gamma_3$, $p = c_1\omega_1 + c_2\omega_2$ and $q = I_3(\omega_1^2 + \omega_2^2)$. Thus by (5.14) the general solution of equation $Z(F) = 0$ is given by (5.12). Now, all the rest is exactly the same as in (a).

Let us stress that in fact we never used the Frobenius theorem. Indeed, the desired partial integral was obtained by direct computation.

Case 2: $c_1 = c_2 = 0$, $c_3 \neq 0$, $I_1 \neq 0$, $I_2 \neq 0$, $I_3 \neq 0$ and U_1 are arbitrary. Now the first integral H_3 is of type (5.4). If a new integral F of this type exists, system (5.6)–(5.7) has at least two non-zero solutions. The condition under which system (5.6)–(5.7) has at least two linearly independent solutions is

$$\operatorname{rank} A \leq 2. \tag{5.15}$$

We compute the determinant D_{44} of the matrix obtained from A by crossing out its last row and last column and obtain

$$D_{44} = -I_1 I_2 I_3^2 (I_1 - I_2) c_3^2 \omega_1 \omega_2^3 \omega_3 [I_1(2I_1 - 3I_3)\omega_1^2 + I_2(2I_2 - 3I_3)\omega_2^2 - 4I_3 c_3 \gamma_3)].$$

Condition (5.15) implies that D_{44} is identically zero. One easily sees that as $c_3 \neq 0$, this is possible only when $I_1 = I_2$, i.e. $\mathcal{I}c \in \mathcal{L}$. Thus a partial integral of the studied type does not exist for system (5.3).

Type 3. Here we look for a first integral of system (5.3) of type 3, $F(\omega_1, \omega_2, \gamma_1, \gamma_3)$, i.e. a first integral that does not depend on ω_3. It satisfies the identity

$$\begin{aligned}
\frac{dF}{dt} = {} & \frac{c_3U_1 + I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1c_3\omega_1\gamma_1 - I_2c_2\omega_2\gamma_3 - I_3c_3\omega_3\gamma_3}{I_1I_2\omega_2}\frac{\partial F}{\partial \omega_1} \\
& + \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\gamma_3 - c_3\gamma_1}{I_2}\frac{\partial F}{\partial \omega_2} \\
& + \frac{-I_1\omega_1\omega_3\gamma_1 - I_2\omega_2^2\gamma_3 - I_3\omega_3^2\gamma_3 + U_1\omega_3}{I_2\omega_2}\frac{\partial F}{\partial \gamma_1} \\
& + \frac{I_1\omega_1^2\gamma_1 + I_2\omega_2^2\gamma_1 + I_3\omega_1\omega_3\gamma_3 - \omega_1U_1}{I_2\omega_2}\frac{\partial F}{\partial \gamma_3} = 0,
\end{aligned}$$

or equivalently

$$I_1I_2\omega_2\frac{dF}{dt} = \omega_3^2Y_1(F) + \omega_3Y_2(F) + Y_3(F) = 0, \tag{5.16}$$

where Y_1, Y_2 and Y_3 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \gamma_1, \gamma_3)$:

$$\begin{aligned}
Y_1 = {} & -I_1I_3\gamma_3\frac{\partial}{\partial \gamma_1}, \\
Y_2 = {} & (\omega_2^2I_2^2 - I_2\omega_2^2I_3 - c_3I_3\gamma_3)\frac{\partial}{\partial \omega_1} - I_1\omega_1\omega_2(-I_3 + I_1)\frac{\partial}{\partial \omega_2} \\
& + (U_1 - I_1\omega_1\gamma_1)I_1\frac{\partial}{\partial \gamma_1} + \omega_1I_3\gamma_3I_1\frac{\partial}{\partial \gamma_3}, \\
Y_3 = {} & (c_3U_1 - c_3I_1\omega_1\gamma_1 - c_2I_2\omega_2\gamma_3)\frac{\partial}{\partial \omega_1} + I_1\omega_2(c_1\gamma_3 - c_3\gamma_1)\frac{\partial}{\partial \omega_2} \\
& - I_1\omega_2^2\gamma_3I_2\frac{\partial}{\partial \gamma_1} + (I_2\omega_2^2\gamma_1 - U_1\omega_1 + I_1\omega_1^2\gamma_1)I_1\frac{\partial}{\partial \gamma_3}.
\end{aligned}$$

As (5.16) is an identity with respect to all the variables and as $Y_1(F)$, $Y_2(F)$ and $Y_3(F)$ do not depend on ω_3, we have

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{5.17}$$

We compute the Lie bracket $Y_4 = [Y_2, Y_3]/I_1$ and obtain

$$\begin{aligned} Y_4 = {} & [I_2^2 c_3 \omega_2^2 \gamma_1 + I_1(I_1 + I_3)c_3\omega_1^2\gamma_1 - 2I_2(I_2 - I_3)c_1\omega_2^2\gamma_3 \\ & + I_2(I_1 - 2I_3)c_2\omega_1\omega_2\gamma_3 + I_3c_3^2\gamma_1\gamma_3 - (I_1 + I_3)U_1c_3\omega_1]\frac{\partial}{\partial\omega_1} \\ & + [I_1I_3c_1\omega_1\omega_2\gamma_3 + I_1I_3c_3\omega_1\omega_2\gamma_1 + I_2(I_3 - I_1)c_2\omega_2^2\gamma_3 - I_3c_3U_1\omega_2]\frac{\partial}{\partial\omega_2} \\ & + [-I_1^2c_3\omega_1\gamma_1^2 - I_1I_2c_2\omega_2\gamma_1\gamma_3 + I_1I_2(I_1 - 3I_3)\omega_1\omega_2^2\gamma_3 + I_1c_3U_1\gamma_1]\frac{\partial}{\partial\gamma_1} \\ & - [(I_1 + I_3)I_1^2\omega_1^3\gamma_1 + (3I_1 - 2I_2 + I_3)I_1I_2\omega_1\omega_2^2\gamma_1 + I_1I_3c_3\omega_1\gamma_1\gamma_3 \\ & - I_2I_3c_2\omega_2\gamma_3^2 - I_1(I_1 + I_3)U_1\omega_1^2 - I_2(I_1 - I_2 + I_3)U_1\omega_2^2]\frac{\partial}{\partial\gamma_3}. \end{aligned}$$

Equations (5.17) imply that

$$Y_4(F) = 0. \tag{5.18}$$

Equations (5.17) and (5.18) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_1}, \frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\gamma_1}, \frac{\partial F}{\partial\gamma_3}\right)$, which do not vanish identically on any open subset of the domain of definition of F, because F is non-constant on any such open subset.

If a new integral F exists then system (5.17)–(5.18) has at least one non-zero solution. Let us consider the 4×4 matrix A whose rows are the coefficients of the vector fields Y_1, Y_2, Y_3 and Y_4. We know that the condition under which system (5.17)–(5.18) has at least one non-zero solution is

$$D = \det(A) \equiv 0. \tag{5.19}$$

We compute

$$D = I_1^2 I_2^2 I_3 \omega_2^2 \gamma_3 \widehat{D}.$$

The expression for $\widehat{D}$ is long and we do not show it here. It is a polynomial in ω_1, ω_2, γ_1 and γ_3 having 26 monomials and thus with 26 coefficients depending on $\mathcal{I}c$ and U_1. It is clear that solving (5.19) is equivalent to finding all values of the parameters $\mathcal{I}c$ and U_1 for which the 26 coefficients of $\widehat{D}$ are zero. To solve this system of 26 equations we proceed as described in Sec. 3.

After three consecutive simplifications of the source system we obtain the reduced system consisting of the following five equations:

$$c_2c_3 = 0, \quad c_1c_3 = 0, \quad (I_1 - I_2)c_3 = 0, \quad (I_1 - I_3)c_2 = 0, \quad (I_2 - I_3)c_1 = 0.$$

We solve these five equations by the MAPLE command `solve` and obtain five solutions. Three of them give the Lagrange case, one the Euler case, and one the kinetic symmetry case.

Thus a partial integral of type 3 does not exist.

Type 4. Now let us study the existence of a first integral of system (5.3) of type 4, i.e. $F(\omega_1, \omega_3, \gamma_1, \gamma_3)$. We have the identity

$$\begin{aligned}\frac{dF}{dt} &= \frac{c_3U_1 + I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1c_3\omega_1\gamma_1 - I_2c_2\omega_2\gamma_3 - I_3c_3\omega_3\gamma_3}{I_1I_2\omega_2}\,\frac{\partial F}{\partial \omega_1}\\ &+ \frac{I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1c_1\omega_1\gamma_1 + I_2c_2\omega_2\gamma_1 + I_3c_1\omega_3\gamma_3 - c_1U_1}{I_2I_3\omega_2}\,\frac{\partial F}{\partial \omega_3}\\ &+ \frac{-I_1\omega_1\omega_3\gamma_1 - I_2\omega_2^2\gamma_3 - I_3\omega_3^2\gamma_3 + U_1\omega_3}{I_2\omega_2}\,\frac{\partial F}{\partial \gamma_1}\\ &+ \frac{I_1\omega_1^2\gamma_1 + I_2\omega_2^2\gamma_1 + I_3\omega_1\omega_3\gamma_3 - \omega_1U_1}{I_2\omega_2}\,\frac{\partial F}{\partial \gamma_3} = 0,\end{aligned}$$

or equivalently

$$I_1I_2I_3\omega_2\frac{dF}{dt} = I_2\omega_2^2Y_1(F) + I_2\omega_2Y_2(F) + (U_1 - I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3)Y_3(F) = 0, \tag{5.20}$$

where Y_1, Y_2 and Y_3 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_3, \gamma_1, \gamma_3)$:

$$\begin{aligned}Y_1 &= \omega_3I_3(I_2 - I_3)\frac{\partial}{\partial\omega_1} + I_1\omega_1(I_1 - I_2)\frac{\partial}{\partial\omega_3} - \gamma_3I_1I_3\frac{\partial}{\partial\gamma_1} + \gamma_1I_1I_3\frac{\partial}{\partial\gamma_3},\\ Y_2 &= c_2\left(-I_3\gamma_3\frac{\partial}{\partial\omega_1} + I_1\gamma_1\frac{\partial}{\partial\omega_3}\right),\\ Y_3 &= I_3c_3\frac{\partial}{\partial\omega_1} - I_1c_1\frac{\partial}{\partial\omega_3} + I_1I_3\omega_3\frac{\partial}{\partial\gamma_1} - I_1I_3\omega_1\frac{\partial}{\partial\gamma_3}.\end{aligned}$$

As (5.20) is an identity with respect to all the variables and as $Y_1(F)$, $Y_2(F)$ and $Y_3(F)$ do not depend on ω_2, we have

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{5.21}$$

We compute the Lie bracket $Y_4 = [Y_1, Y_2]/(I_1I_2I_3)$ and obtain

$$Y_4 = -c_2\left(\gamma_1\frac{\partial}{\partial\omega_1} + \gamma_3\frac{\partial}{\partial\omega_3}\right).$$

Equations (5.21) imply that

$$Y_4(F) = 0. \tag{5.22}$$

Equations (5.21) and (5.22) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_1}, \frac{\partial F}{\partial\omega_3}, \frac{\partial F}{\partial\gamma_1}, \frac{\partial F}{\partial\gamma_3}\right)$, which do not vanish identically on any open subset of the domain of definition of F, because F is non-constant on any such open subset.

If a new integral F exists then system (5.21)–(5.22) has at least one non-zero solution. Let us consider the 4×4 matrix A whose rows are the coefficients of the vector fields Y_1, Y_2, Y_3 and Y_4. We know that the condition under which system (5.21)–(5.22) has at least one non-zero solution is

$$D = \det(A) \equiv 0.$$

We compute

$$D = -I_1^2I_3^2c_2^2\gamma_3\omega_1(I_1\gamma_1^2 + I_3\gamma_3^2).$$

This determinant is not zero if $c_2 \neq 0$. Thus in this case a first integral cannot exist. We should consider the case $c_2 = 0$.

Therefore let $c_2 = 0$. We compute the Lie brackets $Y_5 = [Y_1, Y_3]/(I_1 I_3)$ and $Y_6 = [Y_1, Y_5]$. We have

$$\begin{aligned} Y_5 &= (I_2 - I_3)c_1 \frac{\partial}{\partial \omega_1} - (I_1 - I_2)c_3 \frac{\partial}{\partial \omega_3} \\ &\quad + I_1(I_1 - I_2 - I_3)\omega_1 \frac{\partial}{\partial \gamma_1} - I_3(I_1 + I_2 - I_3)\omega_3 \frac{\partial}{\partial \gamma_3}, \\ Y_6 &= I_3(I_2 - I_3)(I_1 - I_2)c_3 \frac{\partial}{\partial \omega_1} - I_1(I_1 - I_2)(I_2 - I_3)c_1 \frac{\partial}{\partial \omega_3} \\ &\quad - I_1 I_3(I_2^2 - I_1 I_2 + 2I_1 I_3 + I_2 I_3 - 2I_3^2)\omega_3 \frac{\partial}{\partial \gamma_1} \\ &\quad - I_1 I_3(2I_1^2 - I_1 I_2 - 2I_1 I_3 - I_2^2 + I_2 I_3)\omega_1 \frac{\partial}{\partial \gamma_3}. \end{aligned}$$

We consider the system

$$Y_1(F) = 0, \quad Y_3(F) = 0, \quad Y_5(F) = 0, \quad Y_6(F) = 0.$$

Its determinant δ should be zero. We compute

$$\delta = I_1^2 I_3^2 \widehat{\delta},$$

where

$$\begin{aligned} \widehat{\delta} &= I_1^2(I_1 - I_2)(I_1 - I_2 - I_3)(2I_1 - 2I_2 - I_3)c_3\omega_1^3 \\ &\quad - I_1 I_3(I_2 - I_3)(3I_1^2 - I_1 I_2 - 3I_1 I_3 - 2I_2^2 + 2I_2 I_3)c_1\omega_1^2\omega_3 \\ &\quad - I_1 I_3(I_1 - I_2)(2I_1 I_2 - 3I_1 I_3 - 2I_2^2 - I_2 I_3 + 3I_3^2)c_3\omega_1\omega_3^2 \\ &\quad + I_1(2I_1 - 2I_2 - I_3)(I_1 I_2 c_1^2 - I_1 I_3 c_1^2 - I_1 I_3 c_3^2 + I_2 I_3 c_3^2)\omega_1\gamma_3 \\ &\quad + I_3^2(I_2 - I_3)(I_1 + I_2 - I_3)(I_1 + 2I_2 - 2I_3)c_1\omega_3^3 \\ &\quad + I_3(I_1 + 2I_2 - 2I_3)(I_1 I_2 c_1^2 - I_1 I_3 c_1^2 - I_1 I_3 c_3^2 + I_2 I_3 c_3^2)\omega_3\gamma_1. \end{aligned}$$

It is clear that the equation $\delta = 0$ is equivalent to $\widehat{\delta} = 0$. As is seen from the expression for $\widehat{\delta}$, it is a polynomial in the variables ω_1, ω_3, γ_1 and γ_3 having six monomials and thus with six coefficients depending on $\mathcal{I}c$. Thus we should solve a system of six equations with respect to the parameters $\mathcal{I}c$. To solve this system we apply a simplification. After four consecutive simplifications we obtain the reduced system consisting of the following five equations:

$$\begin{aligned} &(I_1 - I_3)c_1c_3 = 0, \quad (I_1 - I_2)(2I_2 - I_3)c_3 = 0, \quad (2I_1 + 2I_2 - 3I_3)(I_1 - I_2)c_3 = 0, \\ &(I_2 - I_3)(2I_2 - I_3)c_1 = 0, \quad (I_1 - I_3)(I_2 - I_3)c_1 = 0. \end{aligned}$$

We solve these five equations by the MAPLE command `solve` and obtain the following six solutions:

$$\begin{aligned} &\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_3 = 0\}, \\ &\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_3 = c_3\}, \\ &\{I_1 = 2I_2,\ I_2 = I_2,\ I_3 = 2I_2,\ c_1 = 0,\ c_3 = c_3\}, \\ &\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_3 = 0\}, \\ &\{I_1 = 2I_2,\ I_2 = I_2,\ I_3 = 2I_2,\ c_1 = c_1,\ c_3 = c_3\}, \\ &\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_3 = c_3\}. \end{aligned}$$

Taking into account that we consider now the case $c_2 = 0$ we see that the first solution leads to the Euler case, and the second and fourth ones to the Lagrange case. The third and fifth solutions give the Kovalevskaya case, and the last one the kinetic symmetry case.

Thus a first integral of type 4 does not exist.

5.2.2. Elimination of $\boldsymbol{\omega_2}$. Let us express ω_2 from the equation $H_1 = U_1$. We have

$$\omega_2 = \frac{U_1 - I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3}{I_2\gamma_2}. \tag{5.23}$$

We put the expression for ω_2 from (5.23) in the Euler–Poisson equations (1.1) and remove the second equation. In this way we obtain

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{(I_2 - I_3)\omega_3[-I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3 + U_1] + c_3 I_2\gamma_2^2 - c_2 I_2\gamma_2\gamma_3}{I_1 I_2\gamma_2},\\
\frac{d\omega_3}{dt} &= \frac{(I_1 - I_2)\omega_1[-I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3 + U_1] + c_2 I_2\gamma_1\gamma_2 - c_1 I_2\gamma_2^2}{I_2 I_3\gamma_2},\\
\frac{d\gamma_1}{dt} &= \frac{I_1\omega_1\gamma_1\gamma_3 + I_2\omega_3\gamma_2^2 + I_3\omega_3\gamma_3^2 - U_1\gamma_3}{I_2\gamma_2},\\
\frac{d\gamma_2}{dt} &= \omega_1\gamma_3 - \omega_3\gamma_1,\\
\frac{d\gamma_3}{dt} &= \frac{-I_1\omega_1\gamma_1^2 - I_2\omega_1\gamma_2^2 - I_3\omega_3\gamma_1\gamma_3 + U_1\gamma_1}{I_2\gamma_2}.
\end{aligned} \tag{5.24}$$

Looking for a first integral of system (5.24) which depends on four variables indicated in brackets, we come to the following five possible cases:

1. $F(\omega_1, \omega_3, \gamma_1, \gamma_2)$ (case (iii)).
2. $F(\omega_1, \omega_3, \gamma_1, \gamma_3)$ (case (ii)).
3. $F(\omega_1, \omega_3, \gamma_2, \gamma_3)$ (case (iii)).
4. $F(\omega_1, \gamma_1, \gamma_2, \gamma_3)$ (case (iv)).
5. $F(\omega_3, \gamma_1, \gamma_2, \gamma_3)$ (case (iv)).

In Sec. 5.2.1 we have already studied cases (i), (ii) and (iii) from Table 5.1. It remains to consider case (iv). Functions of types 4 and 5 belong to this case. We should examine one of these two first integrals, no matter which, because their study is exactly of the same nature. We choose type 4.

Type 4. Let us study the existence of a first integral of system (5.24) of type 4, i.e. $F(\omega_1, \gamma_1, \gamma_2, \gamma_3)$. We have

$$\begin{aligned}
\frac{dF}{dt} = {}& \frac{(I_2 - I_3)\omega_3[-I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3 + U_1] + c_3 I_2\gamma_2^2 - c_2 I_2\gamma_2\gamma_3}{I_1 I_2\gamma_2}\,\frac{\partial F}{\partial\omega_1}\\
&+ \frac{I_1\omega_1\gamma_1\gamma_3 + I_2\omega_3\gamma_2^2 + I_3\omega_3\gamma_3^2 - U_1\gamma_3}{I_2\gamma_2}\,\frac{\partial F}{\partial\gamma_1} + (\omega_1\gamma_3 - \omega_3\gamma_1)\,\frac{\partial F}{\partial\gamma_2}\\
&+ \frac{-I_1\omega_1\gamma_1^2 - I_2\omega_1\gamma_2^2 - I_3\omega_3\gamma_1\gamma_3 + U_1\gamma_1}{I_2\gamma_2}\,\frac{\partial F}{\partial\gamma_3} = 0,
\end{aligned}$$

or equivalently

$$I_1 I_2 \gamma_2 \frac{dF}{dt} = \omega_3^2 Y_1(F) + \omega_3 Y_2(F) + Y_3(F) = 0, \tag{5.25}$$

where Y_1, Y_2 and Y_3 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \gamma_1, \gamma_2, \gamma_3)$:

$$\begin{aligned}
Y_1 &= -I_3(I_2 - I_3)\gamma_3 \frac{\partial}{\partial \omega_1},\\
Y_2 &= (I_2 U_1 - I_2 I_1 \omega_1 \gamma_1 + I_3 I_1 \omega_1 \gamma_1 - I_3 U_1)\frac{\partial}{\partial \omega_1} + I_1(I_2\gamma_2^2 + I_3\gamma_3^2)\frac{\partial}{\partial \gamma_1}\\
&\quad - \gamma_1 I_2 \gamma_2 I_1 \frac{\partial}{\partial \gamma_2} - I_3 \gamma_1 \gamma_3 I_1 \frac{\partial}{\partial \gamma_3},\\
Y_3 &= -I_2\gamma_2(-c_3\gamma_2 + c_2\gamma_3)\frac{\partial}{\partial \omega_1} - \gamma_3 I_1(-I_1\omega_1\gamma_1 + U_1)\frac{\partial}{\partial \gamma_1}\\
&\quad + I_2\gamma_2 I_1 \omega_1 \gamma_3 \frac{\partial}{\partial \gamma_2} + (-I_1\omega_1\gamma_1^2 - I_2\omega_1\gamma_2^2 + U_1\gamma_1)I_1 \frac{\partial}{\partial \gamma_3}
\end{aligned}$$

As (5.25) is an identity with respect to all the variables and as $Y_1(F)$, $Y_2(F)$ and $Y_3(F)$ do not depend on ω_3, we have

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{5.26}$$

Equations (5.26) can be considered as a system of three homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_2}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically on any open subset of the domain of definition of F, because F is non-constant on any such subset.

It is clear that the first integral H_2 is of type 4 and therefore $\operatorname{grad} H_2$ is a solution of system (5.26). If a new integral F exists then system (5.26) has at least two non-zero solutions. This is possible if and only if

$$\operatorname{rank} A \le 2, \tag{5.27}$$

where A is the 3×4 matrix whose rows are the coefficients of the vector fields Y_1, Y_2 and Y_3.

Let us consider the 3×3 matrix A_{123} obtained from A by crossing out its last column. A necessary condition for the fulfillment of (5.27) is

$$D_{123} = \det(A_{123}) = 0.$$

We compute

$$D_{123} = I_1^2 I_2 I_3 (I_2 - I_3)\gamma_2\gamma_3^2(-I_1\omega_1\gamma_1^2 - I_2\omega_1\gamma_2^2 - I_3\omega_1\gamma_3^2 + U_1\gamma_1).$$

It is easily seen that $D_{123} = 0$ is possible if and only if $I_2 = I_3$. Under this last condition we compute the Lie bracket $Y_4 = [Y_2, Y_3]/(I_1 I_3)$ and obtain

$$\begin{aligned}
Y_4 &= 2I_3(-c_3\gamma_2 + c_2\gamma_3)\gamma_1\gamma_2 \frac{\partial}{\partial \omega_1} + I_1\gamma_3[I_1\omega_1(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) - U_1\gamma_1]\frac{\partial}{\partial \gamma_1}\\
&\quad + I_1\gamma_2\gamma_3[\omega_1\gamma_1(I_1 - I_3) - U_1]\frac{\partial}{\partial \gamma_2}\\
&\quad + I_1[(I_3 - 2I_1)\omega_1\gamma_1\gamma_2^2 - I_1\omega_1\gamma_1(\gamma_1^2 + \gamma_3^2) + U_1(\gamma_1^2 + \gamma_2^2)]\frac{\partial}{\partial \gamma_3}.
\end{aligned}$$

Now $Y_1 = 0$ and we consider the following system:

$$Y_2(F) = Y_3(F) = Y_4(F) = 0.$$

For the same reason as above we should require that

$$\operatorname{rank} B \leq 2, \tag{5.28}$$

where B is the 3×4 matrix whose rows are the coefficients of Y_2, Y_3 and Y_4.

We consider the 3×3 matrix B_{123} obtained from B by crossing out its last column. Condition (5.28) implies

$$\widehat{D}_{123} = \det(B_{123}) = 0. \tag{5.29}$$

We compute

$$\widehat{D}_{123} = I_1^2 I_3^2 (c_3\gamma_2 - c_2\gamma_3)\gamma_2^2\gamma_3[-3I_1\omega_1\gamma_1^3 - (2I_1 + I_3)\omega_1\gamma_1(\gamma_2^2 + \gamma_3^2) + U_1(3\gamma_1^2 + \gamma_2^2 + \gamma_3^2)].$$

One immediately sees that the condition (5.29) leads to $c_2 = c_3 = 0$, which together with $I_2 = I_3$ leads to the Lagrange case. Thus a partial integral of type 4 does not exist.

The results from Secs. 5.2.1 and 5.2.2 show that we have completely studied all the four cases of the basis (5.1). Now from Theorem 2.2 we conclude that apart from the Goryachev–Chaplygin case ($I_1 = I_2 = 4I_3$, $(c_1, c_2) \neq (0, 0)$, $c_3 = 0$ or $I_1 = I_3 = 4I_2$, $(c_1, c_3) \neq (0, 0)$, $c_2 = 0$ or $I_2 = I_3 = 4I_1$, $(c_2, c_3) \neq (0, 0)$, $c_1 = 0$), the Euler–Poisson equations restricted to the invariant manifold $\{H_1 = U_1\}$ never have a partial integral depending on at most four variables.

5.3. Invariant manifold $\{H_2 = 0\}$. We will now study what happens on the submanifold $\{H_2 = 0\}$. Here we proceed as in Sec. 5.2. We should stress the easily seen but important fact that now a first integral belonging to case (iv) from Table 5.1 does not exist because all possible eliminations from the equation $H_2 = 0$ are eliminations of some γ_i, $1 \leq i \leq 3$. We consider here the elimination of γ_3. The completely analogous results concerning the elimination of γ_1 or γ_2 follow from Theorem 2.2. But they can also be obtained in exactly the same way as the elimination of γ_3 that we describe below.

Let us express γ_3 from the equation $H_2 = 0$. We obtain

$$\gamma_3 = \sqrt{-\gamma_1^2 - \gamma_2^2}. \tag{5.30}$$

γ_3 is now considered as an algebraic function (see Sec. 4) of (γ_1, γ_2).

Putting the expression for γ_3 from (5.30) in the Euler–Poisson equations (1.1) and removing the sixth equation we get

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{(I_2 - I_3)\omega_2\omega_3 + c_3\gamma_2 - c_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1},\\
\frac{d\omega_2}{dt} &= \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2},\\
\frac{d\omega_3}{dt} &= \frac{(I_1 - I_2)\omega_1\omega_2 + c_2\gamma_1 - c_1\gamma_2}{I_3},\\
\frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2},\\
\frac{d\gamma_2}{dt} &= \omega_1\sqrt{-\gamma_1^2 - \gamma_2^2} - \omega_3\gamma_1.
\end{aligned} \tag{5.31}$$

Looking for a first integral of system (5.31) which depends on at most four variables we come to the following five possible cases:

1. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$ (case (i)).
2. $F(\omega_1, \omega_2, \omega_3, \gamma_2)$ (case (i)).
3. $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$ (case (ii)).
4. $F(\omega_1, \omega_3, \gamma_1, \gamma_2)$ (case (iii)).
5. $F(\omega_2, \omega_3, \gamma_1, \gamma_2)$ (case (iii)).

It suffices to examine functions of types 1, 3 and 5.

Type 1. Here we use the idea from [67] applied for the proof of Theorem 1.1.B there. Let us look for a first integral of system (5.31) that is of type 1, $F(\omega_1, \omega_2, \omega_3, \gamma_1)$, i.e. which does not depend on γ_2. Thus F satisfies the following identity:

$$\begin{aligned}\frac{dF}{dt} = {} & \frac{(I_2 - I_3)\omega_2\omega_3 + c_3\gamma_2 - c_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1}\frac{\partial F}{\partial \omega_1} \\ & + \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\frac{\partial F}{\partial \omega_2} \\ & + \frac{(I_1 - I_2)\omega_1\omega_2 + c_2\gamma_1 - c_1\gamma_2}{I_3}\frac{\partial F}{\partial \omega_3} + \left(\omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}\right)\frac{\partial F}{\partial \gamma_1} = 0,\end{aligned}$$

or equivalently

$$\frac{dF}{dt} = \gamma_2 Y_1(F) + \sqrt{-\gamma_1^2 - \gamma_2^2}\, Y_2(F) + Y_3(F) = 0, \tag{5.32}$$

where Y_1, Y_2 and Y_3 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$:

$$\begin{aligned} Y_1 &= \frac{c_3}{I_1}\frac{\partial}{\partial \omega_1} - \frac{c_1}{I_3}\frac{\partial}{\partial \omega_3} + \omega_3\frac{\partial}{\partial \gamma_1}, \\ Y_2 &= -\frac{c_2}{I_1}\frac{\partial}{\partial \omega_1} + \frac{c_1}{I_2}\frac{\partial}{\partial \omega_2} - \omega_2\frac{\partial}{\partial \gamma_1}, \\ Y_3 &= \frac{(I_2 - I_3)\omega_2\omega_3}{I_1}\frac{\partial}{\partial \omega_1} + \frac{(I_3 - I_1)\omega_1\omega_3 - c_3\gamma_1}{I_2}\frac{\partial}{\partial \omega_2} + \frac{(I_1 - I_2)\omega_1\omega_2 + c_2\gamma_1}{I_3}\frac{\partial}{\partial \omega_3}. \end{aligned} \tag{5.33}$$

Let us write (5.32) in the following way:

$$\gamma_2 Y_1(F) + Y_3(F) = -\sqrt{-\gamma_1^2 - \gamma_2^2}\, Y_2(F).$$

Squaring the last equation we obtain

$$\gamma_2^2[Y_1(F)^2 + Y_2(F)^2] + 2\gamma_2 Y_1(F)Y_3(F) + \gamma_1^2 Y_2(F)^2 + Y_3(F)^2 = 0, \tag{5.34}$$

where $Y_1(F)$, $Y_2(F)$ and $Y_3(F)$ depend only on $(\omega_1, \omega_2, \omega_3, \gamma_1)$.

As (5.32) is an identity with respect to all the variables ω_1, ω_2, ω_3, γ_1 and γ_2, the same concerns (5.34). Moreover, (5.34) is a polynomial with respect to γ_2 because the coefficients of the powers of γ_2 do not depend on γ_2.

Let us fix ω_1, ω_2, ω_3 and $\gamma_1 \neq 0$. We prove that

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{5.35}$$

For this purpose, we examine the polynomial (5.34), studying two cases separately.

Case (A): The first two coefficients of (5.34) vanish. That means that

$$Y_1(F)^2 + Y_2(F)^2 = 0, \quad Y_1(F)Y_3(F) = 0. \tag{5.36}$$

Thus either $Y_1(F) = 0$ or $Y_3(F) = 0$. If $Y_1(F) = 0$, then from the first equation of (5.36) one obtains $Y_2(F) = 0$ and thus from (5.34), $Y_3(F) = 0$. If $Y_3(F) = 0$, then from (5.34) one has $\gamma_1^2 Y_2(F)^2 = 0$. As $\gamma_1^2 \neq 0$, we get $Y_2(F) = 0$ and thus also $Y_1(F) = 0$. Thus in case (A), (5.35) holds.

Case (B): At least one of the first two coefficients of (5.34) is non-vanishing. In this case (5.34) is a first or second order non-zero polynomial in γ_2. For fixed (ω, γ_1) such a polynomial admits at most two roots. But this contradicts the fact that for these (ω, γ_1), (5.34) is identically satisfied for all γ_2.

This proves that Case (B) cannot occur and consequently (5.34) implies (5.35).

Let us compute the Lie bracket $Y_4 = [Y_2, Y_3]$. We obtain

$$\begin{aligned} Y_4 = {} & \frac{(I_2 - I_3)c_1\omega_3}{I_1 I_2} \frac{\partial}{\partial \omega_1} + \frac{(I_1 - I_3)c_2\omega_3 + I_1 c_3 \omega_2}{I_1 I_2} \frac{\partial}{\partial \omega_2} \\ & + \frac{I_1(I_1 - I_2)c_1\omega_1 + I_2(I_2 - 2I_1)c_2\omega_2}{I_1 I_2 I_3} \frac{\partial}{\partial \omega_3} + \frac{(I_3 - I_1)\omega_1\omega_3 - c_3\gamma_1}{I_2} \frac{\partial}{\partial \gamma_1}. \end{aligned}$$

Equations (5.35) imply that $Y_4(F) = 0$ so that we have

$$Y_1(F) = Y_2(F) = Y_3(F) = Y_4(F) = 0. \tag{5.37}$$

Equations (5.37) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_1}\right)$, which do not vanish identically.

As in Sec. 5.2, we should equate to zero the determinant D of the 4×4 matrix A of the coefficients of system (5.37). We compute

$$\begin{aligned} D = {} & \frac{1}{I_1^2 I_2^2 I_3} [I_1 I_2 (I_1 - I_2) c_3^2 \omega_1 \omega_2^3 - I_1 I_2 (2I_1 - I_2 - I_3) c_2 c_3 \omega_1 \omega_2^2 \omega_3 \\ & + I_1 I_2 (I_1 - I_3) c_2^2 \omega_1 \omega_2 \omega_3^2 + I_1 I_2 (I_2 - I_3) c_1 c_3 \omega_2^3 \omega_3 \\ & + I_1 (I_1 - I_3)(I_2 - I_3) c_1 c_2 \omega_1^2 \omega_3^2 + I_1 (I_2 - I_3) c_1 c_2 c_3 \omega_1 \omega_3 \gamma_1 \\ & - I_2 (I_2 - I_3)(I_1 - I_2 + I_3) c_1 c_2 \omega_2^2 \omega_3^2 - I_2 (I_1 - I_2) c_2 c_3^2 \omega_2^2 \gamma_1 \\ & + I_2 (2I_1 - I_2 - I_3) c_2^2 c_3 \omega_2 \omega_3 \gamma_1 - (I_1 I_2 c_1^2 + I_1 I_2 c_2^2 - I_1 I_3 c_1^2 - I_2 I_3 c_2^2) c_2 \omega_3^2 \gamma_1]. \end{aligned}$$

It is identically equal to zero and therefore all of its coefficients should be zeros. D is a polynomial in ω_1, ω_2, ω_3 and γ_1 having ten monomials and thus with ten coefficients depending on $\mathcal{I}c$. It is clear that solving the equation $D = 0$ is equivalent to finding all values of the parameters $\mathcal{I}c$ for which the ten coefficients of D are zero. To solve this system of ten equations we proceed as in Sec. 5.2.

After three consecutive simplifications of the source system we obtain the reduced system having five equations:

$$\begin{aligned} & (I_1 - I_2)c_3 = 0, \quad (I_1 - I_3)c_2 = 0, \quad (I_2 - I_3)c_2 c_3 = 0, \\ & (I_2 - I_3)c_1 c_3 = 0, \quad (I_2 - I_3)c_1 c_2 = 0. \end{aligned}$$

We solve these five equations by the MAPLE command `solve` and obtain the following four solutions:

- $I_1 = I_2$, $c_1 = c_2 = 0$,
- $I_1 = I_3$, $c_1 = c_3 = 0$,
- $I_1 = I_2 = I_3$,
- $c_2 = c_3 = 0$.

The first three of them lead to the Lagrange and kinetic symmetry cases. We should study only the fourth solution.

Let $c_2 = c_3 = 0$. In this case Y_4 depends on Y_1, Y_2 and Y_3 and system (5.35) has a solution $\operatorname{grad} H_3$. However, H_3 is not a fourth integral. Thus, if a fourth integral F exists, system (5.35) has at least two linearly independent solutions. We consider the 3×4 matrix A of the coefficients of this system. It is clear that our problem has a solution if and only if

$$\operatorname{rank} A \leq 2. \tag{5.38}$$

Now we are going to study when (5.38) is fulfilled. For this purpose we calculate all possible determinants of order three which can be obtained from the matrix A. For $1 \leq i \leq 4$ we denote by D_i the determinant of the matrix obtained from A by crossing out its ith column. We have

$$D_1 = -\frac{(I_2 - I_3)c_1}{I_2 I_3}\omega_1\omega_2\omega_3, \qquad D_2 = \frac{(I_2 - I_3)c_1}{I_1 I_3}\omega_2^2\omega_3,$$
$$D_3 = -\frac{(I_2 - I_3)c_1}{I_1 I_2}\omega_2\omega_3^2, \qquad D_4 = \frac{(I_2 - I_3)c_1^2}{I_1 I_2 I_3}\omega_2\omega_3.$$

It is easy to see that the equations $D_i = 0$, $1 \leq i \leq 4$, are satisfied only if either $c_1 = 0$, which together with $c_2 = c_3 = 0$ leads to the Euler case, or $I_2 = I_3$, which leads to the Lagrange case.

Thus a partial integral of type 1, i.e. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$, does not exist.

Type 3. Let us look for a first integral of system (5.31) of type 3, $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$, i.e. one which does not depend on ω_3. Thus F satisfies the following identity:

$$\begin{aligned}\frac{dF}{dt} = {} & \frac{(I_2 - I_3)\omega_2\omega_3 - c_2\sqrt{-\gamma_1^2 - \gamma_2^2} + c_3\gamma_2}{I_1}\,\frac{\partial F}{\partial \omega_1} \\ & + \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\,\frac{\partial F}{\partial \omega_3} \\ & + (\omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2})\frac{\partial F}{\partial \gamma_1} + (\omega_1\sqrt{-\gamma_1^2 - \gamma_2^2} - \omega_3\gamma_1)\frac{\partial F}{\partial \gamma_2} = 0,\end{aligned}$$

which can be presented in the following way:

$$\frac{dF}{dt} = \omega_3 Y_1(F) + Y_2(F) = 0, \tag{5.39}$$

where Y_1 and Y_2 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \gamma_1, \gamma_2)$:

$$\begin{aligned} Y_1 &= \frac{(I_2 - I_3)\omega_2}{I_1}\,\frac{\partial}{\partial \omega_1} + \frac{(I_3 - I_1)\omega_1}{I_2}\,\frac{\partial}{\partial \omega_2} + \gamma_2\frac{\partial}{\partial \gamma_1} - \gamma_1\frac{\partial}{\partial \gamma_2}, \\ Y_2 &= -\frac{c_2\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_2}{I_1}\,\frac{\partial}{\partial \omega_1} + \frac{c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\,\frac{\partial}{\partial \omega_2} \\ & \quad - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}\frac{\partial}{\partial \gamma_1} + \omega_1\sqrt{-\gamma_1^2 - \gamma_2^2}\frac{\partial}{\partial \gamma_2}. \end{aligned}$$

As (5.39) is an identity with respect to all the variables and as $Y_1(F)$ and $Y_2(F)$ do not depend on ω_3 we have

$$Y_1(F) = Y_2(F) = 0. \tag{5.40}$$

We compute the Lie brackets $Y_3 = [Y_1, Y_2]$ and $Y_4 = [Y_1, Y_3]$ and obtain

$$\begin{aligned} Y_3 = {} & \frac{(I_3 - I_2)c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - I_3c_3\gamma_1}{I_1I_2}\,\frac{\partial}{\partial\omega_1} - \frac{(I_1 - I_3)c_2\sqrt{-\gamma_1^2 - \gamma_2^2} + I_3c_3\gamma_2}{I_1I_2}\,\frac{\partial}{\partial\omega_2} \\ & + \frac{(I_1 - I_2 - I_3)\omega_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2}\,\frac{\partial}{\partial\gamma_1} - \frac{(I_1 - I_2 + I_3)\omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1}\,\frac{\partial}{\partial\gamma_2}, \\ Y_4 = {} & \frac{I_3c_3(I_2 - I_1 - I_3)\gamma_2 + (I_1 - I_3)(I_2 - I_3)c_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1^2I_2}\,\frac{\partial}{\partial\omega_1} \\ & + \frac{I_3c_3(I_2 - I_1 + I_3)\gamma_1 - (I_1 - I_3)(I_2 - I_3)c_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1I_2^2}\,\frac{\partial}{\partial\omega_2} \\ & + \frac{(2I_1I_2 + I_2I_3 - 2I_2^2 + I_3^2 - I_1I_3)\omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1I_2}\,\frac{\partial}{\partial\gamma_1} \\ & + \frac{(2I_1^2 - 2I_1I_2 - I_3^2 - I_1I_3 + I_2I_3)\omega_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1I_2}\,\frac{\partial}{\partial\gamma_2}. \end{aligned}$$

Equations (5.40) imply that

$$Y_3(F) = Y_4(F) = 0. \tag{5.41}$$

Equations (5.40) and (5.41) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_1}, \frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\gamma_1}, \frac{\partial F}{\partial\gamma_2}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

If a new integral F exists, system (5.40)–(5.41) has at least one non-zero solution. As in Sec. 5.2 we consider the 4×4 matrix A of the coefficients of this system. The condition under which system (5.40)–(5.41) has at least one non-zero solution is $\operatorname{rank} A \leq 3$.

Therefore we equate to zero the determinant $D = \det(A)$ and study when the identity

$$D \equiv 0 \tag{5.42}$$

is fulfilled. We compute

$$D = \frac{\gamma_1^2 + \gamma_2^2}{I_2^3I_3^3}\widehat{D}, \quad \text{where} \quad \widehat{D} = D_1\sqrt{-\gamma_1^2 - \gamma_2^2} + D_2.$$

The expressions for D_1 and D_2 are polynomials in ω_1, ω_2, γ_1 and γ_2.

It is clear that (5.42) is equivalent to $\widehat{D} = 0$, that is, $D_1\sqrt{-\gamma_1^2 - \gamma_2^2} + D_2 = 0$. If $D_1 = 0$ identically, then $D_2 = 0$ identically too. Let us suppose that $D_1 \neq 0$. Then we have

$$\sqrt{-\gamma_1^2 - \gamma_2^2} = -\frac{D_2}{D_1}. \tag{5.43}$$

Applying Proposition 4.3 to $V = -\gamma_1^2 - \gamma_2^2$ one sees that (5.43) can never occur because $\sqrt{V} \notin \mathbb{C}(\gamma_1, \gamma_2)$. Consequently, $D_1 = D_2 = 0$. Thus we require that all the coefficients of D_1 and D_2 be zero. First we consider D_1. It has six monomials and thus six coefficients depending on $\mathcal{I}c$. We want to find all values of the parameters $\mathcal{I}c$ for which the six coefficients of D_1 are zero, i.e.

$$\begin{aligned}
&I_1^2(I_1-I_3)(2I_1-I_2-2I_3)(I_1-I_2-I_3)c_2=0,\\
&I_1I_2(I_2-I_3)(3I_1^2-3I_1I_2-I_1I_3+2I_2I_3-2I_3^2)c_1=0,\\
&I_1I_2(I_1-I_3)(3I_1I_2-2I_1I_3-3I_2^2+I_2I_3+2I_3^2)c_2=0,\\
&I_1(2I_1^2I_2c_1^2+2I_1^2I_2c_2^2-2I_1^2I_3c_1^2-I_1I_2^2c_1^2-I_1I_2^2c_2^2-I_1I_2I_3c_1^2\\
&\quad-4I_1I_2I_3c_2^2+2I_1I_2I_3c_3^2+2I_1I_3^2c_1^2+I_2^2I_3c_2^2-2I_2^2I_3c_3^2+2I_2I_3^2c_2^2)=0,\\
&I_2^2(I_2-I_3)(I_1-I_2+I_3)(I_1-2I_2+2I_3)c_1=0,\\
&I_2(I_1^2I_2c_1^2+I_1^2I_2c_2^2-I_1^2I_3c_1^2+2I_1^2I_3c_3^2-2I_1I_2^2c_1^2-2I_1I_2^2c_2^2\\
&\quad+I_1I_2I_3c_2^2-2I_1I_2I_3c_3^2+4I_1I_2I_3c_1^2-2I_1I_3^2c_1^2+2I_2^2I_3c_2^2-2I_2I_3^2c_2^2)=0.
\end{aligned}$$

After five consecutive simplifications we obtain the reduced system that consists of seven equations,

$$\begin{aligned}
&(I_1-I_2)c_3=0,\quad (I_1-I_2)c_1c_2=0,\quad (I_1-I_3)(I_2-2I_3)c_2=0,\\
&(I_1-I_3)(I_1-2I_3)c_2=0,\quad (I_2-I_3)(I_2-2I_3)c_1=0,\quad (I_2-I_3)(I_1-2I_3)c_1=0,\\
&(I_2-I_3)(I_2-2I_3)c_2c_3=0.
\end{aligned}$$

Solving this system by the MAPLE command `solve` we obtain six solutions. After removing the solutions that lead to the Euler, Lagrange, Kovalevskaya and kinetic symmetry cases there remains only one solution

$$I_1=2I_3,\ I_2=2I_3,\ I_3,\ c_1,\ c_2,\ c_3 \text{ are arbitrary.}$$

Thus we should only consider the case

$$I_1=I_2=2I_3. \tag{5.44}$$

Under the condition (5.44) we have also $D_2=0$ and therefore the vector fields Y_i, $1\le i\le 4$, are linearly dependent (they satisfy $4Y_4+Y_2=0$). That is why we compute the Lie bracket $Y_5=[Y_2,Y_3]$ and obtain

$$\begin{aligned}
Y_5={}&\frac{\omega_1(c_2\gamma_1+c_1\gamma_2)+2c_3\omega_2\sqrt{-\gamma_1^2-\gamma_2^2}+\omega_2(c_2\gamma_2-c_1\gamma_1)}{4I_3}\,\frac{\partial}{\partial\omega_1}\\
&-\frac{\omega_1(c_1\gamma_1-c_2\gamma_2)+2c_3\omega_1\sqrt{-\gamma_1^2-\gamma_2^2}+\omega_2(c_2\gamma_1+c_1\gamma_2)}{4I_3}\,\frac{\partial}{\partial\omega_2}\\
&+\frac{I_3\omega_1^2+I_3\omega_2^2-c_3\sqrt{-\gamma_1^2-\gamma_2^2}}{2I_3}\left(\gamma_2\frac{\partial}{\partial\gamma_1}-\gamma_1\frac{\partial}{\partial\gamma_2}\right)
\end{aligned}$$

and consider the following four equations:

$$Y_i(F)=0,\ 1\le i\le 3,\quad Y_5(F)=0. \tag{5.45}$$

As above we equate to zero the determinant $\Delta=\det(B)$, where B is the matrix of the coefficients of system (5.45), and study when the identity

$$\Delta\equiv 0$$

is fulfilled. We compute

$$\Delta=-\frac{\sqrt{-\gamma_1^2-\gamma_2^2}}{8I_3^2}\widehat{\Delta},$$

where

$$\begin{aligned}\widehat{\Delta} = c_3\,\big(&c_2\omega_1^2\gamma_1^3 + c_1\omega_1^2\gamma_1^2\gamma_2 + c_2\omega_1^2\gamma_1\gamma_2^2 + c_1\omega_1^2\gamma_2^3 - 2c_1\omega_1\omega_2\gamma_1^3 + 2c_2\omega_1\omega_2\gamma_1^2\gamma_2\\ &- 2c_1\omega_1\omega_2\gamma_1\gamma_2^2 + 2c_2\omega_1\omega_2\gamma_2^3 - c_2\omega_2^2\gamma_1^3 - c_1\omega_2^2\gamma_1^2\gamma_2 - c_2\omega_2^2\gamma_1\gamma_2^2 - c_1\omega_2^2\gamma_2^3\big)\,.\end{aligned}$$

It is clear that the equation $\Delta = 0$ is equivalent to $\widehat{\Delta} = 0$. It is easily seen from the expression for $\widehat{\Delta}$ that $\widehat{\Delta}$ vanishes identically only if $c_3 = 0$ or if $c_1 = c_2 = 0$. Taking into account the condition (5.44) we see that if $c_3 = 0$ we come to the Kovalevskaya case, and if $c_1 = c_2 = 0$ to the Lagrange case.

Thus a partial integral of type 3, i.e. $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$, does not exist.

Type 5. Let us look for a first integral of system (5.31) of type 5, $F(\omega_2, \omega_3, \gamma_1, \gamma_2)$, i.e. one which does not depend on ω_1. Thus F satisfies the identity

$$\begin{aligned}\frac{dF}{dt} = {}& \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\,\frac{\partial F}{\partial \omega_2}\\ &+ \frac{(I_1 - I_2)\omega_1\omega_2 + c_2\gamma_1 - c_1\gamma_2}{I_3}\,\frac{\partial F}{\partial \omega_3}\\ &+ (\omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2})\frac{\partial F}{\partial \gamma_1} + (\omega_1\sqrt{-\gamma_1^2 - \gamma_2^2} - \omega_3\gamma_1)\frac{\partial F}{\partial \gamma_2} = 0,\end{aligned}$$

which can be presented in the following way:

$$\frac{dF}{dt} = \omega_1 Y_1(F) + Y_2(F) = 0, \tag{5.46}$$

where Y_1 and Y_2 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_2)$:

$$\begin{aligned}Y_1 &= \frac{(I_3 - I_1)\omega_3}{I_2}\,\frac{\partial}{\partial \omega_2} + \frac{(I_1 - I_2)\omega_2}{I_3}\,\frac{\partial}{\partial \omega_3} + \sqrt{-\gamma_1^2 - \gamma_2^2}\,\frac{\partial}{\partial \gamma_2},\\ Y_2 &= \frac{c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\,\frac{\partial}{\partial \omega_2} + \frac{c_2\gamma_1 - c_1\gamma_2}{I_3}\,\frac{\partial}{\partial \omega_3}\\ &\quad + (\omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2})\frac{\partial}{\partial \gamma_1} - \omega_3\gamma_1\frac{\partial}{\partial \gamma_2}.\end{aligned}$$

As (5.46) is an identity with respect to all the variables and as $Y_1(F)$ and $Y_2(F)$ do not depend on ω_1, we have

$$Y_1(F) = Y_2(F) = 0. \tag{5.47}$$

We compute the Lie brackets $Y_3 = [Y_1, Y_2]$ and $Y_4 = [Y_1, Y_3]$ and obtain

$$\begin{aligned}Y_3 = {}& \frac{(I_1 - I_3)c_2\gamma_1 - I_1c_1\gamma_2}{I_2I_3}\,\frac{\partial}{\partial \omega_2} + \frac{(I_1 - I_2)c_3\gamma_1 - I_1c_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2I_3}\,\frac{\partial}{\partial \omega_3}\\ &+ \frac{I_2(I_1 - I_2 + I_3)\omega_2\gamma_2 + I_3(I_1 + I_2 - I_3)\omega_3\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2I_3}\,\frac{\partial}{\partial \gamma_1}\\ &- \frac{(I_1 - I_2 + I_3)\omega_2\gamma_1}{I_3}\,\frac{\partial}{\partial \gamma_2},\\ Y_4 = {}& \frac{(I_1 - I_2)(I_1 - I_3)c_3\gamma_1 - I_1(I_1 + I_2 - I_3)c_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2^2I_3}\,\frac{\partial}{\partial \omega_2}\\ &- \frac{(I_1 - I_2)(I_1 - I_3)c_2\gamma_1 - I_1(I_1 - I_2 + I_3)c_1\gamma_2}{I_2I_3^2}\,\frac{\partial}{\partial \omega_3}\end{aligned}$$

$$- \left[\frac{I_1(I_1 - I_2 + I_3) + 2I_3(I_2 - I_3)}{I_2 I_3} \omega_3 \gamma_2 \right.$$
$$\left. - \frac{I_1(I_1 + I_2 - I_3) - 2I_2(I_2 - I_3)}{I_2 I_3} \omega_2 \sqrt{-\gamma_1^2 - \gamma_2^2} \right] \frac{\partial}{\partial \gamma_1}$$
$$+ \frac{I_1(I_1 - I_2 + I_3) + 2I_3(I_2 - I_3)}{I_2 I_3} \omega_3 \gamma_1 \frac{\partial}{\partial \gamma_2}.$$

Equations (5.47) imply that

$$Y_3(F) = Y_4(F) = 0. \tag{5.48}$$

Equations (5.47) and (5.48) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_2}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

If a new integral F exists, system (5.47)–(5.48) has at least one non-zero solution. As in Sec. 5.2 we consider the 4×4 matrix A of the coefficients of this system. The condition under which system (5.47)–(5.48) has at least one non-zero solution is $\operatorname{rank} A \leq 3$.

Therefore we equate to zero the determinant $D = \det(A)$ and study when the identity

$$D \equiv 0 \tag{5.49}$$

is fulfilled. We compute

$$D = \frac{\gamma_1 \sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2^3 I_3^3} \widehat{D}, \quad \text{where} \quad \widehat{D} = D_1 \sqrt{-\gamma_1^2 - \gamma_2^2} + D_2.$$

The expressions for D_1 and D_2 are polynomials in ω_2, ω_3, γ_1 and γ_2.

It is clear that (5.49) is equivalent to $\widehat{D} = 0$, that is, $D_1 \sqrt{-\gamma_1^2 - \gamma_2^2} + D_2 = 0$. If $D_1 = 0$ identically, then $D_2 = 0$ identically too. Let us suppose that $D_1 \neq 0$. Then

$$\sqrt{-\gamma_1^2 - \gamma_2^2} = -\frac{D_2}{D_1}. \tag{5.50}$$

Applying Proposition 4.3 to $V = -\gamma_1^2 - \gamma_2^2$ one sees that (5.50) can never occur because $\sqrt{V} \notin \mathbb{C}(\gamma_1, \gamma_2)$. Consequently, $D_1 = D_2 = 0$. Thus we require that all the coefficients of D_1 and D_2 be zero. First we consider D_2. It has 11 monomials and thus 11 coefficients depending on $\mathcal{I}c$. We want to find all values of the parameters $\mathcal{I}c$ for which the 11 coefficients of D_2 are zero, i.e.

$$I_2^2(I_1 - I_2)(2I_1 - 2I_2 + I_3)(I_1 - I_2 + I_3)c_3 = 0,$$
$$2I_2I_3(I_2 - I_3)(I_1 - I_2)(I_1 + I_2 - I_3)c_1 = 0,$$
$$2I_3^2(I_2 - I_3)(I_1 - I_3)(I_1 + I_2 - I_3)c_1 = 0, \quad I_2I_3(I_2 - I_3)(I_1 - I_3)c_1c_2 = 0,$$
$$I_2I_3(I_2 - I_3)(I_1 - I_2)c_1c_3 = 0, \quad I_3^2(I_1 - I_3)(I_1 + I_2 - I_3)(2I_1 + I_2 - 2I_3)c_2 = 0,$$
$$I_2I_3(2I_1^2 - 4I_1I_2 + 3I_1I_3 + 2I_2^2 - 2I_2I_3)c_1c_3 = 0,$$
$$I_3(2I_1^2I_2c_2^2 - 4I_1I_2I_3c_2^2 - 2I_1I_3^2c_3^2 + 2I_1I_2I_3c_1^2 + 2I_1^2I_3c_3^2 + 2I_2I_3^2c_3^2$$
$$- 2I_1I_2^2c_1^2 + I_1I_2^2c_2^2 - I_2^2I_3c_2^2 - I_2^2I_3c_3^2 + 2I_2I_3^2c_2^2 - I_1I_2I_3c_3^2) = 0,$$
$$I_2I_3(2I_1 - I_3)(I_1 + I_2 - I_3)c_1c_2 = 0,$$
$$I_2I_3(I_1 - I_2)(2I_1^2 - 2I_1I_2 + I_1I_3 + 3I_2I_3 - 3I_3^2)c_3 = 0,$$
$$I_2I_3(I_1 - I_3)(2I_1^2 + I_1I_2 - 2I_1I_3 - 3I_2^2 + 3I_2I_3)c_2 = 0.$$

After six consecutive simplifications we obtain the reduced system that consists of eight equations,

$$(I_2-I_3)c_1=0,\quad (I_2-I_3)c_3c_2=0,\quad (2I_1-I_3)c_3c_1=0,$$
$$(I_2-I_3)(I_1-I_2)c_3=0,\quad (I_1-I_2)(2I_1+2I_2-3I_3)c_3=0,\quad (2I_1-I_3)c_1c_2=0,$$
$$(I_2-I_3)(I_1-I_3)c_2=0,\quad (2I_1-I_3)(I_1-I_3)c_2=0.$$

Solving this system by the MAPLE command `solve` we obtain seven solutions. Removing the solutions that lead to the Euler, Lagrange, Kovalevskaya and kinetic symmetry cases we obtain only one solution

$$I_2=2I_1,\ I_3=2I_1,\ I_1,\ c_1,\ c_2,\ c_3 \text{ are arbitrary.}$$

Thus we should only consider the case

$$I_2=I_3=2I_1. \tag{5.51}$$

Under condition (5.51) we also have $D_1=0$ and therefore the vector fields Y_i, $1\le i\le 4$, are linearly dependent (they satisfy $4Y_4+Y_2=0$). That is why we consider the Lie bracket $Y_5=[Y_2,Y_3]$ and obtain

$$\begin{aligned}Y_5=&\ \frac{c_3\omega_2\gamma_2+\omega_3(2c_1\gamma_1-c_2\gamma_2)+(c_2\omega_2+c_3\omega_3)\sqrt{-\gamma_1^2-\gamma_2^2}}{4I_1}\frac{\partial}{\partial\omega_2}\\ &-\frac{\omega_2(2c_1\gamma_1+c_2\gamma_2)+c_3\omega_3\gamma_2-(c_3\omega_2-c_2\omega_3)\sqrt{-\gamma_1^2-\gamma_2^2}}{4I_1}\frac{\partial}{\partial\omega_3}\\ &+\frac{I_1\omega_2^2+I_1\omega_3^2-c_1\gamma_1}{2I_1}\sqrt{-\gamma_1^2-\gamma_2^2}\frac{\partial}{\partial\gamma_2}\end{aligned}$$

and consider the following four equations:

$$Y_i(F)=0,\ 1\le i\le 3,\quad Y_5(F)=0. \tag{5.52}$$

As above we equate to zero the determinant $\Delta=\det(B)$, where B is the matrix of the coefficients of system (5.52) and study when the identity

$$\Delta\equiv 0$$

is fulfilled. We compute

$$\Delta=-\frac{\gamma_1^2}{8I_1^2}\widehat{\Delta},\quad \text{where}\quad \widehat{\Delta}=\Delta_1\sqrt{-\gamma_1^2-\gamma_2^2}+\Delta_2.$$

The expressions for Δ_1 and Δ_2 are the following polynomials in ω_2, ω_3, γ_1 and γ_2:

$$\Delta_1=c_1\gamma_2(-c_3\omega_2^2+2c_2\omega_2\omega_3+c_3\omega_3^2),\qquad \Delta_2=c_1(\gamma_1^2+\gamma_2^2)(c_2\omega_2^2+2c_3\omega_2\omega_3-c_2\omega_3^2).$$

As $\widehat{\Delta}=0$, by Proposition 4.3 we have $\Delta_1=\Delta_2=0$. As is easily seen, the last equations can be satisfied only in two cases: when $c_1=0$, which together with condition (5.51) leads to the Kovalevskaya case, and when $c_2=c_3=0$, which leads to the Lagrange case. The conclusion is that a partial integral of type 5 does not exist.

5.4. Invariant manifold $\{H_3=U_3\}$. Here we proceed as in Sec. 5.2. We first eliminate ω_3 from the equation

$$H_3=U_3. \tag{5.53}$$

Then we study the elimination of γ_3 from (5.53). The results are presented in the next two subsections.

5.4.1. Elimination of $\boldsymbol{\omega_3}$. We express ω_3 from (5.53) and obtain

$$\omega_3 = \sqrt{\frac{U_3 - I_1\omega_1^2 - I_2\omega_2^2 - 2c_1\gamma_1 - 2c_2\gamma_2 - 2c_3\gamma_3}{I_3}}. \tag{5.54}$$

ω_3 is now considered as an algebraic function of all its variables.

To shorten formulas, we denote the square root (5.54) by Ω_3 so that

$$\omega_3 = \Omega_3. \tag{5.55}$$

Now we insert this form of ω_3 in the Euler–Poisson equations (1.1) and remove the third equation. In this way we obtain the following system of five differential equations:

$$\begin{aligned} \frac{d\omega_1}{dt} &= \frac{(I_2 - I_3)\omega_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3}{I_1}, \\ \frac{d\omega_2}{dt} &= \frac{(I_3 - I_1)\omega_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1}{I_2}, \\ \frac{d\gamma_1}{dt} &= \Omega_3\gamma_2 - \omega_2\gamma_3, \\ \frac{d\gamma_2}{dt} &= \omega_1\gamma_3 - \Omega_3\gamma_1, \\ \frac{d\gamma_3}{dt} &= \omega_2\gamma_1 - \omega_1\gamma_2. \end{aligned} \tag{5.56}$$

There are five possible types of first integrals of this system which depend on at most four variables. They are:

1. $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$ (case (ii)).
2. $F(\omega_1, \omega_2, \gamma_1, \gamma_3)$ (case (iii)).
3. $F(\omega_1, \omega_2, \gamma_2, \gamma_3)$ (case (iii)).
4. $F(\omega_1, \gamma_1, \gamma_2, \gamma_3)$ (case (iv)).
5. $F(\omega_2, \gamma_1, \gamma_2, \gamma_3)$ (case (iv)).

It is then sufficient to examine functions of type 1, 2 and 4. Afterwards, eliminating γ_3, we will be able to study functions belonging to case (i), $F(\omega_1, \omega_2, \omega_3, \gamma_1)$.

Type 1. Let us look for a first integral of system (5.56) that does not depend on γ_3, i.e. of type 1. Let us suppose that $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$ is such a first integral. Then

$$\begin{aligned} I_1I_2\frac{dF}{dt} &= I_2[(I_2 - I_3)\omega_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3]\frac{\partial F}{\partial\omega_1} \\ &\quad + I_1[(I_3 - I_1)\omega_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1]\frac{\partial F}{\partial\omega_2} \\ &\quad + I_1I_2(\Omega_3\gamma_2 - \omega_2\gamma_3)\frac{\partial F}{\partial\gamma_1} + I_1I_2(\omega_1\gamma_3 - \Omega_3\gamma_1)\frac{\partial F}{\partial\gamma_2} = Y_1(F) = 0, \end{aligned} \tag{5.57}$$

where Y_1 is a vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

Equation (5.57) is an identity with respect to all the five variables. F does not depend on γ_3. Thus if we differentiate this identity with respect to γ_3 we again obtain a linear partial differential equation for F. Let us note that from (5.54) and (5.55) it follows that

$$\frac{\partial \Omega_3}{\partial \gamma_3} = -\frac{c_3}{I_3 \Omega_3}.$$

In this way we deduce from (5.57) that

$$\begin{aligned} I_3\Omega_3 \frac{\partial Y_1(F)}{\partial \gamma_3} &= -I_2(I_3c_2\Omega_3 + I_2c_3\omega_2 - I_3c_3\omega_2)\frac{\partial F}{\partial \omega_1} \\ &\quad + I_1(I_1c_3\omega_1 - I_3c_3\omega_1 + I_3c_1\Omega_3)\frac{\partial F}{\partial \omega_2} - I_1I_2(c_3\gamma_2 + I_3\omega_2\Omega_3)\frac{\partial F}{\partial \gamma_1} \\ &\quad + I_1I_2(I_3\omega_1\Omega_3 + c_3\gamma_1)\frac{\partial F}{\partial \gamma_2} = Y_2(F) = 0, \end{aligned} \tag{5.58}$$

where Y_2 is a vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

We differentiate (5.58) and obtain

$$\Omega_3 \frac{\partial Y_2(F)}{\partial \gamma_3} = c_3\left(I_2c_2\frac{\partial F}{\partial \omega_1} - I_1c_1\frac{\partial F}{\partial \omega_2} + I_1I_2\omega_2\frac{\partial F}{\partial \gamma_1} - I_1I_2\omega_1\frac{\partial F}{\partial \gamma_2}\right) = Y_3(F) = 0, \tag{5.59}$$

where Y_3 is a vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

When $c_3 = 0$, Y_3 vanishes identically. Therefore we consider two cases separately: $c_3 \neq 0$ and $c_3 = 0$.

First let $c_3 \neq 0$. Then we compute the Lie bracket $Y_4 = [Y_2, Y_3]/(I_1I_2c_3^2)$ and obtain

$$\begin{aligned} Y_4(F) &= -c_1(I_2 - I_3)\frac{\partial F}{\partial \omega_1} - c_2(I_1 - I_3)\frac{\partial F}{\partial \omega_2} \\ &\quad + I_1\omega_1(I_1 - I_2 - I_3)\frac{\partial F}{\partial \gamma_1} - I_2\omega_2(I_1 - I_2 + I_3) = 0. \end{aligned} \tag{5.60}$$

Equations (5.57)–(5.60) can be considered as a system of three homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_2}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if a new first integral F exists, system (5.57)–(5.60) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D of the coefficients of equations (5.57)–(5.60) is identically zero. We compute this determinant and obtain

$$D = I_1^2 I_2^2 c_3^3 \widehat{D},$$

where

$$\begin{aligned} \widehat{D} &= I_1(I_1 - I_3)(I_1 - I_2 - I_3)\omega_1^3\gamma_2 - I_1(I_2 - I_3)(I_1 - I_2 - I_3)\omega_1^2\omega_2\gamma_1 \\ &\quad - I_2(I_1 - I_3)(I_1 - I_2 + I_3)\omega_1\omega_2^2\gamma_2 + I_1c_2(I_1 - I_2 - I_3)\omega_1\gamma_1^2 \\ &\quad - I_1c_1(I_1 - 2I_3)\omega_1\gamma_1\gamma_2 - I_2c_2(I_1 - I_3)\omega_1\gamma_2^2 + I_2(I_2 - I_3)(I_1 - I_2 + I_3)\omega_2^3\gamma_1 \\ &\quad + I_1c_1(I_2 - I_3)\omega_2\gamma_1^2 + I_2c_2(I_2 - 2I_3)\omega_2\gamma_1\gamma_2 + I_2c_1(I_1 - I_2 + I_3)\omega_2\gamma_2^2. \end{aligned}$$

It is clear that the equation $D = 0$ is equivalent to $\widehat{D} = 0$. $\widehat{D} = 0$ has ten coefficients, so they should all be zeros. In this way we obtain a system of ten equations for the parameters $\mathcal{I}c$.

After three consecutive simplifications we come to the reduced system

$$c_1 = 0, \quad c_2 = 0, \quad I_2 - I_3 = 0, \quad I_1 - I_3 = 0,$$

which obviously leads to a particular case of the kinetic symmetry case. Thus a partial integral of type 1 does not exist when $c_3 \neq 0$.

Let $c_3 = 0$. Now Ω_3 does not depend on γ_3 and $Y_1(F)$ is of the form (see (5.57))

$$Y_1(F) = Z_1(F)\gamma_3 + Z_2(F)\Omega_3, \tag{5.61}$$

where the vector fields Z_1 and Z_2, defined on $\mathbb{C}^4(\omega_1, \omega_2, \gamma_1, \gamma_2)$, are as follows:

$$\begin{aligned} Z_1 &= -I_2 c_2 \frac{\partial}{\partial \omega_1} + I_1 c_1 \frac{\partial}{\partial \omega_2} - I_1 I_2 \omega_2 \frac{\partial}{\partial \gamma_1} + I_1 I_2 \omega_1 \frac{\partial}{\partial \gamma_2}, \\ Z_2 &= I_2(I_2 - I_3)\omega_2 \frac{\partial}{\partial \omega_1} + I_1(I_3 - I_1)\omega_1 \frac{\partial}{\partial \omega_2} + I_1 I_2 \gamma_2 \frac{\partial}{\partial \gamma_1} - I_1 I_2 \gamma_1 \frac{\partial}{\partial \gamma_2}. \end{aligned}$$

Equation (5.61) implies that

$$Z_1(F) = Z_2(F) = 0. \tag{5.62}$$

We compute the Lie brackets $Z_3 = [Z_1, Z_2]/(I_1 I_2)$ and $Z_4 = [Z_2, Z_3]$ and obtain

$$\begin{aligned} Z_3 = {} & (I_2 - I_3)c_1 \frac{\partial}{\partial \omega_1} + (I_1 - I_3)c_2 \frac{\partial}{\partial \omega_2} \\ & - I_1(I_1 - I_2 - I_3)\omega_1 \frac{\partial}{\partial \gamma_1} + I_2(I_1 - I_2 + I_3)\omega_2 \frac{\partial}{\partial \gamma_2}, \\ Z_4 = {} & -I_2 c_2 (I_2 - I_3)(I_1 - I_3) \frac{\partial}{\partial \omega_1} + I_1 c_1 (I_2 - I_3)(I_1 - I_3) \frac{\partial}{\partial \omega_2} \\ & - I_1 I_2 (2I_1 I_2 - I_1 I_3 - 2I_2^2 + I_2 I_3 + I_3^2)\omega_2 \frac{\partial}{\partial \gamma_1} \\ & - I_1 I_2 (2I_1^2 - 2I_1 I_2 - I_1 I_3 + I_2 I_3 - I_3^2)\omega_1 \frac{\partial}{\partial \gamma_2}. \end{aligned}$$

Equations (5.62) imply that

$$Z_3(F) = Z_4(F) = 0. \tag{5.63}$$

As in the case $c_3 \neq 0$, the system of equations (5.62) and (5.63) is a linear homogeneous system that has a non-zero solution. Thus the determinant δ of its coefficients should vanish identically. We compute

$$\delta = I_1^2 I_2^2 \widehat{\delta},$$

where

$$\begin{aligned} \widehat{\delta} = {} & I_1^2 (I_1 - I_3)(2I_1 - I_2 - 2I_3)(I_1 - I_2 - I_3) c_2 \omega_1^3 \\ & + I_1 I_2 (I_2 - I_3)(2I_3 I_2 - 3I_1 I_2 + 3I_1^2 - I_3 I_1 - 2I_3^2) c_1 \omega_1^2 \omega_2 \\ & + I_1 I_2 (I_1 - I_3)(3I_1 I_2 - 2I_1 I_3 + I_2 I_3 - 3I_2^2 + 2I_3^2) c_2 \omega_1 \omega_2^2 \\ & - I_1 (2I_1 - I_2 - 2I_3)(I_1 I_2 c_1^2 + I_1 I_2 c_2^2 - I_1 I_3 c_1^2 - I_2 I_3 c_2^2)\omega_1 \gamma_2 \\ & - I_2^2 (I_2 - I_3)(I_1 - 2I_2 + 2I_3)(I_1 - I_2 + I_3) c_1 \omega_2^3 \\ & - I_2 (I_1 - 2I_2 + 2I_3)(I_1 I_2 c_1^2 + I_1 I_2 c_2^2 - I_1 I_3 c_1^2 - I_2 I_3 c_2^2)\omega_2 \gamma_1. \end{aligned}$$

The equation $\delta = 0$ is equivalent to $\widehat{\delta} = 0$. Thus the six coefficients of $\widehat{\delta}$ should be zeros. In this way we obtain a system of six equations for the parameters $\mathcal{I}c$. After five consecutive simplifications we come to the reduced system consisting of the following five equations:

$$(I_1 - I_2)c_1c_2 = 0, \quad (I_1 - I_3)(I_2 - 2I_3)c_2 = 0, \quad (I_1 - I_3)(I_1 - 2I_3)c_2 = 0,$$
$$(I_2 - I_3)(I_2 - 2I_3)c_1 = 0, \quad (I_2 - I_3)(I_1 - 2I_3)c_1 = 0.$$

We solve them by `solve` and obtain five solutions:

$$\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0\},$$
$$\{I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2\},$$
$$\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0\},$$
$$\{I_1 = 2I_3,\ I_2 = 2I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2\},$$
$$\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2\}.$$

Taking into account that now $c_3 = 0$ we see that the first of these solutions leads to the Euler case, the second and third to the Lagrange case, the fourth to the Kovalevskaya case, and the last one to the kinetic symmetry case.

Thus a partial integral of type 1 does not exist also when $c_3 = 0$.

Type 2. Let us study now a first integral of type 2, i.e. $F(\omega_1, \omega_2, \gamma_1, \gamma_3)$. We have

$$\begin{aligned} I_1I_2\frac{dF}{dt} &= I_2[(I_2 - I_3)\omega_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3]\frac{\partial F}{\partial \omega_1} \\ &\quad + I_1[(I_3 - I_1)\omega_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1]\frac{\partial F}{\partial \omega_2} \\ &\quad + I_1I_2(\Omega_3\gamma_2 - \omega_2\gamma_3)\frac{\partial F}{\partial \gamma_1} + I_1I_2(\omega_2\gamma_1 - \omega_1\gamma_2)\frac{\partial F}{\partial \gamma_3} = Y_1(F) = 0, \end{aligned} \tag{5.64}$$

where Y_1 is a vector field on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

Equation (5.64) is an identity with respect to all the five variables. F does not depend on γ_2. Thus if we differentiate this identity with respect to γ_2 we again obtain a linear partial differential equation for F. Let us note that

$$\frac{\partial \Omega_3}{\partial \gamma_2} = -\frac{c_2}{I_3\Omega_3}.$$

In this way we have

$$\begin{aligned} I_3\Omega_3\frac{\partial Y_1(F)}{\partial \gamma_2} &= I_2(I_3c_3\Omega_3 - I_2c_2\omega_2 + I_3c_2\omega_2)\frac{\partial F}{\partial \omega_1} + I_1(I_1 - I_3)c_2\omega_1\frac{\partial F}{\partial \omega_2} \\ &\quad + I_1I_2(-c_2\gamma_2 + I_3\Omega_3^2)\frac{\partial F}{\partial \gamma_1} - I_1I_2I_3\omega_1\Omega_3\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \end{aligned} \tag{5.65}$$

where Y_2 is a vector field on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

We differentiate (5.65) with respect to γ_2 and obtain

$$\frac{\Omega_3}{I_2}\,\frac{\partial Y_2(F)}{\partial \gamma_2} = -c_2\left(c_3\frac{\partial F}{\partial \omega_1} + 3I_1\Omega_3\frac{\partial F}{\partial \gamma_1} - I_1\omega_1\frac{\partial F}{\partial \gamma_3}\right) = Y_3(F) = 0, \tag{5.66}$$

where Y_3 is a vector field on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

When $c_2 = 0$, Y_3 vanishes identically. Therefore we consider two cases separately: $c_2 \neq 0$ and $c_2 = 0$.

Let first $c_2 \neq 0$. Then we differentiate (5.66) with respect to γ_2 and obtain

$$\frac{I_3\Omega_3}{3I_1c_2^2}\,\frac{\partial Y_3(F)}{\partial \gamma_2} = \frac{\partial F}{\partial \gamma_1} = Y_4(F) = 0. \tag{5.67}$$

Equations (5.64)–(5.67) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if a fourth integral F exists, system (5.64)–(5.67) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D of the coefficients of (5.64)–(5.67) is identically zero. We compute

$$D = I_1^2 I_2 c_2^2 \omega_1 \widehat{D},$$

where

$$\widehat{D} = c_2(I_1 - I_3)\omega_1\gamma_3 - c_3(I_1 - I_2)\omega_2\gamma_1 - c_1(I_2 - I_3)\omega_2\gamma_3.$$

The equation $D = 0$ is equivalent to $\widehat{D} = 0$. Thus, as $c_2 \neq 0$, the first coefficient of $\widehat{D}$ vanishes identically if and only if $I_1 = I_3$. Under this condition the remaining two terms vanish if either $I_2 = I_3$ or $c_1 = c_3 = 0$. The first possibility leads to the kinetic symmetry case, and the second to the Lagrange case.

Thus a partial integral of type 2 does not exist when $c_2 \neq 0$.

Let $c_2 = 0$. Now Ω_3 does not depend on γ_2 and $Y_1(F)$ is of the form (see (5.64))

$$Y_1(F) = Z_1(F)\gamma_2 + Z_2(F)\Omega_3, \tag{5.68}$$

where the vector fields Z_1 and Z_2, defined on $\mathbb{C}^4(\omega_1, \omega_2, \gamma_1, \gamma_3)$, are given as follows:

$$\begin{aligned}
Z_1 &= I_2c_3\frac{\partial}{\partial \omega_1} + I_1I_2\Omega_3\frac{\partial}{\partial \gamma_1} - I_1I_2\omega_1\frac{\partial}{\partial \gamma_3},\\
Z_2 &= I_2(I_2 - I_3)\omega_2\Omega_3\frac{\partial}{\partial \omega_1} + I_1\left[(I_3 - I_1)\omega_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1\right]\frac{\partial}{\partial \omega_2}\\
&\quad - I_1I_2\omega_2\gamma_3\frac{\partial}{\partial \gamma_1} + I_1I_2\omega_2\gamma_1\frac{\partial}{\partial \gamma_3}.
\end{aligned}$$

Equation (5.68) implies that

$$Z_1(F) = Z_2(F) = 0. \tag{5.69}$$

We compute the Lie brackets $Z_3 = [Z_1, Z_2]/(I_1I_2)$ and $Z_4 = [Z_2, Z_3]/I_2$ and obtain

$$\begin{aligned}
Z_3 &= -I_2(I_2 - I_3)c_1\omega_2\frac{\partial}{\partial \omega_1} + \left[I_3c_3\Omega_3(I_3 - 2I_1) + I_1c_1\omega_1(I_1 - 2I_3)\right]\frac{\partial}{\partial \omega_2}\\
&\quad - I_1I_2(I_1 - I_2 - I_3)\omega_1\omega_2\frac{\partial}{\partial \gamma_1} + I_2I_3(I_1 + I_2 - I_3)\omega_2\Omega_3\frac{\partial}{\partial \gamma_3},\\
Z_4 &= a_1\frac{\partial}{\partial \omega_1} + a_2\frac{\partial}{\partial \omega_2} - a_3\frac{\partial}{\partial \gamma_1} - a_4\frac{\partial}{\partial \gamma_2},
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= (I_2 - I_3)[-I_1(2I_1 - I_3)c_3\omega_1^2 + I_1I_3c_1\omega_1\Omega_3 - I_2(3I_1 - I_2 - I_3)c_3\omega_2^2\\
&\qquad - (3I_1 - 2I_3)c_1c_3\gamma_1 - (I_1c_1^2 + 4I_1c_3^2 - 2I_3c_3^2)\gamma_3 + (2I_1 - I_3)c_3U_3],\\
a_2 &= -I_1\omega_2[I_1(2I_1 - 2I_2 - I_3)c_3\omega_1 + I_3(I_1 + 2I_2 - 2I_3)c_1\Omega_3],
\end{aligned}$$

$$\begin{aligned}
a_3 = {} & I_1[I_1(I_1 - I_3)(I_1 - I_2 - I_3)\omega_1^2\Omega_3 + I_1(I_1 - I_2 - I_3)c_3\omega_1\gamma_1 + I_1(I_2 - I_3)c_1\omega_1\gamma_3 \\
& - I_2(I_1I_2 - 2I_1I_3 - I_2^2 - I_2I_3 + 2I_3^2)\omega_2^2\Omega_3 - I_3(2I_1 - I_3)c_3\gamma_3\Omega_3], \\
a_4 = {} & I_1[I_1(I_1 - I_3)(I_1 + I_2 - I_3)\omega_1^3 + I_2(3I_1^2 - 4I_1I_3 - I_2^2 + I_3^2)\omega_1\omega_2^2 \\
& + (I_1^2 + 2I_1I_2 - 2I_1I_3 - 2I_2I_3 + 2I_3^2)c_1\omega_1\gamma_1 + 2(I_1 - I_3)(I_1 + I_2 - I_3)c_3\omega_1\gamma_3 \\
& - (I_1 - I_3)(I_1 + I_2 - I_3)U_3\omega_1 + I_3(I_1 - I_2)c_3\gamma_1\Omega_3 + I_3(I_1 + I_2 - I_3)c_1\gamma_3\Omega_3].
\end{aligned}$$

Equations (5.69) imply that

$$Z_3(F) = Z_4(F) = 0. \tag{5.70}$$

As in the case $c_2 \neq 0$, the system of equations (5.69) and (5.70) is a linear homogeneous system that has a non-zero solution. Thus the determinant δ of its coefficients should vanish identically. We compute

$$\delta = I_1^2 I_2^3 \omega_2^3 \Omega_3 \widehat{\delta},$$

where

$$\widehat{\delta} = I_3\Omega_3 b_1 + I_1\omega_1 b_2.$$

b_1 and b_2 are polynomials in ω_1, ω_2, γ_1 and γ_3 with coefficients that depend on the parameters $\mathcal{I}c$ and U_3. They are given by

$$\begin{aligned}
b_1 = {} & -2I_1c_1(I_2 - I_3)(I_1 - I_3)(2I_1 + I_2 - I_3)\omega_1^2 \\
& - I_2(I_2 - I_3)(I_1 + I_2 - I_3)(I_1 + 2I_2 - 2I_3)c_1\omega_2^2 \\
& - (I_1 + 2I_2 - 2I_3)(I_1I_2c_1^2 - I_1I_3c_1^2 + I_1I_3c_3^2 + 2I_2^2c_1^2 - 4I_2I_3c_1^2 - I_2I_3c_3^2 + 2I_3^2c_1^2)\gamma_1 \\
& - 2c_3c_1(I_2 - I_3)(I_1 + I_2 - I_3)(I_1 + 2I_2 - 2I_3)\gamma_3 \\
& + (I_2 - I_3)(I_1 + I_2 - I_3)(I_1 + 2I_2 - 2I_3)c_1U_3, \\
b_2 = {} & 2I_1(I_1 - I_3)(I_1 - I_2)(I_1 - I_2 - 2I_3)c_3\omega_1^2 \\
& + I_2(I_1 - I_2)(2I_1I_2 - 3I_1I_3 - 2I_2^2 - I_2I_3 + 3I_3^2)c_3\omega_2^2 \\
& + 2(I_1 - I_2)(-3I_1I_3 + 2I_1I_2 + 3I_3^2 - I_2I_3 - 2I_2^2)c_1c_3\gamma_1 \\
& + [(2I_1^2I_2(c_1^2 + 2c_3^2) - 2I_1^2I_3(c_1^2 + 4c_3^2) - 2I_1I_2^2(c_1^2 + 4c_3^2) + I_1I_2I_3(c_1^2 + 8c_3^2) \\
& + I_1I_3^2(c_1^2 + 7c_3^2) + 4I_2^3c_3^2 - 7I_2I_3^2c_3^2)]\gamma_3 \\
& - (I_1 - I_2)(2I_1I_2 - 3I_1I_3 - 2I_2^2 - I_2I_3 + 3I_3^2)c_3U_3.
\end{aligned}$$

The equation $\delta = 0$ is equivalent to $\widehat{\delta} = 0$. $\widehat{\delta}$ depends on the function Ω_3 and it is easy to see that $\Omega_3 \notin \mathbb{C}(\omega_1, \omega_2, \gamma_1, \gamma_3)$. Thus according to Proposition 4.3, the coefficients b_1 and b_2 of $\widehat{\delta}$ should be zeros. In this way we obtain a system of ten equations for the parameters $\mathcal{I}c$ and U_3. After four consecutive simplifications we come to the reduced system consisting of the following five equations:

$$\begin{gathered}
(I_1 - I_3)c_1c_3 = 0, \quad (2I_2 - I_3)(I_1 - I_2)c_3 = 0, \quad (I_1 - I_2)(2I_1 - 3I_3 + 2I_2)c_3 = 0, \\
(2I_2 - I_3)(I_2 - I_3)c_1 = 0, \quad (I_2 - I_3)(I_1 - I_3)c_1 = 0.
\end{gathered}$$

We solve them by using `solve` and obtain six solutions:

$$\begin{gathered}
\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_3 = 0\}, \\
\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_3 = c_3\},
\end{gathered}$$

$$\begin{aligned}
&\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_3 = 0\},\\
&\{I_1 = 2I_2,\ I_2 = I_2,\ I_3 = 2I_2,\ c_1 = 0,\ c_3 = c_3\},\\
&\{I_1 = I_3,\ I_2 = I_3/2,\ I_3 = I_3,\ c_1 = c_1,\ c_3 = c_3\},\\
&\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_3 = c_3\}.
\end{aligned}$$

Taking into account that now $c_2 = 0$ we see that the first solution leads to the Euler case, the second and third solutions lead to the Lagrange case, the fourth and fifth to the Kovalevskaya case, and the last one to the kinetic symmetry case.

Thus a partial integral of type 2 does not exist when $c_2 = 0$ either.

Type 4. Let $F(\omega_1, \gamma_1, \gamma_2, \gamma_3)$ be a new first integral of type 4. Thus we have

$$\begin{aligned}
I_1\frac{dF}{dt} &= [(I_2 - I_3)\omega_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3]\frac{\partial F}{\partial \omega_1} + I_1(\Omega_3\gamma_2 - \omega_2\gamma_3)\frac{\partial F}{\partial \gamma_1}\\
&\quad - I_1(\Omega_3\gamma_1 - \omega_1\gamma_3)\frac{\partial F}{\partial \gamma_2} + I_1(\omega_2\gamma_1 - \omega_1\gamma_2)\frac{\partial F}{\partial \gamma_3} = Y_1(F) = 0, \qquad (5.71)
\end{aligned}$$

where Y_1 is a vector field on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

Equation (5.71) is an identity with respect to all the five variables. F does not depend on ω_2. Thus if we differentiate this identity with respect to ω_2 we again obtain a linear partial differential equation for F. Let us note that

$$\frac{\partial \Omega_3}{\partial \omega_2} = -\frac{I_2\omega_2}{I_3\Omega_3}.$$

In this way we have

$$\begin{aligned}
I_3\Omega_3\frac{\partial Y_1(F)}{\partial \omega_2} &= (I_2 - I_3)(I_3\Omega_3^2 - I_2\omega_2^2)\frac{\partial F}{\partial \omega_1} - I_1(I_2\omega_2\gamma_2 + I_3\gamma_3\Omega_3)\frac{\partial F}{\partial \gamma_1}\\
&\quad + I_1I_2\omega_2\gamma_1\frac{\partial F}{\partial \gamma_2} + I_1I_3\gamma_1\Omega_3\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \qquad (5.72)
\end{aligned}$$

where Y_2 is a vector field on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

We differentiate (5.72) with respect to ω_2 and obtain

$$\begin{aligned}
\Omega_3\frac{\partial Y_2(F)}{\partial \omega_2} &= -4I_2(I_2 - I_3)\omega_2\Omega_3\frac{\partial F}{\partial \omega_1} - I_1I_2(\gamma_2\Omega_3 - \omega_2\gamma_3)\frac{\partial F}{\partial \gamma_1}\\
&\quad + I_1I_2\gamma_1\Omega_3\frac{\partial F}{\partial \gamma_2} - I_1I_2\omega_2\gamma_1\frac{\partial F}{\partial \gamma_3} = Y_3(F) = 0, \qquad (5.73)
\end{aligned}$$

where Y_3 is a vector field on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

Let us note that the first integral H_2 is of type 4, i.e. it satisfies system (5.71)–(5.73). Thus if a new first integral exists, then this system will have two non-zero linearly independent solutions $\operatorname{grad} H_2$ and $\operatorname{grad} F$. This is possible if and only if the 3×4 matrix M of the coefficients of system (5.71)–(5.73) satisfies the condition

$$\operatorname{rank} M \leq 2. \qquad (5.74)$$

We compute the determinant M_{124} obtained from M by crossing out its third column and obtain

$$M_{124} = I_1^2 I_2\gamma_2\widehat{M}_{124},$$

where

$$\widehat{M}_{124} = \Omega_3 b_1 + b_2.$$

Here the coefficients b_1 and b_2 are the polynomials given by

$$\begin{aligned}
b_1 &= (I_2 - I_3)(-I_1\omega_1^3\gamma_2 + 3I_1\omega_1^2\omega_2\gamma_1 + 2I_2\omega_1\omega_2^2\gamma_2 - 2c_1\omega_1\gamma_1\gamma_2 - 2c_2\omega_1\gamma_2^2 - 2c_3\omega_1\gamma_2\gamma_3 \\
&\quad + U_3\omega_1\gamma_2 + 6c_1\omega_2\gamma_1^2 + 6c_2\omega_2\gamma_1\gamma_2 + 6c_3\omega_2\gamma_1\gamma_3 - 3U_3\omega_2\gamma_1), \\
b_2 &= -3I_1(I_2 - I_3)\omega_1^3\omega_2\gamma_3 - I_1c_3\omega_1^2\gamma_1\gamma_2 + I_1c_2\omega_1^2\gamma_1\gamma_3 - 2I_2(I_2 - I_3)\omega_1\omega_2^3\gamma_3 \\
&\quad - 6(I_2 - I_3)c_1\omega_1\omega_2\gamma_1\gamma_3 - 6(I_2 - I_3)c_2\omega_1\omega_2\gamma_2\gamma_3 - 6(I_2 - I_3)c_3\omega_1\omega_2\gamma_3^2 \\
&\quad + 3(I_2 - I_3)U_3\omega_1\omega_2\gamma_3 - 2c_1c_3\gamma_1^2\gamma_2 + 2c_1c_2\gamma_1^2\gamma_3 - 2c_2c_3\gamma_1\gamma_2^2 + 2(c_2^2 - c_3^2)\gamma_1\gamma_2\gamma_3 \\
&\quad + c_3U_3\gamma_1\gamma_2 + 2c_2c_3\gamma_1\gamma_3^2 - c_2U_3\gamma_1\gamma_3.
\end{aligned}$$

Taking into account (5.74), $\widehat{M}_{124}$ should vanish identically. According to Proposition 4.3 the coefficients b_1 and b_2 should be zeros because $\Omega_3 \notin \mathbb{C}(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$. The polynomial b_1 has 11 coefficients and b_2 has 15. We only use $b_2 = 0$. In this way we obtain a system of 15 equations for the parameters $\mathcal{I}c$ and U_3. After two consecutive simplifications we come to the reduced system

$$c_2 = 0, \quad c_3 = 0, \quad I_2 - I_3 = 0.$$

This leads to the Lagrange case, and therefore a partial integral of type 4 does not exist.

5.4.2. Elimination of γ_3. We now study the elimination of γ_3 from equation (5.53). In this section we suppose that $c_3 \neq 0$ because otherwise the elimination is not possible. We obtain

$$\gamma_3 = \frac{U_3 - I_1\omega_1^2 - I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 - 2c_2\gamma_2}{2c_3}. \tag{5.75}$$

To shorten formulas, we denote the right side of (5.75) by Γ_3 so that $\gamma_3 = \Gamma_3$.

Now we put this value of γ_3 in the Euler–Poisson equations (1.1) and remove the sixth equation. In this way we obtain the following system of five differential equations:

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{(I_2 - I_3)\omega_2\omega_3 + c_3\gamma_2 - c_2\Gamma_3}{I_1}, \\
\frac{d\omega_2}{dt} &= \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\Gamma_3 - c_3\gamma_1}{I_2}, \\
\frac{d\omega_3}{dt} &= \frac{(I_1 - I_2)\omega_1\omega_3 + c_2\gamma_1 - c_1\gamma_2}{I_3}, \\
\frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \omega_2\Gamma_3, \\
\frac{d\gamma_2}{dt} &= \omega_1\Gamma_3 - \omega_3\gamma_1.
\end{aligned} \tag{5.76}$$

There are five possible types of first integrals of this system which depend on at most four variables:

1. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$ (case (i)).
2. $F(\omega_1, \omega_2, \omega_3, \gamma_2)$ (case (i)).

3. $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$ (case (ii)).
4. $F(\omega_1, \omega_3, \gamma_1, \gamma_2)$ (case (iii)).
5. $F(\omega_2, \omega_3, \gamma_1, \gamma_2)$ (case (iii)).

As the functions belonging to cases (ii) and (iii) have already been examined, it only remains to study functions belonging to case (i).

Type 1. Let us look for a first integral of system (5.76) of type 1, i.e. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$. Then we have

$$\begin{aligned} I_1I_2I_3\frac{dF}{dt} &= I_2I_3[(I_2-I_3)\omega_2\omega_3 + c_3\gamma_2 - c_2\Gamma_3]\frac{\partial F}{\partial\omega_1} \\ &\quad + I_1I_3[(I_3-I_1)\omega_1\omega_3 + c_1\Gamma_3 - c_3\gamma_1]\frac{\partial F}{\partial\omega_2} \\ &\quad + I_1I_2[(I_1-I_2)\omega_1\omega_2 + c_2\gamma_1 - c_1\gamma_2]\frac{\partial F}{\partial\omega_3} \\ &\quad + I_1I_2I_3(\omega_3\gamma_2 - \omega_2\Gamma_3)\frac{\partial F}{\partial\gamma_1} = Z(F) = 0, \end{aligned} \tag{5.77}$$

where Z is a vector field on $\mathbb{C}^5(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$.

The vector field Z is of the form $Z = 2Y_1\gamma_2 + Y_2$, where the polynomial vector fields Y_1 and Y_2 are defined on $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$ as follows:

$$\begin{aligned} Y_1 &= I_2I_3(c_2^2+c_3^2)\frac{\partial}{\partial\omega_1} - I_1I_3c_1c_2\frac{\partial}{\partial\omega_2} - I_1I_2c_1c_3\frac{\partial}{\partial\omega_3} + I_1I_2I_3(c_2\omega_2 + c_3\omega_3)\frac{\partial}{\partial\gamma_1}, \\ Y_2 &= I_2I_3[c_2(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3) + 2(I_2-I_3)c_3\omega_2\omega_3]\frac{\partial}{\partial\omega_1} \\ &\quad - I_1I_3[c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3) + 2(I_1-I_3)c_3\omega_1\omega_3 + 2c_3^2\gamma_1]\frac{\partial}{\partial\omega_2} \\ &\quad + 2I_1I_2c_3[(I_1-I_2)\omega_1\omega_2 + c_2\gamma_1]\frac{\partial}{\partial\omega_3} \\ &\quad + I_1I_2I_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3)\omega_2\frac{\partial}{\partial\gamma_1}. \end{aligned}$$

Taking into account that (5.77) should be an identity with respect to all the variables and that F does not depend on γ_2, we conclude that

$$Y_1(F) = Y_2(F) = 0. \tag{5.78}$$

We compute the Lie bracket $Y_3 = [Y_1, Y_2]/(2I_1I_2I_3)$ and obtain

$$\begin{aligned} Y_3 &= [I_2I_3c_2(c_2^2+c_3^2)\omega_1 - I_2(I_2-I_3)c_1c_3^2\omega_2 - I_3(I_2-I_3)c_1c_2c_3\omega_3]\frac{\partial}{\partial\omega_1} \\ &\quad + [I_1c_1(I_1c_3^2 - I_3c_2^2 - 2I_3c_3^2)\omega_1 - I_1I_3c_2c_3^2\omega_2 \\ &\quad - I_3(I_1c_2^2 + 2I_1c_3^2 - I_3c_2^2 - I_3c_3^2)c_3\omega_3]\frac{\partial}{\partial\omega_2} \\ &\quad - c_3[I_1(I_1-I_2)c_1c_2\omega_1 - I_2(2I_1c_2^2 + I_1c_3^2 - I_2c_2^2 - I_2c_3^2)\omega_2 - I_1I_2c_3c_2\omega_3]\frac{\partial}{\partial\omega_3} \\ &\quad - I_1[I_2(I_1c_3^2 - I_2c_3^2 - I_3c_2^2 - I_3c_3^2)\omega_1\omega_2 \\ &\quad - I_3(I_1-I_3)c_2c_3\omega_1\omega_3 + (I_2-I_3)c_2c_3^2\gamma_1]\frac{\partial}{\partial\gamma_1}. \end{aligned}$$

Then we compute $Y_4 = [Y_2, Y_3]$. Unfortunately, the expression for Y_4 is too long to be given here.

Equations (5.78) imply that

$$Y_3(F) = Y_4(F) = 0. \tag{5.79}$$

System (5.78)–(5.79) can be considered as a homogeneous linear algebraic system with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_1}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if a fourth integral F exists, system (5.78)–(5.79) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D of the coefficients of this system is identically zero. We compute

$$D = I_1^2 I_2^2 I_3^2 c_3^3 \widehat{D},$$

where $\widehat{D}$ is a very long expression that we cannot give here. This expression is a polynomial in ω_1, ω_2, ω_3 and γ_1 with 79 coefficients, which are polynomials in the parameters $\mathcal{I}c$ and U_3. As $c_3 \neq 0$ the equation $D = 0$ is equivalent to $\widehat{D} = 0$. Thus all the coefficients of $\widehat{D}$ should be zeros and we have to solve the corresponding system of 79 equations.

After three consecutive simplifications we come to the reduced system of six equations

$$(I_2 - I_3)c_2 = 0, \quad (I_1 - I_3)c_2 = 0, \quad (I_1 - I_3)c_1 = 0, \quad (2I_2 - I_3)(I_1 - I_2) = 0,$$
$$(I_1 - I_2)(2I_1 + 2I_2 - 3I_3) = 0, \quad (2I_2 - I_3)(I_2 - I_3)c_1 = 0.$$

Solving these equations by `solve` we obtain four solutions

$$\begin{aligned}
&\{U_3 = U_3,\ I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\},\\
&\{U_3 = U_3,\ I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = c_3\},\\
&\{U_3 = U_3,\ I_1 = 2I_2,\ I_2 = I_2,\ I_3 = 2I_2,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\},\\
&\{U_3 = U_3,\ I_1 = I_3,\ I_2 = I_3/2,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = c_3\}.
\end{aligned}$$

The first solution leads to the Lagrange case, the second to the kinetic symmetry case, and the remaining two solutions to the Kovalevskaya case.

Thus a partial integral of type 1 does not exist.

6. The gyrostat

6.1. The gyrostat equations. These equations are only slightly modified Euler–Poisson equations (1.1):

$$\begin{aligned}
I_1 \frac{d\omega_1}{dt} &= (I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + Mg(c_3\gamma_2 - c_2\gamma_3),\\
I_2 \frac{d\omega_2}{dt} &= (I_3 - I_1)\omega_1\omega_3 + b_1\omega_3 - b_3\omega_1 + Mg(c_1\gamma_3 - c_3\gamma_1),\\
I_3 \frac{d\omega_3}{dt} &= (I_1 - I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + Mg(c_2\gamma_1 - c_1\gamma_2),
\end{aligned} \tag{6.1}$$

$$\begin{aligned}\frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \omega_2\gamma_3,\\ \frac{d\gamma_2}{dt} &= \omega_1\gamma_3 - \omega_3\gamma_1,\\ \frac{d\gamma_3}{dt} &= \omega_2\gamma_1 - \omega_1\gamma_2.\end{aligned} \tag{6.1}$$

Just as the Euler–Poisson equations, we study them in the complex domain, and without any restriction of generality, we assume that $Mg = 1$.

Just as for the Euler–Poisson equations, H_2 and H_3 defined by (1.2) continue to be first integrals of the gyrostat equations (6.1). This is no more true for H_1 defined by (1.2). The area first integral for the gyrostat is

$$H_1 = I_1\omega_1\gamma_1 + I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - b_1\gamma_1 - b_2\gamma_2 - b_3\gamma_3. \tag{6.2}$$

Up to the end of Sec. 6, H_1 is defined by (6.2) and is a first integral of (6.1). The first integrals H_1, H_2 and H_3 are always functionally independent.

Formally the definition of permutational symmetries cannot be applied to the the gyrostat equations because the numbers of variables and of parameters do not coincide. But in fact it is easy to see that all permutational symmetries of the gyrostat equations, like for Euler–Poisson equations, coincide with the symmetric group S_3, where the same permutation is simultaneously applied to variables $\{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ and to the parameters $\{I_1, I_2, I_3\}$, $\{b_1, b_2, b_3\}$ and $\{c_1, c_2, c_3\}$. It is easy to verify that property (2.2) remains true, that is,

$$\begin{aligned}&V_k(\sigma(\omega), \sigma(\gamma), \sigma(I), \sigma(b), \sigma(c)) = \varepsilon V_{\sigma(k)}(\omega, \gamma, I, b, c),\\ &W_k(\sigma(\omega), \sigma(\gamma)) = \varepsilon W_{\sigma(k)}(\omega, \gamma), \qquad 1 \le k \le 3.\end{aligned}$$

Here $\{V_k\}_{1\le k\le 3}$ are the right sides of the first three gyrostat equations (6.1), $\{W_k\}_{1\le k\le 3}$ of the remaining three equations (6.1), and $\varepsilon = \pm 1$ only depends on the choice of $\sigma \in S_3$. The same concerns the analogue of Theorem 2.2. We leave the details to the reader.

The known integrable cases for the real gyrostat equations (6.1) are the same as for the Euler–Poisson equations (1.1) but with some additional restrictions on the constants b_i, $1 \le i \le 3$. Up to permutational symmetry they are the following ones. These cases remain valid also for the complex gyrostat equations.

The *Zhukovskii case*, which is an extension of the Euler case [25, 28], is defined by the condition (1.3) without additional restrictions on b_i, $1 \le i \le 3$. The fourth integral is

$$H_4 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 - 2(I_1b_1\omega_1 + I_2b_2\omega_2 + I_3b_3\omega_3).$$

When $b_1 = b_2 = b_3 = 0$ we recover the fourth integral of the Euler case.

The *Lagrange case* for the gyrostat [25, 28] is defined by the conditions (1.4) and $b_1 = b_2 = 0$. The fourth integral in this case is the same as for the Euler–Poisson equations, i.e.

$$H_4 = \omega_3.$$

The *Yehia case* [25, 87], which is an extension of the Kovalevskaya case, is defined by the conditions (1.5) and $b_1 = b_2 = 0$. The fourth integral in this case is

$$\begin{aligned} H_4 &= [I_3(\omega_1^2-\omega_2^2) - c_1\gamma_1 + c_2\gamma_2]^2 + (2I_3\omega_1\omega_2 - c_1\gamma_2 - c_2\gamma_1)^2 \\ &\quad + 4b_3\gamma_3(c_1\omega_1 + c_2\omega_2) - 2b_3(\omega_1^2+\omega_2^2)(I_3\omega_3 + b_3). \end{aligned} \tag{6.3}$$

When $b_3 = 0$ we recover the Kovalevskaya fourth integral (1.7).

The *kinetic symmetry case* for the gyrostat is defined by the conditions (1.6) together with the condition that the vectors (c_1, c_2, c_3) and (b_1, b_2, b_3) are proportional, i.e.

$$b_1c_3 = b_3c_1, \quad b_2c_3 = b_3c_2, \quad b_2c_1 = b_1c_2,$$

and the fourth integral is the same as for the Euler–Poisson equations, i.e.

$$H_4 = c_1\omega_1 + c_2\omega_2 + c_3\omega_3.$$

Let us note that except for the Yehia case, in all remaining three cases, the fourth integral can be found along the same lines as in [67], where fourth integrals are computed for integrable cases of the Euler–Poisson equations.

This is not so for the Yehia fourth integral because even if $c_2 = 0$, it depends on all variables. When $c_2 = 0$, this fourth integral can be found in [25] and [87]. Comparing formula (1.7) for a fourth integral in the Kovalevskaya case when $c_2 \neq 0$ and when $c_2 = 0$ with formula (6.3) when $c_2 = 0$, it is natural to conjecture that formula (6.3) with an arbitrary c_2 defines a fourth integral in the general Yehia case. A simple MAPLE computation confirms this.

Just as for the Euler–Poisson equations, we will call these four cases *classical* integrable cases.

6.2. The Sretenskii case. In 1963 L. N. Sretenskii discovered an extension of the Goryachev–Chaplygin partial integral (1.8) of the Euler–Poisson equations to the gyrostat case [71, 72].

Now we apply the method used in Sec. 5.2 that led to the successful derivation of the Goryachev–Chaplygin case for the Euler–Poisson equations to the gyrostat equations (6.1). The computations are almost the same, only slightly longer. Therefore we do not give the details here.

We express γ_2 from the equation $H_1 = U_1$, where H_1 is the function given by (6.2), and obtain

$$\gamma_2 = -\frac{(I_1\omega_1 - b_1)\gamma_1 + (I_3\omega_3 - b_3)\gamma_3 + U_1}{I_2\omega_2 - b_2}.$$

We put this expression for γ_2 in the gyrostat equations (6.1) and remove the fifth equation. We study the resulting system of five equations for the existence of a new first integral $F(\omega_1, \omega_2, \omega_3, \gamma_3)$, i.e. one which does not depend on γ_1. For this purpose we compute $\frac{dF}{dt}$ and take only its numerator. It is easily seen that the resulting expression can be represented in the following way:

$$\frac{dF}{dt} = \gamma_1 Y_1(F) + Y_2(F) = 0, \tag{6.4}$$

where Y_1 and Y_2 are vector fields on $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_3)$. As (6.4) is an identity with respect to all the variables and as $Y_1(F)$ and $Y_2(F)$ do not depend on γ_1 we have

$$Y_1(F) = Y_2(F) = 0. \tag{6.5}$$

We compute the Lie brackets $Y_3 = [Y_1, Y_2]$ and $Y_4 = [Y_1, Y_3]$. Taking into account equations (6.5) we find that

$$Y_3(F) = Y_4(F) = 0. \tag{6.6}$$

Equations (6.5) and (6.6) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically on any open subset of the domain of definition of F, because F is non-constant on any such subset.

If a new integral F exists, system (6.5)–(6.6) has at least one non-zero solution. Let us consider the 4×4 matrix A whose columns are the coefficients of the vector fields Y_1, Y_2, Y_3 and Y_4. The condition under which system (6.5)–(6.6) has at least one non-zero solution is

$$\operatorname{rank} A \leq 3.$$

We equate to zero the determinant D of A and study when it is identically equal to zero.

D is a polynomial in ω_1, ω_2, ω_3 and γ_3. We consider the system consisting of the coefficients of D equated to zero. This system has 226 equations in the unknowns U_1, I_i, b_i, c_i, $1 \leq i \leq 3$. After four consecutive simplifications we obtain the reduced system of 28 equations. Solving these equations by the MAPLE command `solve` we obtain nine solutions. Two of them contain zero values of the moments of inertia, two lead to the Lagrange case and three lead to the kinetic symmetry case. Thus only two essential solutions remain:

- $I_1 = 4I_3, \quad I_2 = 4I_3, \quad b_1 = 0, \quad b_2 = 0, \quad c_3 = 0;$
- $c_1 = c_2 = 0.$

Studying them exactly as in Sec. 5.2 we find that the first solution leads to a partial integral H_4 under the additional restriction $U_1 = 0$, which means that $H_1 = 0$. In this case the fourth integral is

$$H_4 = (I_3\omega_3 + b_3)(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3,$$

which is the Sretenskii partial integral of the gyrostat equations (6.1). When $b_3 = 0$ we recover the Goryachev–Chaplygin partial integral. As noted in Sec. 1.3 this result was already announced in [19]. As in the Goryachev–Chaplygin case, the Sretenskii case is also integrable in the sense of Sec. 1.1.

The second solution, when investigated, imposes additional restrictions $I_1 = I_2$ and $b_1 = b_2 = 0$, i.e. leads to the Lagrange case.

6.3. The new complex integrable cases. If we restrict ourselves to the real case, then the 1906 theorem of E. Husson [39, 40] asserts that for the Euler–Poisson equations only in the four classical cases the fourth integral is an algebraic function [3, 20, 24, 62]. The completely analogous assertion for the real gyrostat equations was proved in 1992 by L. Gavrilov [25].

The main result of [67] can be formulated as follows. For the complex Euler–Poisson equations, the fourth integral that does not depend on all variables exists only in the four classical cases.

The theorem below proves that for the complex gyrostat equations (6.1) the analog of the main result of [67] is not true. As a consequence, in the complex setting the analog of the Gavrilov theorem fails. Indeed, in the proof of this theorem we find two new cases of integrability with not just algebraic but polynomial fourth integrals.

Theorem 6.1. *Up to permutational symmetry the complex gyrostat equations* (6.1) *admit exactly two new* (*non-classical*) *integrable cases with a fourth integral which does not depend on all variables. These cases are*

$$I_1 = I_2 = 2I_3,\ b_1 = -i\varepsilon b_2,\ b_3 = 0,\ c_1 = i\varepsilon c_2,\ c_3 = 0, \tag{6.7}$$

where $\varepsilon = \pm 1$. *In both cases, the fourth integral can be found as the product of a quadratic polynomial with the square of a linear polynomial. Indeed,*

(a) *if* $\varepsilon = 1$, $I_1 = I_2 = 2I_3$, $b_1 = -ib_2$, $b_3 = 0$, $c_1 = ic_2$, $c_3 = 0$,
then $H_{4+} = (I_3\omega_1^2 + 2iI_3\omega_1\omega_2 - I_3\omega_2^2 - 2ic_2\gamma_1 + 2c_2\gamma_2)(I_3\omega_1 - iI_3\omega_2 + 2ib_2)^2$,
(b) *if* $\varepsilon = -1$, $I_1 = I_2 = 2I_3$, $b_1 = ib_2$, $b_3 = 0$, $c_1 = -ic_2$, $c_3 = 0$,
then $H_{4-} = (I_3\omega_1^2 - 2iI_3\omega_1\omega_2 - I_3\omega_2^2 + 2ic_2\gamma_1 + 2c_2\gamma_2)(I_3\omega_1 + iI_3\omega_2 - 2ib_2)^2$.

Proof. Let us look for example for a fourth integral F of the gyrostat equations (6.1) that does not depend on ω_3, i.e. $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

We compute the derivative of F with respect to the gyrostat equations and obtain

$$\begin{aligned} I_1 I_2 \frac{dF}{dt} &= I_2[(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\gamma_3]\frac{\partial F}{\partial \omega_1} \\ &\quad + I_1[(I_3 - I_1)\omega_1\omega_3 + b_1\omega_3 - b_3\omega_1 + c_1\gamma_3 - c_3\gamma_1]\frac{\partial F}{\partial \omega_2} \\ &\quad + I_1 I_2\left[(\omega_3\gamma_2 - \omega_2\gamma_3)\frac{\partial F}{\partial \gamma_1} + (\omega_1\gamma_3 - \omega_3\gamma_1)\frac{\partial F}{\partial \gamma_2} + (\omega_2\gamma_1 - \omega_1\gamma_2)\frac{\partial F}{\partial \gamma_3}\right] = 0. \end{aligned}$$

It is easily seen that

$$I_1 I_2 \frac{dF}{dt} = \omega_3 Y_1(F) + Y_2(F) = 0, \tag{6.8}$$

where Y_1 and Y_2 are the following vector fields not depending on ω_3, defined on $\mathbb{C}^5 = \mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$:

$$\begin{aligned} Y_1 &= I_2[(I_2 - I_3)\omega_2 - b_2]\frac{\partial}{\partial \omega_1} + I_1[(I_3 - I_1)\omega_1 + b_1]\frac{\partial}{\partial \omega_2} + I_1 I_2\gamma_2\frac{\partial}{\partial \gamma_1} - I_1 I_2\gamma_1\frac{\partial}{\partial \gamma_2}, \\ Y_2 &= I_2(b_3\omega_2 + c_3\gamma_2 - c_2\gamma_3)\frac{\partial}{\partial \omega_1} + I_1(-b_3\omega_1 + c_1\gamma_3 - c_3\gamma_1)\frac{\partial}{\partial \omega_2} \\ &\quad + I_1 I_2\left[-\omega_2\gamma_3\frac{\partial}{\partial \gamma_1} + \omega_1\gamma_3\frac{\partial}{\partial \gamma_2} + (\omega_2\gamma_1 - \omega_1\gamma_2)\frac{\partial}{\partial \gamma_3}\right]. \end{aligned}$$

As $Y_1(F)$ and $Y_2(F)$ do not depend on ω_3, the identity (6.8) implies that

$$Y_1(F) = Y_2(F) = 0. \tag{6.9}$$

We consider the Lie brackets $Y_3 = [Y_1, Y_2]/(I_1 I_2)$ and $Y_4 = [Y_1, Y_3]$ and obtain

$$\begin{aligned}
Y_3 &= [(I_2 - I_1)b_3\omega_1 - I_3c_3\gamma_1 + (I_3 - I_2)c_1\gamma_3 + b_1b_3]\frac{\partial}{\partial\omega_1}\\
&\quad + [(I_1 - I_2)b_3\omega_2 - I_3c_3\gamma_2 + (I_3 - I_1)c_2\gamma_3 + b_2b_3]\frac{\partial}{\partial\omega_2}\\
&\quad + I_1\gamma_3[(I_1 - I_2 - I_3)\omega_1 - b_1]\frac{\partial}{\partial\gamma_1} + I_2\gamma_3[(I_2 - I_1 - I_3)\omega_2 - b_2]\frac{\partial}{\partial\gamma_2}\\
&\quad + [-I_1(I_1 - I_2 - I_3)\omega_1\gamma_1 - I_2(I_2 - I_1 - I_3)\omega_2\gamma_2 + I_1b_1\gamma_1 + I_2b_2\gamma_2]\frac{\partial}{\partial\gamma_3},\\
Y_4 &= -I_2\bigl[2(I_2 - I_3)(I_1 - I_2)b_3\omega_2 + I_3(-I_2 + I_1 + I_3)c_3\gamma_2\\
&\quad - (I_2 - I_3)(I_1 - I_3)c_2\gamma_3 - (I_1 - 2I_2 + I_3)b_2b_3\bigr]\frac{\partial}{\partial\omega_1}\\
&\quad - I_1\bigl[2(I_1 - I_3)(I_1 - I_2)b_3\omega_1 + I_3(I_1 - I_2 - I_3)c_3\gamma_1\\
&\quad + (I_2 - I_3)(I_1 - I_3)c_1\gamma_3 - (2I_1 - I_2 - I_3)b_1b_3\bigr]\frac{\partial}{\partial\omega_2}\\
&\quad + I_1I_2\Bigl\{\gamma_3[(2I_2I_1 + I_2I_3 - 2I_2^2 + I_3^2 - I_1I_3)\omega_2 - (I_1 - 2I_2 - I_3)b_2]\frac{\partial}{\partial\gamma_1}\\
&\quad + \gamma_3[(I_2I_3 + 2I_1^2 - 2I_2I_1 - I_3^2 - I_1I_3)\omega_1 - (2I_1 - I_2 + I_3)b_1]\frac{\partial}{\partial\gamma_2}\\
&\quad - \bigl[(I_2I_3 + 2I_1^2 - 2I_1I_2 - I_3^2 - I_1I_3)\omega_1\gamma_2 + (2I_1I_2 + I_2I_3 - 2I_2^2 + I_3^2 - I_1I_3)\omega_2\gamma_1\\
&\quad - (I_1 - 2I_2 - I_3)b_2\gamma_1 - (2I_1 - I_2 + I_3)b_1\gamma_2\bigr]\frac{\partial}{\partial\gamma_3}\Bigr\}.
\end{aligned}$$

Equations (6.9) imply that

$$Y_3(F) = Y_4(F) = 0. \tag{6.10}$$

The system (6.9)–(6.10) is a linear homogeneous system in the unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_1}, \frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\gamma_1}, \frac{\partial F}{\partial\gamma_2}, \frac{\partial F}{\partial\gamma_3}\right)$, which do not vanish identically on any open subset of the domain of definition of F, because F is non-constant on any such open subset.

As H_2 is a first integral of the sought type, i.e. it does not depend on ω_3, if a new integral F exists then system (6.9)–(6.10) should have at least two linearly independent non-zero solutions. Let us consider the 4×5 matrix A whose rows are the coefficients of the vector fields Y_1, Y_2, Y_3 and Y_4. The condition under which system (6.9)–(6.10) has at least two non-zero solutions is

$$\operatorname{rank} A \leq 3.$$

We compute all the five 4×4 minors of the matrix A and require that they be identically zero. Denoting them by D_{ijkl}, where the subscript contains the numbers of the relevant columns of A, we see that $D_{1345} = D_{2345} = 0$. Thus it remains to study when the minors D_{1234}, D_{1235} and D_{1245} vanish identically. These three minors are polynomials in ω_1, ω_2, γ_1, γ_2 and γ_3 with coefficients that are polynomials in I_i, b_i and c_i, $1 \leq i \leq 3$. We cannot write here the expressions for D_{1234}, D_{1235} and D_{1245} because they are too long. They have non-zero factors which we remove by setting

$$d_{1234} = \frac{D_{1234}}{I_1^2I_2^2\gamma_3}, \quad d_{1235} = \frac{D_{1235}}{I_1^2I_2^2\gamma_2}, \quad d_{1245} = \frac{D_{1245}}{I_1^2I_2^2\gamma_1}.$$

After this cancellation of non-zero factors it turns out that

$$d_{1234} = -d_{1235} = d_{1245}.$$

We can therefore restrict ourselves to considering the identity $d_{1234} = 0$.

The polynomial d_{1234} has 83 monomials and therefore 83 coefficients which should vanish. We consider the system consisting of the coefficients of d_{1234} equated to zero, i.e. the system of 83 equations in unknowns I_i, b_i and c_i, $1 \le i \le 3$. After six consecutive simplifications we obtain a reduced system consisting of 29 equations. Solving that system using `solve` we obtain the following ten solutions:

(1) $\{I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = 0,\ b_2 = b_2,\ b_3 = 0,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0\}$,
(2) $\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0\}$,
(3) $\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = 0,\ b_3 = 0,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\}$,
(4) $\{I_1 = 0,\ I_2 = 0,\ I_3 = 0,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = 0\}$,
(5) $\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = \frac{c_1 b_2}{c_2},\ b_2 = b_2,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = 0\}$,
(6) $\{I_1 = 2I_3,\ I_2 = 2I_3,\ I_3 = I_3,\ b_1 = -i\varepsilon b_2,\ b_2 = b_2,\ b_3 = 0,\ c_1 = i\varepsilon c_2,\ c_2 = c_2,\ c_3 = 0\}$,
(7) $\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = 0,\ b_2 = 0,\ b_3 = b_3,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = 0\}$,
(8) $\{I_1 = -I_3,\ I_2 = -I_3,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\}$,
(9) $\{I_1 = 2I_3,\ I_2 = 2I_3,\ I_3 = I_3,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = c_3\}$,
(10) $\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = 0,\ b_2 = 0,\ b_3 = b_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\}$,

where $\varepsilon = \pm 1$.

A careful study of this list shows that only three solutions are essential. They are the sixth, seventh and eighth solutions. All other solutions lead to some of the classical cases of integrability of the gyrostat equations. Let us stress that the ninth solution implies $b_1 = b_2 = b_3 = 0$ and therefore the gyrostat equations become the Euler–Poisson equations whose fourth integrals not depending on all variables have been studied in [67]. Note that the sixth solution leads to the condition (6.7). Below we examine the three essential solutions.

Solution 6+: $I_1 = I_2 = 2I_3$, $b_1 = -i\varepsilon b_2$, $b_3 = 0$, $c_1 = i\varepsilon c_2$, $c_3 = 0$, $\varepsilon = 1$.

Under these conditions the vector fields Y_i, $1 \le i \le 4$, are linearly dependent as $d_{1234} = 0$. More precisely, we have

$$(I_3\omega_1 - iI_3\omega_2 - ib_2)Y_4 + I_3^2(I_3\omega_1 - iI_3\omega_2 - 4ib_2)Y_2 + 6I_3^2 b_2 Y_3 = 0.$$

We compute $Y_5 = [Y_2, Y_3]/[4I_3^2(I_3\omega_1 - iI_3\omega_2 - ib_2)]$ and obtain

$$Y_5 = -c_2(\gamma_1 + i\gamma_2)\left(\frac{\partial}{\partial\omega_1} - i\frac{\partial}{\partial\omega_2}\right) - 2I_3(\omega_1 + i\omega_2)\left(\gamma_2\frac{\partial}{\partial\gamma_1} - \gamma_1\frac{\partial}{\partial\gamma_2}\right).$$

Like Y_4, the vector field Y_5 is also linearly dependent on Y_2 and Y_3. Indeed, we have

$$2I_3(I_3\omega_1 - iI_3\omega_2 - ib_2)\gamma_3 Y_5 - (I_3\omega_1\gamma_1 + I_3\omega_2\gamma_2 - ib_2\gamma_1 + b_2\gamma_2)Y_2 \\ + 2I_3(\omega_1\gamma_2 - \omega_2\gamma_1)Y_3 = 0.$$

Moreover, easy computations show that the vector fields Y_i, $1 \le i \le 3$, are linearly independent. Thus the system $Y_i(F) = 0$, $1 \le i \le 3$, is in involution and according to the Frobenius Integrability Theorem it should have two functionally independent solutions.

The first one is the function H_2. Finding another one, functionally independent of H_2, is not feasible with crude use of the MAPLE command `pdsolve`. To overcome this difficulty we add the fourth equation $Y_0(F) = 0$, where $Y_0 = \frac{\partial}{\partial \gamma_3}$. We choose such Y_0 because $Y_0(H_2) \neq 0$. The MAPLE command `pdsolve` applied to the system of four equations $Y_i(F) = 0$, $0 \le i \le 3$, gives the solution

$$F = G\Big[-\frac{(I_3\omega_1^2 + 2iI_3\omega_1\omega_2 - I_3\omega_2^2 - 2ic_2\gamma_1 + 2c_2\gamma_2)(I_3\omega_1 - iI_3\omega_2 + 2ib_2)^2}{2c_2} \Big],$$

where G is an arbitrary smooth function. As a second solution of the system $Y_i(F) = 0$, $1 \le i \le 3$, we take the function

$$H_{4+} = (I_3\omega_1^2 + 2iI_3\omega_1\omega_2 - I_3\omega_2^2 - 2ic_2\gamma_1 + 2c_2\gamma_2)(I_3\omega_1 - iI_3\omega_2 + 2ib_2)^2,$$

which corresponds to $G(x) = -2c_2x$.

Solution 6: $I_1 = I_2 = 2I_3$, $b_1 = -i\varepsilon b_2$, $b_3 = 0$, $c_1 = i\varepsilon c_2$, $c_3 = 0$, $\varepsilon = -1$.

In this case in a completely analogous way we find

$$H_{4-} = (I_3\omega_1^2 - 2iI_3\omega_1\omega_2 - I_3\omega_2^2 + 2ic_2\gamma_1 + 2c_2\gamma_2)(I_3\omega_1 + iI_3\omega_2 - 2ib_2)^2.$$

It is easy to verify that H_{4+} and H_{4-} are functionally independent of the first integrals H_1 (see (6.2)), H_2 and H_3 (see (1.2)) and thus they are fourth integrals of the gyrostat equations (6.1), for $\varepsilon = 1$ and $\varepsilon = -1$ respectively.

Solution 7: $I_1 = I_2 = I_3$, $b_1 = b_2 = 0$, $c_3 = 0$.

As in the previous case, due to the equality $d_{1234} = 0$, the Y_i, $1 \le i \le 4$, are linearly dependent. Indeed, we have

$$(\omega_1\gamma_1 + \omega_2\gamma_2)Y_4 - I_3^3(\omega_1^2 + \omega_2^2)\gamma_3 Y_1 - I_3^3(\omega_1\gamma_2 - \omega_2\gamma_1)Y_3 = 0.$$

We compute $Y_5 = [Y_2, Y_3]/I_3^3$ and obtain

$$\begin{aligned} Y_5 = (\omega_1\gamma_1 + \omega_2\gamma_2)\Big(c_2\frac{\partial}{\partial\omega_1} - c_1\frac{\partial}{\partial\omega_2}\Big) + [I_3(\omega_2^2 + \omega_1^2)\gamma_2 - (b_3\omega_2 - c_2\gamma_3)\gamma_3]\frac{\partial}{\partial\gamma_1} \\ - [I_3(\omega_1^2 + \omega_2^2)\gamma_1 - (b_3\omega_1 - c_1\gamma_3)\gamma_3]\frac{\partial}{\partial\gamma_2} \\ - [(b_3\omega_1 - c_1\gamma_3)\gamma_2 - (b_3\omega_2 - c_2\gamma_3)\gamma_1]\frac{\partial}{\partial\gamma_3}. \end{aligned}$$

As H_2 is a first integral of the sought type, the existence of a fourth integral of the gyrostat equations requires that the vector fields Y_i, $1 \le i \le 3$, and Y_5 be linearly dependent. We compute the determinant V_{1234} consisting of the first four columns of the matrix of the coefficients of these vector fields and obtain

$$V_{1234} = I_3^5 b_3\gamma_3(\omega_1\gamma_1 + \omega_2\gamma_2)^2(c_2\omega_1 - c_1\omega_2).$$

If the vector fields Y_i, $1 \le i \le 3$, and Y_5 are linearly dependent then V_{1234} should be identically zero. It is clear that this happens either if $b_3 = 0$, which leads to the Euler–Poisson equations, or if $c_1 = c_2 = 0$, which leads to the Zhukovskii case. Thus a partial integral for Solution 7 does not exist.

Solution 8: $I_1 = I_2 = -I_3$, $b_3 = 0$, $c_1 = c_2 = 0$.

In this case $Y_4 = -Y_2$. We compute $Y_5 = [Y_2, Y_3]/I_3^2$ and obtain

$$\begin{aligned} Y_5 &= c_3\gamma_3(2I_3\omega_2 + b_2)\frac{\partial}{\partial\omega_1} - c_3\gamma_3(2I_3\omega_1 + b_1)\frac{\partial}{\partial\omega_2} \\ &\quad - I_3(I_3\omega_1^2 + I_3\omega_2^2 + b_1\omega_1 + b_2\omega_2 + 2c_3\gamma_3)\left(\gamma_2\frac{\partial}{\partial\gamma_1} - \gamma_1\frac{\partial}{\partial\gamma_2}\right). \end{aligned}$$

As in Solution 7, we compute the determinant W_{1234} consisting of the first four columns of the matrix of the coefficients of the vector fields Y_i, $1 \le i \le 3$, and Y_5 and we obtain

$$W_{1234} = I_3^4 c_3(I_3\omega_2^2 + I_3\omega_1^2 + b_1\omega_1 + 3c_3\gamma_3 + b_2\omega_2)\gamma_3 w,$$

where

$$\begin{aligned} w &= I_3(b_2\omega_1\gamma_1^2 - 2b_1\omega_1\gamma_1\gamma_2 - b_2\omega_1\gamma_2^2 + b_1\omega_2\gamma_1^2 + 2b_2\omega_2\gamma_1\gamma_2 - b_1\omega_2\gamma_2^2) \\ &\quad + b_2b_1\gamma_1^2 - (b_1^2 - b_2^2)\gamma_1\gamma_2 - b_1b_2\gamma_2^2. \end{aligned}$$

If a new first integral exists then $W_{1234} = 0$ should be fulfilled. To avoid the Zhukovskii case we assume that $c_3 \neq 0$. In that case it is clear that $W_{1234} = 0$ is equivalent to $w = 0$. This is possible if and only if $b_1 = b_2 = 0$, which leads to the Euler–Poisson equations. Thus a new first integral of the sought type does not exist for Solution 8 either.

All the above considerations lead to the conclusion that the gyrostat equations (6.1) admit a fourth integral which does not depend on ω_3 either in certain classical cases or else only when the conditions (6.7) are fulfilled.

Now let us look for a fourth integral F of (6.1) that does not depend on γ_3, i.e. $F = F(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$.

We compute the derivative of F with respect to the gyrostat equations and obtain

$$\begin{aligned} I_1I_2I_3\frac{dF}{dt} &= I_2I_3[(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\gamma_3]\frac{\partial F}{\partial\omega_1} \\ &\quad + I_1I_3[(I_3 - I_1)\omega_1\omega_3 + b_1\omega_3 - b_3\omega_1 + c_1\gamma_3 - c_3\gamma_1]\frac{\partial F}{\partial\omega_2} \\ &\quad + I_1I_2[(I_1 - I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + (c_2\gamma_1 - c_1\gamma_2)]\frac{\partial F}{\partial\omega_3} \\ &\quad + I_1I_2I_3\left[(\omega_3\gamma_2 - \omega_2\gamma_3)\frac{\partial F}{\partial\gamma_1} + (\omega_1\gamma_3 - \omega_3\gamma_1)\frac{\partial F}{\partial\gamma_2}\right] = 0. \end{aligned}$$

It is easily seen that

$$I_1I_2I_3\frac{dF}{dt} = I_3\gamma_3 Z_1(F) + Z_2(F) = 0, \tag{6.11}$$

where Z_1 and Z_2 are the following vector fields not depending on γ_3, defined on $\mathbb{C}^5 = \mathbb{C}^5(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$:

$$\begin{aligned} Z_1 &= -I_2c_2\frac{\partial}{\partial\omega_1} + I_1c_1\frac{\partial}{\partial\omega_2} - I_1I_2\left(\omega_2\frac{\partial}{\partial\gamma_1} - \omega_1\frac{\partial}{\partial\gamma_2}\right), \\ Z_2 &= I_2I_3[(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2]\frac{\partial}{\partial\omega_1} \\ &\quad + I_1I_3[(I_3 - I_1)\omega_1\omega_3 + b_1\omega_3 - b_3\omega_1 - c_3\gamma_1]\frac{\partial}{\partial\omega_2} \end{aligned}$$

$$
\begin{aligned}
&+ I_1I_2[(I_1-I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + c_2\gamma_1 - c_1\gamma_2]\frac{\partial}{\partial\omega_3}\\
&+ I_1I_2I_3\omega_3\left(\gamma_2\frac{\partial}{\partial\gamma_1} - \gamma_1\frac{\partial}{\partial\gamma_2}\right).
\end{aligned}
$$

As $Z_1(F)$ and $Z_2(F)$ do not depend on γ_3, the identity (6.11) implies that

$$Z_1(F) = Z_2(F) = 0. \tag{6.12}$$

We consider the Lie brackets $Z_3 = [Z_1, Z_2]/(I_1I_2)$, $Z_4 = [Z_1, Z_3]$ and $Z_5 = [Z_2, Z_3]/I_3$ and obtain

$$
\begin{aligned}
Z_3 = {}& I_3[(I_2-I_3)c_1\omega_3 + I_2c_3\omega_1 + c_1b_3]\frac{\partial}{\partial\omega_1} + I_3[(I_1-I_3)c_2\omega_3 + I_1c_3\omega_2 + c_2b_3]\frac{\partial}{\partial\omega_2}\\
&+ [I_1(I_1-2I_2)c_1\omega_1 + (I_2-2I_1)I_2c_2\omega_2 - I_2b_2c_2 - I_1b_1c_1]\frac{\partial}{\partial\omega_3}\\
&- I_1I_3[(I_1-I_2-I_3)\omega_1\omega_3 + b_3\omega_1 - b_1\omega_3 + c_3\gamma_1]\frac{\partial}{\partial\gamma_1}\\
&+ I_3I_2[(I_1-I_2+I_3)\omega_2\omega_3 - b_3\omega_2 + b_2\omega_3 - c_3\gamma_2]\frac{\partial}{\partial\gamma_2},\\
Z_4 = {}& -I_2^2I_3c_2c_3\frac{\partial}{\partial\omega_1} + I_1^2I_3c_1c_3\frac{\partial}{\partial\omega_2} - 3I_1I_2(I_1-I_2)c_2c_1\frac{\partial}{\partial\omega_3}\\
&+ I_1I_2I_3[(2I_1-I_2-2I_3)c_2\omega_3 + 2I_1c_3\omega_2 + 2b_3c_2]\frac{\partial}{\partial\gamma_1}\\
&+ I_1I_2I_3[(I_1-2I_2+2I_3)c_1\omega_3 - 2I_2c_3\omega_1 - 2b_3c_1]\frac{\partial}{\partial\gamma_2},\\
Z_5 = {}& a_1\frac{\partial}{\partial\omega_1} + a_2\frac{\partial}{\partial\omega_2} + a_3\frac{\partial}{\partial\omega_3} + a_4\frac{\partial}{\partial\gamma_1} + a_5\frac{\partial}{\partial\gamma_2},
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 = {}& I_2[I_1I_2(I_2-I_3)c_1\omega_1\omega_2 + I_1(I_1-I_2-I_3)b_2c_1\omega_1 + I_2(I_2-I_3)(2I_1-I_2)c_2\omega_2^2\\
&- I_3(2I_2I_3 - I_1I_3 - 2I_2^2 + 2I_1I_2)c_3\omega_2\omega_3\\
&- (2I_1I_2b_2c_2I_2I_3b_2c_2 + I_1I_3b_3c_3 - 2I_2I_3b_3c_3 - 2I_2^2b_2c_2)\omega_2\\
&- I_3(I_2-I_3)(I_1-I_3)c_2\omega_3^2 - I_3(I_1b_3c_2 - 2I_3b_3c_2 + 2I_2b_2c_3 + I_2b_3c_2)\omega_3\\
&+ I_1(I_2-I_3)c_1c_2\gamma_1 + (2I_2I_3c_3^2 - I_1I_2c_1^2 + I_1I_3c_1^2)\gamma_2 - b_2I_1c_1b_1 - I_3b_3^2c_2 - I_2c_2b_2^2],\\
a_2 = {}& I_1[I_1(I_1-I_3)(I_1-2I_2)c_1\omega_1^2 - I_1I_2(I_1-I_3)c_2\omega_1\omega_2\\
&- I_3(2I_1^2 - 2I_1I_2 - 2I_1I_3 + I_2I_3)c_3\omega_1\omega_3\\
&- (2I_1^2b_1c_1 - 2I_1I_2b_1c_1 - I_1I_3b_1c_1 + 2I_1I_3b_3c_3 - I_2I_3b_3c_3)\omega_1\\
&+ I_2(I_1-I_2+I_3)b_1c_2\omega_2 + I_3(I_2-I_3)(I_1-I_3)c_1\omega_3^2\\
&+ I_3(2I_1b_1c_3 + I_1b_3c_1 + I_2b_3c_1 - 2I_3b_3c_1)\omega_3 + (I_1I_2c_2^2 - 2I_1I_3c_3^2 - I_2I_3c_2^2)\gamma_1\\
&- I_2(I_1-I_3)c_1c_2\gamma_2 + I_1b_1^2c_1 + I_2b_2b_1c_2 + I_3b_3^2c_1],\\
a_3 = {}& I_1I_2[-(I_1-I_2)(I_1+I_2)c_3\omega_1\omega_2 + I_1(2I_1-I_2-2I_3)c_2\omega_1\omega_3 + (2I_1b_3c_2 - I_2b_2c_3)\omega_1\\
&+ I_2(I_1-2I_2+2I_3)c_1\omega_2\omega_3 + (I_1b_1c_3 - 2I_2b_3c_1)\omega_2\\
&+ (I_3b_2c_1 - 2I_1b_1c_2 + I_2b_1c_2 - I_1b_2c_1 - I_3b_1c_2 + 2I_2b_2c_1)\omega_3 + c_3(3I_1-I_2)c_2\gamma_1\\
&+ (I_1-3I_2)c_1c_3\gamma_2 + b_3(b_1c_2 - b_2c_1)],
\end{aligned}
$$

$$\begin{aligned}
a_4 = {} & -I_1I_2[I_1(I_1-I_2)(I_1-I_2-I_3)\omega_1^2\omega_2 + I_1(I_1-I_2-I_3)b_2\omega_1^2 \\
& - I_1(2I_1-I_3-2I_2)b_1\omega_1\omega_2 + I_1(I_1-I_2-I_3)c_2\omega_1\gamma_1 - I_1(I_2-I_3)c_1\omega_1\gamma_2 \\
& - I_1b_2b_1\omega_1 + I_3(I_3^2+I_2I_3-I_1I_3+2I_1I_2-2I_2^2)\omega_2\omega_3^2 + I_3(I_1-I_2-2I_3)b_3\omega_2\omega_3 \\
& - I_2(2I_1-I_2)c_2\omega_2\gamma_2 + (I_1b_1^2+I_3b_3^2)\omega_2 - I_3(I_1-2I_2-I_3)b_2\omega_3^2 \\
& + I_3(2I_1-2I_2-I_3)c_3\omega_3\gamma_2 - I_3b_2b_3\omega_3 - I_1b_1c_2\gamma_1 + (I_3b_3c_3-I_2b_2c_2)\gamma_2], \\
a_5 = {} & I_1I_2[I_2(I_1-I_2)(I_1-I_2+I_3)\omega_1\omega_2^2 + I_2(2I_1-2I_2+I_3)b_2\omega_1\omega_2 \\
& - I_3(2I_1^2-2I_1I_2-I_1I_3+I_2I_3-I_3^2)\omega_1\omega_3^2 - I_3(I_1-I_2+2I_3)b_3\omega_1\omega_3 \\
& + I_1(I_1-2I_2)c_1\omega_1\gamma_1 + (I_2b_2^2+I_3b_3^2)\omega_1 - I_2(I_1-I_2+I_3)b_1\omega_2^2 - c_2I_2(I_1-I_3)\omega_2\gamma_1 \\
& - I_2(I_1-I_2+I_3)c_1\omega_2\gamma_2 - I_2b_1b_2\omega_2 + I_3(2I_1+-I_2I_3)b_1\omega_3^2 \\
& - I_3(2I_1-2I_2+I_3)c_3\omega_3\gamma_1 - I_3b_1b_3\omega_3 - (I_1b_1c_1-I_3b_3c_3)\gamma_1 - I_2c_1b_2\gamma_2].
\end{aligned}$$

Equations (6.12) imply that

$$Z_3(F) = Z_4(F) = Z_5(F) = 0. \tag{6.13}$$

If a first integral $F = F(\omega_1,\omega_2,\omega_3,\gamma_1,\gamma_2)$ exists then the system of five equations (6.12) and (6.13) should have a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_1},\frac{\partial F}{\partial\omega_2},\frac{\partial F}{\partial\omega_3},\frac{\partial F}{\partial\gamma_1},\frac{\partial F}{\partial\gamma_2}\right)$. This is possible if and only if the determinant D of the coefficients of that system vanishes identically.

We compute D and obtain a very long polynomial whose presentation here is impossible. But D has a factor $I_1^3I_2^3I_3^2c_3$. Thus we should consider two cases: $c_3 \neq 0$ and $c_3 = 0$.

Let us start with $c_3 \neq 0$. We remove the non-zero factor of D by setting

$$\widehat{D} = \frac{D}{I_1^3I_2^3I_3^2c_3}.$$

The equation $D = 0$ is equivalent to $\widehat{D} = 0$. The polynomial $\widehat{D}$ has 253 coefficients depending on the parameters I_i, b_i and c_i, $1 \le i \le 3$. To satisfy the equation $\widehat{D} = 0$ we should consider the system consisting of the coefficients of $\widehat{D}$ equated to zero, i.e. the system of 253 equations for the parameters. After three consecutive simplifications we obtain a reduced system consisting of seven equations. Solving that system using `solve` we obtain the following two solutions:

$$\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = b_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\},$$

$$\left\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = b_3,\ c_1 = \frac{b_1c_2}{b_2},\ c_2 = c_2,\ c_3 = \frac{b_3c_2}{b_2}\right\}.$$

The second solution leads to the kinetic symmetry case. Thus only the first solution should be studied. For this, let $I_1 = I_2$, $c_1 = c_2 = 0$. Under these conditions we have $Z_4 + 2I_2I_3c_3Z_1 = 0$. We compute $Z_6 = [Z_3, Z_5]/(I_2^3I_3c_3)$ and the determinant M of the coefficients of the vector fields Z_i, $1 \le i \le 3$, Z_5 and Z_6. We know that if the sought first integral exists then $M = 0$. We have

$$M = -I_2^8I_3^2c_3^2(b_2\omega_1 - b_1\omega_2)^2\widehat{M},$$

where

$$\begin{aligned}\widehat{M} = {} & -3I_3(I_2-3I_3)\omega_1^4\omega_3 - 9I_3b_3\omega_1^4 - 2(I_2-11I_3)b_1\omega_1^3\omega_3 - 15I_3c_3\omega_1^3\gamma_1 - 7b_1b_3\omega_1^3 \\ & - 6I_3(I_2-3I_3)\omega_1^2\omega_2^2\omega_3 - 18I_3b_3\omega_1^2\omega_2^2 - 2(I_2-11I_3)b_2\omega_1^2\omega_2\omega_3 - 15I_3c_3\omega_1^2\omega_2\gamma_2 \\ & - 7b_2b_3\omega_1^2\omega_2 + 12b_1^2\omega_1^2\omega_3 - 12b_1c_3\omega_1^2\gamma_1 - 2(I_2-11I_3)b_1\omega_1\omega_2^2\omega_3 - 15I_3c_3\omega_1\omega_2^2\gamma_1 \\ & - 7b_1b_3\omega_1\omega_2^2 + 24b_1b_2\omega_1\omega_2\omega_3 - 12b_2c_3\omega_1\omega_2\gamma_1 - 12b_1c_3\omega_1\omega_2\gamma_2 - 3I_3(I_2-3I_3)\omega_2^4\omega_3 \\ & - 9I_3b_3\omega_2^4 - 2(I_2-11I_3)b_2\omega_2^3\omega_3 - 15I_3c_3\omega_2^3\gamma_2 - 7b_2b_3\omega_2^3 + 12b_2^2\omega_2^2\omega_3 - 12b_2c_3\omega_2^2\gamma_2.\end{aligned}$$

Let us consider, for example, the coefficient of $\omega_1^3\gamma_1$ in $\widehat{M}$, that is, $-15I_3c_3$. But $-15I_3c_3 \neq 0$, therefore $\widehat{M}$ never vanishes. Thus the only possibility to satisfy the equation $M = 0$ is to put $b_1 = b_2 = 0$. Taking into account that now $I_1 = I_2$, $c_1 = c_2 = 0$ we come to the Lagrange case. Thus a new first integral $F = F(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$ does not exist when $c_3 \neq 0$.

We now study the case $c_3 = 0$. Under this condition, the first integral H_3 does not depend on γ_3, i.e. it is of the type sought. If a fourth integral $F = F(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$ exists then system (6.12)–(6.13) should have at least two non-zero solutions. This is possible if and only if the matrix B of its coefficients satisfies $\operatorname{rank} B \leq 3$. This condition means that all four equations of system (6.12)–(6.13) should be linearly dependent. We choose to consider the system

$$Z_i(F) = 0, \quad 1 \leq i \leq 3, \qquad Z_5(F) = 0. \tag{6.14}$$

This choice is appropriate because if we choose, for example, $Z_i(F) = 0$, $1 \leq i \leq 4$, then we come to a great number of cases which should be studied.

We compute all the five 4×4 minors of the 4×5 matrix consisting of the coefficients of system (6.14) and require that they be identically zero. These five minors are polynomials in ω_1, ω_2, γ_1, γ_2 and γ_3 with coefficients that are polynomials in I_i, b_i, $1 \leq i \leq 3$, c_1 and c_2. As before we denote them by Δ_{ijkl} and see that they have some non-zero factors. To remove these factors we introduce the following notations:

$$\delta_{1234} = \frac{\Delta_{1234}}{I_1^2I_2^2I_3}, \qquad \delta_{1235} = \frac{\Delta_{1235}}{I_1^2I_2^2I_3}, \qquad \delta_{1245} = \frac{\Delta_{1245}}{I_1^2I_2^2I_3^2\omega_2},$$
$$\delta_{1345} = \frac{\Delta_{1345}}{I_1^2I_2^3I_3\omega_2}, \qquad \delta_{2345} = \frac{\Delta_{2345}}{I_1^3I_2^2I_3\omega_1}.$$

Let us note that δ_{1234} has a factor c_2 and δ_{1235} has a factor c_1. We have left them intentionally because we do not know whether c_1 or c_2 is zero. But, as we now consider the case $c_3 = 0$, we know that $(c_1, c_2) \neq (0, 0)$ because otherwise we come to the Zhukovskii case.

It turns out that

$$\delta_{1245} = -\delta_{1345} = \delta_{2345}. \tag{6.15}$$

Moreover, if $c_1 \neq 0$ then

$$\frac{\delta_{1235}}{c_1} = -\delta_{1245}, \tag{6.16}$$

independently of the value of c_2. If $c_2 \neq 0$ then

$$\frac{\delta_{1234}}{c_2} = \delta_{1245}, \tag{6.17}$$

independently of the value of c_1. If $c_1 \neq 0$ and $c_2 \neq 0$ we have

$$\frac{\delta_{1234}}{c_2} = -\frac{\delta_{1235}}{c_1}. \tag{6.18}$$

Thus if the identity $\delta_{1245} = 0$ is satisfied then equations (6.14) will be linearly dependent. Indeed, from (6.15) it follows that $\delta_{1345} = \delta_{2345} = 0$ independently of the values of c_1 and c_2.

Let $c_1 \neq 0$ and $c_2 = 0$. Then $\delta_{1234} = 0$ because it has a factor c_2, and $\delta_{1235} = 0$ follows from (6.16).

Let $c_1 \neq 0$ and $c_2 \neq 0$. Then $\delta_{1235} = 0$ follows from (6.16), and $\delta_{1234} = 0$ from (6.18).

Let $c_1 = 0$ and $c_2 \neq 0$. Then $\delta_{1235} = 0$ because it has a factor c_1, and $\delta_{1234} = 0$ follows from (6.17).

The polynomial δ_{1245} has 84 monomials and therefore 84 coefficients which should vanish. We consider the system consisting of the coefficients of δ_{1245} equated to zero, i.e. the system of 84 equations in the unknowns I_i, b_i, $1 \leq i \leq 3$, c_1 and c_2. After eight consecutive simplifications we obtain a reduced system consisting of 15 equations. Using `solve` we obtain the following seven solutions:

$$\{I_1 = 2I_3,\ I_2 = 2I_3,\ I_3 = I_3,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2\},$$
$$\{I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = 0,\ b_2 = b_2,\ b_3 = 0,\ c_1 = 0,\ c_2 = c_2\},$$
$$\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = 0,\ c_1 = 0,\ c_2 = 0\},$$
$$\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = 0,\ b_3 = 0,\ c_1 = c_1,\ c_2 = 0\},$$
$$\{I_1 = 0,\ I_2 = 0,\ I_3 = 0,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2\},$$
$$\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = c_1 b_2/c_2,\ b_2 = b_2,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2\},$$
$$\{I_1 = 2I_3,\ I_2 = 2I_3,\ I_3 = I_3,\ b_1 = -i\varepsilon b_2,\ b_2 = b_2,\ b_3 = 0,\ c_1 = i\varepsilon c_2,\ c_2 = c_2\},$$

where $\varepsilon = \pm 1$.

Examining this list we see that only the last solution is essential. All the other solutions lead either to the classical cases of the gyrostat equations, or to the Euler–Poisson equations, or to the excluded cases with zero moments of inertia.

Taking into account that now $c_3 = 0$ we see that this last solution determines the conditions (6.7).

Under these conditions the vector field Z_4 vanishes identically and Z_5 is linearly dependent on Z_i, $1 \leq i \leq 3$. Moreover, the vector fields Z_i, $1 \leq i \leq 3$, are linearly independent. Thus the system $Z_i(F) = 0$, $1 \leq i \leq 3$, is in involution and according to the Frobenius Integrability Theorem it has two functionally independent solutions. The first one is H_3 and the second one is the fourth integral we look for. But it is not necessary to look for this fourth integral. We should only notice that the fourth integrals H_{4+} from Case 6+ and H_{4-} from Case 6− not only do not depend on ω_3, but also do not depend on γ_3. ■

Thus the problem of characterization of all cases when the complex gyrostat equations have a fourth integral that does not depend on all the variables is solved.

7. Domain of the Sretenskii partial integral

This section is rather isolated from the rest of the work.

7.1. Definition of the domain. Given that the gyrostat equations (6.1) are polynomial with polynomial first integrals, in this section we will restrict ourselves exclusively to polynomial systems of ordinary differential equations and their polynomial first integrals.

Let us consider a polynomial system of ordinary differential equations

$$\frac{dx_i}{dt} = f_i(x), \qquad 1 \leq i \leq n, \tag{7.1}$$

$x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $f_1, \ldots, f_n \in \mathbb{C}[x]$. Considering system (7.1) and its solutions is equivalent to considering the associate polynomial vector field $V(x) = (f_1, \ldots, f_n)$ and its orbits.

A subset of $\mathbb{C}^n$ is V-*invariant* if it is a union of full orbits of the vector field V.

A differentiable function Φ such that the set $\{x;\, \Phi(x) = 0\}$ is a union of orbits of system (7.1) is called an *invariant relation* [24], [46, Chapt. X, §4].

Let $F \in \mathbb{C}[x] \setminus \mathbb{C}$ be some non-constant polynomial that is not a first integral of system (7.1). Let $M \subset \mathbb{C}^n$ be a V-invariant subset such that $F|_M$ is non-constant on any open subset of M. When is $F|_M$ a first integral of system (7.1) (or the vector field V) restricted to M?

To answer this let us compute

$$\frac{dF}{dt}(x) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x) \frac{dx_i}{dt}(x) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x) f_i(x) =: A(x).$$

$A(x)$ is a polynomial, because F and $\{f_i\}_{1 \leq i \leq n}$ are. F is not a first integral of system (7.1) on $\mathbb{C}^n$, so A does not vanishes identically, but $A(x) = 0$ for $x \in M$.

Set $\widehat{A} = \{x \in \mathbb{C}^n; A(x) = 0\}$, $\widehat{A} \subsetneq \mathbb{C}^n$. Thus $M \subset \widehat{A}$ and the problem is reduced to the study of V-invariant subsets of the algebraic subset $\widehat{A}$ of $\mathbb{C}^n$.

Let $M \subset \widehat{A}$ be a V-invariant smooth submanifold such that F is a partial integral of system (7.1) with respect to M. Any such submanifold will be called a *domain* of F. In what follows we will consider exclusively the case when the submanifold M is of codimension one. Nevertheless the case of smaller codimension also deserves the study. In what follows we will be interested in determining the maximal domain of F.

7.2. Determination of the maximal domain. We will now determine the maximal domain of codimension one of the Sretenskii partial integral

$$F = (I_3\omega_3 + b_3)(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3 \tag{7.2}$$

according to the values of all parameters $(I_1, I_2, I_3, b_1, b_2, b_3, c_1, c_2, c_3)$ of the gyrostat equations (6.1). To avoid the Zhukovskii and Lagrange cases, we will suppose that $(c_1, c_2) \neq (0, 0)$.

Let $U \subset \mathbb{C}^6$ be an open subset. We want to find all invariant manifolds

$$\widehat{S} = \{(\omega, \gamma) \in U;\ S(\omega, \gamma) = 0\}, \tag{7.3}$$

where S is a C^1 function defined on U such that on $\widehat{S}$, F is a first integral of (6.1).

We will consider five distinct cases.

Case 1. Let us suppose that $\frac{\partial S}{\partial \gamma_1} \neq 0$ at some point of $\widehat{S}$, and by continuity also in some open subset of U. We express γ_1 from equation (7.3) and find that locally

$$\gamma_1 = \Gamma_1(\omega_1, \omega_2, \omega_3, \gamma_2, \gamma_3). \tag{7.4}$$

We compute $\frac{dF}{dt}$ and replace γ_1 with Γ_1 everywhere:

$$\begin{aligned}\frac{dF}{dt} = {} & \frac{[2(I_3\omega_3 + b_3)\omega_1 - c_1\gamma_3][(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\gamma_3]}{I_1} \\ & + \frac{[2(I_3\omega_3 + b_3)\omega_2 - c_2\gamma_3][(I_3 - I_1)\omega_1\omega_3 - b_3\omega_1 + b_1\omega_3 + c_1\gamma_3 - c_3\Gamma_1]}{I_2} \\ & + (\omega_1^2 + \omega_2^2)[(I_1 - I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + c_2\Gamma_1 - c_1\gamma_2] \\ & - (c_1\omega_1 + c_2\omega_2)(\omega_2\Gamma_1 - \omega_1\gamma_2).\end{aligned}$$

As we suppose that F is a first integral on the invariant manifold (7.3) we have

$$\frac{dF}{dt} = 0. \tag{7.5}$$

We solve (7.5) with respect to Γ_1 and obtain a rational function depending on ω_1, ω_2, ω_3, γ_2 and γ_3:

$$\begin{aligned}\Gamma_1 = {} & \frac{1}{I_1(-I_2c_2\omega_1^2 + I_2c_1\omega_1\omega_2 + 2b_3c_3\omega_2 + 2I_3c_3\omega_2\omega_3 - c_2c_3\gamma_3)}\big[I_1I_2(I_1 - I_2)\omega_1^3\omega_2 \\ & + I_1I_2b_2\omega_1^3 - I_1I_2b_1\omega_1^2\omega_2 + I_1I_2(I_1 - I_2)\omega_1\omega_2^3 + I_1I_2b_2\omega_1\omega_2^2 \\ & - 2I_3(I_1 - I_2)(I_1 + I_2 - I_3)\omega_1\omega_2\omega_3^2 - 2(I_1^2 - I_2^2)b_3\omega_1\omega_2\omega_3 + I_1I_2c_2\omega_1\omega_2\gamma_2 \\ & - 2(I_1 - I_2)b_3^2\omega_1\omega_2 - 2I_2I_3b_2\omega_1\omega_3^2 + 2I_2I_3c_3\omega_1\omega_3\gamma_2 \\ & + (I_1^2 - I_1I_3 - 2I_2I_3)c_2\omega_1\omega_3\gamma_3 - 2I_2b_2b_3\omega_1\omega_3 + 2I_2b_3c_3\omega_1\gamma_2 \\ & + (I_1 - 2I_2)b_3c_2\omega_1\gamma_3 - I_1I_2b_1\omega_2^3 - I_1I_2c_1\omega_2^2\gamma_2 + 2I_1I_3b_1\omega_2\omega_3^2 \\ & + (2I_1I_3 - I_2^2 + I_2I_3)c_1\omega_2\omega_3\gamma_3 + 2I_1b_1b_3\omega_2\omega_3 + (2I_1 - I_2)b_3c_1\omega_2\gamma_3 \\ & - (I_1b_1c_2 - I_2b_2c_1)\omega_3\gamma_3 - I_2c_1c_3\gamma_2\gamma_3 - (I_1 - I_2)c_1c_2\gamma_3^2\big].\end{aligned} \tag{7.6}$$

As $\Gamma_1 = \gamma_1$ we have

$$W = \frac{d\Gamma_1}{dt} - \frac{d\gamma_1}{dt} = 0. \tag{7.7}$$

The function W depends on γ_1 linearly. Indeed, as the expression for Γ_1 from (7.6) does not depend on γ_1, its derivative $\frac{d\Gamma_1}{dt}$ is a linear function of γ_1, which is easily seen taking into account that the right-hand sides of the Euler–Poisson equations (1.1) are linear with respect to γ_1. It may happen that the coefficient of γ_1 in $\frac{d\Gamma_1}{dt}$ is identically zero. This only occurs in the following three cases:

CASE A:

$$I_1 = \frac{I_2(I_2 - I_3)}{2I_3 + I_2}, \quad b_2 = 0, \quad b_3 = 0, \quad c_2 = 0, \quad c_3 = 0.$$

CASE B:

$$I_1 = 2I_3, \quad I_2 = 4I_3, \quad b_2 = 0, \quad c_2 = 0, \quad c_3 = 0.$$

CASE C:

$$I_1 = 4\mathrm{RootOf}(8Z^2+78-51Z)I_3 - 12I_3, \quad I_2 = \mathrm{RootOf}(8Z^2+78-51Z)I_3,$$
$$b_1 = 0, \quad b_2 = 0, \quad b_3 = 0, \quad c_1 = 0, \quad c_3 = 0.$$

In the last case the gyrostat equations (6.1) are reduced to the Euler–Poisson equations (1.1).

The two values of $\mathrm{RootOf}(8Z^2+78-51Z)$ are $\frac{51\pm\sqrt{105}}{16}$. In fact Case C presents two different solutions, where in I_1 and I_2 the same sign $+$ or $-$ appears and so it will be in what follows.

Let F be the Sretenskii partial integral (7.2).

Let us compute $\frac{dF}{dt}$ along the orbits of (6.1). In the above three cases we obtain:

In Case A,

$$\frac{dF}{dt} = -\frac{3I_2I_3\omega_1^3\omega_2}{I_2+2I_3} - b_1\omega_1^2\omega_2 - c_1\omega_1\omega_2\gamma_1 + \frac{12I_3^2(I_2+I_3)\omega_1\omega_2\omega_3^2}{I_2(I_2+2I_3)}$$
$$- \frac{3I_2I_3\omega_1\omega_2^3}{I_2+2I_3} - b_1\omega_2^3 - c_1\omega_2^2\gamma_2 + \frac{2I_3b_1\omega_2\omega_3^2}{I_2} - c_1\omega_2\omega_3\gamma_3.$$

In Case B,

$$\frac{dF}{dt} = -2I_3\omega_1^3\omega_2 - b_1\omega_1^2\omega_2 - 2I_3\omega_1\omega_2^3 + \frac{5I_3\omega_1\omega_2\omega_3^2}{2} + 3b_3\omega_1\omega_2\omega_3 + \frac{b_3^2\omega_1\omega_2}{2I_3}$$
$$- c_1\omega_1\omega_2\gamma_1 - b_1\omega_2^3 - c_1\omega_2^2\gamma_2 + \frac{b_1\omega_2\omega_3^2}{2} - c_1\omega_2\omega_3\gamma_3 + \frac{b_1b_3\omega_2\omega_3}{2I_3},$$

In Case C,

$$\frac{dF}{dt} = 3\left(\frac{51\pm\sqrt{105}}{16} - 4\right)I_3\omega_1^3\omega_2 + c_2\omega_1^2\gamma_1 + 3\left(\frac{51\pm\sqrt{105}}{16} - 4\right)I_3\omega_1\omega_2^3$$
$$+ c_2\omega_1\omega_2\gamma_2 + \frac{I_3\omega_1\omega_2\omega_3^2}{2}.$$

In all these cases, $\frac{dF}{dt}$ does not vanish identically and thus these cases are outside of the domain of the Sretenskii partial integral and we will ignore them.

Outside of these three cases the MAPLE command `degree` shows that the degree of W with respect to γ_1 is 1.

We solve equation (7.7) with respect to γ_1 and obtain

$$\gamma_1 = \widehat{\Gamma}_1(\omega_1, \omega_2, \omega_3, \gamma_2, \gamma_3).$$

The expression for $\widehat{\Gamma}_1$ is too long to be written here and we skip it. Let us note however that W is a rational function of all the variables (ω, γ), whose numerator has 296 monomials and whose denominator is

$$I_1^2I_3(I_2c_2\omega_1^2 - I_2c_1\omega_1\omega_2 - 2b_3c - 3\omega_2 - 2I_3c_3\omega_2\omega_3 + c_2c_3\gamma_3)^2.$$

Thus $\Gamma_1 - \widehat{\Gamma}_1 = 0$. The function $\Gamma_1 - \widehat{\Gamma}_1$ is a rational function of ω_1, ω_2, ω_3, γ_2 and γ_3. We only consider its numerator which we denote by D. We want to know when D is identically zero with respect to all the variables ω_1, ω_2, ω_3, γ_2 and γ_3. To this end we compute the coefficients of D. There are 513 of them. We should find the conditions on the parameters $\mathcal{I}c$ at which all of these 513 coefficients are zero.

We apply simplification to the resulting system of 513 equations. After three consecutive simplifications we come to the reduced system consisting of nine equations:

$$\begin{gathered}c_3 = 0, \quad b_2c_2 = 0, \quad b_1c_2 = 0, \quad (I_2 - 4I_3)c_2 = 0, \quad (I_1 - 4I_3)c_2 = 0,\\ b_2c_1 = 0, \quad b_1c_1 = 0, \quad (I_2 - 4I_3)c_1 = 0, \quad (I_1 - 4I_3)c_1 = 0.\end{gathered} \tag{7.8}$$

We solve it by applying the MAPLE command `solve` and obtain two solutions. The first one is $c_1 = c_2 = c_3 = 0$ and we discard it because it leads to the Zhukovskii case. The second solution is

$$I_1 = I_2 = 4I_3, \quad b_1 = b_2 = 0, \quad c_3 = 0. \tag{7.9}$$

Now D vanishes identically. Taking into account (7.4) we compute γ_1 from (7.6) under condition (7.9) and obtain

$$\gamma_1 = -\frac{4I_3\omega_2\gamma_2 + I_3\omega_3\gamma_3 - b_3\gamma_3}{4I_3\omega_1},$$

that is,

$$4I_3\omega_1\gamma_1 + 4I_3\omega_2\gamma_2 + I_3\omega_3\gamma_3 - b_3\gamma_3 = 0.$$

Let us note that the last equation is actually nothing other than $H_1 = 0$ (see (6.2)) when $I_1 = I_2 = 4I_3$, $b_1 = b_2 = 0$. $\{H_1 = 0\}$ is an invariant manifold. Finally we conclude that when $\frac{\partial S}{\partial \gamma_1} \neq 0$ at some point of U, when $I_1 = I_2 = 4I_3$, $b_1 = b_2 = 0$, $(c_1, c_2) \neq (0, 0)$, $c_3 = 0$, $\{H_1 = 0\}$ is the sought maximal invariant manifold. Thus we remain in the framework of the Sretenskii case.

The gyrostat equations (6.1) admit permutational symmetry (see Sec. 2)

$$\sigma_4 = \{(2, 1, 3), (2, 1, 3)\}.$$

The function F (see (7.2)) and also the first integral H_1 are σ_4-invariant. Thus the solution of the problem about the maximal invariant manifold S when $\frac{\partial S}{\partial \gamma_2} \neq 0$ at some point is exactly the same as in the just studied case when $\frac{\partial S}{\partial \gamma_1} \neq 0$ at some point of U.

Case 2. Let us suppose now that $\frac{\partial S}{\partial \gamma_1}$ and $\frac{\partial S}{\partial \gamma_2}$ vanish identically on U, which means that S does not depend on γ_1 or γ_2. Thus

$$S = S(\omega_1, \omega_2, \omega_3, \gamma_3).$$

Let us suppose that $\frac{\partial S}{\partial \gamma_3} \neq 0$ at some point of $\widehat{S}$, and by continuity also in some open subset of U. As before we express γ_3 from equation (7.3) and obtain

$$\gamma_3 = \Gamma_3(\omega_1, \omega_2, \omega_3).$$

We compute $\frac{dF}{dt}$, replace γ_3 with Γ_3 everywhere and obtain

$$\begin{aligned}\frac{dF}{dt} = {} & \frac{[2(I_3\omega_3 + b_3)\omega_1 - c_1\Gamma_3][(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\Gamma_3]}{I_1}\\ & + \frac{[2(I_3\omega_3 + b_3)\omega_2 - c_2\Gamma_3][(I_3 - I_1)\omega_1\omega_3 - b_3\omega_1 + b_1\omega_3 + c_1\Gamma_3 - c_3\gamma_1]}{I_2}\\ & + (\omega_1^2 + \omega_2^2)[(I_1 - I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + c_2\gamma_1 - c_1\gamma_2]\\ & - (c_1\omega_1 + c_2\omega_2)(\omega_2\gamma_1 - \omega_1\gamma_2).\end{aligned}$$

As we suppose that F is a partial integral with respect to the invariant manifold $\{S = 0\}$, we have $\frac{dF}{dt} = 0$. Let us denote the numerator of $\frac{dF}{dt}$ by J. In this way we obtain the following equation for Γ_3:

$$\begin{aligned} J = {} & -(I_1 - I_2)c_1c_2\Gamma_3^2 + \bigl[(I_1^2 - I_1I_3 - 2I_2I_3)c_2\omega_1\omega_3 + (I_1 - 2I_2)b_3c_2\omega_1 \\ & + (2I_1I_3 - I_2^2 + I_2I_3)c_1\omega_2\omega_3 + (2I_1 - I_2)b_3c_1\omega_2 + (I_2c_1b_2 - I_1b_1c_2)\omega_3 \\ & + I_1c_2c_3\gamma_1 - I_2c_1c_3\gamma_2\bigr]\Gamma_3 + I_1I_2(I_1 - I_2)\omega_1^3\omega_2 + I_1I_2b_2\omega_1^3 - I_1I_2b_1\omega_1^2\omega_2 \\ & + I_1I_2c_2\omega_1^2\gamma_1 + I_1I_2(I_1 - I_2)\omega_1\omega_2^3 + I_1I_2b_2\omega_1\omega_2^2 \\ & - 2I_3(I_1 - I_2)(I_1 + I_2 - I_3)\omega_1\omega_2\omega_3^2 - 2(I_1^2 - I_2^2)b_3\omega_1\omega_2\omega_3 + I_1I_2c_2\omega_1\omega_2\gamma_2 \\ & + (-I_1I_2c_1\gamma_1 + 2I_2b_3^2 - 2I_1b_3^2)\omega_1\omega_2 - 2I_2I_3b_2\omega_1\omega_3^2 + 2I_2I_3c_3\omega_1\omega_3\gamma_2 \\ & - 2I_2b_2b_3\omega_1\omega_3 + 2I_2b_3c_2\omega_1\gamma_2 - I_1I_2b_1\omega_2^3 - I_1I_2c_1\omega_2^2\gamma_2 + 2I_1I_3b_1\omega_2\omega_3^2 \\ & - 2I_1(I_3c_3\gamma_1 - b_1b_3)\omega_2\omega_3 - 2I_1b_3c_3\omega_2\gamma_1 = 0. \end{aligned}$$

As $\Gamma_3 = \Gamma_3(\omega_1, \omega_2, \omega_3)$, after differentiation of J with respect to γ_1 and γ_2 we have

$$\begin{aligned} \frac{\partial J}{\partial \gamma_1} &= I_1I_2c_2\omega_1^2 - I_1I_2c_1\omega_1\omega_2 - 2I_1I_3c_3\omega_2\omega_3 - 2I_1b_3c_3\omega_2 + I_1c_2c_3\Gamma_3 = 0, \\ \frac{\partial J}{\partial \gamma_2} &= I_1I_2c_2\omega_1\omega_2 + 2I_2I_3c_3\omega_1\omega_3 + 2I_2b_3c_3\omega_1 - I_1I_2c_1\omega_2^2 - I_2c_1c_3\Gamma_3 = 0. \end{aligned} \tag{7.10}$$

Let us first suppose that c_1, c_2 and c_3 are all different from zero. Then excluding Γ_3 from (7.10) we obtain

$$(c_1\omega_2 - c_2\omega_1)(I_2c_1\omega_1 + I_1c_2\omega_2 + 2I_3c_3\omega_3 + 2b_3c_3) = 0,$$

which is obviously impossible.

Let now $c_1 = 0$. Then from the second equation of (7.10) we have the identity

$$I_1I_2c_2\omega_1\omega_2 + 2I_2I_3c_3\omega_1\omega_3 + 2I_2b_3c_3\omega_1 = 0,$$

which is possible only when $c_2 = c_3 = 0$. But this is the Zhukovskii case.

Let now $c_2 = 0$. From $(7.10)_1$ we have

$$-I_1I_2c_1\omega_1\omega_2 - 2I_1I_3c_3\omega_2\omega_3 - 2I_1b_3c_3\omega_2 = 0,$$

which is possible only when $c_1 = c_3 = 0$, i.e. again we are in the Zhukovskii case.

Finally, let $c_3 = 0$. Equations (7.10) give

$$I_1I_2(c_2\omega_1 - c_1\omega_2)\omega_1 = 0, \qquad I_1I_2(c_2\omega_1 - I_1I_2c_1\omega_2)\omega_2 = 0.$$

The above two conditions can be fulfilled only when $c_1 = c_2 = 0$, i.e. only in the Zhukovskii case.

The conclusion is that the sought function $S(\omega_1, \omega_2, \omega_3, \gamma_3)$ does not exist.

Case 3. Let us suppose now that all $\frac{\partial S}{\partial \gamma_i}$, $i = 1, 2, 3$, vanish identically on U, which means that S does not depend on γ_1, γ_2 or γ_3. Thus

$$S = S(\omega_1, \omega_2, \omega_3).$$

Let us suppose that $\frac{\partial S}{\partial \omega_3} \neq 0$ at some point of $\widehat{S}$, and thus also in some open subset of U. We express ω_3 from equation (7.3) and obtain

$$\omega_3 = \Omega_3(\omega_1, \omega_2). \tag{7.11}$$

We compute $\frac{dF}{dt}$ and replace ω_3 with Ω_3 everywhere:

$$\begin{aligned}\frac{dF}{dt} = {} & \frac{[2(I_3\Omega_3 + b_3)\omega_1 - c_1\gamma_3][(I_2 - I_3)\omega_2\Omega_3 + b_3\omega_2 - b_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3]}{I_1} \\ & + \frac{[2(I_3\Omega_3 + b_3)\omega_2 - c_2\gamma_3][(I_3 - I_1)\omega_1\Omega_3 - b_3\omega_1 + b_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1]}{I_2} \\ & + (\omega_1^2 + \omega_2^2)[(I_1 - I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + c_2\gamma_1 - c_1\gamma_2] \\ & - (c_1\omega_1 + c_2\omega_2)(\omega_2\gamma_1 - \omega_1\gamma_2).\end{aligned}$$

As above, $\frac{dF}{dt} = 0$. Denote the numerator of $\frac{dF}{dt}$ by K. We obtain the following equation for Ω_3:

$$\begin{aligned}K = {} & [-2I_3(I_1 - I_2)(I_1 + I_2 - I_3)\omega_1\omega_2 - 2I_2I_3b_2\omega_1 + 2I_1I_3b_1\omega_2]\Omega_3^2 \\ & + \big[-2b_3(I_1^2 - I_2^2)\omega_1\omega_2 + 2I_2I_3c_3\omega_1\gamma_2 + (I_1^2 - I_1I_3 - 2I_2I_3)c_2\omega_1\gamma_3 - 2I_2b_2b_3\omega_1 \\ & - 2I_1I_3c_3\omega_2\gamma_1 + (2I_1I_3 - I_2^2 + I_2I_3)c_1\omega_2\gamma_3 + 2I_1b_1b_3\omega_2 + (I_2b_2c_1 - I_1b_1c_2)\gamma_3\big]\Omega_3 \\ & + I_1I_2(I_1 - I_2)\omega_1^3\omega_2 + I_1I_2b_1\omega_1^3 - I_1I_2b_1\omega_1^2\omega_2 + I_1I_2c_2\omega_1^2\gamma_1 + I_1I_2(I_1 - I_2)\omega_1\omega_2^3 \\ & + I_1I_2b_2\omega_1\omega_2^2 - I_1I_2c_1\omega_1\omega_2\gamma_1 + I_1I_2c_2\omega_1\omega_2\gamma_2 - 2(I_1 - I_2)b_3^2\omega_1\omega_2 + 2I_2b_3c_3\omega_1\gamma_2 \\ & + (I_1 - 2I_2)b_3c_2\omega_1\gamma_3 - I_1I_2b_1\omega_2^3 - I_1I_2c_1\omega_2^2\gamma_2 - 2I_1b_3c_3\omega_2\gamma_1 + (2I_1 - I_2)b_3c_1\omega_2\gamma_3 \\ & + I_1c_2c_3\gamma_1\gamma_3 - I_2c_1c_3\gamma_2\gamma_3 - (I_1 - I_2)c_1c_2\gamma_3^2 = 0.\end{aligned}$$

As $\Omega_3 = \Omega_3(\omega_1, \omega_2)$, after differentiation of K with respect to γ_1 and γ_2 we have

$$\begin{aligned}\frac{\partial K}{\partial \gamma_1} &= I_1I_2c_2\omega_1^2 - I_1I_2c_1\omega_1\omega_2 - 2I_1b_3c_3\omega_2 + I_1c_2c_3\gamma_3 - 2I_1I_3c_3\omega_2\Omega_3 = 0, \\ \frac{\partial K}{\partial \gamma_2} &= I_1I_2c_2\omega_1\omega_2 - I_1I_2c_1\omega_2^2 + 2I_2b_3c_3\omega_1 - I_2c_1c_3\gamma_3 + 2I_2I_3c_3\omega_1\Omega_3 = 0.\end{aligned} \tag{7.12}$$

If we suppose that $c_3 \neq 0$ then we come to the Lagrange case. Indeed, as Ω_3 depends only on ω_1 and ω_2, from $(7.12)_1$ it follows that $c_2 = 0$, because otherwise Ω_3 would depend on γ_3 too. For the same reason, $(7.12)_2$ gives $c_1 = 0$. However, under the condition $c_1 = c_2 = 0$ and $c_3 \neq 0$ both equations of (7.12) lead to the conclusion that $\Omega_3 = -b_3/I_3$, i.e., according to (7.11), $\omega_3 = -b_3/I_3$. In the case from $(6.1)_3$ we obtain $I_1 = I_2$, $b_1 = b_2 = 0$ and $c_1 = c_2 = 0$. Thus we come to the Lagrange case.

Let now $c_3 = 0$. Then the equations of (7.12) become

$$I_1I_2(c_2\omega_1 - c_1\omega_2)\omega_1 = 0, \qquad I_1I_2(c_2\omega_1 - c_1\omega_2)\omega_2 = 0.$$

The above two equations should be identities, which is possible only when $c_1 = c_2 = 0$, i.e. we come to the Zhukovskii case.

The conclusion is that the sought function $S(\omega_1, \omega_2, \omega_3)$ does not exist.

Case 4. Let now the function S be

$$S = S(\omega_1, \omega_2)$$

and $\frac{\partial S}{\partial \omega_2} \neq 0$ at some point of $\widehat{S}$ and therefore in some open subset of $\widehat{S}$. We express ω_2 from (7.3) and obtain

$$\omega_2 = \Omega_2(\omega_1). \tag{7.13}$$

We compute $\frac{dF}{dt}$, replace ω_2 with Ω_2 everywhere and obtain

$$\begin{aligned}\frac{dF}{dt} &= \frac{[2(I_3\omega_3+b_3)\omega_1-c_1\gamma_3][(I_2-I_3)\Omega_2\omega_3+b_3\Omega_2-b_2\omega_3+c_3\gamma_2-c_2\gamma_3]}{I_1}\\ &\quad+\frac{[2(I_3\omega_3+b_3)\Omega_2-c_2\gamma_3][(I_3-I_1)\omega_1\omega_3-b_3\omega_1+b_1\omega_3+c_1\gamma_3-c_3\gamma_1]}{I_2}\\ &\quad+(\omega_1^2+\Omega_2^2)[(I_1-I_2)\omega_1\Omega_2+b_2\omega_1-b_1\Omega_2+c_2\gamma_1-c_1\gamma_2]\\ &\quad-(c_1\omega_1+c_2\Omega_2)(\Omega_2\gamma_1-\omega_1\gamma_2).\end{aligned}$$

As above, $\frac{dF}{dt}=0$. Denote the numerator of $\frac{dF}{dt}$ by L. In this way we obtain the following equation for Ω_2:

$$\begin{aligned}L &= I_1I_2[(I_1-I_2)\omega_1-b_1]\Omega_2^3+I_1I_2(b_2\omega_1-c_1\gamma_2)\Omega_2^2\\ &\quad+\big[I_1I_2(I_1-I_2)\omega_1^3-I_1I_2b_1\omega_1^2-2I_3(I_1-I_2)(I_1+I_2-I_3)\omega_1\omega_3^2-2(I_1^2-I_2^2)b_3\omega_1\omega_3\\ &\quad-I_1I_2c_1\omega_1\gamma_1+I_1I_2c_2\omega_1\gamma_2-2(I_1-I_2)b_3^2\omega_1+2I_1I_3b_1\omega_3^2-2I_1I_3c_3\omega_3\gamma_1\\ &\quad+(2I_1I_3-I_2^2+I_2I_3)c_1\omega_3\gamma_3+2I_1b_3b_1\omega_3-2I_1b_3c_3\gamma_1+(2I_1-I_2)b_3c_1\gamma_3\big]\Omega_2\\ &\quad+I_1I_2b_2\omega_1^3+I_1I_2c_2\omega_1^2\gamma_1-2I_2I_3b_2\omega_1\omega_3^2+2I_2I_3c_3\omega_1\omega_3\gamma_2\\ &\quad+(I_1^2-I_1I_3-2I_2I_3)c_2\omega_1\omega_3\gamma_3-2I_2b_2b_3\omega_1\omega_3+2I_2b_3c_3\omega_1\gamma_2+(I_1-2I_2)b_3c_2\omega_1\gamma_3\\ &\quad+(I_2b_2c_1-I_1b_1c_2)\omega_3\gamma_3+I_1c_2c_3\gamma_1\gamma_3-I_2c_1c_3\gamma_2\gamma_3-(I_1-I_2)c_1c_2\gamma_3^2=0.\end{aligned}$$

As $\Omega_2=\Omega_2(\omega_1)$, after differentiation of L with respect to γ_1, γ_2 and γ_3 we have

$$\begin{aligned}\frac{\partial L}{\partial\gamma_1} &= I_1c_2(I_2\omega_1^2+c_3\gamma_3)-I_1(I_2c_1\omega_1+2I_3c_3\omega_3+2b_3c_3)\Omega_2=0,\\ \frac{\partial L}{\partial\gamma_2} &= I_2c_3(2I_3\omega_1\omega_3+2b_3\omega_1-c_1\gamma_3)+I_1I_2c_2\omega_1\Omega_2-I_1I_2c_1\Omega_2^2=0,\\ \frac{\partial L}{\partial\gamma_3} &= (I_1^2-I_1I_3-2I_2I_3)c_2\omega_1\omega_3+(I_1-2I_2)b_3c_2\omega_1-I_1c_2b_1\omega_3\\ &\quad+I_2c_1b_2\omega_3+I_1c_2c_3\gamma_1-I_2c_1c_3\gamma_2-2(I_1-I_2)c_1c_2\gamma_3\\ &\quad+\big[(2I_1I_3-I_2^2+I_2I_3)c_1\omega_3+(2I_1-I_2)b_3\big]\Omega_2=0.\end{aligned}\tag{7.14}$$

From $(7.14)_1$ it follows that $c_2\neq 0$. Indeed, if $c_2=0$ then either $c_1=c_3=0$, which is the Zhukovskii case, or Ω_2 vanishes identically, which contradicts (7.13). From the same equation it is seen that $c_3=0$ because if $c_3\neq 0$ then this equation contains only one monomial depending on γ_3 which cannot be canceled because Ω_2 depends on ω_1 only. Thus $c_3=0$.

In that case $(7.14)_1$ can be rewritten as follows:

$$I_1I_2\omega_1(c_2\omega_1-c_1\Omega_2)=0.\tag{7.15}$$

This imposes the restriction $c_1\neq 0$ because otherwise $I_1I_2c_2\omega_1^2=0$, which is impossible. We solve (7.15) with respect to Ω_2 and obtain

$$\Omega_2=\Omega_2(\omega_1)=\frac{c_2\omega_1}{c_1}.$$

Under this condition $(7.14)_2$ is satisfied, and $(7.14)_3$ becomes

$$(I_1-I_2)(I_1+I_3+I_2)c_2\omega_1\omega_3+3(I_1-I_2)b_3c_2\omega_1+(I_2b_2c_1-I_1b_1c_2)\omega_3-2(I_1-I_2)c_1c_2\gamma_3=0.$$

Taking into account that $c_1 \neq 0$ and $c_2 \neq 0$, the last item of the above equation leads to $I_1 = I_2$. Thus

$$I_2(b_2c_1 - b_1c_2)\omega_3 = 0.$$

In this way we come to the case

$$I_1 = I_2, \quad c_1 \neq 0, \quad c_2 \neq 0, \quad c_3 = 0, \quad b_2c_1 - b_1c_2 = 0.$$

In this case L vanishes identically and therefore we can take as S the function

$$S = c_1\omega_2 - c_2\omega_1.$$

So far we have not yet examined when $\{S = 0\} = \{c_1\omega_2 - c_2\omega_1 = 0\}$ is an invariant manifold. To do this we compute

$$\frac{dS}{dt} = \frac{(c_1\omega_1 + c_2\omega_2)[(I_3 - I_2)c_2\omega_3 - b_3c_2] + (c_1^2 + c_2^2)(b_2\omega_3 + c_2\gamma_3)}{I_2c_2}.$$

It is seen that in the generic case, $\{S = 0\}$ is not an invariant manifold. But let us put

$$c_1^2 + c_2^2 = 0,$$

i.e. either $c_1 = ic_2$ or $c_1 = -ic_2$. Let us first consider the case $c_1 = ic_2$. We have

$$S = c_2S_1, \quad S_1 = i\omega_2 - \omega_1$$

and

$$\frac{dS_1(t)}{dt} = \frac{i[(I_2 - I_3)\omega_3(t) + b_3](i\omega_2(t) - \omega_1(t))}{I_2} = S_1(t)\frac{i[(I_2 - I_3)\omega_3(t) + b_3]}{I_2}.$$

This equation admits the zero solution $S_1(t) = 0$ for all t. Thus from the unicity of solutions for this equation one sees that if $S_1(t_0) = 0$ for some t_0, then $S_1(t) = 0$ for all t. In other words, if for some t_0, $i\omega_2(t_0) - \omega_1(t_0) = 0$, then $i\omega_2(t) - \omega_1(t) = 0$ for all t. But this is precisely the invariance of the manifold $\widehat{S_1} = \{S_1 = 0\}$. In this way we come to the conclusion that when

$$I_1 = I_2, \quad c_1 = ic_2 \neq 0, \quad c_3 = 0, \quad b_2c_1 - b_1c_2 = 0,$$

the gyrostat equations (6.1) have an invariant manifold $\{i\omega_2 - \omega_1 = 0\}$.

The case $c_1 = -ic_2$ is considered in the same way. The difference is that now $S = c_2S_2$, where $S_2 = -i\omega_2 - \omega_1$ and the invariant manifold is $\{i\omega_2 + \omega_1 = 0\}$.

But F is not a partial integral of the gyrostat equations (6.1) with respect to the invariant manifolds $\widehat{S_1} = \{S_1 = 0\}$ and $\widehat{S_2} = \{S_2 = 0\}$. In fact, it is easy to see that on them F vanishes identically.

Finally, we conclude that the codimension one maximal domain of the Sretenskii partial integral (7.2) coincides with the manifold $\{H_1 = 0\}$ under the conditions

$$I_1 = I_2 = 4I_3, \quad b_1 = b_2 = 0, \quad (c_1, c_2) \neq (0, 0), \quad c_3 = 0.$$

In addition, when

$$I_1 = I_2, \quad b_2c_1 - b_1c_2 = 0, \quad c_1^2 + c_2^2 = 0, \quad c_3 = 0,$$

we found two invariant relations for (6.1):

$$S_1 = i\omega_2 - \omega_1 \text{ when } c_1 = ic_2 \quad \text{and} \quad S_2 = -i\omega_2 - \omega_1 \text{ when } c_1 = -ic_2.$$

We will prove that S_1 is functionally independent of the first integrals H_1, H_2 and H_3. For this purpose we consider the Jacobi matrix J of H_1, H_2, H_3 and S_1. We prove that $\operatorname{rank} J = 4$. Indeed, computing the determinant J_{14} obtained from J by crossing out the first and fourth columns we obtain

$$J_{14} = -4iI_3(I_2\omega_2\omega_3\gamma_3 - I_3\omega_3^2\gamma_2 - c_2\gamma_3^2 - b_2\omega_3\gamma_3 + b_3\omega_3\gamma_2).$$

It is clearly seen that J_{14} never vanishes identically. Thus H_1, H_2, H_3 and S_1 are functionally independent.

The study of the functional independence of H_1, H_2, H_3 and S_2 is the same. The only difference is that now the value of the determinant is $-J_{14}$.

Case 5. Finally, let $S = S(\omega, \gamma) = S(\omega_1)$, where S does not vanish identically. It is easy to see that then $\widehat{S} = \{(\omega, \gamma) \in U;\ S(\omega_1) = 0\}$ is a five-dimensional submanifold if and only if $\widehat{S} = \{(\omega, \gamma) \in U;\ \omega_1 \in \Omega_1\}$, where Ω_1 is the set of zeros of S. In the complex case, Ω_1 is at most countable. In the real case, Ω_1 is a subset of $\mathbb{R}$ that does not contain any open interval. In both cases $\frac{d\omega_1}{dt} = 0$ and from $(6.1)_1$ one obtains

$$(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\gamma_3 = 0$$

for all ω_2, ω_3, γ_2 $\gamma_3 \in \mathbb{C}$. Thus $I_2 = I_3$, $b_2 = b_3 = 0$ and $c_2 = c_3 = 0$ and we recover the Lagrange case. Thus in both cases, complex and real, the sought function $S(\omega_1)$ does not exist.

Let us recall that when $b_1 = b_2 = b_3 = 0$ in (6.1), we recover the Euler–Poisson equations (1.1). The Sretenskii case of partial integrability becomes the Goryachev–Chaplygin case of partial integrability and the Sretenskii partial integral (7.2) becomes the Goryachev–Chaplygin partial integral (1.8).

Consequently, from the above, one deduces immediately that the maximal domain of the Goryachev–Chaplygin partial integral (1.8) is

$$\{H_1 = I_1\omega_1\gamma_1 + I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 = 0\},$$

where $I_1 = I_2 = 4I_3$, $(c_1, c_2) \neq (0, 0)$, $c_3 = 0$.

8. Four-dimensional invariant manifolds. New integrals on $\{H_i = U_i,\ H_j = U_j\}$, $1 \leq i < j \leq 3$

8.1. Extraction procedure. In this section we study the existence of a partial integral of the Euler–Poisson equations (1.1) with respect to the invariant complex four-dimensional level manifold $\{H_i = U_i,\ H_j = U_j\}$, $1 \leq i < j \leq 3$. We study when it depends on at most three variables.

Let us fix i and j, $1 \leq i < j \leq 3$. According to (2.5),

$$M(U_0, U_i, U_j, \mathcal{I}c) = \{x \in \mathbb{C}^6;\ H_i((\omega, \gamma), \mathcal{I}c) = U_i,\ H_j((\omega, \gamma), \mathcal{I}c) = U_j\},$$

where $(\omega, \gamma) = (\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3)$.

As in Sec. 5 we search for all functions $F = F(s_1, s_2, s_3)$ where $(s_1, s_2, s_3) \in (\omega, \gamma)$, of class C^1, such that $\operatorname{grad} F$ does not vanish identically on each open subset of

$M(U_0, U_i, U_j, \mathcal{I}c)$, which are partial integrals of the Euler–Poisson equations (1.1) with respect to $M(U_0, U_i, U_j, \mathcal{I}c)$. Indeed, F is a first integral of Euler–Poisson equations (1.1) restricted to $M(U_0, U_i, U_j, \mathcal{I}c)$.

We follow the same procedure as in Sec. 5.1. As in Sec. 5.1 the order of the variables s_i, $1 \le i \le 3$, in $F(s_1, s_2, s_3)$ is irrelevant for F to be a first integral.

We have exactly 20 different three-element subsets of (ω, γ) and thus 20 cases of functions of three elements to examine. We will now describe an extraction procedure based on permutational symmetries which reduces the above 20 cases to only six.

These 20 functions of three variables (up to the order of variables) are shown in Table 8.1.

Table 8.1

Functions	Case
$F(\omega_1, \omega_2, \omega_3)$	(i)
$F(\omega_1, \omega_2, \gamma_3)$, $F(\omega_1, \omega_3, \gamma_2)$, $F(\omega_2, \omega_3, \gamma_1)$	(ii)
$F(\omega_1, \omega_2, \gamma_1)$, $F(\omega_1, \omega_3, \gamma_1)$, $F(\omega_2, \omega_3, \gamma_2)$, $F(\omega_1, \omega_2, \gamma_2)$, $F(\omega_1, \omega_3, \gamma_3)$, $F(\omega_2, \omega_3, \gamma_3)$	(iii)
$F(\omega_1, \gamma_1, \gamma_2)$, $F(\omega_1, \gamma_1, \gamma_3)$, $F(\omega_2, \gamma_2, \gamma_3)$, $F(\omega_2, \gamma_1, \gamma_2)$, $F(\omega_3, \gamma_1, \gamma_3)$, $F(\omega_3, \gamma_2, \gamma_3)$	(iv)
$F(\omega_3, \gamma_1, \gamma_2)$, $F(\omega_2, \gamma_1, \gamma_3)$, $F(\omega_1, \gamma_2, \gamma_3)$	(v)
$F(\gamma_1, \gamma_2, \gamma_3)$	(vi)

It is easy to see that under the group of permutational symmetries (2.3) of the Euler–Poisson equations for every case (i)–(vi) from Table 8.1 each function from a fixed case can be transformed into all remaining functions from the same case.

Thus in virtue of Theorem 2.2 we can restrict ourselves to the study of only six functions each of which belongs to a different case from Table 8.1 and is chosen arbitrarily from the functions of this case.

We will call such six functions F_i, $1 \le i \le 6$, (up to the order of variables) a *basis*.

8.2. Invariant manifold $\{H_1 = U_1,\ H_2 = U_2\}$. Here we continue the study of the existence of a partial integral of the Euler–Poisson equations (1.1) with respect to the complex four-dimensional level manifold

$$\{H_1 = U_1,\ H_2 = U_2\}, \tag{8.1}$$

supposing that this first integral depends on at most three variables. To this end we shall use the same approach as in Sec. 5.

In the following, when we refer to "a suitable open set" in the space $\mathbb{C}^4(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in Sec. 8 where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \in \{\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3\}$ or in the space $\mathbb{C}^3(\alpha_1, \alpha_2, \alpha_3)$ in Sec. 9 where $\{\alpha_1, \alpha_2, \alpha_3\} \in \{\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3\}$, we mean an open set such that all functions of the above variables never vanish on it when this is necessary for a proof; for example, if such a function appears in some denominator or when we need to have a holomorphic branch of roots of some of these functions. We will use this terminology without further mention.

8.2.1. Elimination of γ_2 and γ_3. Using the Maple command `solve` we express γ_2 and γ_3 from the equations $H_1 = U_1$ and $H_2 = U_2$ and obtain the following solution:

$$\gamma_2 = \frac{I_1\omega_1\gamma_1 + I_3\omega_3 R - U_1}{I_2\omega_2}, \qquad \gamma_3 = R, \tag{8.2}$$

where R is a root of

$$Q(x) = Ax^2 + Bx + C = 0,$$

that is,

$$Q(R) = AR^2 + BR + C = 0 \tag{8.3}$$

and $A = A(\omega_2, \omega_3)$, $B = B(\omega_1, \omega_3, \gamma_1)$ and $C = C(\omega_1, \omega_2, \gamma_1)$ are the following polynomials:

$$\begin{aligned} A &= I_2^2\omega_2^2 + I_3^2\omega_3^2, \qquad B = 2I_3\omega_3(I_1\omega_1\gamma_1 - U_1), \\ C &= I_1^2\omega_1^2\gamma_1^2 - 2I_1U_1\omega_1\gamma_1 - I_2^2U_2\omega_2^2 + I_2^2\omega_2^2\gamma_1^2 + U_1^2. \end{aligned} \tag{8.4}$$

Here Maple does not give an explicit formula for R but expresses R as a root of the following quadratic polynomial:

$$\begin{aligned} &\mathrm{RootOf}((I_3^2\omega_3^2 + I_2^2\omega_2^2)Z^2 + 2I_3\omega_3(I_1\omega_1\gamma_1 - U_1)Z \\ &\qquad\qquad + I_1^2\omega_1^2\gamma_1^2 - 2I_1U_1\omega_1\gamma_1 - I_2^2U_2\omega_2^2 + I_2^2\omega_2^2\gamma_1^2 + U_1^2) \\ &\qquad = \mathrm{RootOf}(AZ^2 + BZ + C). \end{aligned}$$

Thus we can say that R is a root of (8.3) where the coefficients A, B and C are defined by (8.4).

Let us consider the four-dimensional vector space $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$ and a point $(\omega_1, \omega_2, \omega_3, \gamma_1) \in \mathbb{C}^4$ with $\omega_i \neq 0$, $i = 1, 2, 3$, $\gamma_1 \neq 0$.

All our considerations are local. Thus from the beginning we can restrict ourselves to a suitable open set Ω in the space $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$.

By their very definition the first integrals are not constant on any open subset of their domain of definition. As we consider C^1 first integrals, this means that their gradients are non-zero on any open subset of their domain of definition.

We put the values of γ_2 and γ_3 from (8.2) in the Euler–Poisson equations (1.1) and remove the fifth and sixth equations. In this way we get the following system of four equations in the unknowns ω_1, ω_2, ω_3 and γ_1:

$$\begin{aligned} \frac{d\omega_1}{dt} &= \frac{1}{I_1I_2\omega_2}[I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1c_3\omega_1\gamma_1 - (I_2c_2\omega_2 + I_3c_3\omega_3)R + c_3U_1], \\ \frac{d\omega_2}{dt} &= \frac{1}{I_2}[(I_3 - I_1)\omega_1\omega_3 + c_1R - c_3\gamma_1], \\ \frac{d\omega_3}{dt} &= \frac{1}{I_2I_3\omega_2}[I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1c_1\omega_1\gamma_1 + I_2c_2\omega_2\gamma_1 + I_3c_1\omega_3R - c_1U_1], \\ \frac{d\gamma_1}{dt} &= \frac{1}{I_2\omega_2}[-I_1\omega_1\omega_3\gamma_1 - (I_2\omega_2^2 + I_3\omega_3^2)R + \omega_3U_1]. \end{aligned} \tag{8.5}$$

Here we study whether system (8.5) has a first integral that depends on at most three variables among $(\omega_1, \omega_2, \omega_3, \gamma_1)$. Thus we should investigate the following four types of first integrals:

1. $F(\omega_1, \omega_2, \omega_3)$ (case (i)).
2. $F(\omega_1, \omega_2, \gamma_1)$ (case (iii)).
3. $F(\omega_1, \omega_3, \gamma_1)$ (case (iii)).
4. $F(\omega_2, \omega_3, \gamma_1)$ (case (ii)).

Then, as in Sec. 5 it suffices to examine functions of types 1, 2 and 4 respectively.

Type 1. Suppose that the sought first integral F is of type 1, i.e. $F = F(\omega_1, \omega_2, \omega_3)$. As F is a first integral of system (8.5) we have

$$\begin{aligned}\frac{dF}{dt} &= \frac{1}{I_1 I_2 \omega_2}[I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1 c_3 \omega_1 \gamma_1 - (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)R + c_3 U_1]\frac{\partial F}{\partial \omega_1} \\ &\quad + \frac{1}{I_2}[(I_3 - I_1)\omega_1\omega_3 + c_1 R - c_3\gamma_1]\frac{\partial F}{\partial \omega_2} \\ &\quad + \frac{1}{I_2 I_3 \omega_2}[I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1 c_1 \omega_1 \gamma_1 + I_2 c_2 \omega_2 \gamma_1 + I_3 c_1 \omega_3 R - c_1 U_1]\frac{\partial F}{\partial \omega_3} = 0,\end{aligned}$$

or equivalently

$$I_1 I_2 I_3 \omega_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.6}$$

where Y_1 is a vector field defined on Ω.

Equation (8.6) should be an identity with respect to all the four variables ω_1, ω_2, ω_3 and γ_1. As F does not depend on γ_1 its partial derivatives will not depend on γ_1 either. Thus if we differentiate identity (8.6) with respect to γ_1 we shall obtain again a linear partial differential equation for F. We obtain

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial \gamma_1} &= -I_3\left[I_1 c_3 \omega_1 + (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)\frac{\partial R}{\partial \gamma_1}\right]\frac{\partial F}{\partial \omega_1} + I_1 I_3 \omega_2 \left(c_1 \frac{\partial R}{\partial \gamma_1} - c_3\right)\frac{\partial F}{\partial \omega_2} \\ &\quad + I_1\left(I_1 c_1 \omega_1 + I_2 c_2 \omega_2 + I_3 c_1 \omega_3 \frac{\partial R}{\partial \gamma_1}\right)\frac{\partial F}{\partial \omega_3} = 0,\end{aligned}$$

i.e.

$$\frac{\partial Y_1(F)}{\partial \gamma_1} = Y_2(F) = 0, \tag{8.7}$$

where Y_2 is a vector field defined on Ω.

We differentiate (8.7) once more with respect to γ_1 and obtain

$$\frac{\partial Y_2(F)}{\partial \gamma_1} = I_3 \frac{\partial^2 R}{\partial \gamma_1^2}\left[-(I_2 c_2 \omega_2 + I_3 c_3 \omega_3)\frac{\partial F}{\partial \omega_1} + I_2 c_1 \omega_2 \frac{\partial F}{\partial \omega_2} + I_1 c_1 \omega_3 \frac{\partial F}{\partial \omega_3}\right] = 0,$$

i.e.

$$\frac{1}{I_3}\frac{\partial Y_2(F)}{\partial \gamma_1} = Y_3(F) = 0, \tag{8.8}$$

where Y_3 is a vector field defined on Ω.

Let us suppose first that

$$\frac{\partial^2 R}{\partial \gamma_1^2} \neq 0. \tag{8.9}$$

Equations (8.6)–(8.8) can be considered as a system of three homogeneous linear algebraic equations with the unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right)$, which do not vanish identically, because F is non-constant on any open subset of Ω.

Thus, if a fourth integral F exists, system (8.6)–(8.8) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D composed of the coefficients of this system is identically zero. We compute this determinant and obtain

$$D = I_1^2 I_2 I_3 \omega_2^2 \frac{\partial^2 R}{\partial \gamma_1^2} D_1 D_2,$$

where

$$\begin{aligned} D_1 &= (I_1 c_1 \omega_1 + I_2 c_2 \omega_2 + I_3 c_3 \omega_3), \\ D_2 &= (I_2 - I_1) c_3 \omega_1 \omega_2 + (I_1 - I_3) c_2 \omega_1 \omega_3 + (I_3 - I_2) c_1 \omega_2 \omega_3. \end{aligned} \tag{8.10}$$

Note that D depends neither on R nor on $\frac{\partial R}{\partial \gamma_1}$. Taking into account (8.9) it is clear that $D \equiv 0$ if and only if at least one of the expressions (8.10) is identically zero. It is easily seen that this happens only in the Euler, Lagrange and kinetic symmetry cases.

Thus the restriction (8.9) leads to nothing new and we now suppose that

$$\frac{\partial^2 R}{\partial \gamma_1^2} = 0. \tag{8.11}$$

In that case only equations (8.6) and (8.7) remain because $Y_3 \equiv 0$.

Let us study whether there are values of $\mathcal{I}c$, U_1 and U_2 at which (8.11) is fulfilled. For this purpose we differentiate (8.3) twice with respect to γ_1. Taking into account that the polynomial A from (8.4) does not depend on γ_1 we have

$$\frac{\partial Q}{\partial \gamma_1} = \frac{\partial B}{\partial \gamma_1} R + \frac{\partial C}{\partial \gamma_1} + \frac{dQ}{dR} \frac{\partial R}{\partial \gamma_1} = 0 \tag{8.12}$$

and

$$\frac{\partial^2 Q}{\partial \gamma_1^2} = \frac{\partial^2 B}{\partial \gamma_1^2} R + \frac{\partial B}{\partial \gamma_1} \frac{\partial R}{\partial \gamma_1} + \frac{\partial^2 C}{\partial \gamma_1^2} + \frac{\partial}{\partial \gamma_1}\left(\frac{dQ}{dR}\right) \frac{\partial R}{\partial \gamma_1} + \frac{dQ}{dR} \frac{\partial^2 R}{\partial \gamma_1^2} = 0. \tag{8.13}$$

First we prove that if R is a root of (8.3), then $\frac{dQ}{dR} \neq 0$. For this purpose we apply Proposition 4.1 to Q. We consider the resultant ρ of $Q(R)$ and $\frac{dQ}{dR}$ and prove that $\rho \neq 0$. Indeed,

$$\rho = A(4AC - B^2).$$

As we are interested in cases where ρ vanishes identically with respect to ω_1, ω_2, ω_3 and γ_1 only, and as A never vanishes identically, we do not consider ρ but $\widehat{\rho} = 4AC - B^2$ instead. Putting in $\widehat{\rho}$ the expressions for A, B and C (see (8.4)) we obtain

$$\widehat{\rho} = 4 I_2^2 \omega_2^2 (I_1^2 \omega_1^2 \gamma_1^2 + I_2^2 \omega_2^2 \gamma_1^2 + I_3^2 \omega_3^2 \gamma_1^2 - 2 I_1 \omega_1 \gamma_1 U_1 - I_2^2 \omega_2^2 U_2 - I_3^2 \omega_3^2 U_2 + U_1^2), \tag{8.14}$$

which, as one can easily see, never vanishes identically. Thus we can express $\frac{\partial R}{\partial \gamma_1}$ from (8.12) and then determine $\frac{\partial^2 R}{\partial \gamma_1^2}$ from (8.13) as follows:

$$\frac{\partial^2 R}{\partial \gamma_1^2} = \frac{I_2^2 \omega_2^2 S}{[(I_2^2 \omega_2^2 + I_3^2 \omega_3^2) R + I_3 \omega_3 (I_1 \omega_1 \gamma_1 - U_1)]^3}, \tag{8.15}$$

where

$$\begin{aligned} S = &-(I_2^2 \omega_2^2 + I_3^2 \omega_3^2)(I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2) R^2 \\ &+ 2 I_3 \omega_3 (I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2)(U_1 - I_1 \omega_1 \gamma_1) R \\ &- I_1^4 \omega_1^4 \gamma_1^2 + 2 I_1^3 \omega_1^3 \gamma_1 U_1 - 2 I_1^2 I_2^2 \omega_1^2 \omega_2^2 \gamma_1^2 - I_1^2 I_3^2 \omega_1^2 \omega_3^2 \gamma_1^2 - I_1^2 \omega_1^2 U_1^2 \\ &+ 2 I_1 I_2^2 \omega_1 \omega_2^2 \gamma_1 U_1 + 2 I_1 I_3^2 \omega_1 \omega_3^2 \gamma_1 U_1 - I_2^4 \omega_2^4 \gamma_1^2 - I_2^2 I_3^2 \omega_2^2 \omega_3^2 \gamma_1^2 - I_3^2 \omega_3^2 U_1^2. \end{aligned}$$

Equations (8.15) and (8.11) imply that $S = 0$. Taking into account that $Q = 0$ (see (8.3)) we can assert that

$$(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2)Q + S = I_2^2\omega_2^2[U_1^2 - U_2(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2)]$$

is also zero. This is possible if and only if

$$U_1 = U_2 = 0. \tag{8.16}$$

Let us consider the case defined by (8.16). In this case R can be represented in the form $R = P\gamma_1$, where P is a root of

$$(I_2^2\omega_2^2 + I_3^2\omega_3^2)P^2 + 2I_1I_3\omega_1\omega_3 P + I_1^2\omega_1^2 + I_2^2\omega_2^2 = 0. \tag{8.17}$$

This follows very easily from (8.3) if we take into account condition (8.16). Indeed, it suffices to divide (8.3) by γ_1^2 and denote R/γ_1 by P. Moreover, being a root of (8.17), P is a homogeneous function of degree zero and depends only on ω_1, ω_2 and ω_3.

Thus the vector field Y_1 (see (8.6)) is linear with respect to γ_1 and can be represented as

$$Y_1 = K_1\gamma_1 + K_2,$$

where K_1 and K_2 are vector fields defined on Ω by the formulas

$$\begin{aligned} K_1 &= -I_3(I_1c_3\omega_1 + I_2c_2\omega_2P + I_3c_3\omega_3P)\frac{\partial}{\partial\omega_1} + I_1I_3\omega_2(c_1P - c_3)\frac{\partial}{\partial\omega_2} \\ &\quad + I_1(I_1c_1\omega_1 + I_2c_2\omega_2 + I_3c_1\omega_3P)\frac{\partial}{\partial\omega_3}, \\ K_2 &= \omega_2\Big[I_2I_3\omega_2\omega_3(I_2 - I_3)\frac{\partial}{\partial\omega_1} - I_1I_3\omega_1\omega_3(I_1 - I_3)\frac{\partial}{\partial\omega_2} \\ &\quad + I_1I_2\omega_1\omega_2(I_1 - I_2)\frac{\partial}{\partial\omega_3}\Big]. \end{aligned}$$

Equation (8.6) and the fact that F does not depend on γ_1 imply

$$K_1(F) = K_2(F) = 0. \tag{8.18}$$

The function $I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$ is non-constant on all open subsets of $\mathbb{C}_4(\omega_1, \omega_2, \omega_3, \gamma_1)$. Thus without any restriction of generality one can suppose that on our suitable open set, $I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \neq 0$.

In order to simplify formulas let us put

$$\alpha = I_2^2\omega_2^2 + I_3^2\omega_3^2, \quad \beta = \sqrt{-I_1^2\omega_1^2 - I_2^2\omega_2^2 - I_3^2\omega_3^2}, \quad \mathrm{Re}(\beta) > 0. \tag{8.19}$$

Equation (8.17) has two roots

$$P = \frac{-I_1I_3\omega_1\omega_3 + \varepsilon I_2\omega_2\beta}{\alpha},$$

where $\varepsilon = \pm1$. Substituting this value of P in the expression for K_1 we obtain

$$\begin{aligned} \frac{\alpha K_1}{\omega_2} &= -I_2I_3[I_1\omega_1(I_2c_3\omega_2 - I_3c_2\omega_3) + \varepsilon\beta(I_2c_2\omega_2 + I_3c_3\omega_3)]\frac{\partial}{\partial\omega_1} \\ &\quad + I_1I_3[-I_1I_3c_1\omega_1\omega_3 - c_3\alpha + \varepsilon\beta I_2c_1\omega_2]\frac{\partial}{\partial\omega_2} \\ &\quad + I_1I_2[I_1I_2c_1\omega_1\omega_2 + c_2\alpha + \varepsilon\beta I_3c_1\omega_3]\frac{\partial}{\partial\omega_3}. \end{aligned}$$

As we are interested in equations (8.18) we remove all the non-zero factors from K_1 and K_2, i.e. we shall work with the vector fields

$$Z_1 = \frac{\alpha K_1}{\omega_2} \quad \text{and} \quad Z_2 = \frac{K_2}{\omega_2}.$$

Therefore instead of (8.18) we consider the following equations:

$$Z_1(F) = Z_2(F) = 0. \tag{8.20}$$

We compute the Lie bracket $K_3 = [Z_1, Z_2]$ and obtain

$$\begin{aligned}
\frac{K_3}{I_1 I_2 I_3} = {} & [-(I_1 - I_2) I_1 I_2 I_3 c_2 \omega_1^2 \omega_2 - (I_1 - I_3) I_1 I_2 I_3 c_3 \omega_1^2 \omega_3 \\
& + (I_2 - I_3) I_1 I_2^2 c_1 \omega_1 \omega_2^2 - (I_2 - I_3) I_1 I_3^2 c_1 \omega_1 \omega_3^2 \\
& + \varepsilon\beta (I_1 - I_2) I_2 I_3 c_3 \omega_1 \omega_2 - \varepsilon\beta (I_1 - I_3) I_2 I_3 c_2 \omega_1 \omega_3 \\
& + (I_2 - I_3) I_2^3 c_2 \omega_2^3 + 2\,\varepsilon\beta (I_2 - I_3) I_2 I_3 c_1 \omega_2 \omega_3 - (I_2 - I_3) I_3^3 c_3 \omega_3^3] \frac{\partial}{\partial \omega_1} \\
& + [-(I_2 - I_3) I_1^3 c_1 \omega_1^2 \omega_2 - (I_1 - I_3) I_1 I_2^2 c_2 \omega_1 \omega_2^2 \\
& - (I_1 I_2 - 2\, I_1 I_3 + I_2 I_3) I_1 I_3 c_3 \omega_1 \omega_2 \omega_3 - 2(I_1 - I_3) I_1 I_3^2 c_2 \omega_1 \omega_3^2 \\
& + (I_2 - I_3) I_1 I_3^2 c_1 \omega_2 \omega_3^2 + \varepsilon\beta (I_1 - I_3) I_2 I_3 c_2 \omega_2 \omega_3 \\
& + \varepsilon\beta (I_1 - I_3) I_3^2 c_3 \omega_3^2] \frac{\partial}{\partial \omega_2} \\
& + [(I_2 - I_3) I_1^3 c_1 \omega_1^2 \omega_3 - 2(I_1 - I_2) I_1 I_2^2 c_3 \omega_1 \omega_2^2 \\
& + (2 I_1 I_2 - I_1 I_3 - I_2 I_3) I_1 I_2 c_2 \omega_1 \omega_2 \omega_3 - (I_1 - I_2) I_1 I_3^2 c_3 \omega_1 \omega_3^2 \\
& - (I_2 - I_3) I_1 I_2^2 c_1 \omega_2^2 \omega_3 - \varepsilon\beta (I_1 - I_2) I_2^2 c_2 \omega_2^2 - \varepsilon\beta (I_1 - I_2) I_2 I_3 c_3 \omega_2 \omega_3] \frac{\partial}{\partial \omega_3}.
\end{aligned}$$

We consider the vector field

$$Z_3 = \frac{K_3}{I_1 I_2 I_3}$$

instead of K_3.

Equations (8.20) imply that $Z_3(F) = 0$. In this way we obtain the following three equations to determine F:

$$Z_1(F) = Z_2(F) = Z_3(F) = 0. \tag{8.21}$$

Equations (8.21) can be considered as a system of three homogeneous linear algebraic equations with the unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if a fourth integral F exists, system (8.21) has a non-zero solution $\operatorname{grad} F$. We know that this is possible if and only if the determinant D composed of the coefficients of (8.21) is identically zero. We compute this determinant and find that on Ω,

$$D = I_1 I_2 I_3 (f_1 \beta + f_2), \tag{8.22}$$

where f_1 and f_2 are the following polynomials in ω_1, ω_2 and ω_3:

$$\begin{aligned}\frac{f_1}{\varepsilon} = {} & -(I_1-I_2)(I_1-I_3)I_1^2I_2^3c_1c_2\omega_1^3\omega_2^3 - (I_1-I_2)(I_1-I_3)I_1^2I_2^2I_3c_1c_3\omega_1^3\omega_2^2\omega_3 \\ & - (I_1-I_2)(I_1-I_3)I_1^2I_2I_3^2c_1c_2\omega_1^3\omega_2\omega_3^2 - (I_1-I_2)(I_1-I_3)I_1^2I_3^3c_1c_3\omega_1^3\omega_3^3 \\ & + (I_1-I_2)[(I_2-I_3)I_1c_1^2 - (I_1-I_3)I_2c_2^2 - (I_1-I_2)I_3c_3^2]I_1I_2^3\omega_1^2\omega_2^4 \\ & - 2\,(I_1-I_2)(I_2-I_3)I_1^2I_2^2I_3c_2c_3\omega_1^2\omega_2^3\omega_3 \\ & + I_1I_2I_3[(I_2-I_3)^2I_1^2c_1^2 + (I_1-I_3)(I_1I_2-3I_1I_3+2\,I_2I_3)I_2c_2^2 \\ & - (I_1-I_2)(3I_1I_2-I_1I_3-2I_2I_3)I_3c_3^2]\omega_1^2\omega_2^2\omega_3^2 \\ & + 2(I_1-I_3)(I_2-I_3)I_1^2I_2I_3^2c_2c_3\omega_1^2\omega_2\omega_3^3 \\ & - (I_1-I_3)[(I_2-I_3)I_1c_1^2 + (I_1-I_3)I_2c_2^2 + (I_1-I_2)I_3c_3^2]I_1I_3^3\omega_1^2\omega_3^4 \\ & + (I_1-I_2)(I_2-I_3)I_1I_2^4c_1c_2\omega_1\omega_2^5 + (I_1-I_2)(I_2-I_3)I_1I_2^2I_3^2c_1c_2\omega_1\omega_2^3\omega_3^2 \\ & - (I_1-I_3)(I_2-I_3)I_1I_2^2I_3^2c_1c_3\omega_1\omega_2^2\omega_3^3 - (I_1-I_3)(I_2-I_3)I_1I_3^4c_1c_3\omega_1\omega_3^5 \\ & - (I_1-I_2)(I_2-I_3)I_2^4I_3c_2c_3\omega_2^5\omega_3 \\ & + (I_2-I_3)[(I_2-I_3)I_1c_1^2 + (I_1-I_3)I_2c_2^2 - (I_1-I_2)I_3c_3^2]I_2^3I_3\omega_2^4\omega_3^2 \\ & + (I_2-I_3)^2I_1I_2^2I_3^2c_2c_3\omega_2^3\omega_3^3 \\ & + (I_2-I_3)[(I_2-I_3)I_1c_1^2 + (I_1-I_3)I_2c_2^2 - (I_1-I_2)I_3c_3^2]I_2I_3^3\omega_2^2\omega_3^4 \\ & + (I_1-I_3)(I_2-I_3)I_2I_3^4c_2c_3\omega_2\omega_3^5,\end{aligned}$$

$$\begin{aligned}\frac{f_2}{\beta^2} = {} & -(I_1-I_2)I_2^2I_1c_3c_1(-I_1I_3+I_1I_2+\varepsilon^2I_1I_3-\varepsilon^2I_2I_3)\omega_1^2\omega_2^3 \\ & + \varepsilon^2(I_1-I_2)(-I_3+I_1)I_1I_2^2I_3c_2c_1\omega_1^2\omega_2^2\omega_3 \\ & - \varepsilon^2(I_1-I_2)(I_1-I_3)I_1I_2I_3^2c_3c_1\omega_1^2\omega_2\omega_3^2 \\ & + (I_1-I_3)(I_1I_3-I_2I_1+\varepsilon^2I_1I_2-\varepsilon^2I_2I_3)I_1I_3^2c_2c_1\omega_1^2\omega_3^3 \\ & - (I_1-I_2)(I_2-I_3)I_1I_2^3c_2c_3\omega_1\omega_2^4 \\ & - 2(I_1-I_2)(I_2-I_3)I_1I_2^2I_3(\varepsilon^2c_1^2+c_3^2)\omega_1\omega_2^3\omega_3 \\ & + I_1(I_2-I_3)(I_1I_2+I_1I_3-2I_2I_3)I_2I_3c_2c_3\omega_1\omega_2^2\omega_3^2 \\ & - 2(I_2-I_3)(I_1-I_3)I_1I_2I_3^2(\varepsilon^2c_1^2+c_2^2)\omega_1\omega_2\omega_3^3 \\ & - (I_1-I_3)(I_2-I_3)I_1I_3^3c_3c_2\omega_1\omega_3^4 - \varepsilon^2(I_1-I_2)(I_2-I_3)I_2^3I_3c_2c_1\omega_2^4\omega_3 \\ & - (I_2-I_3)(I_1I_2-I_1I_3+\varepsilon^2I_1I_3-\varepsilon^2I_2I_3)I_2^2I_3c_1c_3\omega_2^3\omega_3^2 \\ & - (I_2-I_3)(I_1I_3-I_1I_2+\varepsilon^2I_1I_2-\varepsilon^2I_2I_3)I_2I_3^2c_1c_2\omega_2^2\omega_3^3 \\ & - \varepsilon^2(I_1-I_3)(I_2-I_3)I_2I_3^3c_1c_3\omega_2\omega_3^4.\end{aligned}$$

As $I_i \neq 0$, $1 \leq i \leq 3$, from (8.22) one deduces that $f_1\beta + f_2 = 0$. If $f_1 = 0$ identically, then $f_2 = 0$ identically too. Let us suppose that $f_1 \neq 0$. Then (8.22) is equivalent to

$$\beta = -\frac{f_2}{f_1}. \tag{8.23}$$

Applying Proposition 4.3 to $\beta^2 = -I_1^2\omega_1^2 - I_2^2\omega_2^2 - I_3^2\omega_3^2$ one sees that (8.23) can never occur because $\beta \notin \mathbb{C}(\omega_1, \omega_2, \omega_3)$. Consequently,

$$f_1 = f_2 = 0.$$

Note that ε appears in f_1 as a factor and in f_2 only as ε^2. We replace ε^2 with 1 everywhere in f_2. As β^2 can never be identically zero, we require that all the 18 coefficients of f_1/ε and all the 13 coefficients of f_2/β^2 be zero, i.e. we obtain a system of 31 equations for the parameters $\mathcal{I}c$.

After three consecutive simplifications we obtain a reduced system that contains eight equations:

$$
\begin{aligned}
&(I_2 - I_3)c_2c_3 = 0, \quad (I_1 - I_3)c_1c_3 = 0, \quad (I_1 - I_2)(I_1 - I_3)(I_2 - I_3)c_3 = 0,\\
&(I_1 - I_2)(I_1 - I_3)(c_2^2 + c_3^2) = 0, \quad (I_1 - I_2)(I_2 - I_3)c_1c_2 = 0,\\
&(I_1 - I_2)(I_1 + I_2 - 2I_3)c_1c_2 = 0,\\
&(I_2 - I_3)[(I_2 - I_3)c_1^2 + (I_1 - I_3)c_2^2 + (I_2 - I_1)c_3^2] = 0,\\
&(I_1 - I_3)(I_2 - I_3)(c_1^2 + c_2^2) = 0.
\end{aligned}
$$

We solve these eight equations by using `solve` and obtain a set of eight solutions. We discard the solutions that lead to the Euler, Lagrange and kinetic symmetry cases and obtain only three new solutions:

I. $I_1 = I_2$, $c_1 = \pm ic_2$, $c_3 = 0$,
II. $I_1 = I_3$, $c_1 = \pm ic_3$, $c_2 = 0$,
III. $I_2 = I_3$, $c_1 = 0$, $c_2 = \pm ic_3$.

We describe here only solution I, because solutions II and III are obtained from it by permutational symmetries σ_2 and σ_3, respectively.

We consider two cases:

1. $c_1 = ic_2$ with $\varepsilon = 1$ and with $\varepsilon = -1$.
2. $c_1 = -ic_2$ with $\varepsilon = 1$ and with $\varepsilon = -1$.

Let us remark that the above situation is exactly the one we met in Sec. 7.2 when looking for invariant manifolds.

Case 1. Let $I_1 = I_2$, $c_1 = ic_2$, $c_3 = 0$ and $\varepsilon = 1$. Now the vector field Z_3 is linearly dependent on Z_1 and Z_2 and therefore the local solvability of system (8.21) around any point $(\omega_1, \omega_2, \omega_3) \neq (0, 0, 0)$ follows from the Frobenius Integrability Theorem. Moreover, system (8.5) is quasi-homogeneous (we recall that $R = P\gamma_1$ and P is a homogeneous function of degree zero). Thus from [49] it follows that the sought first integral F can be chosen to be a homogeneous function of $(\omega_1, \omega_2, \omega_3)$. But in fact we shall find F by a crude computation, without any use of the Frobenius Integrability Theorem or the results of [49]. Nevertheless the above facts guide our approach to the problem.

Let us add to equations (8.20) the Euler "homogeneity equation"

$$\omega_1 \frac{\partial F}{\partial \omega_1} + \omega_2 \frac{\partial F}{\partial \omega_2} + \omega_3 \frac{\partial F}{\partial \omega_3} = F. \tag{8.24}$$

Dividing equations (8.20) and (8.24) by F we obtain a system of three linear partial differential equations for the function $V = \log F$. We solve this system as a linear inhomogeneous algebraic system with respect to the partial derivatives of V and obtain

$$\begin{aligned}
\frac{\partial V}{\partial \omega_1} &= \frac{(I_2^2\omega_1\omega_2 - iI_2^2\omega_2^2 - iI_3^2\omega_3^2 + I_3\omega_3\beta)\omega_1}{(I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta)(\omega_1 - i\omega_2)},\\
\frac{\partial V}{\partial \omega_2} &= \frac{(I_2^2\omega_1\omega_2 - iI_2^2\omega_2^2 - iI_3^2\omega_3^2 + I_3\omega_3\beta)\omega_2}{(I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta)(\omega_1 - i\omega_2)},\\
\frac{\partial V}{\partial \omega_3} &= \frac{iI_3(I_3\omega_1\omega_3 - \omega_2\beta)}{I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta}.
\end{aligned}$$

Now, by the standard procedure, we find the function V.

We integrate $\frac{\partial V}{\partial \omega_i}$, $1 \le i \le 3$, with respect to ω_i and in this way we obtain three expressions

$$\begin{aligned}
V &= \int \frac{(I_2^2\omega_1\omega_2 - iI_2^2\omega_2^2 - iI_3^2\omega_3^2 + I_3\omega_3\beta)\omega_1}{(I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta)(\omega_1 - i\omega_2)}\, d\omega_1 + G_1(\omega_2,\omega_3),\\
V &= \int \frac{(I_2^2\omega_1\omega_2 - iI_2^2\omega_2^2 - iI_3^2\omega_3^2 + I_3\omega_3\beta)\omega_2}{(I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta)(\omega_1 - i\omega_2)}\, d\omega_2 + G_2(\omega_1,\omega_3),\\
V &= \int \frac{iI_3(I_3\omega_1\omega_3 - \omega_2\beta)}{I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta}\, d\omega_3 + G_3(\omega_1,\omega_2),
\end{aligned}$$

where G_1, G_2 and G_3 are arbitrary functions of the corresponding variables.

The first two expressions for V are too complicated and we do not use them. We only use the third expression which can be rewritten as follows:

$$V = \operatorname{csgn}(I_3) i \arctan \frac{\operatorname{csgn}(I_3) I_3\omega_3}{\beta} + G_3(\omega_1,\omega_2),$$

where the function $\operatorname{csgn}(z)$ is used to determine in which half-plane (“left” or “right”) the complex number z lies. It is defined by

$$\operatorname{csgn}(z) = \begin{cases} 1 & \text{if } \operatorname{Re}(z) > 0,\\ -1 & \text{if } \operatorname{Re}(z) < 0,\\ \operatorname{sgn}(\operatorname{Im}(z)) & \text{if } \operatorname{Re}(z) = 0. \end{cases}$$

As arctan is an odd function we can write

$$V = i \arctan \frac{I_3\omega_3}{\beta} + G_3(\omega_1,\omega_2). \tag{8.25}$$

In order to determine $G_3(\omega_1,\omega_2)$ we differentiate (8.25) with respect to ω_i, $1 \le i \le 3$, and obtain

$$\begin{aligned}
\frac{\partial V}{\partial \omega_1} &= -\frac{iI_3\omega_1\omega_3}{(\omega_1^2+\omega_2^2)\beta} + \frac{\partial G_3}{\partial \omega_1},\\
\frac{\partial V}{\partial \omega_2} &= -\frac{iI_3\omega_2\omega_3}{(\omega_1^2+\omega_2^2)\beta} + \frac{\partial G_3}{\partial \omega_2},\\
\frac{\partial V}{\partial \omega_3} &= \frac{iI_3}{\beta}.
\end{aligned}$$

We know that V satisfies system (8.20) so that

$$Z_1(V) = Z_2(V) = 0. \tag{8.26}$$

System (8.26) is a system of two linear partial differential equations for the function $G_3(\omega_1,\omega_2)$. We solve it as a linear inhomogeneous algebraic system in the unknowns $\frac{\partial G_3}{\partial \omega_1}$

and $\frac{\partial G_3}{\partial \omega_2}$ and obtain

$$\frac{\partial G_3}{\partial \omega_1} = \frac{\omega_1}{\omega_1^2+\omega_2^2}, \qquad \frac{\partial G_3}{\partial \omega_2} = \frac{\omega_2}{\omega_1^2+\omega_2^2}.$$

After integration these expressions lead to

$$G_3(\omega_1,\omega_2) = \tfrac{1}{2}\log(\omega_1^2+\omega_2^2) + C,$$

where C is a constant which can be considered to be zero because it is added to a first integral. Thus from (8.25) we have

$$V = i\arctan\frac{I_3\omega_3}{\beta} + \frac{1}{2}\log(\omega_1^2+\omega_2^2).$$

As $V = \log F$ we have

$$F = \exp V = \frac{-I_3\omega_3 + i\beta}{\pm I_2}.$$

We remove the constant denominator and change the sign of the function. Let us denote the function obtained by F_1. We have

$$F_1 = I_3\omega_3 - i\beta.$$

F_1 satisfies equations (8.20), which means that in Case 1 with $\varepsilon = 1$, F_1 is a first integral of system (8.5).

Let now $\varepsilon = -1$. The considerations are exactly as above, but now

$$V = -i\arctan\frac{I_3\omega_3}{\beta} + \frac{1}{2}\log(\omega_1^2+\omega_2^2)$$

and $F = \exp V$ is

$$F = \frac{I_2(\omega_1^2+\omega_2^2)}{I_3\omega_3 - i\beta}.$$

The function F can be simplified. Indeed, as according to (8.19),

$$\beta^2 = -I_2^2(\omega_1^2+\omega_2^2) - I_3^2\omega_3^2,$$

we have

$$\omega_1^2+\omega_2^2 = -\frac{I_3^2\omega_3^2+\beta^2}{I_2^2} = -\frac{(I_3\omega_3+i\beta)(I_3\omega_3-i\beta)}{I_2^2}.$$

We put the value for $\omega_1^2+\omega_2^2$ in the above expression for F and obtain

$$F = -\frac{I_3\omega_3+i\beta}{I_2}.$$

By removing the constant denominator I_2 and changing the sign of F we obtain a new function

$$F_2 = I_3\omega_3 + i\beta.$$

F_2 satisfies equations (8.20). Thus in Case 1 with $\varepsilon = -1$, F_2 is a first integral of system (8.5).

Case 2. Let $I_1 = I_2$, $c_1 = -ic_2$, $c_3 = 0$. The only difference in this case in comparison with Case 1 is that when $\varepsilon = 1$, F_2 is a first integral of system (8.5) and when $\varepsilon = -1$ the first integral is F_1.

The functional independence of these partial integrals from H_3 follows from the fact that they do not depend on γ_1 while H_3 does. Computation by Maple shows that the time derivatives $\frac{\partial F_1}{\partial t}$ and $\frac{\partial F_2}{\partial t}$ are both not identically 0 for Euler–Poisson equations (1.1). Thus F_1 and F_2 are partial integrals.

Let us note that the partial integrals F_1 and F_2 are algebraic without being rational. This is a new fact. Up to now, the known first integrals or partial integrals not depending on all the variables have been polynomials.

Type 2. Let us study the existence of a first integral of type 2, i.e. $F = F(\omega_1, \omega_2, \gamma_1)$. Now we have

$$\begin{aligned}\frac{dF}{dt} &= \frac{1}{I_1 I_2 \omega_2}[I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1 c_3 \omega_1 \gamma_1 - (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)R + c_3 U_1]\frac{\partial F}{\partial \omega_1}\\ &+ \frac{1}{I_2}[(I_3 - I_1)\omega_1\omega_3 + c_1 R - c_3\gamma_1]\frac{\partial F}{\partial \omega_2}\\ &+ \frac{1}{I_2\omega_2}[-I_1\omega_1\omega_3\gamma_1 - (I_2\omega_2^2 + I_3\omega_3^2)R + \omega_3 U_1]\frac{\partial F}{\partial \gamma_1} = 0,\end{aligned}$$

or equivalently

$$I_1 I_2 \omega_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.27}$$

where Y_1 is a vector field defined on Ω.

Equation (8.27) should be an identity with respect to all four variables $(\omega_1, \omega_2, \omega_3, \gamma_1)$. Similarly to the consideration of a first integral of type 1, if we differentiate (8.27) with respect to ω_3 we will again obtain a linear partial differential equation for F. We obtain

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial \omega_3} &= \left[I_2(I_2 - I_3)\omega_2^2 - I_3 c_3 R - (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)\frac{\partial R}{\partial \omega_3}\right]\frac{\partial F}{\partial \omega_1}\\ &- I_1\omega_2\left[(I_1 - I_3)\omega_1 - c_1\frac{\partial R}{\partial \omega_3}\right]\frac{\partial F}{\partial \omega_2}\\ &- I_1\left[I_1\omega_1\gamma_1 + 2I_3\omega_3 R + (I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial R}{\partial \omega_3} - U\right]\frac{\partial F}{\partial \gamma_1} = 0,\end{aligned}$$

i.e.

$$\frac{\partial Y_1(F)}{\partial \omega_3} = Y_2(F) = 0, \tag{8.28}$$

where Y_2 is a vector field defined on Ω.

After differentiating (8.28) with respect to ω_3 we obtain

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial \omega_3} &= -\left[2I_3c_3\frac{\partial R}{\partial \omega_3} + (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)\frac{\partial^2 R}{\partial \omega_3^2}\right]\frac{\partial F}{\partial \omega_1} + I_1 c_1 \omega_2 \frac{\partial^2 R}{\partial \omega_3^2}\frac{\partial F}{\partial \omega_2}\\ &- I_1\left[2I_3 R + 4I_3\omega_3\frac{\partial R}{\partial \omega_3} + (I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial^2 R}{\partial \omega_3^2}\right]\frac{\partial F}{\partial \gamma_1} = 0,\end{aligned}$$

i.e.

$$\frac{\partial Y_2(F)}{\partial \omega_3} = Y_3(F) = 0, \tag{8.29}$$

where Y_3 is a vector field defined on Ω.

We already know that if a fourth integral F exists, system (8.27)–(8.29) has a non-zero solution $\operatorname{grad} F$, and that is possible if and only if the determinant D of the coefficients of this system is identically zero.

We compute D and obtain a long expression which we do not write out here. Let us note that D has a non-zero factor $I_1^2\omega_2$ and we denote

$$\widehat{D}(R) = \frac{D(R)}{I_1^2\omega_2}.$$

It is clear that the equation $D(R) = 0$ is equivalent to $\widehat{D}(R) = 0$.

The derivatives $\frac{\partial R}{\partial \omega_3}$ and $\frac{\partial^2 R}{\partial \omega_3^2}$ appear in $\widehat{D}(R)$. To determine them we differentiate (8.3) with respect to ω_3 two times. Taking into account that the polynomial C from (8.4) does not depend on ω_3 we obtain

$$\frac{\partial Q}{\partial \omega_3} = \frac{\partial A}{\partial \omega_3}R^2 + \frac{\partial B}{\partial \omega_3}R + \frac{dQ}{dR}\frac{\partial R}{\partial \omega_3} = 0 \tag{8.30}$$

and

$$\begin{aligned}\frac{\partial^2 Q}{\partial \omega_3^2} = \frac{\partial^2 A}{\partial \omega_3^2}R^2 + 2R\frac{\partial A}{\partial \omega_3}\frac{\partial R}{\partial \omega_3} + \frac{\partial^2 B}{\partial \omega_3^2}R + \frac{\partial B}{\partial \omega_3}\frac{\partial R}{\partial \omega_3}\\ + \frac{\partial}{\partial \omega_3}\left(\frac{dQ}{dR}\right)\frac{\partial R}{\partial \omega_3} + \frac{dQ}{dR}\frac{\partial^2 R}{\partial \omega_3^2} = 0.\end{aligned} \tag{8.31}$$

As in the investigation of a first integral of type 1, by Proposition 4.1 we prove that if R is a root of (8.3), then $\frac{dQ}{dR} \neq 0$. Of course, the resultant ρ of the polynomials $Q(R)$ and $\frac{dQ}{dR}$ is the same and we conclude that $\frac{dQ}{dR} \neq 0$ (see (8.14)).

Thus we can correctly determine $\frac{\partial R}{\partial \omega_3}$ from (8.30) and put its value in (8.31). Then we easily determine $\frac{\partial^2 R}{\partial \omega_3^2}$ from (8.31). This can be done correctly because the coefficient of $\frac{\partial^2 R}{\partial \omega_3^2}$ is also $\frac{dQ}{dR}$.

Then we put the resulting values for the derivatives of R in the expression for $\widehat{D}(R)$ and deduce that $\widehat{D}(R)$ has a non-zero factor $8I_3R$ and denominator $\left(\frac{dQ}{dR}\right)^3$. We set

$$\delta(R) = \frac{\left(\frac{dQ}{dR}\right)^3}{8I_3R}\widehat{D}(R),$$

where $\delta(R)$ is a polynomial in R of degree 5 with coefficients which are polynomials in ω_1, ω_2, ω_3 and γ_1. It is clear that the equation $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$.

We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_1, \omega_2, \gamma_1)$ of system (8.5) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in the polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_1, \omega_2, \omega_3, \gamma_1)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using the Maple command `rem` we divide δ by Q and obtain a remainder which is a polynomial r of the form

$$r(x) = \frac{I_2^4\omega_2^4}{I_2^2\omega_2^2 + I_3^2\omega_3^2}(a_0x + a_1),$$

where a_0 and a_1 are polynomials in ω_1, ω_2, ω_3 and γ_1.

According to Proposition 4.2 we know that if R is a root of (8.3), then a_0 and a_1 should be identically zero. We use a_0 only because this turns out to be sufficient for our purposes.

The polynomial a_0 has 81 coefficients. Thus we should equate all of them to zero. In this way we obtain a system of 81 equations for $\mathcal{I}c$, U_1 and U_2. After two consecutive simplifications we obtain a reduced system which is very simple:

$$c_3 = 0, \quad (I_1 - I_3)c_2 = 0, \quad (I_2 - I_3)c_1 = 0.$$

The solutions are obvious:

$$\begin{array}{ll}
\{c_1 = 0,\, c_2 = 0,\, c_3 = 0\}, & U_1,\, U_2,\, I_1,\, I_2 \text{ and } I_3 \text{ are parameters},\\
\{I_2 = I_3,\, c_2 = 0,\, c_3 = 0\}, & U_1,\, U_2,\, I_1,\, I_3 \text{ and } c_1 \text{ are parameters},\\
\{I_1 = I_3,\, c_1 = 0,\, c_3 = 0\}, & U_1,\, U_2,\, I_2,\, I_3 \text{ and } c_2 \text{ are parameters},\\
\{I_1 = I_3,\, I_2 = I_3,\, c_3 = 0\}, & U_1,\, U_2,\, I_3,\, c_1 \text{ and } c_2 \text{ are parameters}.
\end{array}$$

It is easily seen that these solutions lead to the Euler, Lagrange and kinetic symmetry cases, respectively. Thus the sought partial integral of type 2 cannot exist.

Type 4. Finally, let us investigate the possibilities for the existence of a first integral of type 4, $F(\omega_2, \omega_3, \gamma_1)$, i.e. when it does not depend on ω_1.

As F is a first integral of system (8.5), we have

$$\begin{aligned}
\frac{dF}{dt} &= \frac{1}{I_2}[(I_3 - I_1)\omega_1\omega_3 + c_1 R - c_3\gamma_1]\frac{\partial F}{\partial \omega_2}\\
&\quad + \frac{1}{I_2 I_3 \omega_2}[I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1 c_1 \omega_1 \gamma_1 + I_2 c_2 \omega_2 \gamma_1 + I_3 c_1 \omega_3 R - c_1 U_1]\frac{\partial F}{\partial \omega_3}\\
&\quad + \frac{1}{I_2\omega_2}[-I_1\omega_1\omega_3\gamma_1 - (I_2\omega_2^2 + I_3\omega_3^2)R + \omega_3 U_1]\frac{\partial F}{\partial \gamma_1} = 0,
\end{aligned}$$

or equivalently

$$I_2 I_3 \omega_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.32}$$

where Y_1 is a vector field defined on Ω.

Equation (8.32) should be an identity with respect to all four variables $(\omega_1, \omega_2, \omega_3, \gamma_1)$. As in the previous considerations, taking into account that F does not depend on ω_1, differentiating (8.32) with respect to ω_1 we obtain again a linear partial differential equation for F:

$$\begin{aligned}
\frac{\partial Y_1(F)}{\partial \omega_1} &= I_3\omega_2\left[(I_3 - I_1)\omega_3 + c_1\frac{\partial R}{\partial \omega_1}\right]\frac{\partial F}{\partial \omega_2}\\
&\quad + \left[I_2(I_1 - I_2)\omega_2^2 + I_1 c_1 \gamma_1 + I_3 c_1 \omega_3 \frac{\partial R}{\partial \omega_1}\right]\frac{\partial F}{\partial \omega_3}\\
&\quad + I_3\left[-I_1\omega_3\gamma_1 - (I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial R}{\partial \omega_1}\right]\frac{\partial F}{\partial \gamma_1} = Y_2(F) = 0,
\end{aligned} \tag{8.33}$$

where Y_2 is a vector field defined on Ω.

We differentiate identity (8.33) with respect to ω_1 and obtain

$$\frac{1}{I_3}\,\frac{\partial Y_2(F)}{\partial \omega_1} = \frac{\partial^2 R}{\partial \omega_1^2}\left[c_1\omega_2\frac{\partial F}{\partial \omega_2} + c_1\omega_3\frac{\partial F}{\partial \omega_3} - (I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial F}{\partial \gamma_1}\right] = Y_3(F) = 0, \qquad (8.34)$$

where Y_3 is a vector field defined on Ω.

Let us first suppose that $\frac{\partial^2 R}{\partial \omega_1^2} \neq 0$. In that case, if a first integral F exists then system (8.32)–(8.34) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D of the coefficients of this system is identically zero.

Let us compute D. We have

$$D = I_2 I_3 \omega_2^2 \frac{\partial^2 R}{\partial \omega_1^2}\widehat{D},$$

where

$$\begin{aligned}\widehat{D} = {} & I_2(I_1 - I_2)c_3\omega_2^3\gamma_1 - I_2(I_1 - I_3)c_2\omega_2^2\omega_3\gamma_1 + I_3(I_1 - I_2)c_3\omega_2\omega_3^2\gamma_1 \\ & + (I_2 - I_3)c_1U_1\omega_2\omega_3 + I_1c_1c_3\omega_2\gamma_1^2 - I_3(I_1 - I_3)c_2\omega_3^3\gamma_1 - I_1c_1c_2\gamma_1^2\omega_3.\end{aligned}$$

As $I_2 I_3 \omega_2^2 \frac{\partial^2 R}{\partial \omega_1^2} \neq 0$ we use $\widehat{D}$ instead D.

The polynomial $\widehat{D}$ has 7 coefficients. Equating all of them to zero we obtain a system of 7 equations for $\mathcal{I}c$ and U_1. After two consecutive simplifications we obtain a reduction (see Sec. 3) of this system that consists of 6 equations:

$$\begin{aligned} & c_1c_3 = 0, \quad (I_1 - I_2)c_3 = 0, \quad c_1c_2 = 0, \quad (I_1 - I_3)c_2 = 0, \\ & (I_2 - I_3)c_2c_3 = 0, \quad (I_2 - I_3)c_1U_1 = 0. \end{aligned} \qquad (8.35)$$

A simple case analysis leads to a set of six solutions given by `solve`:

$$\begin{aligned} & \{U_1 = 0,\ I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\}, \\ & \{U_1 = U_1,\ I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\}, \\ & \{U_1 = U_1,\ I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = c_3\}, \\ & \{U_1 = U_1,\ I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0\}, \\ & \{U_1 = U_1,\ I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\}, \\ & \{U_1 = U_1,\ I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0\}. \end{aligned}$$

This list should be understood as follows. If an equation $U_i = U_i$ or $I_i = I_i$ or $c_i = c_i$, $1 \le i \le 3$, appears we should consider the corresponding parameter to be an arbitrary complex number. For example, let us consider the third row. There U_1, I_3, c_2 and c_3 are arbitrary complex numbers but I_1, I_2 and c_1 have fixed values. Some of these fixed values can depend on the chosen value of some arbitrary parameter, as in this example I_1 and I_2 depend on the arbitrary fixed value of I_3.

We discard the solutions that lead to the Euler, Lagrange and kinetic symmetry cases and obtain only one new solution

$$\{U_1 = 0,\ I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\}.$$

Let us study this solution. We have $U_1 = 0$, $c_2 = c_3 = 0$ and $I_i \neq 0$, $1 \le i \le 3$, and c_1 are arbitrary parameters. In this case $\widehat{D} = 0$. Thus the vector fields $Y_i \neq 0$, $1 \le i \le 3$, (see (8.32)–(8.34)) are linearly dependent. More precisely, the following equation holds:

$$Y_1 = \omega_1 Y_2 + I_3\left(R - \omega_1 \frac{\partial R}{\partial \omega_1}\right) Y_3.$$

We discard the vector field Y_1 because it is a linear combination of Y_2 and Y_3 and compute the Lie bracket $Z = [Y_2, Y_3]$:

$$\begin{aligned}
Z = I_3 c_1 \omega_2 &\left[(I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial^2 R}{\partial\omega_1\partial\gamma_1} - c_1\omega_2 \frac{\partial^2 R}{\partial\omega_1\partial\omega_2} - c_1\omega_3\frac{\partial^2 R}{\partial\omega_1\partial\omega_3} + (I_1 - I_3)\omega_3\right]\frac{\partial}{\partial\omega_2}\\
&+ c_1\left\{ I_3\omega_3\left[(I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial^2 R}{\partial\omega_1\partial\gamma_1} - c_1\omega_2\frac{\partial^2 R}{\partial\omega_1\partial\omega_2} - c_1\omega_3\frac{\partial^2 R}{\partial\omega_1\partial\omega_3}\right]\right.\\
&+ \left. I_2^2\omega_2^2 + I_1 I_3\omega_3^2 + I_1 c_1\gamma_1\right\}\frac{\partial}{\partial\omega_3}\\
&- I_3\left\{(I_2\omega_2^2 + I_3\omega_3^2)\left[(I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial^2 R}{\partial\omega_1\partial\gamma_1} - c_1\omega_2\frac{\partial^2 R}{\partial\omega_1\partial\omega_2} - c_1\omega_3\frac{\partial^2 R}{\partial\omega_1\partial\omega_3}\right]\right.\\
&+ \left. I_2(I_1 - 2I_2 + 2I_3)\omega_2^2\omega_3 + I_1 I_3\omega_3^3 + I_1 c_1\omega_3\gamma_1\right\}\frac{\partial}{\partial\gamma_1}.
\end{aligned}$$

As we already know, if a first integral F of the sought type exists, then it should satisfy the system

$$Y_2(F) = Y_3(F) = Z(F) = 0$$

and the determinant of the coefficients of that system should be identically zero. We compute this determinant and obtain the expression

$$-I_2 I_3(I_2 - I_3)c_1\omega_2^3\omega_3[I_2\omega_2^2(3I_1 - 2I_2) + I_3\omega_3^2(3I_1 - 2I_3) + 4I_1c_1\gamma_1].$$

It is easily seen that this expression can be identically zero only if $I_2 = I_3$ or if $c_1 = 0$. The first possibility leads to the Lagrange case and the second to the Euler case.

Thus if we suppose that $\frac{\partial^2 R}{\partial\omega_1^2} \neq 0$, then a first integral of type 4, i.e. $F(\omega_2, \omega_3, \gamma_1)$, does not exist.

Let us suppose now that $\frac{\partial^2 R}{\partial\omega_1^2} = 0$. In that case we have

$$R = f(\omega_2, \omega_3, \gamma_1)\omega_1 + g(\omega_2, \omega_3, \gamma_1), \tag{8.36}$$

where f and g are arbitrary smooth functions not depending on ω_1.

We put the value of R from (8.36) in (8.3) and obtain

$$\begin{aligned}
Q = {} & [(I_2^2\omega_2^2 + I_3^2\omega_3^2)f^2 + 2I_1I_3\omega_3\gamma_1 f + I_1^2\gamma_1^2]\omega_1^2\\
&+ 2[(I_2^2\omega_2^2 + I_3^2\omega_3^2)gf - I_3U_1\omega_3 f + I_1I_3\omega_3\gamma_1 g - I_1U_1\gamma_1]\omega_1\\
&+ (I_2^2\omega_2^2 + I_3^2\omega_3^2)g^2 + I_2^2\omega_2^2\gamma_1^2 - U_2\omega_2^2 - 2I_3U_1\omega_3 g + I_2^2U_1^2 = 0,
\end{aligned}$$

that is, Q is a polynomial of second degree in ω_1 with coefficients depending on ω_2, ω_3 and γ_1. As $Q = 0$, the three coefficients should be zeros.

We equate to zero the coefficient of ω_1^2 in Q and determine f from the resulting equation. We have

$$f_1 = \frac{-iI_1\gamma_1}{iI_3\omega_3 - I_2\omega_2}, \quad f_2 = \frac{-iI_1\gamma_1}{iI_3\omega_3 + I_2\omega_2}.$$

With these values of f we equate to zero the coefficient of ω_1 in Q and determine g as follows:

$$g_1 = \frac{iU_1}{iI_3\omega_3 - I_2\omega_2}, \quad g_2 = \frac{iU_1}{iI_3\omega_3 + I_2\omega_2}.$$

Using the values f_1 and g_1 we equate to zero the constant term of Q developed in powers of ω_1, that is, the value of Q when $\omega_1 = 0$, and obtain

$$J_2^2\omega_2^2(\gamma_1^2 - U_2) = 0.$$

Taking into account that $I_2^2\omega_2^2$ and $\gamma_1^2 - U_2$ never vanish identically, we conclude that the last equality cannot be fulfilled.

The same result is obtained using f_2 and g_2.

Thus a first integral of type 4, i.e. $F(\omega_2, \omega_3, \gamma_1)$, does not exist in the case when $\frac{\partial^2 R}{\partial \omega_1^2} = 0$ either.

8.2.2. Elimination of $\boldsymbol{\omega_1}$ and $\boldsymbol{\gamma_1}$. We eliminate the variables ω_1 and γ_1 from the equations $H_1 = U_1$, $H_2 = U_2$ and obtain the following solution:

$$\omega_1 = -\frac{I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1}{I_1\sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}}, \qquad \gamma_1 = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}. \tag{8.37}$$

Further to simplify formulas we denote

$$\Gamma = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}.$$

As all our considerations are local we can restrict ourselves to some suitable open set $\Omega \subset \mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

We put the values of ω_1 and γ_1 from (8.37) in the Euler–Poisson equations (1.1) and remove the first and fourth equations. In this way we obtain the following system of four equations in the unknowns ω_2, ω_3, γ_2 and γ_3:

$$\begin{aligned}
\frac{d\omega_2}{dt} &= \frac{(I_1 - I_3)(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\omega_3 - I_1c_3\Gamma^2 + I_1c_1\gamma_3\Gamma}{I_1I_2\Gamma}, \\
\frac{d\omega_3}{dt} &= -\frac{(I_1 - I_2)(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\omega_2 - I_1c_2\Gamma^2 + I_1c_1\gamma_2\Gamma}{I_1I_3\Gamma}, \\
\frac{d\gamma_2}{dt} &= -\frac{(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_3 + I_1\omega_3\Gamma^2}{I_1\Gamma}, \\
\frac{d\gamma_3}{dt} &= \frac{(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_2 + I_1\omega_2\Gamma^2}{I_1\Gamma}.
\end{aligned} \tag{8.38}$$

Here we study whether system (8.38) has a first integral that depends on at most three variables among $(\omega_2, \omega_3, \gamma_2, \gamma_3)$. Thus we should investigate the following four types of first integrals:

1. $F(\omega_2, \omega_3, \gamma_2)$ (case (iii)).
2. $F(\omega_2, \omega_3, \gamma_3)$ (case (iii)).
3. $F(\omega_2, \gamma_2, \gamma_3)$ (case (iv)).
4. $F(\omega_3, \gamma_2, \gamma_3)$ (case (iv)).

As partial integrals belonging to case (iii) were already excluded in Sec. 8.2.1, we will now study if a function of type 3, belonging to case (iv), can be a partial integral of the Euler–Poisson equations (1.1).

Type 3. Let us study the existence of a first integral F of type 3, i.e. $F = F(\omega_2, \gamma_2, \gamma_3)$. Being a first integral of system (8.38), F satisfies the equation

$$\begin{aligned}\frac{dF}{dt} = {} & \frac{(I_1 - I_3)\omega_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1) - I_1c_3\Gamma^2 + I_1c_1\gamma_3\Gamma}{I_1I_2\Gamma}\frac{\partial F}{\partial \omega_2} \\ & - \frac{\gamma_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1) + I_1\omega_3\Gamma^2}{I_1\Gamma}\frac{\partial F}{\partial \gamma_2} \\ & + \frac{\gamma_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1) + I_1\omega_2\Gamma^2}{I_1\Gamma}\frac{\partial F}{\partial \gamma_3} = 0,\end{aligned}$$

or equivalently

$$I_1I_2\Gamma\frac{dF}{dt} = Y(F) = 0, \tag{8.39}$$

where Y is a vector field defined on Ω.

The left-hand side of (8.39), i.e. $Y(F)$, is a polynomial in ω_3 of degree 2 with coefficients depending on the parameters $\mathcal{I}c$, U_1, U_2 and the variables $(\omega_2, \gamma_2, \gamma_3)$.

Let us write

$$Y(F) = Y_1(F)\omega_3^2 + Y_2(F)\omega_3 + Y_3(F),$$

where Y_i, $1 \le i \le 3$, are the following vector fields:

$$\begin{aligned}Y_1 &= I_3\gamma_3(I_1 - I_3)\frac{\partial}{\partial \omega_2}, \\ Y_2 &= (I_1 - I_3)(I_2\omega_2\gamma_2 - U_1)\frac{\partial}{\partial \omega_2} + I_2[I_1\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1U_2]\frac{\partial}{\partial \gamma_2} + I_2I_3\gamma_2\gamma_3\frac{\partial}{\partial \gamma_3}, \\ Y_3 &= I_1(c_3\gamma_2^2 + c_3\gamma_3^2 + c_1\gamma_3\Gamma - c_3U_2)\frac{\partial}{\partial \omega_2} - I_2\gamma_3(I_2\omega_2\gamma_2 - U_1)\frac{\partial}{\partial \gamma_2} \\ &\quad - I_2[(I_1 - I_2)\omega_2\gamma_2^2 + I_1\omega_2\gamma_3^2 - I_1U_2\omega_2 + U_1\gamma_2]\frac{\partial}{\partial \gamma_3},\end{aligned}$$

defined on Ω.

$Y(F)$ should be identically zero with respect to all four variables ω_2, ω_3, γ_2 and γ_3. As $Y_i(F)$, $1 \le i \le 3$, do not depend on ω_3, we have the following three equations:

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{8.40}$$

If a first integral $F = F(\omega_2, \gamma_2, \gamma_3)$ exists then system (8.40) has a non-zero solution $\operatorname{grad} F$. We know that this is possible if and only if the determinant D of system (8.40) is identically zero. We compute

$$D = I_1I_2^2I_3(I_1 - I_3)\gamma_3\Gamma^2[(I_1 - I_2)\omega_2\gamma_2^2 + (I_1 - I_3)\omega_2\gamma_3^2 - I_1U_2\omega_2 + \gamma_2U_1] = 0.$$

One can easily see that $D \equiv 0$ if and only if

$$I_1 = I_3. \tag{8.41}$$

Let us study this case. Now $Y_1 = 0$ but simple computations show that the vector fields Y_2 and Y_3 are always linearly independent. We compute the Lie bracket $Z =$

$[Y_2, Y_3]/(I_2 I_3)$ and obtain

$$\begin{aligned}Z = {} & 2I_3\gamma_2\Gamma(c_1\gamma_3 - c_3\Gamma)\frac{\partial}{\partial\omega_2} + I_2(I_2U_2\omega_2 - U_1\gamma_2)\gamma_3\frac{\partial}{\partial\gamma_2}\\ & + I_2\{[(I_3 - I_2)\Gamma^2 - I_2U_2]\omega_2\gamma_2 - U_1\gamma_3^2 + U_1U_2\}\frac{\partial}{\partial\gamma_3}.\end{aligned}$$

Equations $(8.40)_{2,3}$ imply that $Z(F) = 0$ and we come to the following system for F:

$$Y_2(F) = Y_3(F) = Z(F) = 0. \tag{8.42}$$

As above we should study when the determinant $\widehat{D}$ of system (8.42) is identically zero. We compute $\widehat{D} = d_1 d_2 d_3$, where

$$\begin{aligned}d_1 &= I_2^3 I_3^3 \Gamma^2,\\ d_2 &= (c_1\gamma_3 - c_3\Gamma)\Gamma,\\ d_3 &= (I_3 - I_2)\omega_2\gamma_2^3 - (2I_2 + I_3)U_2\omega_2\gamma_2 + 2U_1\gamma_2^2 + U_1U_2.\end{aligned}$$

It is clear that d_1 never vanishes identically. If d_2 vanishes then $c_1 = c_3 = 0$, which together with condition (8.41) leads to the Lagrange case. The third factor d_3 is zero if and only if $I_2 = I_3$ and $U_1 = U_2 = 0$. But taking into account (8.41) this is a particular case of the kinetic symmetry case. Thus a partial integral of type 3 does not exist.

It only remains to study the functions belonging to cases (v) and (vi).

8.2.3. Elimination of $\boldsymbol{\omega_1}$ and $\boldsymbol{\gamma_2}$.

Solving the equations $H_1 = U_1$, $H_2 = U_2$ with respect to ω_1 and γ_2 we obtain

$$\omega_1 = -\frac{I_2\omega_2\sqrt{-\gamma_1^2 - \gamma_3^2 + U_2} + I_3\omega_3\gamma_3 - U_1}{I_1\gamma_1}, \qquad \gamma_1 = \sqrt{-\gamma_1^2 - \gamma_3^2 + U_2}. \tag{8.43}$$

To simplify formulas we denote

$$\Gamma = \sqrt{-\gamma_1^2 - \gamma_3^2 + U_2}.$$

As before, we restrict ourselves to some suitable open set $\Omega \subset \mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

We put the values of ω_1 and γ_2 from (8.43) in the Euler–Poisson equations (1.1) and remove the first and fifth equations. In this way we obtain the following system of four equations in the unknowns ω_2, ω_3, γ_1 and γ_3:

$$\begin{aligned}\frac{d\omega_2}{dt} &= \frac{(I_1 - I_3)(I_3\omega_3\gamma_3 + I_2\omega_2\Gamma - U_1)\omega_3 - (I_1c_3\gamma_1 - I_1c_1\gamma_3)\gamma_1}{I_1I_2\gamma_1},\\ \frac{d\omega_3}{dt} &= -\frac{(I_1 - I_2)(I_3\omega_3\gamma_3 + I_2\omega_2\Gamma - U_1)\omega_2 - (I_1c_2\gamma_1 - I_1c_1\Gamma)\gamma_1}{I_1I_3\gamma_1},\\ \frac{d\gamma_1}{dt} &= \omega_3\Gamma - \omega_2\gamma_3,\\ \frac{d\gamma_3}{dt} &= \frac{(I_1\gamma_1^2 + I_2\Gamma^2)\omega_2 + I_3\omega_3\gamma_3\Gamma - U_1\Gamma}{I_1\gamma_1}.\end{aligned} \tag{8.44}$$

Here we look for first integrals of system (8.44) of the following four types:

1. $F(\omega_2, \omega_3, \gamma_1)$ (case (ii)).
2. $F(\omega_2, \omega_3, \gamma_3)$ (case (iii)).
3. $F(\omega_2, \gamma_1, \gamma_3)$ (case (v)).
4. $F(\omega_3, \gamma_1, \gamma_3)$ (case (iv)).

The functions from cases (ii), (iii) and (iv) were already examined. It remains only to examine case (v).

Type 3. Let us study the existence of a partial integral $F(\omega_2, \gamma_1, \gamma_3)$ of type 3 belonging to case (v). Then we have

$$\frac{dF}{dt} = \frac{(I_1 - I_3)(I_3\omega_3\gamma_3 + I_2\omega_2\Gamma - U_1)\omega_3 - (I_1c_3\gamma_1 - I_1c_1\gamma_3)\gamma_1}{I_1I_2\gamma_1}\frac{\partial F}{\partial \omega_2}$$
$$+ (\omega_3\Gamma - \omega_2\gamma_3)\frac{\partial F}{\partial \gamma_1} + \frac{(I_1\gamma_1^2 + I_2\Gamma^2)\omega_2 + I_3\omega_3\gamma_3\Gamma - U_1\Gamma}{I_1\gamma_1}\frac{\partial F}{\partial \gamma_3} = 0,$$

or equivalently

$$I_1I_2\gamma_1\frac{dF}{dt} = Y(F) = 0,$$

where Y is a vector field defined on Ω.

$Y(F)$ is a polynomial in ω_3 of degree 2 with coefficients depending on the parameters $\mathcal{I}c$, U_1, U_2 and the variables ω_2, γ_1 and γ_3.

Let us write $Y(F)$ in the following way:

$$Y(F) = Y_1(F)\omega_3^2 + Y_2(F)\omega_3 + Y_3(F),$$

where Y_i, $1 \leq i \leq 3$, are

$$Y_1 = I_3(I_1 - I_3)\gamma_3\frac{\partial}{\partial \omega_2},$$
$$Y_2 = (I_1 - I_3)(I_2\omega_2\Gamma - U_1)\frac{\partial}{\partial \omega_2} + I_1I_2\gamma_1\Gamma\frac{\partial}{\partial \gamma_1} + I_2I_3\gamma_3\Gamma\frac{\partial}{\partial \gamma_3},$$
$$Y_3 = I_1\gamma_1(c_1\gamma_3 - c_3\gamma_1)\frac{\partial}{\partial \omega_2} - I_1I_2\omega_2\gamma_1\gamma_3\frac{\partial}{\partial \gamma_1} + I_2[(I_2\Gamma^2 + I_1\gamma_1^2)\omega_2 - U_1\Gamma]\frac{\partial}{\partial \gamma_3}.$$

$Y(F)$ should be identically zero with respect to all four variables ω_2, ω_3, γ_2 and γ_3. As $Y_i(F)$, $1 \leq i \leq 3$, do not depend on ω_3 we have

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{8.45}$$

If a first integral $F = F(\omega_2, \gamma_2, \gamma_3)$ exists then the determinant D of system (8.45) is identically zero. We compute

$$D = I_1I_2^2I_3(I_1 - I_3)\gamma_1\gamma_3\Gamma\{\omega_2[(I_1 - I_2)\gamma_1^2 + (I_3 - I_2)\gamma_3^2 + I_2U_2] - U_1\Gamma\} = 0.$$

It is easily seen that D vanishes identically if either

$$I_1 = I_3 \tag{8.46}$$

or the expression in curly brackets vanishes. This expression is a linear function of ω_2 and we should require that its two coefficients vanish. But this leads to kinetic symmetry case with the additional restriction $U_1 = U_2 = 0$.

Thus we study only the case (8.46). Now $Y_1 = 0$ but simple computations show that outside of the particular case ($U_1 = U_2 = 0$) of kinetic symmetry, the vector fields Y_2 and Y_3 are always linearly independent. We compute the Lie bracket $Z = [Y_2, Y_3]$ and obtain

$$\begin{aligned}\frac{Z\Gamma}{I_2I_3} &= 2I_3\gamma_1(c_1\gamma_3 - c_3\gamma_1)\Gamma^2\frac{\partial}{\partial\omega_2} + I_2\gamma_1\gamma_3\Gamma[(I_2 - I_3)\omega_2\Gamma - U_1]\frac{\partial}{\partial\gamma_1}\\ &\quad + I_2\Gamma[\omega_2\Gamma(I_3\gamma_1^2 - I_2\gamma_1^2 - I_2U_2) - U_1\gamma_3^2 + U_1U_2]\frac{\partial}{\partial\gamma_3}.\end{aligned}$$

Equations $(8.45)_{2,3}$ imply that $Z(F) = 0$ and thus

$$\widehat{Z}(F) = \frac{Z\Gamma}{I_2I_3} = 0.$$

In this way we come to the following system for F:

$$Y_2(F) = Y_3(F) = \widehat{Z}(F) = 0. \tag{8.47}$$

As above we should find the cases when the determinant $\widehat{D}$ of system (8.47) is identically zero. We obtain $\widehat{D} = d_1d_2$, where

$$\begin{aligned}d_1 &= I_2^2I_3^2\gamma_1^2(c_1\gamma_3 - c_3\gamma_1)\Gamma^2,\\ d_2 &= [(I_3 - I_2)\gamma_1^2 + (I_3 - I_2)\gamma_3^2 + 3I_2U_2]\omega_2\Gamma - U_1(2\Gamma^2 + U_2).\end{aligned}$$

It is clear that d_1 vanishes identically only when $c_1 = c_3 = 0$, which together with condition (8.46) leads to the Lagrange case. As d_2 is a linear function of ω_2, the condition $d_2 = 0$ is fulfilled if and only if its two coefficients with respect to ω_2 vanish identically. Thus $I_2 = I_3$ and $U_1 = U_2 = 0$. Taking into account (8.46) this is a particular case of the kinetic symmetry case. Thus a partial integral of type 3 does not exist.

8.2.4. First integrals $F(\gamma_1, \gamma_2, \gamma_3)$. Finally, it remains to study the existence of a partial integral $F(\gamma_1, \gamma_2, \gamma_3)$, which cannot be done by elimination of variables as above.

We have $H_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = U_2$, thus $\gamma_1 = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}$ and then

$$F(\gamma_1, \gamma_2, \gamma_3) = F(\sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}, \gamma_2, \gamma_3) = \widetilde{F}(\gamma_2, \gamma_3).$$

Our problem is now reduced to the study of partial integrals of the form $\widetilde{F} = \widetilde{F}(\gamma_2, \gamma_3)$ with respect to the submanifold $\{H_1 = U_1\}$. Absence of such partial integrals follows from Sec. 8.2.2 where the absence of partial integrals of more general form $F(\omega_i, \gamma_2, \gamma_3)$, $i = 2, 3$, is proved for all U_1 and U_2.

8.3. Invariant manifold $\{H_1 = U_1,\ H_3 = U_3\}$. Here we study the existence of a partial integral of the Euler–Poisson equations (1.1) with respect to the complex four-dimensional level manifold

$$\{H_1 = U_1,\ H_3 = U_3\}, \tag{8.48}$$

supposing that this partial integral depends on at most three variables.

8.3.1. Elimination of ω_1 and ω_2. In the same way as in Sec. 8.2.1 we express ω_1 and ω_2 from the equations $H_1 = U_1$ and $H_3 = U_3$ and obtain the following solution:

$$\omega_1 = -\frac{I_2R\gamma_2 + I_3\omega_3\gamma_3 - U_1}{I_1\gamma_1}, \qquad \omega_2 = R, \tag{8.49}$$

where R is a root of $Q(x) = Ax^2 + Bx + C = 0$, that is,

$$Q(R) = AR^2 + BR + C = 0. \tag{8.50}$$

Here the functions $A = A(\gamma_1, \gamma_2)$, $B = B(\omega_3, \gamma_2, \gamma_3)$ and $C = C(\omega_3, \gamma_1, \gamma_2, \gamma_3)$ are the following polynomials:

$$\begin{aligned} &A = I_2(I_1\gamma_1^2 + I_2\gamma_2^2), \qquad B = 2I_2\gamma_2(I_3\omega_3\gamma_3 - U_1),\\ &C = I_1I_3\omega_3^2\gamma_1^2 + I_3^2\omega_3^2\gamma_3^2 - 2I_3U_1\omega_3\gamma_3 + 2I_1c_1\gamma_1^3 + 2I_1c_2\gamma_1^2\gamma_2\\ &\qquad + 2I_1c_3\gamma_1^2\gamma_3 - I_1U_3\gamma_1^2 + U_1^2. \end{aligned} \tag{8.51}$$

We put the values of ω_1 and ω_2 from (8.49) in the Euler–Poisson equations (1.1) and remove the first and second equations. In this way we obtain the following system of four equations in the unknowns ω_3, γ_1, γ_2 and γ_3:

$$\begin{aligned} \frac{d\omega_3}{dt} &= -\frac{(I_1 - I_2)(I_2\gamma_2R + I_3\omega_3\gamma_3 - U_1)R - I_1(\gamma_1c_2 - c_1\gamma_2)\gamma_1}{I_1I_3\gamma_1},\\ \frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \gamma_3R,\\ \frac{d\gamma_2}{dt} &= -\frac{(I_2R\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_3 + I_1\omega_3\gamma_1^2}{I_1\gamma_1},\\ \frac{d\gamma_3}{dt} &= \frac{(I_1\gamma_1^2 + I_2\gamma_2^2)R + (I_3\omega_3\gamma_3 - U_1)\gamma_2}{I_1\gamma_1}. \end{aligned} \tag{8.52}$$

Now we study the existence of a first integral of system (8.52) that depends on at most three variables among $(\omega_3, \gamma_1, \gamma_2, \gamma_3)$. Thus we should investigate the following four types of first integrals:

1. $F(\omega_3, \gamma_1, \gamma_2)$ (case (iv)).
2. $F(\omega_3, \gamma_1, \gamma_3)$ (case (iv)).
3. $F(\omega_3, \gamma_2, \gamma_3)$ (case (iv)).
4. $F(\gamma_1, \gamma_2, \gamma_3)$ (case (vi)).

Then, as in Sec. 5 it suffices to examine functions of types 1 and 4.

Type 1. Let us consider the existence of a first integral F of system (8.52) which is of type 1, i.e. $F = F(\omega_3, \gamma_1, \gamma_2)$. Thus

$$\begin{aligned} \frac{dF}{dt} &= -\frac{(I_1 - I_2)(I_2\gamma_2R + I_3\omega_3\gamma_3 - U_1)R - I_1(\gamma_1c_2 - c_1\gamma_2)\gamma_1}{I_1I_3\gamma_1}\frac{\partial F}{\partial\omega_3}\\ &\quad + (\omega_3\gamma_2 - \gamma_3R)\frac{\partial F}{\partial\gamma_1} - \frac{(I_2R\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_3 + I_1\omega_3\gamma_1^2}{I_1\gamma_1}\frac{\partial F}{\partial\gamma_2} = 0. \end{aligned}$$

We rewrite the above equation as

$$I_1I_3\gamma_1\frac{dF}{dt} = Y_1(F) = 0, \tag{8.53}$$

where Y_1 is a vector field on a suitable open set $\Omega \subset \mathbb{C}^4(\omega_3, \gamma_1, \gamma_2, \gamma_3)$.

We differentiate identity (8.53) with respect to γ_3 and obtain a linear partial differential equation for F:

$$\begin{aligned} \frac{\partial Y_1(F)}{\partial\gamma_3} &= (I_1 - I_2)\left[-2I_2\gamma_2R\frac{\partial R}{\partial\gamma_3} - I_3\omega_3R - (I_3\omega_3\gamma_3 - U_1)\frac{\partial R}{\partial\gamma_3}\right]\frac{\partial F}{\partial\omega_3}\\ &\quad - I_1I_3\gamma_1\left(R + \gamma_3\frac{\partial R}{\partial\gamma_3}\right)\frac{\partial F}{\partial\gamma_1}\\ &\quad - I_3\left[2I_3\omega_3\gamma_3 + I_2\gamma_2\left(R + \gamma_3\frac{\partial R}{\partial\gamma_3}\right) - U_1\right]\frac{\partial F}{\partial\gamma_2} = Y_2(F) = 0, \end{aligned} \tag{8.54}$$

where Y_2 is a vector field defined on Ω.

After differentiating identity (8.54) with respect to γ_3 we obtain

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial\gamma_3} &= (I_1-I_2)\biggl[-2I_2\gamma_2R\frac{\partial^2R}{\partial\gamma_3^2}-2I_2\gamma_2\biggl(\frac{\partial R}{\partial\gamma_3}\biggr)^2\\ &\quad -2I_3\omega_3\frac{\partial R}{\partial\gamma_3}-(I_3\omega_3\gamma_3-U_1)\frac{\partial^2R}{\partial\gamma_3^2}\biggr]\frac{\partial F}{\partial\omega_3}\\ &\quad -I_1I_3\gamma_1\biggl(2\frac{\partial R}{\partial\gamma_3}+\gamma_3\frac{\partial^2R}{\partial\gamma_3^2}\biggr)\frac{\partial F}{\partial\gamma_1}\\ &\quad -I_3\biggl(2I_2\gamma_2\frac{\partial R}{\partial\gamma_3}+I_2\gamma_2\gamma_3\frac{\partial^2R}{\partial\gamma_3^2}+2I_3\omega_3\biggr)\frac{\partial F}{\partial\gamma_2}=Y_3(F)=0,\end{aligned}\qquad(8.55)$$

where Y_3 is a vector field defined on Ω.

If a first integral F exists, then the linear system (8.53)–(8.55) has a non-zero solution $\operatorname{grad}F=\bigl(\frac{\partial F}{\partial\omega_3},\frac{\partial F}{\partial\gamma_1},\frac{\partial F}{\partial\gamma_2}\bigr)$, which is possible if and only if the determinant D of the coefficients of this system is identically zero.

We compute D. It has a non-zero factor $I_1I_3^2\gamma_1$ and hence we work with

$$\widehat{D}=\frac{D}{I_1I_3^2\gamma_1}.$$

The expression for $\widehat{D}$ is

$$\begin{aligned}\widehat{D}&=a_1R^3+a_2R^2\frac{\partial R}{\partial\gamma_3}+a_3R^2\frac{\partial^2R}{\partial\gamma_3^2}+a_4R^2+a_5R\biggl(\frac{\partial R}{\partial\gamma_3}\biggr)^2+a_6R\frac{\partial R}{\partial\gamma_3}\\ &\quad+a_7R\frac{\partial^2R}{\partial\gamma_3^2}+a_8R+a_9\biggl(\frac{\partial R}{\partial\gamma_3}\biggr)^3+a_{10}\biggl(\frac{\partial R}{\partial\gamma_3}\biggr)^2+a_{11}\frac{\partial R}{\partial\gamma_3}+a_{12}\frac{\partial^2R}{\partial\gamma_3^2},\end{aligned}\qquad(8.56)$$

where

$$\begin{aligned}a_1&=-2I_2I_3(I_1-I_2)\omega_3\gamma_2,\quad a_2=-2I_2(I_1-I_2)\gamma_2(U_1-3I_3\omega_3\gamma_3),\\ a_3&=-I_2(I_1-I_2)\gamma_2(-2\omega_3\gamma_1^2I_1-2\omega_3\gamma_2^2I_2+\gamma_3U_1),\quad a_4=2I_3(I_1-I_2)U_1\omega_3,\\ a_5&=2(I_1-I_2)I_2\gamma_2(-\omega_3\gamma_1^2I_1-\omega_3\gamma_2^2I_2-3I_3\omega_3\gamma_3^2+2\gamma_3U_1),\\ a_6&=2(I_1-I_2)(-2I_2I_3\omega_3^2\gamma_2^2-2I_3U_1\omega_3\gamma_3+U_1^2),\\ a_7&=(I_1-I_2)(-U_1\omega_3\gamma_1^2I_1-3\omega_3\gamma_2^2I_2U_1-I_3\omega_3\gamma_3^2U_1+U_1^2\gamma_3+4I_3\omega_3^2\gamma_3\gamma_2^2I_2),\\ a_8&=-2I_3\omega_3(-I_3\omega_3^2I_2\gamma_2+I_3\omega_3^2I_1\gamma_2-I_1\gamma_1^2c_2+c_1\gamma_2I_1\gamma_1),\\ a_9&=-2I_2(I_1-I_2)\gamma_2\gamma_3(-\omega_3\gamma_1^2I_1-\omega_3\gamma_2^2I_2-I_3\omega_3\gamma_3^2+\gamma_3U_1),\\ a_{10}&=-2(I_1-I_2)(-U_1\omega_3\gamma_1^2I_1-2I_3\omega_3^2\gamma_3\gamma_2^2I_2-I_3\omega_3\gamma_3^2U_1+U_1^2\gamma_3),\\ a_{11}&=2(-I_1\gamma_1^2c_2I_3\gamma_3\omega_3+I_1I_3^2\omega_3^3\gamma_3\gamma_2-c_1\gamma_2I_1\gamma_1U_1+c_1\gamma_2I_1\gamma_1I_3\gamma_3\omega_3\\ &\qquad+I_1\gamma_1^2c_2U_1-I_2I_3^2\omega_3^3\gamma_3\gamma_2),\\ a_{12}&=(U_1-2I_3\omega_3\gamma_3)(-U_1I_2\omega_3\gamma_2+U_1I_1\omega_3\gamma_2-c_1\gamma_2I_1\gamma_1\gamma_3-I_3\omega_3^2I_1\gamma_3\gamma_2\\ &\qquad+I_1\gamma_1^2c_2\gamma_3+I_3\omega_3^2I_2\gamma_3\gamma_2).\end{aligned}$$

To determine the derivatives $\frac{\partial R}{\partial\gamma_3}$ and $\frac{\partial^2R}{\partial\gamma_3^2}$ we differentiate (8.50) with respect to γ_3 two times. Taking into account that the polynomial A from (8.51) does not depend on

γ_3 we obtain

$$\frac{\partial Q}{\partial\gamma_3} = \frac{\partial B}{\partial\gamma_3}R + \frac{\partial C}{\partial\gamma_3} + \frac{dQ}{dR}\frac{\partial R}{\partial\gamma_3} = 0 \tag{8.57}$$

and

$$\frac{\partial^2 Q}{\partial\gamma_3^2} = \frac{\partial^2 B}{\partial\gamma_3^2}R + \frac{\partial B}{\partial\gamma_3}\frac{\partial R}{\partial\gamma_3} + \frac{\partial^2 C}{\partial\gamma_3^2} + \frac{\partial}{\partial\gamma_3}\left(\frac{dQ}{dR}\right)\frac{\partial R}{\partial\gamma_3} + \frac{dQ}{dR}\frac{\partial^2 R}{\partial\gamma_3^2} = 0. \tag{8.58}$$

By Proposition 4.1 we prove that if R is a root of (8.50), then $\frac{dQ}{dR} \neq 0$. For this purpose we consider the resultant $\rho = A(4AC - B^2)$ of the polynomials $Q(R)$ and $\frac{dQ}{dR}$ and prove that $\rho \neq 0$. As A never vanishes identically we do not consider ρ but $\widehat{\rho} = 4AC - B^2$. Putting in $\widehat{\rho}$ the expressions for A, B and C (see (8.51)) we obtain

$$\begin{aligned}\widehat{\rho} = 4I_1I_2\gamma_1^2[&I_3\omega_3^2(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2) + 2(I_1\gamma_1^2 + I_2\gamma_2^2)(c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3)\\ &- 2I_3U_1\omega_3\gamma_3 - U_3I_1\gamma_1^2 - I_2U_3\gamma_2^2 + U_1^2],\end{aligned}$$

which never vanishes identically because it contains a monomial $4I_1^2I_2I_3\omega_3^2\gamma_1^4$.

Thus $\frac{dQ}{dR} \neq 0$ and $\frac{\partial R}{\partial\gamma_3}$ can be correctly determined from (8.57). Then from (8.58) we determine $\frac{\partial^2 R}{\partial\gamma_3^2}$ and put the resulting values for the derivatives of R in the expression for $\widehat{D}$ (see (8.56)). In this way we obtain

$$\widehat{D}(R) = \frac{8I_2\delta(R)}{\left(\frac{dQ}{dR}\right)^3},$$

where $\delta(R)$ is a polynomial of R of degree 6 with coefficients depending on ω_3, γ_1, γ_2 and γ_3.

It is clear that the equation $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_3, \gamma_1, \gamma_2)$ of system (8.52) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in the polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_3, \gamma_1, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

By the MAPLE command `rem` we compute the remainder r of dividing $\delta(x)$ by $Q(x)$. The remainder is of the form

$$r(R) = \frac{I_1^2\gamma_1^4}{I_1\gamma_1^2 + I_2\gamma_2^2}(a_0x + a_1),$$

where a_0 and a_1 are polynomials in ω_3, γ_1, γ_2 and γ_3.

According to Proposition 4.2 if R is a root of (8.3), then a_0 and a_1 should be identically zero. We shall use a_1 only.

The polynomial a_1 has 210 coefficients. Thus we should equate all of them to zero. In this way we obtain a system of 210 equations for the parameters $\mathcal{I}c$, U_1 and U_3. The reduced system (see Sec. 3) obtained after two consecutive simplifications is very simple:

$$c_1 = 0, \quad c_2 = 0, \quad I_1 - I_2 = 0.$$

These equations lead to the Lagrange case. Thus the sought partial integral of type 1 does not exist.

Type 4. The study of the existence of a first integral of type 4 is considerably different. Indeed, suppose that $F = F(\gamma_1, \gamma_2, \gamma_3)$ is a first integral of system (8.52). Then we have

$$\begin{aligned}\frac{dF}{dt} &= (\omega_3\gamma_2 - \gamma_3 R)\frac{\partial F}{\partial \gamma_1} - \frac{(I_2R\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_3 + I_1\omega_3\gamma_1^2}{I_1\gamma_1}\frac{\partial F}{\partial \gamma_2} \\ &\quad + \frac{(I_1\gamma_1^2 + I_2\gamma_2^2)R + (I_3\omega_3\gamma_3 - U_1)\gamma_2}{I_1\gamma_1}\frac{\partial F}{\partial \gamma_3} = 0,\end{aligned}$$

which we rewrite as

$$I_1\gamma_1\frac{dF}{dt} = Y_1(F) = 0, \tag{8.59}$$

where Y_1 is a vector field defined on Ω.

After differentiating (8.59) with respect to ω_3 one obtains again a linear partial differential equation for F,

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial \omega_3} &= I_1\gamma_1\left(\gamma_2 - \gamma_3\frac{\partial R}{\partial \omega_3}\right)\frac{\partial F}{\partial \gamma_1} - \left(I_2\gamma_2\gamma_3\frac{\partial R}{\partial \omega_3} + I_3\gamma_3^2 + I_1\gamma_1^2\right)\frac{\partial F}{\partial \gamma_2} \\ &\quad + \left[(I_1\gamma_1^2 + I_2\gamma_2^2)\frac{\partial R}{\partial \omega_3} + I_3\gamma_2\gamma_3\right]\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0,\end{aligned} \tag{8.60}$$

where Y_2 is defined on Ω.

System (8.59)–(8.60) has a solution. This is the first integral H_2. In order to have one more solution this system should consist of dependent equations. Let us study when this is possible.

We compute the determinant D_{23} of the square matrix obtained from the 2×3 matrix of the coefficients of system (8.59)–(8.60) by crossing out its first column. The result is

$$D_{23} = I_1\gamma_1^2\left[(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2)\left(R - \omega_3\frac{\partial R}{\partial \omega_3}\right) - U_1\left(\gamma_2 - \gamma_3\frac{\partial R}{\partial \omega_3}\right)\right].$$

The expression for D_{23} depends on $\frac{\partial R}{\partial \omega_3}$. We determine it by differentiating (8.50) with respect to ω_3. The polynomial A does not depend on ω_3 (see (8.51)) and therefore

$$\frac{\partial Q}{\partial \omega_3} = \frac{\partial B}{\partial \omega_3}R + \frac{\partial C}{\partial \omega_3} + \frac{dQ}{dR}\frac{\partial R}{\partial \omega_3} = 0.$$

As in the study of a first integral of type 1, using Proposition 4.1 we prove that if R is a root of (8.50) then $\frac{dQ}{dR} \neq 0$ and obtain from the above equation

$$\frac{\partial R}{\partial \omega_3} = -\frac{I_3[\gamma_3(I_2R\gamma_2 - U_1) + \omega_3(I_1\gamma_1^2 + I_3\gamma_3^2)]}{I_2[R(I_1\gamma_1^2 + I_2\gamma_2^2) + \gamma_2(I_3\omega_3\gamma_3 - U_1)]}.$$

We put this value of $\frac{\partial R}{\partial \omega_3}$ in the expression for D_{23} and obtain

$$D_{23} = \frac{I_1\gamma_1^2}{I_2[R(I_1\gamma_1^2 + I_2\gamma_2^2) + \gamma_2(I_3\omega_3\gamma_3 - U_1)]}\widehat{D}_{23},$$

where

$$\begin{aligned}\widehat{D}_{23} &= I_2(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2)[(I_1\gamma_1^2 + I_2\gamma_2^2)R^2 + 2\gamma_2(I_3\omega_3\gamma_3 - U_1)R] \\ &\quad + I_3(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2)[(I_1\gamma_1^2 + I_3\gamma_3^2)\omega_3^2 - 2U_1\gamma_3\omega_3] + U_1^2(I_2\gamma_2^2 + I_3\gamma_3^2).\end{aligned}$$

It is clear that $D_{23} = 0$ is equivalent to

$$\widehat{D}_{23} = 0. \tag{8.61}$$

If a first integral $F = F(\gamma_1, \gamma_2, \gamma_3)$ exists then (8.61) is fulfilled. Thus R is a simultaneous root of (8.50) and (8.61). In that case $\widehat{D}_{23}$ and Q as polynomials in R should have a zero resultant.

Let us denote the resultant by ρ. We obtain

$$\rho = I_1^2 I_2^2 \gamma_1^4 (I_1\gamma_1^2 + I_2\gamma_2^2)^2 \widehat{\rho}^2,$$

where

$$\widehat{\rho} = (2c_1\gamma_1 + 2c_2\gamma_2 + 2c_3\gamma_3 + U_3)(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2) + U_1^2.$$

The equation $\rho = 0$ implies $\widehat{\rho} = 0$. It is easily seen that the latter happens only if $c_1 = c_2 = c_3 = 0$ and $U_1 = U_3 = 0$, which is a particular case of the Euler case. Thus a partial integral of type 4, $F(\gamma_1, \gamma_2, \gamma_3)$, does not exist.

8.3.2. Elimination of $\boldsymbol{\omega_1}$ and $\boldsymbol{\gamma_1}$. Here we should study two cases: when $c_1 \neq 0$ and when $c_1 = 0$. This is necessary because when we express γ_1 from the equation $H_1 = U_1$ we obtain

$$\gamma_1 = -\frac{I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1}{I_1\omega_1} \tag{8.62}$$

independently of c_1. But putting γ_1 from (8.62) in the equation $H_3 = U_3$, two different case for determining ω_1 arise. When $c_1 \neq 0$ the equation for ω_1 is of degree 3 while when $c_1 = 0$ it is of degree 2. Therefore, to avoid any confusion, we consider the two cases separately.

Case A: $c_1 \neq 0$. In the same way as in Sec. 8.2.1, taking into account the value of γ_1 from (8.62) we solve the equation $H_3 = U_3$ with respect to ω_1 and obtain

$$\omega_1 = R,$$

where R is a root of

$$Q(x) = I_1^2 x^3 + Ax + B = 0,$$

that is,

$$Q(R) = I_1^2 R^3 + AR + B = 0 \tag{8.63}$$

and $A = A(\omega_2, \omega_3, \gamma_2, \gamma_3)$ and $B = B(\omega_2, \omega_3, \gamma_2, \gamma_3)$ are the following polynomials:

$$\begin{aligned} A &= I_1(I_2\omega_2^2 + I_3\omega_3^2 + 2c_2\gamma_2 + 2c_3\gamma_3 - U_3), \\ B &= -c_1(2I_2\omega_2\gamma_2 + 2I_3\omega_3\gamma_3 - 2U_1). \end{aligned} \tag{8.64}$$

In this way we come to the following values of the eliminated variables:

$$\omega_1 = R, \quad \gamma_1 = -\frac{I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1}{I_1 R}. \tag{8.65}$$

We put the values of ω_1 and γ_1 from (8.65) in the Euler–Poisson equations (1.1), remove the first and fourth equations and obtain the following system of four equations in the unknowns ω_2, ω_3, γ_2 and γ_3:

$$\begin{aligned}
\frac{d\omega_2}{dt} &= -\frac{I_1(I_1-I_3)\omega_3 R^2 - I_1 c_1\gamma_3 R - c_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_2 R},\\
\frac{d\omega_3}{dt} &= \frac{I_1(I_1-I_2)\omega_2 R^2 - I_1 c_1\gamma_2 R - c_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_3 R},\\
\frac{d\gamma_2}{dt} &= \frac{I_1\gamma_3 R^2 + \omega_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 R},\\
\frac{d\gamma_3}{dt} &= -\frac{I_1\gamma_2 R^2 + \omega_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 R}.
\end{aligned} \tag{8.66}$$

Case B: $c_1 = 0$. We solve the equation $H_3 = U_3$ with respect to ω_1 for the value of γ_1 given from (8.62) and obtain $\omega_1 = R$, where R is a root of $Q(x) = I_1 x^2 + B$, that is,

$$Q(R) = I_1 R^2 + B, \tag{8.67}$$

and $B(\omega_2, \omega_3, \gamma_2, \gamma_3)$ is the polynomial $B = I_2\omega_2^2 + I_3\omega_3^2 + 2c_2\gamma_2 + 2c_3\gamma_3 - U_3$.

In fact, the values of the eliminated variables are determined as in Case A, i.e. by formula (8.65) but R is a root of a different equation.

The restricted Euler–Poisson equations are

$$\begin{aligned}
\frac{d\omega_2}{dt} &= -\frac{I_1(I_1-I_3)\omega_3 R^2 - c_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_2 R},\\
\frac{d\omega_3}{dt} &= \frac{I_1(I_1-I_2)\omega_2 R^2 - c_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_3 R},\\
\frac{d\gamma_2}{dt} &= \frac{I_1\gamma_3 R^2 + \omega_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 R},\\
\frac{d\gamma_3}{dt} &= -\frac{I_1\gamma_2 R^2 + \omega_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 R}.
\end{aligned} \tag{8.68}$$

To study the existence of first integrals of systems (8.66) and (8.68) that depend on at most three variables among ω_2, ω_3, γ_2 and γ_3 we should consider the following four types of first integrals:

1. $F(\omega_2, \omega_3, \gamma_2)$ (case (iii)).
2. $F(\omega_2, \omega_3, \gamma_3)$ (case (iii)).
3. $F(\omega_2, \gamma_2, \gamma_3)$ (case (iv)).
4. $F(\omega_3, \gamma_2, \gamma_3)$ (case (iv)).

Thus we should study first integrals of type 1 only.

Case A.1. Here we consider Case A, i.e. $c_1 \neq 0$, and study the existence of a first integral of type 1, i.e. $F = F(\omega_2, \omega_3, \gamma_2)$. Thus

$$\begin{aligned}
\frac{dF}{dt} = &-\frac{I_1(I_1-I_3)\omega_3 R^2 - I_1 c_1\gamma_3 R - c_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_2 R}\,\frac{\partial F}{\partial\omega_2}\\
&+\frac{I_1(I_1-I_2)\omega_2 R^2 - I_1 c_1\gamma_2 R - c_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_3 R}\,\frac{\partial F}{\partial\omega_3}\\
&+\frac{I_1\gamma_3 R^2 + \omega_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 R}\,\frac{\partial F}{\partial\gamma_2} = 0,
\end{aligned}$$

which is equivalent to

$$I_1 I_2 I_3 R \frac{dF}{dt} = Y_1(F) = 0, \tag{8.69}$$

where Y_1 is a vector field defined on a suitable open set $\Omega \subset \mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

We differentiate identity (8.69) with respect to γ_3 and obtain again a linear partial differential equation for F:

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial \gamma_3} &= I_3\left[-2I_1(I_1 - I_3)\omega_3 R \frac{\partial R}{\partial \gamma_3} + I_1 c_1 R + I_1 c_1 \gamma_3 \frac{\partial R}{\partial \gamma_3} + I_3 c_3 \omega_3\right] \frac{\partial F}{\partial \omega_2} \\ &\quad + I_2\left[2I_1(I_1 - I_2)\omega_2 R \frac{\partial R}{\partial \gamma_3} - I_1 c_1 \gamma_2 \frac{\partial R}{\partial \gamma_3} - I_3 c_2 \omega_3\right] \frac{\partial F}{\partial \omega_3} \\ &\quad + I_2 I_3 \left(I_1 R^2 + 2I_1 \gamma_3 R \frac{\partial R}{\partial \gamma_3} + I_3 \omega_3^2\right) \frac{\partial F}{\partial \gamma_2} = Y_2(F) = 0, \end{aligned} \tag{8.70}$$

where Y_2 is a vector field on Ω.

After differentiating identity (8.70) with respect to γ_3 we obtain

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial \gamma_3} &= I_1 I_3 \left[-2(I_1 - I_3)\omega_3 R \frac{\partial^2 R}{\partial \gamma_3^2} - 2(I_1 - I_3)\omega_3 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 \right. \\ &\quad \left. + 2c_1 \frac{\partial R}{\partial \gamma_3} + c_1 \gamma_3 \frac{\partial^2 R}{\partial \gamma_3^2}\right] \frac{\partial F}{\partial \omega_2} \\ &\quad + I_1 I_2 \left[2(I_1 - I_2)\omega_2 R \frac{\partial^2 R}{\partial \gamma_3^2} + 2(I_1 - I_2)\omega_2 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 - c_1 \gamma_2 \frac{\partial^2 R}{\partial \gamma_3^2}\right] \frac{\partial F}{\partial \omega_3} \\ &\quad + 2I_1 I_2 I_3 \left[\gamma_3 R \frac{\partial^2 R}{\partial \gamma_3^2} + \gamma_3 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 + 2R \frac{\partial R}{\partial \gamma_3}\right] \frac{\partial F}{\partial \gamma_2} = Y_3(F) = 0, \end{aligned} \tag{8.71}$$

where Y_3 is a vector field on Ω.

If a first integral F exists, system (8.69)–(8.71) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_2}\right)$, which is possible if and only if the determinant D of its coefficients is identically zero.

We compute D. It has a non-zero factor $I_1^2 I_2^2 I_3^2$ and we denote

$$\widehat{D} = \frac{D}{I_1^2 I_2^2 I_3^2}.$$

The expression for $\widehat{D}$ is

$$\begin{aligned}\widehat{D} &= a_1 R^4 \frac{\partial R}{\partial \gamma_3} + a_2 R^4 \frac{\partial^2 R}{\partial \gamma_3^2} + a_3 R^3 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 + a_4 R^3 \frac{\partial R}{\partial \gamma_3} + a_5 R^3 \frac{\partial^2 R}{\partial \gamma_3^2} \\ &\quad + a_6 R^2 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 + a_7 R^2 \frac{\partial R}{\partial \gamma_3} + a_8 R^2 \frac{\partial^2 R}{\partial \gamma_3^2} + a_9 R \left(\frac{\partial R}{\partial \gamma_3}\right)^2 + a_{10} R \frac{\partial R}{\partial \gamma_3} \\ &\quad + a_{11} R \frac{\partial^2 R}{\partial \gamma_3^2} + a_{12} \left(\frac{\partial R}{\partial \gamma_3}\right)^3 + a_{13} \left(\frac{\partial R}{\partial \gamma_3}\right)^2, \end{aligned}$$

where

$$a_1 = -2I_1 c_1 \omega_2 (I_1 - I_2), \qquad a_2 = I_1 c_1 [(-I_1 + I_2)\omega_2 \gamma_3 + (I_1 - I_3)\omega_3 \gamma_2],$$

$$a_3 = -2I_1 c_1 [(-I_1 + I_2)\omega_2 \gamma_3 + (I_1 - I_3)\omega_3 \gamma_2],$$

$$\begin{aligned}
a_4 &= 2[2I_3(I_2 - I_1)c_3\omega_2\omega_3 + 2I_3(I_1 - I_3)c_2\omega_3^2 + I_1c_1^2\gamma_2],\\
a_5 &= -2I_2(I_1 - I_2)c_3\omega_2^2\gamma_2 + 2I_2(-I_3 + I_1)c_2\omega_2\omega_3\gamma_2 - 2I_3(I_1 - I_2)c_3\omega_2\omega_3\gamma_3\\
&\quad + 2(I_1 - I_2)c_3U_1\omega_2 + 2I_3(-I_3 + I_1)c_2\omega_3^2\gamma_3 - 2(-I_3 + I_1)c_2U_1\omega_3 + I_1c_1^2\gamma_2\gamma_3,\\
a_6 &= 6[(I_2 - I_1)c_3\omega_2 + (I_1 - I_3)c_2\omega_3](-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1),\\
a_7 &= -2c_1(-I_3(I_1 - I_2)\omega_2\omega_3^2 - c_2I_2\omega_2\gamma_2 - 2c_3I_3\omega_3\gamma_2 + c_2U_1),\\
a_8 &= -c_1\big[-2I_2(I_1 - I_2)\omega_2^2\omega_3\gamma_2 - I_3(I_1 - I_2)\omega_2\omega_3^2\gamma_3 + 2(I_1 - I_2)U_1\omega_2\omega_3\\
&\quad - I_2c_3\omega_2\gamma_2^2 - I_2c_2\omega_2\gamma_2\gamma_3 - I_3(-I_3 + I_1)\omega_3^3\gamma_2 - 2I_3c_3\omega_3\gamma_2\gamma_3 + c_3U_1\gamma_2 + c_2U_1\gamma_3\big],\\
a_9 &= 2c_1\big[-I_2(I_1 - I_2)\omega_2^2\omega_3\gamma_2 - 2I_3(I_1 - I_2)\omega_2\omega_3^2\gamma_3 + U_1(I_1 - I_2)\omega_2\omega_3 - 2c_3I_2\omega_2\gamma_2^2\\
&\quad + I_2c_2\omega_2\gamma_2\gamma_3 + I_3(-I_3 + I_1)\omega_3^3\gamma_2 - I_3c_3\omega_3\gamma_2\gamma_3 + 2c_3U_1\gamma_2 - c_2U_1\gamma_3\big],\\
a_{10} &= -2I_3c_1^2\omega_3^2\gamma_2, \qquad a_{11} = c_1^2\omega_3\gamma_2(-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1),\\
a_{12} &= 2c_1[(I_2 - I_1)\omega_2\omega_3\gamma_3 + (-I_3 + I_1)\omega_3^2\gamma_2 + c_3\gamma_2\gamma_3 - c_2\gamma_3^2](-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1),\\
a_{13} &= -2c_1^2\omega_3\gamma_2(-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1).
\end{aligned}$$

The first and second derivatives of R with respect to γ_3 appear in D. To determine them we differentiate equation (8.63) twice with respect to γ_3 and obtain

$$\frac{\partial Q}{\partial\gamma_3} = \frac{\partial A}{\partial\gamma_3}R + \frac{\partial B}{\partial\gamma_3} + \frac{dQ}{dR}\,\frac{\partial R}{\partial\gamma_3} = 0 \tag{8.72}$$

and

$$\frac{\partial^2 Q}{\partial\gamma_3^2} = \frac{\partial^2 A}{\partial\gamma_3^2}R + \frac{\partial A}{\partial\gamma_3}\,\frac{\partial R}{\partial\gamma_3} + \frac{\partial^2 B}{\partial\gamma_3^2} + \frac{\partial}{\partial\gamma_3}\left(\frac{dQ}{dR}\right)\frac{\partial R}{\partial\gamma_3} + \frac{dQ}{dR}\,\frac{\partial^2 R}{\partial\gamma_3^2} = 0. \tag{8.73}$$

We prove that if R is a root of (8.63), then $\frac{dQ}{dR} \neq 0$. For this purpose we consider the resultant $\rho = I_1^4(4A^3 + 27I_1^2B^2)$ of the polynomials $Q(R)$ and $\frac{dQ}{dR} = 3I_1^2R^2 + A$ and prove that $\rho \neq 0$. Indeed, putting in ρ the expressions for A and B from (8.64) we obtain a polynomial which we do not give here but it never vanishes identically as it has a monomial $4I_1^3I_2^3\omega_2^6$.

Thus, by Proposition 4.1, $\frac{dQ}{dR} \neq 0$ and $\frac{\partial R}{\partial\gamma_3}$ can be correctly determined from (8.72). Then by (8.73) we determine $\frac{\partial^2 R}{\partial\gamma_3^2}$, put the resulting values for the derivatives of R in the expression for $\widehat{D}(R)$ and obtain

$$\widehat{D}(R) = \frac{L\delta(R)}{\left(\frac{dQ}{dR}\right)^3},$$

where $L = 4(I_1c_3R - I_3c_1\omega_3)$ and $\delta(R)$ is a polynomial in R of degree 8 with coefficients depending on ω_2, ω_3, γ_2 and γ_3.

We prove that L does not vanish identically provided that R is a root of (8.63). Indeed, if $c_3 = 0$ then $L = -4I_3c_1\omega_3$ and as $c_1 \neq 0$ then L could not vanish identically. If $c_3 \neq 0$ and we suppose that $L = 0$ then we have $R = I_3c_1\omega_3/(I_1c_3)$. Simple computations show that this value of R cannot be a root of (8.63). Thus we can work with $\widehat{D}$ instead of D because L and $\frac{dQ}{dR}$ are not zeros.

Thus the equation $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \omega_3, \gamma_2)$ of system (8.66) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in the polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

By the MAPLE command `rem` we compute the remainder r of dividing $\delta(x)$ by $Q(x)$:

$$r(x) = 2I_1c_1a_0x^2 + a_1x + 4c_1(-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1)a_2,$$

where the coefficients a_0, a_1 and a_2 are polynomials in ω_2, ω_3, γ_2 and γ_3.

According to Proposition 4.2 if R is a root of (8.63), then, as $2I_1c_1$ is not zero and $4c_1(-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1)$ does not vanish identically, we have $a_0 = a_1 = a_2 = 0$ identically. We use only the last equation $a_2 = 0$.

a_2 is a polynomial with 51 monomials and therefore with 51 coefficients. We equate them to zero and apply simplification to the resulting system. After three consecutive simplifications we come to the reduced system that consists of only one equation 1=0. Thus a first integral of type 1, $F = F(\omega_2, \omega_3, \gamma_2)$, does not exist when $c_1 \neq 0$.

Case B.1. Here we consider Case B, i.e. $c_1 = 0$, and study the existence of a first integral of system (8.68) which is of type 1, $F(\omega_2, \omega_3, \gamma_2)$.

In the same way as in Case A we obtain a system

$$Y_1(F) = Y_2(F) = Y_3(F) = 0, \tag{8.74}$$

where Y_i, $1 \leq i \leq 3$, are vector fields defined on Ω.

System (8.74) coincides with (8.69)–(8.71) if we substitute $c_1 = 0$ in the last equation. The existence of the sought first integral is possible if and only if the determinant D of the coefficients of system (8.74) is identically zero.

Let us compute this determinant. We obtain

$$D(R) = 2I_1^2I_2^2I_3^2R^2[(I_2 - I_1)c_3\omega_2 + (I_1 - I_3)c_2\omega_3]\widehat{D}(R), \tag{8.75}$$

where

$$\widehat{D}(R) = (I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\left[R\frac{\partial^2 R}{\partial\gamma_3^2} - 3\left(\frac{\partial R}{\partial\gamma_3}\right)^2\right] + 2I_3\omega_3R\frac{\partial R}{\partial\gamma_3}.$$

Taking into account that now $R = 0$ cannot be a root of (8.67) and that $c_1 = 0$ we easily see that the factor before $\widehat{D}(R)$ in (8.75) can vanish identically only in the Euler, Lagrange and kinetic symmetry cases. Thus the equation $D(R) = 0$ is equivalent to $\widehat{D}(R) = 0$.

We compute the derivatives of R (see (8.67)) with respect to γ_3 and obtain

$$\frac{\partial R}{\partial\gamma_3} = -\frac{c_3}{I_1R}, \qquad \frac{\partial^2 R}{\partial\gamma_3^2} = -\frac{c_3^2}{I_1^2R^3}$$

and determine

$$\widehat{D}(R) = -2c_3\frac{I_1I_3\omega_3R^2 + 2c_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1^2R^2}.$$

Let us suppose that $c_3 \neq 0$. It is clear that now $\widehat{D}(R)$ never vanishes identically and consequently the sought partial integral does not exist.

Let $c_3 = 0$. In that case $\widehat{D} = 0$ and therefore equations (8.74) are linearly dependent. More precisely, $Y_3 \equiv 0$ because if $c_3 = 0$ then R does not depend on γ_3 but every item of Y_3 contains either $\frac{\partial R}{\partial \gamma_3}$ or $\frac{\partial^2 R}{\partial \gamma_3^2}$ (see (8.71) under the condition $c_1 = 0$).

Thus we have only two partial differential equations for the first integral F:

$$Y_1(F) = 0, \quad Y_2(F) = 0. \tag{8.76}$$

Easy computations show that these equations are independent unless $c_2 = 0$ which leads either to the Euler case or to $I_1 = I_3$, the Lagrange case.

We compute the Lie bracket $Z = [Y_1, Y_2]$ taking into account that R is a root of (8.67). Equations (8.76) imply that

$$\begin{aligned} Z(F) &= 3(I_1 - I_3)I_2 I_3^2 c_2 \omega_3 (I_2\omega_2^2 + I_3\omega_3^2 + 2c_2\gamma_2 - U_3)\frac{\partial F}{\partial \omega_2} \\ &\quad - I_2^2 I_3 c_2 [I_2 I_1 \omega_2^3 + I_3(I_1 - I_2)\omega_2\omega_3^2 \\ &\quad + c_2(2I_1 - I_2)\omega_2\gamma_2 - I_1 U_3 \omega_2 + c_2 U_1]\frac{\partial F}{\partial \omega_3} \\ &\quad + I_2^2 I_3^2 \omega_3 [-I_2(2I_1 - I_2 - 2I_3)\omega_2^3 - 2I_3(I_1 - I_3)\omega_2\omega_3^2 \\ &\quad - c_2(4I_1 - I_2 - 4I_3)\omega_2\gamma_2 + (2I_1 - I_2 - 2I_3)U_3\omega_2 + c_2 U_1]\frac{\partial F}{\partial \gamma_2} = 0. \end{aligned} \tag{8.77}$$

The determinant δ composed from the coefficients of equations (8.76) and (8.77) should be identically zero. We compute

$$\delta = \delta_1 \delta_2,$$

where

$$\begin{aligned} \delta_1 &= I_2^3 I_3^3 (I_1 - I_3) c_2 \omega_3 (I_2\omega_2^2 + I_3\omega_3^2 + 2c_2\gamma_2 - U_3)^2, \\ \delta_2 &= -I_2(2I_1 - 3I_2)\omega_2^3 - 2I_3(I_1 - I_3)\omega_2\omega_3^2 - 2c_2(2I_1 - I_2)\omega_2\gamma_2 \\ &\quad + U_3(2I_1 - 3I_2)\omega_2 + 4c_2 U_1. \end{aligned}$$

It is easily seen that δ_1 can vanish identically only in the Euler and Lagrange cases. The expression for δ_2 contains a monomial $2I_3(I_1 - I_3)\omega_2\omega_3^2$ and therefore the minimal requirement for δ_2 to vanish identically is $I_1 = I_3$, the Lagrange case.

Thus a first integral of type 1, $F = F(\omega_2, \omega_3, \gamma_2)$, does not exist when $c_1 = 0$.

8.3.3. Elimination of $\boldsymbol{\omega_1}$ and $\boldsymbol{\gamma_2}$.

In the same way as in Sec. 8.2.1 we solve the equations $H_1 = U_1$ and $H_3 = U_3$ with respect to ω_1 and γ_2 and obtain

$$\omega_1 = R, \qquad \gamma_2 = -\frac{I_1\gamma_1 R + I_3\omega_3\gamma_3 - U_1}{I_2\omega_2}, \tag{8.78}$$

where R is a root of $Q(x) = Ax^2 + Bx + C = 0$, that is,

$$Q(R) = AR^2 + BR + C = 0. \tag{8.79}$$

The functions $A = A(\omega_2)$, $B = B(\gamma_1)$ and $C = C(\omega_2, \omega_3, \gamma_1, \gamma_3)$ are the following polynomials:

$$\begin{aligned} A &= I_1 I_2 \omega_2, \qquad B = -2 I_1 c_2 \gamma_1, \\ C &= I_2^2 \omega_2^3 + I_2 I_3 \omega_2 \omega_3^2 + 2 I_2 c_1 \omega_2 \gamma_1 + 2 I_2 c_3 \omega_2 \gamma_3 \\ &\quad - I_2 U_3 \omega_2 - 2 I_3 c_2 \omega_3 \gamma_3 + 2 c_2 U_1. \end{aligned} \tag{8.80}$$

We put the values of ω_1 and γ_2 from (8.78) in the Euler–Poisson equations (1.1) and remove its first and fifth equations. In this way we obtain the following system of four equations in the unknowns ω_2, ω_3, γ_1 and γ_3:

$$\begin{aligned} \frac{d\omega_2}{dt} &= -\frac{(I_1 - I_3)\omega_3 R - c_1 \gamma_3 + c_3 \gamma_1}{I_2}, \\ \frac{d\omega_3}{dt} &= \frac{I_2 (I_1 - I_2) \omega_2^2 R + I_2 c_2 \omega_2 \gamma_1 + c_1 (I_1 \gamma_1 R + I_3 \omega_3 \gamma_3 - U_1)}{I_2 I_3 \omega_2}, \\ \frac{d\gamma_1}{dt} &= -\frac{I_1 \omega_3 \gamma_1 R + I_3 \omega_3^2 \gamma_3 - U_1 \omega_3 + I_2 \omega_2^2 \gamma_3}{I_2 \omega_2}, \\ \frac{d\gamma_3}{dt} &= \frac{I_2 \omega_2^2 \gamma_1 + (I_1 \gamma_1 R + I_3 \omega_3 \gamma_3 - U_1) R}{I_2 \omega_2} \end{aligned} \tag{8.81}$$

We consider the following four possible types of first integrals of system (8.81) that depend on at most three variables among ω_2, ω_3, γ_1 and γ_3:

1. $F(\omega_2, \omega_3, \gamma_1)$ (case (ii)).
2. $F(\omega_2, \omega_3, \gamma_3)$ (case (iii)).
3. $F(\omega_2, \gamma_1, \gamma_3)$ (case (v)).
4. $F(\omega_3, \gamma_1, \gamma_3)$ (case (iv)).

As we have already studied cases (iii) and (iv), now we should consider first integrals of types 1 and 3.

Type 1. Let us start with a first integral of type 1, $F = F(\omega_2, \omega_3, \gamma_1)$. We have

$$\begin{aligned} \frac{dF}{dt} = &-\frac{(I_1 - I_3)\omega_3 R - c_1 \gamma_3 + c_3 \gamma_1}{I_2} \frac{\partial F}{\partial \omega_2} \\ &+ \frac{I_2 (I_1 - I_2) \omega_2^2 R + I_2 c_2 \omega_2 \gamma_1 + c_1 (I_1 \gamma_1 R + I_3 \omega_3 \gamma_3 - U_1)}{I_2 I_3 \omega_2} \frac{\partial F}{\partial \omega_3} \\ &- \frac{I_1 \omega_3 \gamma_1 R + I_3 \omega_3^2 \gamma_3 - U_1 \omega_3 + I_2 \omega_2^2 \gamma_3}{I_2 \omega_2} \frac{\partial F}{\partial \gamma_1} = 0, \end{aligned}$$

which is equivalent to

$$I_2 I_3 \omega_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.82}$$

where Y_1 is a vector field defined on a suitable open set $\Omega \subset \mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

We differentiate identity (8.82) with respect to γ_3 and obtain again a linear partial differential equation for F:

$$\begin{aligned} \frac{\partial Y_1(F)}{\partial \gamma_3} = I_3 &\left[(I_3 - I_1) \omega_2 \omega_3 \frac{\partial R}{\partial \gamma_3} + c_1 \omega_2 \right] \frac{\partial F}{\partial \omega_2} \\ &+ \left[I_2 (I_1 - I_2) \omega_2^2 \frac{\partial R}{\partial \gamma_3} + I_1 c_1 \gamma_1 \frac{\partial R}{\partial \gamma_3} + I_3 c_1 \omega_3 \right] \frac{\partial F}{\partial \omega_3} \\ &- I_3 \left(I_1 \omega_3 \gamma_1 \frac{\partial R}{\partial \gamma_3} + I_2 \omega_2^2 + I_3 \omega_3^2 \right) \frac{\partial F}{\partial \gamma_1} = Y_2(F) = 0. \end{aligned} \tag{8.83}$$

We differentiate $Y_2(F)$ with respect to γ_3 and obtain

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial\gamma_3} &= \frac{\partial^2 R}{\partial\gamma_3^2}\Big[I_3(I_3-I_1)\omega_2\omega_3\frac{\partial F}{\partial\omega_2} + (I_1I_2\omega_2^2 - I_2^2\omega_2^2 + I_1c_1\gamma_1)\frac{\partial F}{\partial\omega_3}\\ &\qquad - I_1I_3\omega_3\gamma_1\frac{\partial F}{\partial\gamma_1}\Big] = Y_3(F) = 0. \qquad (8.84)\end{aligned}$$

Above, Y_2 and Y_3 are vector fields defined on Ω.

If a first integral $F = F(\omega_2,\omega_3,\gamma_1)$ exists, system (8.82)–(8.84) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\omega_3}, \frac{\partial F}{\partial\gamma_1}\right)$. This is possible if and only if the determinant D of its coefficients is identically zero.

We compute

$$D(R) = -I_2I_3^2\omega_2^2\frac{\partial^2 R}{\partial\gamma_3^2}\Delta,$$

where

$$\begin{aligned}\Delta &= (I_2-I_3)c_1U_1\omega_2\omega_3 + I_1c_1\gamma_1^2(c_3\omega_2 - c_2\omega_3)\\ &\quad + (I_2\omega_2^2 + I_3\omega_3^2)[(I_1-I_2)c_3\omega_2 - (I_1-I_3)c_2\omega_3]\gamma_1.\end{aligned}$$

Let us first suppose that

$$\frac{\partial^2 R}{\partial\gamma_3^2} \neq 0. \qquad (8.85)$$

In that case $D(R) = 0$ if and only if $\Delta = 0$. The polynomial Δ has seven coefficients and they should be zeros, i.e. we have a system of seven equations for the parameters $\mathcal{I}c$ and U_1.

After two consecutive simplifications we come to a reduced system that consists of the following six equations:

$$\begin{aligned}&c_3c_1 = 0, \quad (I_1-I_2)c_3 = 0, \quad c_1c_2 = 0, \quad (I_1-I_3)c_2 = 0,\\ &(I_2-I_3)c_2c_3 = 0, \quad (I_2-I_3)c_1U_1 = 0.\end{aligned}$$

This system coincides with (8.35) and therefore has the same six solutions. After removing the solutions that lead to the Euler, Lagrange and kinetic symmetry cases we obtain only one new solution

$$\{U_1 = 0,\ I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\},$$

which is impossible because the condition $c_2 = c_3 = 0$ contradicts (8.85). Indeed, equation (8.79) has no root that depends on γ_3 because the coefficients A, B and C (see (8.80)) do not depend on γ_3 when $c_2 = c_3 = 0$ and therefore R does not depend on γ_3 either.

Thus a partial integral of type 1, $F = F(\omega_2,\omega_3,\gamma_1)$, does not exist when condition (8.85) is fulfilled.

Let us study what happens when

$$\frac{\partial^2 R}{\partial\gamma_3^2} = 0. \qquad (8.86)$$

Equation (8.86) implies that $R = M(\omega_2,\omega_3,\gamma_1)\gamma_3 + N(\omega_2,\omega_3,\gamma_1)$, where M and N are some functions not depending on γ_3. Now (8.79) gives

$$Q = AM^2\gamma_3^2 + M\gamma_3(2AN+B) + AN^2 + BN + C = 0.$$

Thus Q is a polynomial in γ_3 of degree 2. The coefficient of γ_3^2 is AM^2. As $Q = 0$ we have $AM^2 = 0$. As A cannot be zero (see (8.80)) there remains only the possibility $M = 0$ and equation (8.79) is transformed into the form

$$Q = AN^2 + BN + C = 0,$$

i.e. Q is already a polynomial in γ_3 of degree 1. Its leading coefficient is $2(I_2c_3\omega_2 - I_3c_2\omega_3)$ (see the expression for C in (8.80)). This coefficient should be identically zero. Thus $c_2 = c_3 = 0$.

As the function B vanishes under this condition, R takes the simple form

$$R = \frac{\sqrt{I_1(-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3)}}{I_1}. \tag{8.87}$$

Now $Y_3(F) \equiv 0$ and the equations $Y_i(F) = 0$, $1 \le i \le 2$, are obtained from (8.82) and (8.83) where we put $c_2 = c_3 = 0$ and $\frac{\partial R}{\partial \gamma_3} = 0$, i.e.

$$\begin{aligned}
Y_1(F) = {} & -I_3\omega_2[(I_1 - I_3)\omega_3 R - c_1\gamma_3]\frac{\partial F}{\partial \omega_2} \\
& + [I_2(I_1 - I_2)\omega_2^2 R + c_1(I_1\gamma_1 R + I_3\omega_3\gamma_3 - U_1)]\frac{\partial F}{\partial \omega_3} \\
& - I_3[(I_2\omega_2^2 + I_3\omega_3^2)\gamma_3 + (I_1\gamma_1 R - U_1)\omega_3]\frac{\partial F}{\partial \gamma_1} = 0, \\
Y_2(F) = {} & I_3c_1\omega_2\frac{\partial F}{\partial \omega_2} + I_3c_1\omega_3\frac{\partial F}{\partial \omega_3} - I_3(I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial F}{\partial \gamma_1} = 0,
\end{aligned}$$

where R is taken from (8.87).

Further we work with the vector fields

$$Z_1 = Y_1 - \gamma_3 Y_2, \quad Z_2 = Y_2, \quad Z_3 = \frac{[Z_1, Z_2]}{I_3}.$$

We compute

$$\begin{aligned}
Z_3 = {} & I_3(I_1 - I_3)c_1\omega_2\omega_3 R\frac{\partial}{\partial \omega_2} + c_1[(I_2^2\omega_2^2 + I_1I_3\omega_3^2 + I_1c_1\gamma_1)R - c_1U_1]\frac{\partial}{\partial \omega_3} \\
& + I_3\omega_3[(2I_2^2\omega_2^2 - 2I_2I_3\omega_2^2 - I_1I_2\omega_2^2 - I_1I_3\omega_3^2 - I_1c_1\gamma_1)R + c_1U_1]\frac{\partial}{\partial \gamma_1}.
\end{aligned}$$

In this way we obtain the following system of three equations for F:

$$Z_i(F) = 0, \qquad 1 \le i \le 3,$$

and we know that the determinant δ of the coefficients of this system should vanish identically with respect to ω_2, ω_3 and γ_1. Computing this determinant, we obtain

$$I_1^2\delta = \delta_1\delta_2,$$

where

$$\begin{aligned}
\delta_1 = {} & I_1I_2I_3^2(I_2 - I_3)c_1\omega_2^3\omega_3 \\
\delta_2 = {} & [I_2(2I_2 - 3I_1)\omega_2^2 + I_3(2I_3 - 3I_1)\omega_3^2 - 4I_1c_1\gamma_1](-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3) \\
& + 4c_1U_1\sqrt{I_1(-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3)}
\end{aligned}$$

δ_1 vanishes identically if either $I_2 = I_3$, which leads to the Lagrange case, or $c_1 = 0$, which leads to the Euler case. Thus we suppose that $I_2 \neq I_3$ and $c_1 \neq 0$ and should study when δ_2 vanishes.

According to Proposition 4.3 applied to $V = I_1(-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3)$ we conclude that $U_1 = 0$. Thus

$$\delta_2 = [I_2(2I_2 - 3I_1)\omega_2^2 + I_3(2I_3 - 3I_1)\omega_3^2 - 4I_1c_1\gamma_1](-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3)$$

and it is easily seen that if $c_1 \neq 0$ then neither the first factor in the square brackets nor the second one vanishes independently of the values of the moments of inertia.

Thus a partial integral of type 1, $F(\omega_2, \omega_3, \gamma_1)$, does not exist.

Type 3. We turn to the consideration of a first integral of type 3, $F(\omega_2, \gamma_1, \gamma_3)$. Thus

$$\begin{aligned}\frac{dF}{dt} = &-\frac{(I_1 - I_3)\omega_3 R - c_1\gamma_3 + c_3\gamma_1}{I_2}\frac{\partial F}{\partial \omega_2} \\ &- \frac{I_1\omega_3\gamma_1 R + I_2\omega_2^2\gamma_3 + I_3\omega_3^2\gamma_3 - U_1\omega_3}{I_2\omega_2}\frac{\partial F}{\partial \gamma_1} \\ &+ \frac{I_2\omega_2^2\gamma_1 + (I_1\gamma_1 R + I_3\omega_3\gamma_3 - U_1)R}{I_2\omega_2}\frac{\partial F}{\partial \gamma_3} = 0,\end{aligned}$$

which is equivalent to

$$I_2\omega_2\frac{dF}{dt} = Y_1(F) = 0, \tag{8.88}$$

where Y_1 is a vector field defined on a suitable Ω.

We differentiate (8.88) twice with respect to ω_3 and obtain

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial \omega_3} &= \omega_2(I_3 - I_1)\left(\omega_3\frac{\partial R}{\partial \omega_3} + R\right)\frac{\partial F}{\partial \omega_2} \\ &- \left[I_1\gamma_1\left(\omega_3\frac{\partial R}{\partial \omega_3} + R\right) + 2I_3\omega_3\gamma_3 - U_1\right]\frac{\partial F}{\partial \gamma_1} \\ &+ \left[I_3\gamma_3\left(\omega_3\frac{\partial R}{\partial \omega_3} + R\right) + 2I_1\gamma_1 R\frac{\partial R}{\partial \omega_3} - U_1\frac{\partial R}{\partial \omega_3}\right]\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0\end{aligned} \tag{8.89}$$

and

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial \omega_3} &= \omega_2(I_3 - I_1)\left(\omega_3\frac{\partial^2 R}{\partial \omega_3^2} + 2\frac{\partial R}{\partial \omega_3}\right)\frac{\partial F}{\partial \omega_2} \\ &- \left[I_1\gamma_1\left(\omega_3\frac{\partial^2 R}{\partial \omega_3^2} + 2\frac{\partial R}{\partial \omega_3}\right) + 2I_3\gamma_3\right]\frac{\partial F}{\partial \gamma_1} \\ &+ \left[I_3\gamma_3\left(\omega_3\frac{\partial^2 R}{\partial \omega_3^2} + 2\frac{\partial R}{\partial \omega_3}\right) + 2I_1\gamma_1\left(\frac{\partial R}{\partial \omega_3}\right)^2\right. \\ &\left. + (2I_1\gamma_1 R - U_1)\frac{\partial^2 R}{\partial \omega_3^2}\right]\frac{\partial F}{\partial \gamma_3} = Y_3(F) = 0,\end{aligned} \tag{8.90}$$

where Y_2 and Y_3 are vector fields defined on Ω.

If a first integral $F(\omega_2, \gamma_1, \gamma_3)$ exists, system (8.88)–(8.90) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_3}\right)$. This is possible if and only if the determinant $D(R)$ of its coefficients is identically zero.

$D(R)$ has a factor ω_2 and we denote

$$\widehat{D}(R) = \frac{D(R)}{\omega_2}.$$

The expression for $\widehat{D}(R)$ is

$$\begin{aligned}\widehat{D}(R) &= a_1 R^3 + a_2 R^2 \frac{\partial R}{\partial \omega_3} + a_3 R^2 \frac{\partial^2 R}{\partial \omega_3^2} + a_4 R^2 + a_5 R \left(\frac{\partial R}{\partial \omega_3}\right)^2 + a_6 R \frac{\partial^2 R}{\partial \omega_3^2} \\ &\quad + a_7 R \frac{\partial^2 R}{\partial \omega_3^2} + a_8 R + a_9 \left(\frac{\partial R}{\partial \omega_3}\right)^3 + a_{10} \left(\frac{\partial R}{\partial \omega_3}\right)^2 + a_{11} \frac{\partial R}{\partial \omega_3} + a_{12} \frac{\partial^2 R}{\partial \omega_3^2}, \qquad (8.91)\end{aligned}$$

where

$$\begin{aligned}
a_1 &= 2I_1 I_3 (I_1 - I_3)\gamma_1\gamma_3, \qquad a_2 = 2I_1(I_1 - I_3)(-3I_3\omega_3\gamma_3 + U_1)\gamma_1, \\
a_3 &= I_1[-2I_2(I_1 - I_3)\omega_2^2\gamma_3 + U_1(I_1 - I_3)\omega_3 + 2I_1c_3\gamma_1^2 - 2I_1c_1\gamma_1\gamma_3]\gamma_1, \\
a_4 &= -2I_3(I_1 - I_3)U_1\gamma_3 \\
a_5 &= 2I_1[I_2(I_1 - I_3)\omega_2^2\gamma_3 + 3I_3(I_1 - I_3)\omega_3^2\gamma_3 - 2U_1(I_1 - I_3)\omega_3 - I_1c_3\gamma_1^2 + I_1c_1\gamma_1\gamma_3]\gamma_1 \\
a_6 &= 4I_3(I_1 - I_3)U_1\omega_3\gamma_3 - 4I_1I_3c_3\gamma_1^2\gamma_3 + 4I_1I_3c_1\gamma_1\gamma_3^2 - 2(I_1 - I_3)U_1^2, \\
a_7 &= I_2(I_1 - I_3)U_1\omega_2^2\gamma_3 + I_3(I_1 - I_3)U_1\omega_3^2\gamma_3 + 4I_1I_3c_3\omega_3\gamma_1^2\gamma_3 - 4I_1I_3c_1\omega_3\gamma_1\gamma_3^2 \\
&\quad - (I_1 - I_3)U_1^2\omega_3 - 3I_1c_3U_1\gamma_1^2 + 3I_1c_1U_1\gamma_1\gamma_3, \\
a_8 &= 2I_3\gamma_3[I_2(I_1 - I_3)\omega_2^2\gamma_1 - I_3c_3\gamma_1\gamma_3 + I_3c_1\gamma_3^2], \\
a_9 &= 2I_1\omega_3\gamma_1[-I_2(I_1 - I_3)\omega_2^2\gamma_3 - I_3(I_1 - I_3)\omega_3^2\gamma_3 \\
&\quad + (I_1 - I_3)U_1\omega_3 + I_1c_3\gamma_1^2 - I_1c_1\gamma_1\gamma_3], \\
a_{10} &= -2I_2(I_1 - I_3)U_1\omega_2^2\gamma_3 - 2I_3(I_1 - I_3)U_1\omega_3^2\gamma_3 + 4I_1I_3c_3\omega_3\gamma_1^2\gamma_3 \\
&\quad - 4I_1I_3c_1\omega_3\gamma_1\gamma_3^2 + 2(I_1 - I_3)U_1^2\omega_3, \\
a_{11} &= -2I_2I_3(I_1 - I_3)\omega_2^2\omega_3\gamma_1\gamma_3 + 2I_2(I_1 - I_3)U_1\omega_2^2\gamma_1 + 2I_3^2c_3\omega_3\gamma_1\gamma_3^2 - 2I_3^2c_1\omega_3\gamma_3^3, \\
a_{12} &= (U_1 - 2I_3\omega_3\gamma_3)[I_2(I_1 - I_3)\omega_2^2\omega_3\gamma_1 - I_3c_3\omega_3\gamma_1\gamma_3 + I_3c_1\omega_3\gamma_3^2 + c_3U_1\gamma_1 - c_1U_1\gamma_3].
\end{aligned}$$

$\widehat{D}(R)$ contains $\frac{\partial R}{\partial \omega_3}$ and $\frac{\partial^2 R}{\partial \omega_3^2}$. We determine them by the same method we used for a first integral of type 2 and obtain

$$\begin{aligned}
\frac{\partial R}{\partial \gamma_1} &= -\frac{I_3(I_2\omega_2\omega_3 - c_2\gamma_3)}{I_1(I_2\omega_2 R - c_2\gamma_1)}, \\
\frac{\partial^2 R}{\partial \gamma_1^2} &= -\frac{I_2I_3\omega_2}{I_1^2(I_2\omega_2 R - c_2\gamma_1)^3}(I_2^2I_3\omega_2^2\omega_3^2 - 2I_2I_3c_2\omega_2\omega_3\gamma_3 + I_1I_2^2R^2\omega_2^2 \\
&\quad - 2I_1I_2c_2\omega_2\gamma_1 R + I_1c_2^2\gamma_1^2 + I_3c_2^2\gamma_3^2).
\end{aligned}$$

We put these values in (8.91) and obtain

$$\widehat{D}(R) = \frac{I_3}{I_1^2(I_2\omega_2 R - c_2\gamma_1)^3}\delta(R),$$

where $\delta(R)$ is a polynomial in R of degree 6 with coefficients that are polynomials in the variables ω_2, ω_3, γ_1, γ_3 and the parameters $\mathcal{I}c$ and U_1.

It is clear that the equation $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \gamma_1, \gamma_3)$ of system (8.81) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in the polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_1, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

By the Maple command `rem` we compute the remainder $r(x)$ from the division of $\delta(x)$ by $Q(x)$. It is of the form

$$r(x) = \frac{1}{I_2^2\omega_2^2}(a_0 x + a_1),$$

where the coefficients a_0 and a_1 are polynomials in ω_2, ω_3, γ_1 and γ_3 and the parameters $\mathcal{I}c$, U_1 and U_3.

According to Proposition 4.2, if R is a root of (8.79), then we have $a_0 = a_1 = 0$ identically. Although a_0 has 84 coefficients we use a_1 which has 160 ones. This is because if we use $a_0 = 0$ then the reduced system has one solution $U_1 = 0$, $c_2 = 0$ which should be studied separately whereas only two consecutive simplifications on the system with 160 equations coming from $a_1 = 0$ lead to the reduced system

$$c_1 = 0, \quad c_3 = 0, \quad I_1 - I_3 = 0,$$

which immediately implies the Lagrange case and leads to the conclusion that a partial integral of type 3, $F(\omega_2, \gamma_1, \gamma_3)$, does not exist.

8.3.4. Elimination of $\boldsymbol{\gamma_2}$ and $\boldsymbol{\gamma_3}$. As in Sec. 8.2.1, we solve the equations $H_1 = U_1$ and $H_3 = U_3$ with respect to γ_2 and γ_3 and obtain

$$\begin{aligned}\gamma_2 &= \frac{I_3\omega_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3) - 2I_1c_3\omega_1\gamma_1 + 2c_3U_1}{2(I_2c_3\omega_2 - I_3c_2\omega_3)},\\ \gamma_3 &= -\frac{I_2\omega_2(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3) - 2I_1c_2\omega_1\gamma_1 + 2c_2U_1}{2(I_2c_3\omega_2 - I_3c_2\omega_3)}.\end{aligned} \tag{8.92}$$

Let us note that the elimination of γ_2 and γ_3 from the equations $H_1 = U_1$ and $H_3 = U_3$ is possible only if

$$(c_2, c_3) \neq (0, 0). \tag{8.93}$$

Further we suppose that this condition is fulfilled.

We put the values of γ_2 and γ_3 from (8.92) in the Euler–Poisson equations (1.1) and remove its fifth and sixth equations. In this way we obtain the following system of four equations in the unknowns ω_1, ω_2, ω_3 and γ_1:

$$\begin{aligned}\frac{d\omega_1}{dt} &= \frac{1}{2I_1(I_2c_3\omega_2 - I_3c_2\omega_3)}[I_1I_2c_2\omega_1^2\omega_2 + I_1I_3c_3\omega_1^2\omega_3 + I_2^2c_2\omega_2^3\\ &\quad + I_3^2c_3\omega_3^3 + 2I_2c_1c_2\omega_2\gamma_1 + I_2(2I_2 - I_3)c_3\omega_2^2\omega_3 + 2I_3c_1c_3\omega_3\gamma_1\\ &\quad - I_3(I_2 - 2I_3)c_2\omega_2\omega_3^2 - I_2c_2U_3\omega_2 - I_3c_3U_3\omega_3\\ &\quad - 2I_1(c_2^2 + c_3^2)\omega_1\gamma_1 + 2(c_2^2 + c_3^2)U_1],\\ \frac{d\omega_2}{dt} &= \frac{1}{2I_2(I_2c_3\omega_2 - I_3c_2\omega_3)}[-I_1I_2c_1\omega_1^2\omega_2 - I_2^2c_1\omega_2^3 - I_2I_3c_1\omega_2\omega_3^2\\ &\quad + 2I_1c_1c_2\omega_1\gamma_1 - 2I_2(I_1 - I_3)c_3\omega_1\omega_2\omega_3 + 2I_3c_2c_3\omega_3\gamma_1\\ &\quad + 2I_3(I_1 - I_3)c_2\omega_1\omega_3^2 + I_2c_1U_3\omega_2 - 2I_2(c_1^2 + c_3^2)\omega_2\gamma_1 - 2c_1c_2U_1],\end{aligned} \tag{8.94}$$

$$
\begin{aligned}
\frac{d\omega_3}{dt} &= \frac{1}{2I_3(I_2c_3\omega_2 - I_3c_2\omega_3)}[-I_1I_3c_1\omega_1^2\omega_3 - I_2I_3c_1\omega_2^2\omega_3 - I_3^2c_1\omega_3^3 \\
&\quad + 2I_1c_1c_3\omega_1\gamma_1 + 2I_2c_2c_3\omega_2\gamma_1 + 2I_2(I_1 - I_2)c_3\omega_1\omega_2^2 \\
&\quad - 2I_3(I_1 - I_2)c_2\omega_1\omega_2\omega_3 + I_3c_1U_3\omega_3 - 2I_3(c_1^2 + c_2^2)\omega_3\gamma_1 - 2c_1c_3U_1], \\
\frac{d\gamma_1}{dt} &= \frac{1}{2(I_2c_3\omega_2 - I_3c_2\omega_3)}[I_1I_2\omega_1^2\omega_2^2 + I_1I_3\omega_1^2\omega_3^2 + I_2^2\omega_2^4 + 2I_2I_3\omega_2^2\omega_3^2 \\
&\quad + I_3^2\omega_3^4 - 2I_1c_2\omega_1\omega_2\gamma_1 - 2I_1c_3\omega_1\omega_3\gamma_1 + 2I_2c_1\omega_2^2\gamma_1 + 2I_3c_1\omega_3^2\gamma_1 \\
&\quad - I_2U_3\omega_2^2 - I_3U_3\omega_3^2 + 2c_2U_1\omega_2 + 2c_3U_1\omega_3].
\end{aligned}
$$

We consider the following four possible types of first integrals of system (8.94) that depend on at most three variables among ω_1, ω_2, ω_3 and γ_1:

1. $F(\omega_1, \omega_2, \omega_3)$ (case (i)).
2. $F(\omega_1, \omega_2, \gamma_1)$ (case (iii)).
3. $F(\omega_1, \omega_3, \gamma_1)$ (case (iii)).
4. $F(\omega_2, \omega_3, \gamma_1)$ (case (ii)).

The only case not studied yet for the invariant manifold (8.48) is case (i). Thus here we should study the existence of a first integral of type 1 only.

Type 1. Let us consider a first integral of type 1, $F(\omega_1, \omega_2, \omega_3)$. Thus

$$
2I_1I_2I_3(I_2c_3\omega_2 - I_3c_2\omega_3)\frac{dF}{dt} = Z(F) = 0, \tag{8.95}
$$

where Z is a vector field defined on a suitable open set $\Omega \subset \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$.

Note that the right-hand sides of (8.94) are linear functions of γ_1. Thus, as F does not depend on γ_1, $Z(F)$ is also a linear function of γ_1, i.e. $Z(F) = \gamma_1 Y_1(F) + Y_2(F)$. Equation (8.95) which is an identity with respect to ω_1, ω_2, ω_3 and γ_1 implies that the coefficients $Y_1(F)$ and $Y_2(F)$ should vanish. The vector fields Y_1 and Y_2 are given by the following expressions:

$$
\begin{aligned}
Y_1 &= 2I_2I_3[I_2c_1c_2\omega_2 + I_3c_1c_3\omega_3 - I_1(c_2^2 + c_3^2)\omega_1]\frac{\partial}{\partial\omega_1} \\
&\quad + 2I_1I_3[I_1c_1c_2\omega_1 + I_3c_2c_3\omega_3 - I_2(c_1^2 + c_3^2)\omega_2]\frac{\partial}{\partial\omega_2} \\
&\quad + 2I_1I_2[2I_1c_1c_3\omega_1 + 2I_2c_2c_3\omega_2 - 2I_3(c_1^2 + c_2^2)\omega_3]\frac{\partial}{\partial\omega_3}, \\
Y_2 &= I_2I_3[I_2c_2\omega_2(I_1\omega_1^2 + I_2\omega_2^2) + I_3c_3\omega_3(I_1\omega_1^2 + I_3\omega_3^2) + I_2(2I_2 - I_3)c_3\omega_2^2\omega_3 \\
&\quad - I_3(I_2 - 2I_3)c_2\omega_2\omega_3^2 - I_2c_2U_3\omega_2 - I_3c_3U_3\omega_3 + 2(c_2^2 + c_3^2)U_1]\frac{\partial}{\partial\omega_1} \\
&\quad + I_1I_3[-I_2c_1\omega_2(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) - 2I_2(I_1 - I_3)c_3\omega_1\omega_2\omega_3 \\
&\quad + 2I_3(I_1 - I_3)c_2\omega_1\omega_3^2 + I_2c_1U_3\omega_2 - 2c_1c_2U_1]\frac{\partial}{\partial\omega_2} \\
&\quad + I_1I_2[-I_3c_1\omega_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2I_2(I_1 - I_2)c_3\omega_1\omega_2^2 \\
&\quad - 2I_3(I_1 - I_2)c_2\omega_1\omega_2\omega_3 + I_3c_1U_3\omega_3 - 2c_1c_3U_1]\frac{\partial}{\partial\omega_3}.
\end{aligned}
$$

We compute the Lie bracket $Y_3 = -[Y_1, Y_2]/(2I_1I_2I_3)$. The expression for Y_3 is long and we do not write it here.

We consider the equations

$$Y_i(F) = 0, \qquad 1 \le i \le 3. \tag{8.96}$$

If a first integral $F(\omega_1, \omega_2, \omega_3)$ exists, system (8.96) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right)$. This is possible if and only if the determinant D of its coefficients is identically zero.

The expression for D is too long to be given here but our computations show that

$$D = 4I_1I_2I_3(I_2c_3\omega_2 - I_3c_2\omega_3)^2\widehat{D}.$$

As the factor in front of $\widehat{D}$ never vanishes identically because of (8.93), the equation $D = 0$ is equivalent to $\widehat{D} = 0$. $\widehat{D}$ is a polynomial in ω_1, ω_2 and ω_3 with 37 coefficients depending on the parameters $\mathcal{I}c$, U_1 and U_3.

Thus we should solve the system obtained by equating the 37 coefficients of $\widehat{D}$ to zero. After four consecutive simplifications we obtain the following simple reduced system:

$$c_3(I_1 - I_2) = 0, \quad c_2(I_1 - I_3) = 0, \quad c_1(I_2 - I_3) = 0, \quad c_2c_3(I_2 - I_3) = 0.$$

We solve this system by `solve` and obtain the following five solutions with arbitrary values of U_1 and U_3:

$$\begin{aligned}
&\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0\},\\
&\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\},\\
&\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\},\\
&\{I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0\},\\
&\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = c_3\}.
\end{aligned}$$

Taking into account condition (8.93), we remove the first and second solutions. The remaining three solutions lead either to the Lagrange case or to kinetic symmetry.

Thus a partial integral of type 1, $F(\omega_1, \omega_2, \omega_3)$, does not exist.

8.4. Invariant manifold $\{H_2 = U_2,\ H_3 = U_3\}$. Here we study the existence of a partial integral of the Euler–Poisson equations (1.1) with respect to the complex four-dimensional level manifold

$$\{H_2 = U_2,\ H_3 = U_3\}, \tag{8.97}$$

supposing that this first integral depends on at most three variables. U_2 and U_3 are arbitrary complex numbers, fixed once and for all.

Let us stress that the elimination of ω_1 and ω_2 on the invariant manifold (8.97) is impossible.

8.4.1. Elimination of ω_1 and γ_1. We express γ_1 from the equation $H_2 = U_2$ and obtain

$$\gamma_1 = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}. \tag{8.98}$$

Then we put γ_1 from (8.98) in the equation $H_3 = U_3$ and as in Sec. 8.2.1 solve it by the MAPLE command `solve`. In this way we obtain

$$\omega_1 = R, \tag{8.99}$$

where R is a root of the equation

$$Q(x) = I_1 x^2 + B = 0, \tag{8.100}$$

that is,

$$Q(R) = I_1 R^2 + B = 0, \tag{8.101}$$

where $B = B(\omega_2, \omega_3, \gamma_2, \gamma_3)$ is the function

$$B = I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\sqrt{-\gamma_2^2 - \gamma_3^2 + U_2} + 2c_2\gamma_2 + 2c_3\gamma_3 - U_3. \tag{8.102}$$

R and B are algebraic functions defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$. Equation (8.100) has only simple roots because the function B does not vanish identically.

Further, to simplify the notations, we put

$$\Gamma = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}.$$

We put the values of γ_1 and ω_1 from (8.98) and (8.99) in the Euler–Poisson equations (1.1), remove the first and fourth equations and obtain the following system of four differential equations in the unknowns ω_2, ω_3, γ_2 and γ_3:

$$\begin{aligned} \frac{d\omega_2}{dt} &= \frac{1}{I_2}[(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\Gamma], & \frac{d\gamma_2}{dt} &= \gamma_3 R - \omega_3\Gamma, \\ \frac{d\omega_3}{dt} &= \frac{1}{I_3}[(I_1 - I_2)\omega_2 R - c_1\gamma_2 + c_2\Gamma], & \frac{d\gamma_3}{dt} &= -\gamma_2 R + \omega_2\Gamma. \end{aligned} \tag{8.103}$$

We want to study the existence of a first integral of system (8.103) that depends on at most three variables among ω_2, ω_3, γ_2 and γ_3. The following four types of first integrals are possible:

1. $F(\omega_2, \omega_3, \gamma_2)$ (case (iii)).
2. $F(\omega_2, \omega_3, \gamma_3)$ (case (iii)).
3. $F(\omega_2, \gamma_2, \gamma_3)$ (case (iv)).
4. $F(\omega_3, \gamma_2, \gamma_3)$ (case (iv)).

As in Sec. 5, we consider here only types 1 and 3.

Type 1. Let us suppose that there exists a first integral of type 1, $F(\omega_2, \omega_3, \gamma_2)$. Then

$$I_2 I_3 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.104}$$

where Y_1 is the vector field

$$\begin{aligned} Y_1 = I_3[(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\Gamma]&\frac{\partial}{\partial\omega_2} \\ + I_2[(I_1 - I_2)\omega_2 R - c_1\gamma_2 + c_2\Gamma]&\frac{\partial}{\partial\omega_3} \\ + I_2 I_3[\gamma_3 R - \omega_3\Gamma]&\frac{\partial}{\partial\gamma_2}, \end{aligned}$$

defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

As F does not depend on γ_3, if we differentiate identity (8.104) with respect to γ_3 we obtain again a linear partial differential equation for F:

$$\begin{aligned}\Gamma\frac{\partial Y_1(F)}{\partial\gamma_3} &= I_3\left[\frac{\partial R}{\partial\gamma_3}(I_3-I_1)\omega_3\Gamma + c_1\Gamma + c_3\gamma_3\right]\frac{\partial F}{\partial\omega_2}\\ &\quad + I_2\left[\frac{\partial R}{\partial\gamma_3}(I_1-I_2)\omega_2\Gamma - c_2\gamma_3\right]\frac{\partial F}{\partial\omega_3}\\ &\quad + I_2I_3\left[\frac{\partial R}{\partial\gamma_3}\gamma_3\Gamma + R\Gamma + \omega_3\gamma_3\right]\frac{\partial F}{\partial\gamma_2} = Y_2(F) = 0, \qquad (8.105)\end{aligned}$$

where Y_2 is a vector field defined on $\mathbb{C}^4(\omega_2,\omega_3,\gamma_2,\gamma_3)$.

After differentiating identity (8.105) with respect to γ_3 we obtain

$$\begin{aligned}\Gamma\frac{\partial Y_2(F)}{\partial\gamma_3} &= I_3\left[\frac{\partial^2 R}{\partial\gamma_3^2}(I_3-I_1)\omega_3\Gamma^2 - \frac{\partial R}{\partial\gamma_3}(I_3-I_1)\omega_3\gamma_3 - c_1\gamma_3 + c_3\Gamma\right]\frac{\partial F}{\partial\omega_2}\\ &\quad + I_2\left[\frac{\partial^2 R}{\partial\gamma_3^2}(I_1-I_2)\omega_2\Gamma^2 - \frac{\partial R}{\partial\gamma_3}(I_1-I_2)\omega_2\gamma_3 - c_2\Gamma\right]\frac{\partial F}{\partial\omega_3}\\ &\quad + I_2I_3\left[\frac{\partial^2 R}{\partial\gamma_3^2}\gamma_3\Gamma^2 + \frac{\partial R}{\partial\gamma_3}(2\Gamma^2-\gamma_3^2) - R\gamma_3 + \omega_3\Gamma\right]\frac{\partial F}{\partial\gamma_2}\\ &= Y_3(F) = 0, \qquad (8.106)\end{aligned}$$

where Y_3 is a vector field defined on $\mathbb{C}^4(\omega_2,\omega_3,\gamma_2,\gamma_3)$.

If a first integral F exists, the linear system (8.104)–(8.106) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\omega_3}, \frac{\partial F}{\partial\gamma_2}\right)$. This is possible if and only if the determinant $D(R)$ of its coefficients is identically zero on $\mathbb{C}^4(\omega_2,\omega_3,\gamma_2,\gamma_3)$.

The expression for $D(R)$ is too long to be shown here. $D(R)$ has a non-zero factor $I_2^2I_3^2$ so we remove it and denote

$$\widehat{D}(R) = \frac{D(R)}{I_2^2I_3^2}.$$

$\widehat{D}(R)$ contains the partial derivatives $\frac{\partial R}{\partial\gamma_3}$ and $\frac{\partial^2 R}{\partial\gamma_3^2}$ as well. To determine them we use equation (8.101) which we differentiate twice with respect to γ_3 and obtain two equations for the derivatives of R:

$$\begin{aligned}\frac{\partial Q(R)}{\partial\gamma_3} &= 2I_1R\frac{\partial R}{\partial\gamma_3} + \frac{\partial B}{\partial\gamma_3} = 0,\\ \frac{\partial^2 Q(R)}{\partial\gamma_3^2} &= 2I_1\left(\frac{\partial R}{\partial\gamma_3}\right)^2 + 2I_1R\frac{\partial^2 R}{\partial\gamma_3^2} + \frac{\partial^2 B}{\partial\gamma_3^2} = 0.\end{aligned}$$

Determining $\frac{\partial R}{\partial\gamma_3}$ and $\frac{\partial^2 R}{\partial\gamma_3^2}$ from these two equations is possible if and only if $2I_1R \neq 0$. It is clear that this is always so because $R = 0$ is not a root of (8.101). Thus the derivatives of R can be found. We obtain

$$\begin{aligned}\frac{\partial R}{\partial\gamma_3} &= \frac{c_1\gamma_3 - c_3\Gamma}{I_1R\Gamma},\\ \frac{\partial^2 R}{\partial\gamma_3^2} &= -\frac{(c_1^2\gamma_3^2 + c_3^2\Gamma^2)\Gamma - 2c_1c_3\gamma_3\Gamma^2 + I_1c_1R^2(\gamma_2^2 - U_2)}{I_1^2R^3\Gamma^3}.\end{aligned}$$

We put the values for the derivatives of R in the expression for $\widehat{D}(R)$ and obtain

$$\widehat{D}(R) = \frac{\delta(R)}{I_1^2 R^3},$$

where δ is a polynomial in R of degree 5, whose coefficients are algebraic functions of $(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

It is clear that $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \omega_3, \gamma_2)$ of system (8.103) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in the polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using the `rem` command we compute the remainder r of dividing $\delta(x)$ by $Q(x)$. We obtain

$$r(x) = a_0 x + a_1,$$

where $a_i = a_i(\omega_2, \omega_3, \gamma_2, \gamma_3)$, $i = 0, 1$, depend linearly on Γ.

According to Proposition 4.2, a_0 and a_1 should vanish identically with respect to ω_2, ω_3, γ_2 and γ_3. We use only a_0 which suffices for our aims. We have

$$a_0 = b_0 \Gamma + b_1,$$

where b_0 and b_1 are polynomials in ω_2, ω_3, γ_2 and γ_3.

According to Proposition 4.3, the coefficients b_0 and b_1 should vanish identically because $\Gamma \notin \mathbb{C}(\gamma_2, \gamma_3)$. The polynomial b_0 has 30 coefficients and b_1 has 68 coefficients. Equating all of them to zero we obtain 98 equations for the parameters $\mathcal{I}c$, U_2 and U_3.

After three consecutive simplifications we come to a reduced system that consists of the following four equations:

$$c_1 = 0, \quad c_3(I_1 - I_2) = 0, \quad c_2(I_1 - I_3) = 0, \quad c_2 c_3 (I_2 - I_3) = 0.$$

We solve this system by using `solve` and obtain four solutions, all of them with arbitrary values of U_2 and U_3:

$$\begin{aligned}
&I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0;\\
&I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3;\\
&I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0;\\
&I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = c_3.
\end{aligned}$$

The first solution is the Euler case, the second and third are the Lagrange case and the fourth solution is a particular case of the kinetic symmetry case.

Thus a partial integral of type 1, $F(\omega_2, \omega_3, \gamma_2)$, does not exist.

Type 3. The study of a new first integral of type 3, $F(\omega_2, \gamma_2, \gamma_3)$, follows the algorithm already described in the considerations concerning a first integral of type 1. There are some differences of course. For example, computing the vector fields Y_2 and Y_3 requires differentiation with respect to ω_3 instead of γ_3. By the way, as is seen below, this considerably simplifies the computations because the differentiation does not affect the function Γ.

Let us suppose that there exists a first integral of type 3, $F(\omega_2, \gamma_2, \gamma_3)$. Then we have

$$I_2\frac{dF}{dt} = Y_1(F) = 0, \tag{8.107}$$

where Y_1 is the vector field

$$Y_1 = [(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\Gamma]\frac{\partial}{\partial\omega_2} + I_2(\gamma_3 R - \omega_3\Gamma)\frac{\partial}{\partial\gamma_2} - I_2(\gamma_2 R - \omega_2\Gamma)\frac{\partial}{\partial\gamma_3},$$

defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

As in the study of first integrals of type 1, we differentiate identity (8.107) with respect to ω_3 and obtain

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial\omega_3} &= (I_3 - I_1)\left(\frac{\partial R}{\partial\omega_3}\omega_3 + R\right)\frac{\partial F}{\partial\omega_2} + I_2\left(\frac{\partial R}{\partial\omega_3}\gamma_3 - \Gamma\right)\frac{\partial F}{\partial\gamma_2}\\ &\quad - I_2\frac{\partial R}{\partial\omega_3}\gamma_2\frac{\partial F}{\partial\gamma_3} = Y_2(F) = 0,\end{aligned} \tag{8.108}$$

where Y_2 is a vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

After differentiating identity (8.108) with respect to ω_3 we obtain

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial\omega_3} &= (I_3 - I_1)\left[\frac{\partial^2 R}{\partial\omega_3^2}\omega_3 + 2\frac{\partial R}{\partial\omega_3}\right]\frac{\partial F}{\partial\omega_2} + I_2\frac{\partial^2 R}{\partial\omega_3^2}\gamma_3\frac{\partial F}{\partial\gamma_2}\\ &\quad - I_2\frac{\partial^2 R}{\partial\omega_3^2}\gamma_2\frac{\partial F}{\partial\gamma_3} = Y_3(F) = 0,\end{aligned} \tag{8.109}$$

where Y_3 is a vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

As in the investigation of first integrals of type 1, we require that the determinant $D(R)$ of the coefficients of system (8.107)–(8.109) be identically zero. Computing it we see that it has a non-zero factor $I_2^2\Gamma$. We remove it and denote

$$\widehat{D}(R) = \frac{D(R)}{I_2^2\Gamma}.$$

In this way we obtain

$$\begin{aligned}\widehat{D}(R) = &-\left\{[(I_1 - I_3)\omega_2\omega_3 + c_3\gamma_2]\frac{\partial^2 R}{\partial\omega_3^2} + 2\omega_2(I_1 - I_3)\frac{\partial R}{\partial\omega_3}\right\}\Gamma + c_1\gamma_2\gamma_3\frac{\partial^2 R}{\partial\omega_3^2}\\ &+ 2\gamma_2(I_1 - I_3)R\frac{\partial R}{\partial\omega_3} + (\omega_2\gamma_3 - \omega_3\gamma_2)(I_1 - I_3)\left[2\left(\frac{\partial R}{\partial\omega_3}\right)^2 - \frac{\partial^2 R}{\partial\omega_3^2}R\right].\end{aligned}$$

$\widehat{D}(R)$ contains the partial derivatives $\frac{\partial R}{\partial\omega_3}$ and $\frac{\partial^2 R}{\partial\omega_3^2}$. We use equation (8.101) to determine them. To this end we differentiate (8.101) twice with respect to ω_3 and obtain two equations for the sought derivatives of R:

$$\begin{aligned}\frac{\partial Q}{\partial\omega_3} &= 2I_1 R\frac{\partial R}{\partial\omega_3} + \frac{\partial B}{\partial\omega_3} = 0,\\ \frac{\partial^2 Q}{\partial\omega_3^2} &= 2I_1\left(\frac{\partial R}{\partial\omega_3}\right)^2 + 2I_1 R\frac{\partial^2 R}{\partial\omega_3^2} + \frac{\partial^2 B}{\partial\omega_3^2} = 0.\end{aligned}$$

As we have mentioned when studying first integrals of type 1, $R = 0$ cannot be a root of (8.101) and therefore the partial derivatives of R can be correctly determined from the above equations. We put the value of B taken from (8.102) in these equations and solve them. The solution is

$$\frac{\partial R}{\partial \omega_3} = -\frac{I_3\omega_3}{I_1 R}, \qquad \frac{\partial^2 R}{\partial \omega_3^2} = -\frac{I_3(I_3\omega_3^2 + I_1R^2)}{I_1^2R^3}.$$

We put the above values of $\frac{\partial R}{\partial \omega_3}$ and $\frac{\partial^2 R}{\partial \omega_3^2}$ in the expression for $\widehat{D}(R)$ and obtain

$$\widehat{D}(R) = \frac{I_3\delta(R)}{I_1^2R^3},$$

where $\delta(R)$ is the following polynomial in R of degree 3:

$$\begin{aligned}\delta(R) = {}& -I_1(I_1 - I_3)(3\omega_3\gamma_2 - \omega_2\gamma_3)R^3 + I_1[3(I_1 - I_3)\omega_2\omega_3\Gamma + (c_3\Gamma - c_1\gamma_3)\gamma_2]R^2 \\ & - 3I_3(I_1 - I_3)\omega_3^2(\omega_3\gamma_2 - \omega_2\gamma_3)R + I_3\omega_3^2[(I_1 - I_3)\omega_2\omega_3\Gamma + (c_3\Gamma - c_1\gamma_3)\gamma_2].\end{aligned}$$

It is clear that $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \gamma_2, \gamma_3)$ of system (8.103) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides $\delta(x)$.

Using the `rem` command we compute the remainder r of dividing $\delta(x)$ by $Q(x)$. We obtain

$$r(x) = (a_0 + b_0\Gamma)x + a_1 + b_1\Gamma,$$

where

$$\begin{aligned}a_0 &= (I_3 - I_1)(I_2\omega_2^3\gamma_3 - 3I_2\omega_2^2\omega_3\gamma_2 - 2I_3\omega_2\omega_3^2\gamma_3 + 2c_2\omega_2\gamma_3\gamma_2 + 2c_3\omega_2\gamma_3^2 - U_3\omega_2\gamma_3 \\ &\quad - 6c_2\omega_3\gamma_2^2 - 6\omega_3c_3\gamma_2\gamma_3 + 3U_3\omega_3\gamma_2), \\ b_0 &= 2(I_3 - I_1)c_1(\omega_2\gamma_3 - 3\omega_3\gamma_2), \\ a_1 &= c_1[I_2\omega_2^2\gamma_2\gamma_3 - 6c_1(I_1 - I_3)\omega_2\omega_3\Gamma^2 + 2c_3\gamma_2^3 + 2c_2\gamma_2^2\gamma_3 + 4c_3\gamma_2\gamma_3^2 \\ &\quad - U_3\gamma_2\gamma_3 - 2c_3U_2\gamma_2], \\ b_1 &= -I_2c_3\omega_2^2\gamma_2 - (I_1 - I_3)(3I_2\omega_2^2 + 2I_3\omega_3^2 + 6c_2\gamma_2 + 6c_3\gamma_3 - 3U_3)\omega_2\omega_3 \\ &\quad - 2c_2c_3\gamma_2^2 + 2(c_1^2 - c_3^2)\gamma_2\gamma_3 + c_3U_3\gamma_2.\end{aligned}$$

According to Proposition 4.2, all the coefficients of the remainder r should vanish identically with respect to ω_2, ω_3, γ_2 and γ_3. We use only the coefficient $a_1 + b_1\Gamma$, which is sufficient for our aims.

According to Proposition 4.3, the coefficients a_1 and b_1 should vanish identically because $\Gamma \notin \mathbb{C}(\gamma_2, \gamma_3)$. We use only b_1. It has nine coefficients. Equating all of them to zero we obtain nine equations for the parameters $\mathcal{I}c$, U_2 and U_3:

$$\begin{gathered}2I_3(I_3 - I_1) = 0, \quad 3I_2(I_3 - I_1) = 0, \quad -I_2c_3 = 0, \quad 3(I_1 - I_3)U_3, \quad c_3U_3 = 0, \\ 2c_2c_3 = 0, \quad 2(c_1^2 - c_3^2) = 0, \quad 6(I_3 - I_1)c_2 = 0, \quad 6(I_3 - I_1)c_3 = 0.\end{gathered}$$

It is very easy to see that these equations imply that

$$c_1 = 0, \quad c_3 = 0, \quad I_1 - I_3 = 0,$$

which obviously leads to the Lagrange case.

Thus a partial integral of type 3, $F(\omega_2, \gamma_2, \gamma_3)$, does not exist.

8.4.2. Elimination of ω_1 and γ_2. As in Sec. 8.4.1, we express γ_2 from the equation $H_2 = U_2$ and obtain

$$\gamma_2 = \sqrt{-\gamma_1^2 - \gamma_3^2 + U_2}. \tag{8.110}$$

Then we put γ_2 from (8.110) in the equation $H_3 = U_3$ and just as in Sec. 8.2.1 solve it by using `solve`. In this way we obtain

$$\omega_1 = R, \tag{8.111}$$

where R is a root of

$$Q(x) = I_1 x^2 + B = 0, \tag{8.112}$$

that is,

$$Q(R) = I_1 R^2 + B = 0, \tag{8.113}$$

and $B = B(\omega_2, \omega_3, \gamma_1, \gamma_3)$ is the following function:

$$B = I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 + 2c_2\sqrt{-\gamma_1^2 - \gamma_3^2 + U_2} + 2c_3\gamma_3 - U_3.$$

R and B are algebraic functions defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$. Equation (8.112) has only simple roots because the function B does not vanish identically.

Further, to simplify the notations, we put

$$\Gamma = \sqrt{-\gamma_1^2 - \gamma_3^2 + U_2}.$$

We put the values of γ_2 and ω_1 from (8.110) and (8.111) in the Euler–Poisson equations (1.1), remove the first and fifth equations and obtain the following system of four differential equations in the unknowns ω_2, ω_3, γ_1 and γ_3:

$$\begin{aligned} \frac{d\omega_2}{dt} &= \frac{1}{I_2}[(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\gamma_1], & \frac{d\gamma_1}{dt} &= \omega_3\Gamma - \omega_2\gamma_3, \\ \frac{d\omega_3}{dt} &= \frac{1}{I_3}[(I_1 - I_2)\omega_2 R + c_1\Gamma - c_2\gamma_1], & \frac{d\gamma_3}{dt} &= \omega_2\gamma_1 - R\Gamma. \end{aligned} \tag{8.114}$$

We want to study the existence of a first integral of (8.114) that depends on at most three variables among ω_2, ω_3, γ_1 and γ_3 and that is functionally independent of H_1 restricted to the invariant manifold (8.97). The following types of first integrals are possible:

1. $F(\omega_2, \omega_3, \gamma_1)$ (case (ii)).
2. $F(\omega_2, \omega_3, \gamma_3)$ (case (iii)).
3. $F(\omega_2, \gamma_1, \gamma_3)$ (case (v)).
4. $F(\omega_3, \gamma_1, \gamma_3)$ (case (iv)).

As cases (iii) and (iv) were already examined, it remains to examine cases (ii) and (v).

Type 1. Let us suppose that there exists a first integral of type 1, $F(\omega_2, \omega_3, \gamma_1)$. Then

$$I_2 I_3 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.115}$$

where Y_1 is the vector field

$$\begin{aligned} Y_1 = {} & I_3[(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\gamma_1]\frac{\partial}{\partial\omega_2} \\ & + I_2[(I_1 - I_2)\omega_2 R + c_1\Gamma - c_2\gamma_1]\frac{\partial}{\partial\omega_3} + I_2 I_3(\omega_3\Gamma - \omega_2\gamma_3)\frac{\partial}{\partial\gamma_1}, \end{aligned}$$

defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

As F does not depend on γ_3, if we differentiate identity (8.115) with respect to γ_3 we obtain again a linear partial differential equation for F:

$$\begin{aligned}\Gamma\frac{\partial Y_1(F)}{\partial\gamma_3} &= I_3\left[\frac{\partial R}{\partial\gamma_3}(I_3-I_1)\omega_3+c_1\right]\Gamma\frac{\partial F}{\partial\omega_2}+I_2\left[\frac{\partial R}{\partial\gamma_3}(I_1-I_2)\omega_2\Gamma+c_1\gamma_3\right]\frac{\partial F}{\partial\omega_3}\\ &\quad - I_2I_3[\omega_2\Gamma+\omega_3\gamma_3]\frac{\partial F}{\partial\gamma_1}=Y_2(F)=0, \qquad (8.116)\end{aligned}$$

where Y_2 is a vector field defined on $\mathbb{C}^4(\omega_2,\omega_3,\gamma_1,\gamma_3)$.

After differentiating (8.116) with respect to γ_3 we obtain

$$\begin{aligned}\Gamma\frac{\partial Y_2(F)}{\partial\gamma_3} &= I_3\left[\frac{\partial^2 R}{\partial\gamma_3^2}(I_3-I_1)\omega_3\Gamma^2-\frac{\partial R}{\partial\gamma_3}(I_3-I_1)\omega_3\gamma_3-c_1\gamma_3\right]\frac{\partial F}{\partial\omega_2}\\ &\quad + I_2\left[\frac{\partial^2 R}{\partial\gamma_3^2}(I_1-I_2)\omega_2\Gamma^2-\frac{\partial R}{\partial\gamma_3}(I_1-I_2)\omega_2\gamma_3+c_1\Gamma\right]\frac{\partial F}{\partial\omega_3}\\ &\quad + I_2I_3\left(\omega_2\gamma_3-\omega_3\Gamma\right)\frac{\partial F}{\partial\gamma_1}=Y_3(F)=0, \qquad (8.117)\end{aligned}$$

where Y_3 is a vector field defined on $\mathbb{C}^4(\omega_2,\omega_3,\gamma_1,\gamma_3)$.

If a first integral F exists, the linear system (8.115)–(8.117) has a non-zero solution $\operatorname{grad}F=\left(\frac{\partial F}{\partial\omega_2},\frac{\partial F}{\partial\omega_3},\frac{\partial F}{\partial\gamma_1}\right)$. This is possible if and only if the determinant $D(R)$ of its coefficients is identically zero on $\mathbb{C}^4(\omega_2,\omega_3,\gamma_1,\gamma_3)$.

The expression for $D(R)$ has a non-zero factor $I_2^2I_3^2$ so we remove it and denote

$$\widehat{D}(R)=\frac{D(R)}{I_2^2I_3^2}.$$

In this way we obtain

$$\begin{aligned}\widehat{D}(R) &= -\frac{\partial^2 R}{\partial\gamma_3^2}[c_3(I_1-I_2)\omega_2^2\gamma_1\Gamma-c_1(I_2-I_3)\omega_2\omega_3\gamma_1^2+c_3(I_1-I_2)\omega_2\omega_3\gamma_1\gamma_3\\ &\quad -c_2(I_1-I_3)\omega_2\omega_3\gamma_1\Gamma+(I_2-I_3)c_1U_2\omega_2\omega_3-c_2(I_1-I_3)\omega_3^2\gamma_1\gamma_3]\Gamma^2\\ &\quad -\frac{\partial R}{\partial\gamma_3}\omega_3[c_3(I_2-I_1)\omega_2\gamma_1-c_1(I_2-I_3)\omega_2\gamma_3+c_2(I_1-I_3)\omega_3\gamma_1](U_2-\gamma_1^2)\\ &\quad -c_1[R(I_2-I_3)\omega_2\omega_3+c_3\omega_2\gamma_1-c_2\omega_3\gamma_1](U_2-\gamma_1^2)\end{aligned}$$

$\widehat{D}(R)$ contains the partial derivatives $\frac{\partial R}{\partial\gamma_3}$ and $\frac{\partial^2 R}{\partial\gamma_3^2}$. To determine them we use equation (8.113), which we differentiate twice with respect to γ_3 and in the same way as in Sec. 8.4.1 we obtain

$$\begin{aligned}\frac{\partial R}{\partial\gamma_3} &= \frac{c_2\gamma_3-c_3\Gamma}{I_1R\Gamma},\\ \frac{\partial^2 R}{\partial\gamma_3^2} &= -\frac{(c_2^2\gamma_3^2+c_3^2\Gamma^2)\Gamma-2c_2c_3\gamma_3\Gamma^2+I_1c_2R^2(\gamma_1^2-U_2)}{I_1^2R^3\Gamma^3}.\end{aligned}$$

We put the resulting values for the derivatives of R in the expression for $\widehat{D}(R)$ and obtain

$$\widehat{D}(R)=\frac{\delta(R)}{I_1^2R^3},$$

where δ is a long polynomial of R of degree 4, whose coefficients are algebraic functions of $(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

The identity $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \omega_3, \gamma_1)$ of system (8.114) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_1, \gamma_3)$, the polynomial $Q(x)$ divides $\delta(x)$.

Using the `rem` command we compute the remainder r of dividing $\delta(x)$ by $Q(x)$:

$$r(x) = a_0 x + a_1, \tag{8.118}$$

where $a_i = a_i(\omega_2, \omega_3, \gamma_1, \gamma_3)$, $i = 0, 1$, depend linearly on Γ.

According to Proposition 4.2, a_0 and a_1 should vanish identically with respect to ω_2, ω_3, γ_1 and γ_3. We use only a_1 which suffices for our aims. We have

$$a_1 = b_0 \Gamma + b_1,$$

where b_0 and b_1 are polynomials in ω_2, ω_3, γ_1 and γ_3.

According to Proposition 4.3, the coefficients b_0 and b_1 should vanish identically because $\Gamma \notin \mathbb{C}(\gamma_1, \gamma_3)$.

We use only the polynomial b_1. It has 48 coefficients. Equating all of them to zero we obtain a system of 48 equations for the parameters $\mathcal{I}c$, U_2 and U_3.

After four consecutive simplifications we come to a reduced system that consists of the following four equations:

$$c_1(I_2 - I_3) = 0, \quad c_2(I_1 - I_3) = 0 \quad c_3(I_1 - I_2) = 0, \quad c_2 c_3(I_2 - I_3) = 0.$$

We solve this system using `solve` and obtain five solutions, all of them with arbitrary values of U_2 and U_3:

$$\begin{aligned}
&I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0;\\
&I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0;\\
&I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3;\\
&I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0;\\
&I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = c_3.
\end{aligned}$$

The first solution is the Euler case, the next three are the Lagrange case and the last one is the kinetic symmetry case.

Thus a partial integral of type 1, $F(\omega_2, \omega_3, \gamma_1)$, does not exist.

Type 3. Let us suppose that there exists a first integral of type 3, $F(\omega_2, \gamma_1, \gamma_3)$. The independence of the first integral of ω_3 considerably simplifies the computations because there is no need to differentiate the function Γ.

So, let $F(\omega_2, \gamma_1, \gamma_3)$ be a first integral of system (8.114). Then we have

$$I_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.119}$$

where Y_1 is the vector field

$$Y_1 = [(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\gamma_1]\frac{\partial}{\partial \omega_2} + I_2(\omega_3\Gamma - \omega_2\gamma_3)\frac{\partial}{\partial \gamma_1} + I_2(\omega_2\gamma_1 - R\Gamma)\frac{\partial}{\partial \gamma_3},$$

defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

We differentiate (8.119) with respect to ω_3 and obtain

$$\frac{\partial Y_1(F)}{\partial \omega_3} = (I_3 - I_1)\left(\frac{\partial R}{\partial \omega_3}\omega_3 + R\right)\frac{\partial F}{\partial \omega_2} + I_2\Gamma\frac{\partial F}{\partial \gamma_1} - I_2\frac{\partial R}{\partial \omega_3}\Gamma\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \quad (8.120)$$

where Y_2 is a vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

After differentiating (8.120) with respect to ω_3 we obtain

$$\frac{\partial Y_2(F)}{\partial \omega_3} = (I_3 - I_1)\left(\frac{\partial^2 R}{\partial \omega_3^2}\omega_3 + 2\frac{\partial R}{\partial \omega_3}\right)\frac{\partial F}{\partial \omega_2} - I_2\frac{\partial^2 R}{\partial \omega_3^2}\Gamma\frac{\partial F}{\partial \gamma_3} = Y_3(F) = 0, \quad (8.121)$$

where Y_3 is a vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

The existence of a first integral $F(\omega_2, \gamma_1, \gamma_3)$ implies that the determinant $D(R)$ of the coefficients of system (8.119)–(8.121) is identically zero. Computing $D(R)$ we see that it has a non-zero factor $I_2^2\Gamma$. We remove it and denote

$$\widehat{D}(R) = \frac{D(R)}{I_2^2\Gamma}.$$

In this way we obtain

$$\begin{aligned}\widehat{D}(R) = &-\Bigg\{[(I_1 - I_3)\omega_3 R - c_3\gamma_1 + c_1\gamma_3]\frac{\partial^2 R}{\partial \omega_3^2} \\ &- 2\omega_3(I_1 - I_3)\left(\frac{\partial R}{\partial \omega_3}\right)^2 + 2(I_1 - I_3)R\frac{\partial R}{\partial \omega_3}\Bigg\}\Gamma \\ &+ \omega_2(I_1 - I_3)\left[(\omega_3\gamma_1 + \gamma_3 R)\frac{\partial^2 R}{\partial \omega_3^2} - 2\gamma_3\left(\frac{\partial R}{\partial \omega_3}\right)^2 + 2\gamma_1\frac{\partial R}{\partial \omega_3}\right].\end{aligned}$$

$\widehat{D}(R)$ contains the partial derivatives $\frac{\partial R}{\partial \omega_3}$ and $\frac{\partial^2 R}{\partial \omega_3^2}$. We use (8.113) to determine them. We differentiate (8.113) twice with respect to ω_3 and in the same way as in Sec. 8.4.1 obtain

$$\frac{\partial R}{\partial \omega_3} = -\frac{I_3\omega_3}{I_1 R}, \qquad \frac{\partial^2 R}{\partial \omega_3^2} = -\frac{I_3(I_3\omega_3^2 + I_1 R^2)}{I_1^2 R^3}.$$

We put the above values of $\frac{\partial R}{\partial \omega_3}$ and $\frac{\partial^2 R}{\partial \omega_3^2}$ in the expression for $\widehat{D}(R)$ and obtain

$$\widehat{D}(R) = \frac{I_3\delta(R)}{I_1^2 R^3},$$

where δ is the following polynomial in R of degree 3:

$$\begin{aligned}\delta(R) = &\ I_1(I_1 - I_3)(3\omega_3\Gamma - \omega_2\gamma_3)R^3 + I_1[3(I_3 - I_1)\omega_2\omega_3\gamma_1 - (c_3\gamma_1 - c_1\gamma_3)\Gamma]R^2 \\ &+ 3I_3(I_1 - I_3)\omega_3^2(\omega_3\Gamma - \omega_2\gamma_3)R + I_3\omega_3^2[(I_3 - I_1)\omega_2\omega_3\gamma_1 - \Gamma(c_3\gamma_1 - c_1\gamma_3)],\end{aligned}$$

whose coefficients are algebraic functions of $(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

The identity $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \gamma_1, \gamma_3)$ of system (8.114) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_1, \gamma_3)$, the polynomial $Q(x)$ divides $\delta(x)$.

Using MAPLE we divide δ by Q and obtain a remainder which is a polynomial r of the form (8.118) with coefficients $a_i = a_i(\omega_2, \omega_3, \gamma_1, \gamma_3)$, $i = 0, 1$, which depend linearly on Γ.

According to Proposition 4.2, a_0 and a_1 should vanish identically with respect to ω_2, ω_3, γ_1 and γ_3. We use only a_1 which suffices for our aims. We have

$$a_1 = b_0\Gamma + b_1,$$

where b_0 and b_1 are polynomials in ω_2, ω_3, γ_1 and γ_3.

According to Proposition 4.3, the coefficients b_0 and b_1 should vanish identically because $\Gamma \notin \mathbb{C}(\gamma_1, \gamma_3)$.

We use only the polynomial b_0. It has eight coefficients. Equating all of them to zero we obtain a system of eight equations for the parameters $\mathcal{I}c$, U_2 and U_3:

$$\begin{aligned} c_1^2 - c_3^2 = 0, \quad I_2c_3 = 0, \quad I_2c_1 = 0 \quad (I_1 - I_3)c_2 = 0,\\ c_3U_3 = 0, \quad c_3c_1 = 0, \quad c_1U_3 = 0, \quad c_1c_3 = 0. \end{aligned}$$

After two consecutive simplifications we come to the reduced system

$$c_1 = 0, \quad c_3 = 0, \quad (I_1 - I_3)c_2 = 0,$$

which leads either to the Euler case or to the Lagrange case.

Thus a partial integral of type 3, $F(\omega_2, \gamma_1, \gamma_3)$, does not exist.

8.4.3. Elimination of $\boldsymbol{\gamma_2}$ and $\boldsymbol{\gamma_3}$. Let us note that the elimination of γ_2 and γ_3 from the equations $H_2 = U_2$ and $H_3 = U_3$ is possible only if

$$(c_2, c_3) \neq (0, 0). \tag{8.122}$$

Hereafter we suppose that this condition is always fulfilled.

We start with the case of $c_2 \neq 0$ and c_3 arbitrary. The elimination is made much as in Sec. 8.4.1. First we express γ_2 from the equation $H_3 = U_3$ and put the resulting value of γ_2 in the equation $H_2 = U_2$ from which we find γ_3. In this way we have

$$\gamma_2 = -\frac{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3 + 2c_3R}{2c_2}, \qquad \gamma_3 = R, \tag{8.123}$$

where, if $c_2^2 + c_3^2 \neq 0$, R is a root of

$$Q(x) = 4(c_2^2 + c_3^2)x^2 + Bx + C = 0,$$

that is,

$$Q(R) = 4(c_2^2 + c_3^2)R^2 + BR + C = 0. \tag{8.124}$$

If $c_2^2 + c_3^2 = 0$ R is a root of

$$Q(x) = Bx + C = 0,$$

that is,

$$Q(R) = BR + C = 0. \tag{8.125}$$

The functions $B = B(\omega_2, \omega_3, \gamma_1, \gamma_3)$ and $C = C(\omega_2, \omega_3, \gamma_1, \gamma_3)$ are the following polynomials:

$$\begin{aligned} B &= 4c_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3),\\ C &= (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)^2 + 4c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)\gamma_1\\ &\quad - 2U_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 4(c_1^2 + c_2^2)\gamma_1^2 - 4c_1U_3\gamma_1 - 4c_2^2U_2 + U_3^2. \end{aligned} \tag{8.126}$$

Let us note that if $c_2^2+c_3^2=0$ then $c_3\neq 0$ because if $c_3=0$ then condition (8.122) will not be satisfied. Consequently, $B\neq 0$ and therefore (8.125) is well defined.

We put the values of γ_2 and γ_3 from (8.123) in the Euler–Poisson equations (1.1) and remove the fifth and sixth equations. In this way we obtain the following system of four equations in unknowns ω_1, ω_2, ω_3 and γ_1:

$$\begin{aligned}\frac{d\omega_1}{dt}&=\frac{1}{2I_1c_2}[-c_3(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)+2(I_2-I_3)c_2\omega_2\omega_3\\&\quad-2c_1c_3\gamma_1-2(c_2^2+c_3^2)R+c_3U_3],\\\frac{d\omega_2}{dt}&=\frac{1}{I_2}[(I_3-I_1)\omega_1\omega_3+c_1R-c_3\gamma_1],\\\frac{d\omega_3}{dt}&=\frac{1}{2I_3c_2}[c_1(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)+2(I_1-I_2)c_2\omega_1\omega_2\\&\quad+c_1(2c_3R-U_3)+2(c_1^2+c_2^2)\gamma_1],\\\frac{d\gamma_1}{dt}&=\frac{1}{2c_2}[(-I_1\omega_1^2-I_2\omega_2^2-I_3\omega_3^2-2c_1\gamma_1-2c_3R+U_3)\omega_3-2c_2\omega_2R].\end{aligned}\tag{8.127}$$

We consider the following four possible types of first integrals of system (8.127) that depend on at most three variables among ω_1, ω_2, ω_3 and γ_1:

1. $F(\omega_1,\omega_2,\omega_3)$ (case (i)).
2. $F(\omega_1,\omega_2,\gamma_1)$ (case (iii)).
3. $F(\omega_1,\omega_3,\gamma_1)$ (case (iii)).
4. $F(\omega_2,\omega_3,\gamma_1)$ (case (iii)).

As case (iii) was already examined, it remains only to examine case (i).

Type 1. Let us consider a first integral of type 1, i.e. $F(\omega_1,\omega_2,\omega_3)$. We have

$$2I_1I_2I_3c_2\frac{dF}{dt}=Y_1(F)=0,\tag{8.128}$$

where the vector field Y_1, defined on $\mathbb{C}^4(\omega_1,\omega_2,\omega_3,\gamma_1)$, is

$$\begin{aligned}Y_1&=I_2I_3\big[-c_3(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)+2(I_2-I_3)c_2\omega_2\omega_3-2c_1c_3\gamma_1\\&\quad-2(c_2^2+c_3^2)R+c_3U_3\big]\frac{\partial}{\partial\omega_1}+2I_1I_3c_2[(I_3-I_1)\omega_1\omega_3+c_1R-c_3\gamma_1]\frac{\partial}{\partial\omega_2}\\&\quad+I_1I_2\big[c_1(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)+2(I_1-I_2)c_2\omega_1\omega_2\\&\quad+c_1(2c_3R-U_3)+2(c_1^2+c_2^2)\gamma_1\big]\frac{\partial}{\partial\omega_3}.\end{aligned}$$

We differentiate identity (8.128) with respect to γ_1 and obtain again a linear partial differential equation for F:

$$\begin{aligned}\frac{1}{2}\frac{\partial Y_1(F)}{\partial\gamma_1}&=-I_2I_3\left[(c_2^2+c_3^2)\frac{\partial R}{\partial\gamma_1}+c_1c_3\right]\frac{\partial F}{\partial\omega_1}+I_1I_3c_2\left(c_1\frac{\partial R}{\partial\gamma_1}-c_3\right)\frac{\partial F}{\partial\omega_2}\\&\quad+I_1I_2\left(c_1c_3\frac{\partial R}{\partial\gamma_1}+c_1^2+c_2^2\right)\frac{\partial F}{\partial\omega_3}=Y_2(F)=0,\end{aligned}\tag{8.129}$$

where Y_2 is a vector field defined on $\mathbb{C}^4(\omega_1,\omega_2,\omega_3,\gamma_1)$.

The derivative of $Y_2(F)$ with respect to γ_1 has a factor $\frac{\partial^2 R}{\partial \gamma_1^2}$. Crude computations show that for the two roots of equation (8.124), i.e. when $c_2^2 + c_3^2 \neq 0$, and also for the single root of equation (8.125), i.e. when $c_2^2 + c_3^2 = 0$, one has

$$\frac{\partial^2 R}{\partial \gamma_1^2} \neq 0.$$

In this way differentiating (8.129) with respect to γ_1 we obtain

$$\begin{aligned}\left(\frac{\partial^2 R}{\partial \gamma_1^2}\right)^{-1} \frac{\partial Y_2(F)}{\partial \gamma_1} &= -I_2 I_3 (c_2^2 + c_3^2) \frac{\partial F}{\partial \omega_1} + I_1 I_3 c_1 c_2 \frac{\partial F}{\partial \omega_2} + I_1 I_2 c_1 c_3 \frac{\partial F}{\partial \omega_3} \\ &= Y_3(F) = 0, \end{aligned} \tag{8.130}$$

where Y_3 is a vector field defined on $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$.

Instead of Y_1 we consider $Y_4 = Y_1 - 2RY_3$, which implies that $Y_4(F) = 0$. We obtain

$$\begin{aligned} Y_4(F) = {} & I_2 I_3 [-c_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2(I_2 - I_3) c_2 \omega_2 \omega_3 - 2c_1 c_3 \gamma_1 + c_3 U_3] \frac{\partial F}{\partial \omega_1} \\ & + 2 I_1 I_3 c_2 [(I_3 - I_1)\omega_1 \omega_3 - c_3 \gamma_1] \frac{\partial F}{\partial \omega_2} + I_1 I_2 \big[c_1 (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \\ & + 2(I_1 - I_2) c_2 \omega_1 \omega_2 - c_1 U_3 + 2(c_1^2 + c_2^2)\gamma_1\big] \frac{\partial F}{\partial \omega_3} = 0. \end{aligned} \tag{8.131}$$

Note that Y_4 does not depend on R.

Instead of Y_2 we consider $Y_5 = Y_2 - Y_3 \frac{\partial R}{\partial \gamma_1}$, which also does not depend on R. We have

$$Y_5(F) = -I_2 I_3 c_1 c_3 \frac{\partial F}{\partial \omega_1} - I_1 I_3 c_2 c_3 \frac{\partial F}{\partial \omega_2} + I_1 I_2 (c_1^2 + c_2^2) \frac{\partial F}{\partial \omega_3} = 0. \tag{8.132}$$

We compute the Lie bracket $Y_6 = [Y_3, Y_4]/(2I_1 I_2 I_3)$. We know that $Y_6(F) = 0$ so we have

$$\begin{aligned} Y_6(F) = {} & [I_2 I_3 (c_3^2 + c_2^2) c_3 \omega_1 + I_2 (I_2 - 2I_3) c_1 c_2 c_3 \omega_2 + I_3 (I_2 c_2^2 - I_3 c_2^2 - I_2 c_3^2) c_1 \omega_3] \frac{\partial F}{\partial \omega_1} \\ & - (I_1 - I_3) c_2 [I_1 c_1 c_3 \omega_1 - I_3 (c_3^2 + c_2^2) \omega_3] \frac{\partial F}{\partial \omega_2} + \big[I_1 (I_1 c_2^2 - 2I_2 c_2^2 - I_2 c_3^2) c_1 \omega_1 \\ & + I_2 (I_1 c_1^2 + I_2 c_2^2 - I_1 c_2^2 + I_2 c_3^2 - I_1 c_3^2) c_2 \omega_2 + I_2 I_1 c_1^2 c_3 \omega_3\big] \frac{\partial F}{\partial \omega_3} = 0. \end{aligned} \tag{8.133}$$

Thus we have obtained four linear homogeneous equations in the unknowns grad $F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right)$, that is, system (8.130)–(8.133). If a first integral F exists, system (8.130)–(8.133) has a non-zero solution. This is possible if and only if

$$\operatorname{rank} M < 3, \tag{8.134}$$

where M is the 4×3 matrix of the coefficients of system (8.130)–(8.133).

Let us compute the determinant D_{345} that consists of the coefficients of Y_3, Y_4 and Y_5. It should be identically zero because of requirement (8.134).

We compute

$$D_{345} = -2 I_1^2 I_2^2 I_3^2 c_2^2 [c_3 (I_1 - I_2) \omega_1 \omega_2 + c_2 (I_3 - I_1) \omega_1 \omega_3 + c_1 (I_2 - I_3) \omega_2 \omega_3] \delta_{345},$$

where

$$\delta_{345} = c_1^2 + c_2^2 + c_3^2.$$

The expression in square brackets vanishes identically only in the kinetic symmetry case and in the Lagrange case $I_1 = I_3$, $c_1 = c_3 = 0$. We have $-2I_1^2 I_2^2 I_3^2 c_2^2 \neq 0$. Thus $D_{345} = 0$ is equivalent to $\delta_{345} = 0$.

Now we compute the determinant D_{346} that consists of the coefficients of Y_3, Y_4 and Y_6. It should be identically zero too (see (8.134)). We have $D_{346} = I_1 I_2 I_3 c_2^2 \delta_{346}$, where

$$\begin{aligned}
\delta_{346} = & -I_1^3 c_1^2 c_2 c_3 (I_2 - I_3)\omega_1^3 \\
& - I_1 I_2 c_1 c_3 [(I_1 - I_2)(2I_1 - 3I_3)(c_2^2 + c_3^2) - I_1(I_2 - I_3)c_1^2]\omega_1^2\omega_2 \\
& + I_1 I_3 c_1 c_2 [(I_1 - I_3)(2I_1 - 3I_2)(c_2^2 + c_3^2) + I_1(I_2 - I_3)c_1^2]\omega_1^2\omega_3 \\
& + I_1 I_2 c_1^2 c_2 c_3 (I_1 I_2 - 3I_1 I_3 - 2I_2^2 + 4I_2 I_3)\omega_1\omega_2^2 \\
& + 2I_1 I_2 I_3 c_1^2 [(-I_1 + 2I_2 - I_3)c_3^2 + (I_1 + I_2 - 2I_3)c_2^2]\omega_1\omega_2\omega_3 \\
& + I_1 I_3 c_1^2 c_2 c_3 (3I_1 I_2 - I_1 I_3 - 4I_2 I_3 + 2I_3^2)\omega_1\omega_3^2 \\
& - 2I_1^2 c_1 c_2 c_3 (c_1^2 + c_2^2 + c_3^2)(I_2 - I_3)\omega_1\gamma_1 + I_1^2 (I_2 - I_3) c_1^2 c_2 c_3 U_3 \omega_1 \\
& - I_2^2 c_1 c_3 [I_3(I_1 - I_2)(c_2^2 + c_3^2) - I_1(I_2 - I_3)c_1^2]\omega_2^3 \\
& - I_2 I_3 c_1 c_2 [(-3I_1 I_2 + 2I_1 I_3 + 2I_2^2 - I_2 I_3)(c_2^2 + c_3^2) + I_1(I_2 - I_3)c_1^2]\omega_3\omega_2^2 \\
& - I_2 I_3 c_1 c_3 [(3I_1 I_3 - 2I_1 I_2 + I_2 I_3 - 2I_3^2)(c_2^2 + c_3^2) + I_1(I_2 - I_3)c_1^2]\omega_3^2\omega_2 \\
& - 2I_2 c_3 (c_3^2 + c_1^2 + c_2^2)[I_3(I_1 - I_2)(c_2^2 + c_3^2) - I_1(I_2 - I_3)c_1^2]\omega_2\gamma_1 \\
& + I_2 c_1 c_3 U_3 [I_3(I_1 - I_2)(c_2^2 + c_3^2) - I_1(I_2 - I_3)c_1^2]\omega_2 \\
& + I_3^2 c_1 c_2 [I_2(I_1 - I_3)(c_2^2 + c_3^2) + I_1(I_2 - I_3)c_1^2]\omega_3^3 \\
& + 2I_3 c_2 (c_1^2 + c_2^2 + c_3^2)[I_2(I_1 - I_3)(c_2^2 + c_3^2) + I_1(I_2 - I_3)c_1^2]\omega_3\gamma_1 \\
& - I_3 c_1 c_2 U_3 [I_2(I_1 - I_3)(c_2^2 + c_3^2) + I_1(I_2 - I_3)c_1^2]\omega_3.
\end{aligned}$$

As $I_1 I_2 I_3 c_2^2 \neq 0$ we see that $D_{346} = 0$ is equivalent to $\delta_{346} = 0$.

Thus we should find conditions under which the polynomials δ_{345} and δ_{346} vanish identically with respect to the variables $(\omega_1, \omega_2, \omega_3, \gamma_1)$. This means finding the values of the parameters $\mathcal{I}c$ and U_3 for which all the coefficients of δ_{345} and δ_{346} are zero.

The polynomial δ_{345} has only one coefficient and δ_{346} has 16 coefficients. In this way we obtain a system of 17 equations. To solve it we apply simplification. After four consecutive simplifications we obtain the reduced system

$$\begin{aligned}
& c_3^2 + c_1^2 + c_2^2 = 0, \quad (I_2 - I_3)c_1 = 0, \quad (I_1 - I_3)c_1 = 0, \\
& (I_2 - I_3)(c_2^2 + c_3^2) = 0, \quad (I_1 - I_3)(c_2^2 + c_3^2) = 0.
\end{aligned}$$

The MAPLE command `solve` gives two solutions for an arbitrary value of U_3:

$$\begin{aligned}
& \{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = \mathrm{RootOf}(Z^2 + 1)c_2\}, \\
& \{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = \mathrm{RootOf}(Z^2 + c_2^2 + c_3^2),\ c_2 = c_2,\ c_3 = c_3\}.
\end{aligned}$$

The second solution is a particular case of the kinetic symmetry case so we discard it. We have to consider the first solution. Thus

$$c_1 = 0, \quad c_3 = \varepsilon i c_2,$$

where $\varepsilon = \pm 1$. We consider here only the case $\varepsilon = 1$ because the final result is the same when $\varepsilon = -1$.

Now $Y_3(F)$ and $Y_6(F)$ are identically zero and therefore condition (8.134) is fulfilled. We have

$$\begin{aligned} Y_4(F) = {}& -iI_2I_3c_2[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2i(I_2 - I_3)\omega_2\omega_3 - U_3]\frac{\partial F}{\partial \omega_1} \\ & - 2I_1I_3c_2[(I_1 - I_3)\omega_1\omega_3 + ic_2\gamma_1]\frac{\partial F}{\partial \omega_2} + 2I_1I_2c_2[(I_1 - I_2)\omega_1\omega_2 + c_2\gamma_1]\frac{\partial F}{\partial \omega_3}, \\ Y_5(F) = {}& -I_1c_2^2\left(iI_3\frac{\partial F}{\partial \omega_2} - I_2\frac{\partial F}{\partial \omega_3}\right). \end{aligned}$$

We compute

$$\begin{aligned} Y_7(F) = \frac{[Y_4(F), Y_5(F)]}{2I_1I_2I_3c_2^3} = {}& [I_2(I_2 - 2I_3)\omega_2 - iI_3(I_3 - 2I_2)\omega_3]\frac{\partial F}{\partial \omega_1} \\ & - I_1(I_1 - I_3)\omega_1\frac{\partial F}{\partial \omega_2} + iI_1(I_1 - I_2)\omega_1\frac{\partial F}{\partial \omega_3}. \end{aligned}$$

We compute the determinant Δ of the coefficients of the equations $Y_4(F)$, $Y_5(F)$ and $Y_7(F)$ and obtain $\Delta = I_1^2I_2I_3c_2^3\omega_1\widetilde{\Delta}$ where

$$\begin{aligned} \widetilde{\Delta} = {}& iI_1^2(I_2 - I_3)\omega_1^2 + iI_2(2I_2^2 - 4I_2I_3 - I_1I_2 + 3I_3I_1)\omega_2^2 \\ & - 2I_2I_3(2I_1 - I_2 - I_3)\omega_2\omega_3 - iI_3(2I_3^2 - 4I_2I_3 - I_3I_1 + 3I_1I_2)\omega_3^2 - iI_1(I_2 - I_3)U_3. \end{aligned}$$

As $I_1^2I_2I_3c_2^3\omega_1 \neq 0$, we require that $\widetilde{\Delta} = 0$. Looking at the coefficient of ω_1^2 in the expression for $\widetilde{\Delta}$ we see that $I_2 = I_3$ should be fulfilled. Under this condition we obtain

$$\widetilde{\Delta} = 2I_3^2(I_1 - I_3)(\omega_2 + i\omega_3)^2.$$

Thus $\widetilde{\Delta} = 0$ only if $I_1 = I_2 = I_3$, i.e. we come to the kinetic symmetry case. Consequently, a partial integral of type 1, $F(\omega_1, \omega_2, \omega_3)$, does not exist when $c_2 \neq 0$.

Let us consider the case $c_2 = 0$. In this case, according to condition (8.122), $c_3 \neq 0$. First we express γ_3 from the equation $H_3 = U_3$ and put the value of γ_3 in the equation $H_2 = U_2$, from which we find γ_2. In this way we have

$$\gamma_2 = \frac{\widehat{R}}{2c_3}, \qquad \gamma_3 = -\frac{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3}{2c_3}, \tag{8.135}$$

where $\widehat{R}$ is a root of $Q(x) = x^2 + \widehat{C} = 0$, that is,

$$Q(\widehat{R}) = \widehat{R}^2 + \widehat{C} = 0. \tag{8.136}$$

The function $\widehat{C} = \widehat{C}(\omega_2, \omega_3, \gamma_1, \gamma_3)$ is the following polynomial:

$$\begin{aligned} \widehat{C} = {}& (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)^2 + 4c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)\gamma_1 \\ & - 2U_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 4(c_1^2 + c_3^2)\gamma_1^2 - 4c_1U_3\gamma_1 - 4c_3^2U_2 + U_3^2. \end{aligned}$$

We put the values of γ_2 and γ_3 from (8.135) in the Euler–Poisson equations (1.1) and remove the fifth and sixth equations. In this way we obtain the following system of four equations in the unknowns ω_1, ω_2, ω_3 and γ_1:

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{2(I_2-I_3)\omega_2\omega_3+\widehat{R}}{2I_1},\\
\frac{d\omega_2}{dt} &= -\frac{1}{2I_2c_3}[c_1(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)+2(I_1-I_3)c_3\omega_1\omega_3\\
&\quad +2(c_1^2+c_3^2)\gamma_1-c_1U_3],\\
\frac{d\omega_3}{dt} &= \frac{2(I_1-I_2)c_3\omega_1\omega_2-c_1\widehat{R}}{2I_3c_3},\\
\frac{d\gamma_1}{dt} &= \frac{1}{2c_3}[(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2+2c_1\gamma_1-U_3)\omega_2+\omega_3\widehat{R}].
\end{aligned} \tag{8.137}$$

Let $c_2=0$ and suppose $F(\omega_1,\omega_2,\omega_3)$ is a first integral of type 1. We have

$$2I_1I_2I_3c_3\frac{dF}{dt}=Y_1(F)=0, \tag{8.138}$$

where the vector field Y_1, defined on $\mathbb{C}^4(\omega_1,\omega_2,\omega_3,\gamma_1)$, is

$$\begin{aligned}
Y_1 &= I_2I_3c_3[2(I_2-I_3)\omega_2\omega_3+\widehat{R}]\frac{\partial}{\partial\omega_1}\\
&\quad +I_1I_3[-c_1(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)+2c_3(I_3-I_1)\omega_1\omega_3-2(c_1^2+c_3^2)\gamma_1+c_1U_3]\frac{\partial}{\partial\omega_2}\\
&\quad +I_1I_2[2(I_1-I_2)c_3\omega_1\omega_2-c_1\widehat{R}]\frac{\partial}{\partial\omega_3}.
\end{aligned}$$

We differentiate identity (8.138) with respect to γ_1 and obtain again a linear partial differential equation for F:

$$\frac{\partial Y_1(F)}{\partial\gamma_1}=I_2I_3c_3\frac{\partial\widehat{R}}{\partial\gamma_1}\frac{\partial F}{\partial\omega_1}-2I_1I_3(c_1^2+c_3^2)\frac{\partial F}{\partial\omega_2}-I_1I_2c_1\frac{\partial\widehat{R}}{\partial\gamma_1}\frac{\partial F}{\partial\omega_3}=Y_2(F)=0,$$

where Y_2 is a vector field defined on $\mathbb{C}^4(\omega_1,\omega_2,\omega_3,\gamma_1)$.

The derivative of $Y_2(F)$ with respect to γ_1 has a factor $I_2\frac{\partial^2\widehat{R}}{\partial\gamma_1^2}$. We have verified that for the two roots of equation (8.136) this derivative is not zero. Thus differentiating (8.129) with respect to γ_1 we obtain

$$\frac{1}{I_2}\left(\frac{\partial^2\widehat{R}}{\partial\gamma_1^2}\right)^{-1}\frac{\partial Y_2(F)}{\partial\gamma_1}=I_3c_3\frac{\partial F}{\partial\omega_1}-I_1c_1\frac{\partial F}{\partial\omega_3}=Y_3(F)=0, \tag{8.139}$$

where Y_3 is a vector field defined on $\mathbb{C}^4(\omega_1,\omega_2,\omega_3,\gamma_1)$.

Instead of Y_1 we consider $Y_4=Y_1-I_2RY_3$, for which $Y_4(F)=0$. We obtain

$$\begin{aligned}
Y_4(F) &= 2I_2I_3(I_2-I_3)c_3\omega_2\omega_3\frac{\partial F}{\partial\omega_1}-I_1I_3\big[c_1(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)\\
&\quad +2(I_1-I_3)c_3\omega_1\omega_3+2(c_1^2+c_3^2)\gamma_1-c_1U_3\big]\frac{\partial F}{\partial\omega_2}\\
&\quad +2I_1I_2(I_1-I_2)c_3\omega_1\omega_2\frac{\partial F}{\partial\omega_3}=0.
\end{aligned} \tag{8.140}$$

Note that Y_4 does not depend on $\widehat{R}$.

Instead of Y_2 we consider

$$Y_5 = \frac{I_2 Y_3 \frac{\partial \widehat{R}}{\partial \gamma_1} - Y_2}{2I_1 I_3},$$

which does not depend on $\widehat{R}$ either. We have

$$Y_5(F) = (c_1^2 + c_3^2)\frac{\partial F}{\partial \omega_2} = 0. \tag{8.141}$$

We compute the Lie bracket $Y_6 = \frac{[Y_3, Y_4]}{2I_1 I_3}$. We know that $Y_6(F) = 0$ so we have

$$\begin{aligned} Y_6(F) = {} & -I_2(I_2 - I_3)c_1 c_3 \omega_2 \frac{\partial F}{\partial \omega_1} \\ & + \big[I_1(I_1 - 2I_3)c_1 c_3 \omega_1 + I_3(I_1 c_1^2 - I_1 c_3^2 + I_3 c_3^2)\omega_3\big]\frac{\partial F}{\partial \omega_2} \\ & + I_2(I_1 - I_2)c_3^2 \omega_2 \frac{\partial F}{\partial \omega_3} = 0. \end{aligned} \tag{8.142}$$

Thus we have obtained four linear homogeneous equations in the unknowns $\operatorname{grad} F = \big(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\big)$, that is, system (8.139)–(8.142). If a first integral F exists, system (8.139)–(8.142) has a non-zero solution. This is possible if and only if

$$\operatorname{rank} M < 3, \tag{8.143}$$

where M is the 4×3 matrix of the coefficients of system (8.139)–(8.142).

Let us compute the determinant D_{345} that consists of the coefficients of Y_3, Y_4 and Y_5. It should be identically zero because of requirement (8.143).

We compute

$$D_{345} = -2I_1 I_2 I_3 c_3 \omega_2 [c_3(I_1 - I_2)\omega_1 + c_1(I_2 - I_3)\omega_3]\delta_{345},$$

where

$$\delta_{345} = c_1^2 + c_3^2.$$

As now $c_2 = 0$, according to (8.122) we have $c_3 \neq 0$. Thus the expression in the square brackets vanishes identically only in the kinetic symmetry case and in the Lagrange case $I_1 = I_2$, $c_1 = c_2 = 0$. We have $-2I_1 I_2 I_3 c_3 \omega_2 \neq 0$. Thus $D_{345} = 0$ is equivalent to $\delta_{345} = 0$.

We also compute the determinant D_{346} of the coefficients of Y_3, Y_4 and Y_6. It should be identically zero too (see (8.134)). We have $D_{346} = I_1 I_2 I_3 c_3 \omega_2 \delta_{346}$, where

$$\begin{aligned} \delta_{346} = {} & I_1[I_1(I_2 - I_3)c_1^2 - (2I_1 - 3I_3)(I_1 - I_2)c_3^2]c_1\omega_1^2 - 2I_1 I_3(I_1 + I_3 - 2I_2)c_1^2 c_3 \omega_1 \omega_3 \\ & + I_2[I_3(I_2 - I_1)c_3^2 + I_1(I_2 - I_3)c_1^2]c_1\omega_2^2 - I_3\big[I_1(I_2 - I_3)c_1^2 + I_3(3I_1 - 2I_3)c_3^2 \\ & + I_2(I_3 - 2I_1)c_3^2\big]c_1\omega_3^2 + 2(c_1^2 + c_3^2)[I_1(I_2 - I_3)c_1^2 + I_3(I_2 - I_1)c_3^2]\gamma_1 \\ & - c_1[I_1(I_2 - I_3)c_1^2 + I_3(I_2 - I_1)c_3^2]U_3. \end{aligned}$$

As $I_1 I_2 I_3 c_3 \omega_2 \neq 0$, we see that $D_{346} = 0$ is equivalent to $\delta_{346} = 0$.

Thus we should find conditions under which the polynomials δ_{345} and δ_{346} vanish identically with respect to $(\omega_1, \omega_2, \omega_3, \gamma_1)$. This means finding the values of the parameters $\mathcal{I}c$ and U_3 for which all the coefficients of δ_{345} and δ_{346} are zero.

The polynomial δ_{345} has only one coefficient and δ_{346} has six coefficients. In this way we obtain a system of seven equations. To solve it we apply simplification. After three consecutive simplifications we obtain the reduced system

$$I_2 - I_3 = 0, \quad I_1 - I_3 = 0, \quad c_1^2 + c_3^2 = 0.$$

This system obviously leads to the kinetic symmetry case. Thus a partial integral of type 1, $F(\omega_1, \omega_2, \omega_3)$, does not exist when $c_2 = 0$ either.

8.4.4. First integrals $F(\gamma_1, \gamma_2, \gamma_3)$. Finally, it remains to study the existence of a partial integral $F(\gamma_1, \gamma_2, \gamma_3)$, which cannot be studied by elimination of variables as above.

We proceed here in the same way as in Sec. 8.2.4.

We have $F(\gamma_1, \gamma_2, \gamma_3) = \widetilde{F}(\gamma_2, \gamma_3)$. Our problem is now reduced to the study of partial integrals of the form $\widetilde{F} = \widetilde{F}(\gamma_2, \gamma_3)$ on the submanifold $\{H_3 = U_3\}$. Absence of such partial integrals follows from Sec. 8.4.1 where the absence of partial integrals of more general form $F(\omega_i, \gamma_2, \gamma_3)$, $i = 2, 3$, is proved for all U_2 and U_3.

This concludes the description of the four-dimensional invariant manifolds.

9. Three-dimensional invariant manifold $\{H_1 = U_1,\ H_2 = U_2,\ H_3 = U_3\}$

9.1. Extraction procedure. In this section we study the existence of a partial integral of the Euler–Poisson equations (1.1) with respect to the invariant complex three-dimensional level manifold

$$\{H_1 = U_1,\ H_2 = U_2,\ H_3 = U_3\},$$

which depends on at most two variables.

According to (2.5),

$$\begin{aligned} &M(U_0, U_1, U_2, U_3, \mathcal{I}c) \\ &\qquad = \{x \in \mathbb{C}^6;\ H_1((\omega,\gamma), \mathcal{I}c) = U_1,\ H_2((\omega,\gamma), \mathcal{I}c) = U_2,\ H_3((\omega,\gamma), \mathcal{I}c) = U_3\}, \end{aligned}$$

where $(\omega, \gamma) = (\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3)$.

We search for all functions F of two variables $F = F(s_1, s_2)$ where $(s_1, s_2) \in (\omega, \gamma)$, of class C^1, such that $\operatorname{grad} F$ does not vanish identically on any open subset of the manifold $M(U_0, U_1, U_2, U_3, \mathcal{I}c)$, which are partial integrals of the Euler–Poisson equations (1.1) with respect to this manifold.

As in Sec. 5.1 the order of the variables s_i, $1 \le i \le 2$, in $F(s_1, s_2)$ is irrelevant for F to be a first integral.

We have exactly 15 different two-element subsets of (ω, γ) and thus 15 cases of functions of two elements to examine. We will now describe an extraction procedure based on permutational symmetries which reduces the above 15 cases to only four.

These 15 functions of two variables (up to the order of variables) are shown in Table 9.1. This table can be easily obtained directly like Table 5.1. But it can also be easily deduced from Table 5.1 and vice versa.

As in Sec. 8, let us stress that the permutational symmetries act on the variables (ω, γ) and the parameters $\mathcal{I}c$ but not on the constants U_1, U_2, U_3 that define the manifold $M(U_0, U_1, U_2, U_3, \mathcal{I}c)$.

Table 9.1

Functions	Case
$F(\gamma_i, \gamma_j)$, $1 \le i < j \le 3$	(i)
$F(\omega_1, \gamma_1)$, $F(\omega_2, \gamma_2)$, $F(\omega_3, \gamma_3)$	(ii)
$F(\omega_3, \gamma_2)$, $F(\omega_2, \gamma_3)$, $F(\omega_1, \gamma_3)$, $F(\omega_3, \gamma_1)$, $F(\omega_2, \gamma_1)$, $F(\omega_1, \gamma_2)$	(iii)
$F(\omega_i, \omega_j)$, $1 \le i < j \le 3$	(iv)

It is easy to see that under the group of permutational symmetries (2.3) of the Euler–Poisson equations for every case (i)–(iv) from Table 9.1 the first function from the case can be transformed into all remaining functions from the same case.

Thus in virtue of Theorem 2.2 we can restrict ourselves to the study of only four functions, each belonging to a different case from Table 9.1 and chosen arbitrarily from the functions of this case.

As in Secs. 5 and 8, we will call such four functions F_i, $1 \le i \le 4$, (up to the order of variables) a *basis*.

9.2. Elimination of ω_1, ω_2, γ_1. Here we study the existence of a partial integral of the Euler–Poisson equations (1.1) after expressing the variables ω_1, ω_2 and γ_1 from the equations

$$H_i = U_i, \quad 1 \le i \le 3. \tag{9.1}$$

First we express γ_1 from (9.1) for $i = 2$ and obtain

$$\gamma_1 = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}. \tag{9.2}$$

Further, to simplify the notations, we put

$$\Gamma = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}.$$

Then, using `solve`, we express ω_1 and ω_2 from (9.1) for $i = 1, 3$ and obtain the following solution:

$$\omega_1 = R, \quad \omega_2 = -\frac{I_1 R \Gamma + I_3 \omega_3 \gamma_3 - U_1}{I_2 \gamma_2}, \tag{9.3}$$

where $R = R(\omega_3, \gamma_2, \gamma_3)$ is a root of $Q(x) = Ax^2 + Bx + C = 0$, that is,

$$Q(R) = AR^2 + BR + C = 0. \tag{9.4}$$

Here $A = A(\gamma_2, \gamma_3)$, $B = B(\omega_3, \gamma_2, \gamma_3)$ and $C = C(\omega_3, \gamma_2, \gamma_3)$ are the following functions:

$$\begin{aligned} A &= I_1[(I_2 - I_1)\gamma_2^2 - I_1\gamma_3^2 + I_1 U_2], \\ B &= 2I_1\Gamma(I_3\omega_3\gamma_3 - U_1), \\ C &= I_3\omega_3^2(I_2\gamma_2^2 + I_3\gamma_3^2) - 2I_3\omega_3\gamma_3 U_1 \\ &\quad + I_2\gamma_2^2(2c_2\gamma_2 + 2c_3\gamma_3 + 2c_1\Gamma - U_3) + U_1^2. \end{aligned} \tag{9.5}$$

R, A, B and C are algebraic functions defined on $\mathbb{C}^3(\omega_3, \gamma_2, \gamma_3)$.

We put the values of γ_1, ω_1 and ω_2 from (9.2) and (9.3) in the Euler–Poisson equations (1.1) and remove the first, second and fourth equations. In this way we obtain the following system of three equations in the unknowns ω_3, γ_2 and γ_3:

$$\begin{aligned}
\frac{d\omega_3}{dt} &= \frac{1}{I_2 I_3 \gamma_2}[I_1(I_2-I_1)\Gamma R^2 + (I_2-I_1)(I_3\omega_3\gamma_3 - U_1)R + I_2\gamma_2(c_2\Gamma - c_1\gamma_2)],\\
\frac{d\gamma_2}{dt} &= \gamma_3 R - \omega_3\Gamma, \\
\frac{d\gamma_3}{dt} &= \frac{1}{I_2\gamma_2}[(I_1\gamma_2^2 - I_2\gamma_2^2 + I_1\gamma_3^2 - I_1U_2)R - \Gamma(I_3\omega_3\gamma_3 - U_1)].
\end{aligned} \tag{9.6}$$

Now we study whether system (9.6) has a first integral that depends on at most two variables among $(\omega_3, \gamma_2, \gamma_3)$. Thus we should investigate the following three types of first integrals:

1. $F(\gamma_2, \gamma_3)$ (case (i)).
2. $F(\omega_3, \gamma_3)$ (case (ii)).
3. $F(\omega_3, \gamma_2)$ (case (iii)).

Then, as in Secs. 5 and 8 we should examine the three types given above because they belong to different cases (see Table 9.1).

Let us fix $U_2 \in \mathbb{C}$. Let us consider a suitable open set $\Omega \subset \mathbb{C}^3(\omega_3, \gamma_2, \gamma_3)$ contained in the domain of definition of F.

From now on we consider system (9.6) and the first integral F only on Ω. System (9.6) restricted to Ω has C^1 right-hand sides.

We always suppose that the first integrals considered are not constant on any open subset of their domain of definition. As we consider C^1 first integrals, this means that their gradients do not vanish identically on any open subset of their domain of definition.

Type 1. Let us consider the existence of a first integral F of system (9.6) which is of type 1, i.e. $F = F(\gamma_2, \gamma_3)$. Thus we have

$$\frac{dF}{dt} = \frac{d\gamma_2}{dt}\frac{\partial F}{\partial \gamma_2} + \frac{d\gamma_3}{dt}\frac{\partial F}{\partial \gamma_3} = Y_1(F) = 0, \tag{9.7}$$

where Y_1 is a vector field defined on Ω.

Equation (9.7) should be an identity with respect to all the three variables $(\omega_3, \gamma_2, \gamma_3)$. As F does not depend on ω_3, its partial derivatives will not depend on ω_3 either. Thus if we differentiate identity (9.7) with respect to ω_3 we shall obtain again a linear partial differential equation for F:

$$\frac{\partial Y_1(F)}{\partial \omega_3} = \frac{\partial}{\partial \omega_3}\left(\frac{d\gamma_2}{dt}\right)\frac{\partial F}{\partial \gamma_2} + \frac{\partial}{\partial \omega_3}\left(\frac{d\gamma_3}{dt}\right)\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \tag{9.8}$$

where Y_2 is a vector field defined on Ω.

Equations (9.7) and (9.8) can be considered as a system of two homogeneous linear algebraic equations with the unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \gamma_2}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if a first integral F exists, system (9.7)–(9.8) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant Δ of this linear system satisfies $\Delta \equiv 0$ provided that R is a root of (9.4).

We compute this determinant and obtain

$$\begin{aligned}\Delta(R) &= \frac{d\gamma_2}{dt}\frac{\partial}{\partial\omega_3}\left(\frac{d\gamma_3}{dt}\right) - \frac{d\gamma_3}{dt}\frac{\partial}{\partial\omega_3}\left(\frac{d\gamma_2}{dt}\right)\\ &= \frac{\Gamma}{I_2\gamma_2}\left\{[(I_1-I_2)\gamma_2^2+(I_1-I_3)\gamma_3^2-I_1U_2]\left(R-\omega_3\frac{\partial R}{\partial\omega_3}\right)+U_1\gamma_3\frac{\partial R}{\partial\omega_3}-U_1\Gamma\right\}.\end{aligned}$$

As we are interested in the case $\Delta = 0$ we remove the denominator $I_2\gamma_2$ and the non-zero factor Γ and denote

$$\Delta(R) = \frac{\Gamma}{I_2\gamma_2}\delta(R),$$

where

$$\delta(R) = [(I_1-I_2)\gamma_2^2+(I_1-I_3)\gamma_3^2-I_1U_2]\left(R-\omega_3\frac{\partial R}{\partial\omega_3}\right)-U_1\gamma_3\frac{\partial R}{\partial\omega_3}+U_1\Gamma. \tag{9.9}$$

As $\delta(R)$ contains $\frac{\partial R}{\partial\omega_3}$ we should determine this derivative. We use equation (9.4). We differentiate it with respect to ω_3 and, as A does not depend on ω_3 (see (9.5)), we obtain

$$\frac{\partial Q}{\partial\omega_3} = (2AR+B)\frac{\partial R}{\partial\omega_3}+\frac{\partial B}{\partial\omega_3}R+\frac{\partial C}{\partial\omega_3} = 0. \tag{9.10}$$

Determining $\frac{\partial R}{\partial\omega_3}$ from the last equation is possible only if $\frac{dQ(R)}{dR} = 2AR+B \neq 0$. Using Proposition 4.1 we prove that if R is a root of (9.4) then $2AR+B=0$ only in a very particular case

$$I_1 = I_2 = I_3, \quad c_1 = c_2 = c_3 = 0, \quad U_1 = U_2 = U_3 = 0. \tag{9.11}$$

Indeed, let us compute the resultant ρ of $Q(R)$ and $2AR+B$ with respect to R. We obtain

$$\rho = A(4AC-B^2).$$

As we are interested only in the cases when ρ vanishes identically with respect to ω_3, γ_2 and γ_3, and as A never vanishes identically, we consider $\widehat{\rho} = 4AC-B^2$ instead of ρ. We compute $\widehat{\rho}$ with values of A, B and C from (9.5) and obtain

$$\widehat{\rho} = 4I_1I_2\gamma_2^2(a_0\Gamma+a_1),$$

where

$$\begin{aligned}a_0 &= 2c_1(I_2\gamma_2^2-I_1\gamma_2^2-I_1\gamma_3^2+I_1U_2),\\ a_1 &= -I_3(I_1-I_2)\omega_3^2\gamma_2^2-I_3(I_1-I_3)\omega_3^2\gamma_3^2+I_1I_3U_2\omega_3^2-2I_3U_1\omega_3\gamma_3\\ &\quad -2(I_1-I_2)c_2\gamma_2^3-2(I_1-I_2)c_3\gamma_2^2\gamma_3+(I_1-I_2)U_3\gamma_2^2-2I_1c_2\gamma_2\gamma_3^2\\ &\quad +2I_1c_2U_2\gamma_2-2I_1c_3\gamma_3^3+I_1U_3\gamma_3^2+2I_1c_3U_2\gamma_3+U_1^2-I_1U_2U_3.\end{aligned}$$

According to Proposition 4.3, if $\widehat{\rho}=0$ then $a_0 = a_1 = 0$ because $\Gamma \notin \mathbb{C}(\gamma_2,\gamma_3)$. $a_0 = 0$ is possible if and only if $c_1 = 0$. One immediately sees that $a_1 = 0$ will be true if and only if $I_1 = I_2 = I_3$, $c_2 = c_3 = 0$ and $U_1 = U_2 = U_3 = 0$, i.e. we come to condition (9.11). Thus apart from this case the equations $Q(R) = 0$ and $2AR+B = 0$ have no common roots, i.e. if $Q(R) = 0$ then $2AR+B \neq 0$.

Thus determining $\frac{\partial R}{\partial \omega_3}$ from (9.10) is possible and we obtain

$$\frac{\partial R}{\partial \omega_3} = -\frac{\frac{\partial B}{\partial \omega_3} R + \frac{\partial C}{\partial \omega_3}}{2AR + B}$$

and put it in the expression (9.9) for $\delta(R)$. The non-zero expression $2AR + B$ appears as a denominator of $\delta(R)$ and we denote

$$\delta(R) = \frac{\widetilde{\delta}(R)}{2AR + B},$$

where

$$\begin{aligned}\widetilde{\delta}(R) = {} & [(I_1 - I_2)\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1 U_2]\left[(2AR + B)R + \omega_3\left(R\frac{\partial B}{\partial \omega_3} + \frac{\partial C}{\partial \omega_3}\right)\right] \\ & + U_1\gamma_3\left(R\frac{\partial B}{\partial \omega_3} + \frac{\partial C}{\partial \omega_3}\right) + U_1(2AR + B)\Gamma.\end{aligned}$$

After replacing A, B and C with their values from (9.5) we obtain

$$\begin{aligned}\widetilde{\delta}(R) = 2\{ & [(I_1 - I_2)\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1 U_2][I_1(I_1 U_2 + I_2\gamma_2^2 - I_1\gamma_2^2 - I_1\gamma_3^2)R^2 \\ & + 2I_1(I_3\omega_3\gamma_3 - U_1)R\Gamma] + I_2 I_3(I_1 - I_2)\omega_3^2\gamma_2^4 + I_3(I_1 I_2 + I_1 I_3 - 2I_2 I_3)\omega_3^2\gamma_2^2\gamma_3^2 \\ & - I_1 I_2 I_3 U_2\omega_3^2\gamma_2^2 + I_3^2(I_1 - I_3)\omega_3^2\gamma_3^4 - I_1 I_3^2 U_2\omega_3^2\gamma_3^2 - 2I_3(I_1 - I_2)U_1\omega_3\gamma_2^2\gamma_3 \\ & - 2I_3(I_1 - I_3)U_1\omega_3\gamma_3^3 + 2I_1 I_3 U_1 U_2\omega_3\gamma_3 + I_1 U_1^2\gamma_2^2 + (I_1 - I_3)U_1^2\gamma_3^2 - I_1 U_1^2 U_2\}.\end{aligned}$$

Let us record the following observation. The expression in the last square brackets above is

$$I_1(I_1 U_2 + I_2\gamma_2^2 - I_1\gamma_2^2 - I_1\gamma_3^2)R^2 + 2I_1(I_3\omega_3\gamma_3 - U_1)R\Gamma = AR^2 + BR = Q(R) - C$$

and, as $Q(R) = 0$ (cf. (9.4)), we can replace this expression with $-C$. In this way $\widetilde{\delta}$ is a function that does not depend on R:

$$\widetilde{\delta} = b_0\Gamma + b_1,$$

where

$$\begin{aligned}b_0 &= -4I_2 c_1\gamma_2^2[(I_1 - I_2)\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1 U_2], \\ b_1 &= 2I_2\gamma_2^2[2c_2(I_2 - I_1)\gamma_2^3 + 2c_3(I_2 - I_1)\gamma_2^2\gamma_3 + 2c_2(I_3 - I_1)\gamma_2\gamma_3^2 + 2c_3(I_3 - I_1)\gamma_3^3 \\ &\quad + (I_1 - I_2)U_3\gamma_2^2 + (I_1 - I_3)U_3\gamma_3^2 + 2I_1 c_2 U_2\gamma_2 + 2I_1 c_3 U_2\gamma_3 + U_1^2 - I_1 U_2 U_3].\end{aligned}$$

According to Proposition 4.3, if $\widetilde{\delta} = 0$ then $b_0 = b_1 = 0$ because $\Gamma \notin \mathbb{C}(\gamma_2, \gamma_3)$. $b_0 = 0$ is possible either if $I_1 = I_2 = I_3$ and $U_2 = 0$, which is a particular case of the kinetic symmetry, or when $c_1 = 0$.

Let $c_1 = 0$. We consider $b_1 = 0$. As $c_1 = 0$ we should have $(c_2, c_3) \neq (0, 0)$ to avoid the Euler case. First let us suppose that $c_2 \neq 0$. Then the vanishing of the coefficients of γ_2^3 and $\gamma_2\gamma_3^2$ of b_1 leads to the kinetic symmetry case. Let us suppose now that $c_3 \neq 0$. Then the coefficients of γ_3^3 and $\gamma_2^2\gamma_3$ lead to the same case. Consequently, a partial integral of type 1, $F(\gamma_2, \gamma_3)$, does not exist.

Type 2. Let us study the existence of a first integral of type 2. That means looking for a first integral of system (9.6) which does not depend on γ_2, i.e. $F(\omega_3, \gamma_3)$.

In fact, the investigations go along the same lines but, of course, the expressions are different. Now we have

$$\frac{dF}{dt} = \frac{d\omega_3}{dt}\frac{\partial F}{\partial \omega_3} + \frac{d\gamma_3}{dt}\frac{\partial F}{\partial \gamma_3} = Y_1(F) = 0, \tag{9.12}$$

where Y_1 is a vector field defined on Ω.

Equation (9.12) should be an identity with respect to all three variables $(\omega_3, \gamma_2, \gamma_3)$. As F does not depend on γ_2, its partial derivatives will not depend on γ_2 either. Thus if we differentiate identity (9.12) with respect to γ_2 we shall obtain again a linear partial differential equation for F:

$$\frac{\partial Y_1(F)}{\partial \gamma_2} = \frac{\partial}{\partial \gamma_2}\left(\frac{d\omega_3}{dt}\right)\frac{\partial F}{\partial \omega_3} + \frac{\partial}{\partial \gamma_2}\left(\frac{d\gamma_3}{dt}\right)\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \tag{9.13}$$

where Y_2 is a vector field defined on Ω.

Equations (9.12) and (9.13) can be considered as a system of two homogeneous linear algebraic equations with the unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if an integral F exists, system (9.12)–(9.13) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant $\Delta(R)$ composed of the coefficients of this system satisfies $\Delta(R) \equiv 0$ provided that R is a root of (9.4).

We compute this determinant and obtain a long expression which we do not give here. We only mention that $\Delta(R)$ has a non-zero denominator $I_2^2 I_3 \gamma_2^2 \Gamma$ and we denote

$$\Delta(R) = \frac{\widehat{\Delta}(R)}{I_2^2 I_3 \gamma_2^2 \Gamma}.$$

Thus $\Delta(R) = 0$ is equivalent to $\widehat{\Delta}(R) = 0$.

$\widehat{\Delta}(R)$ depends on $\frac{\partial R}{\partial \gamma_2}$. To determine this derivative we use the same steps as in the case of the first integral of type 1 and obtain

$$\frac{\partial R}{\partial \gamma_2} = -\frac{\frac{\partial A}{\partial \gamma_2}R^2 + \frac{\partial B}{\partial \gamma_2}R + \frac{\partial C}{\partial \gamma_2}}{2AR + B}.$$

We put it in the expression for $\widehat{\Delta}(R)$. After this substitution the non-zero expression $2AR + B$ appears as a denominator of $\widehat{\Delta}(R)$ and we denote

$$\widehat{\Delta}(R) = \frac{\widetilde{\Delta}(R)}{2AR + B}.$$

The identity $\widehat{\Delta}(R) = 0$ is equivalent to $\widetilde{\Delta}(R) = 0$ but $\widetilde{\Delta}(R)$ depends on the functions A, B and C from (9.5) and their derivatives with respect to γ_2. We put these functions in the expression for $\widetilde{\Delta}(R)$ and find that $\widetilde{\Delta}(R)$ has a denominator Γ. We denote

$$\widetilde{\Delta}(R) = \frac{\delta(R)}{\Gamma}.$$

The identity $\widetilde{\Delta}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_3, \gamma_3)$ of system (9.6) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_3, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides $\delta(x)$.

Using MAPLE we divide δ by Q and obtain a remainder which is a polynomial r of the form

$$r = \frac{r_0 x + r_1}{[(I_2 - I_1)\gamma_2^2 - I_1\gamma_3^2 + I_1 U_2]^2},$$

where

$$r_0 = r_{01}\Gamma + r_{02} \quad \text{and} \quad r_1 = r_{11}\Gamma + r_{12},$$

Here r_{01}, r_{02}, r_{11} and r_{12} are polynomials in the variables ω_3, γ_2, γ_3 and the parameters $\mathcal{I}c$ and U_i, $1 \le i \le 3$.

According to Propositions 4.2 we have $r_0 = r_1 = 0$. Then by Proposition 4.3 we conclude that $r_{01} = r_{02} = r_{11} = r_{12} = 0$ because $\Gamma \notin \mathbb{C}(\gamma_2, \gamma_3)$. It turns out that for our aims the equation $r_{11} = 0$ is sufficient. This equation will be identically satisfied if and only if all the coefficients of r_{11} are zero. The polynomial r_{11} has 109 coefficients. We should find all values of the parameters $\mathcal{I}c$ and U_i, $1 \le i \le 3$, for which the 109 coefficients are zero. After four consecutive simplifications we obtain a reduced system of only three very simple equations:

$$I_1 - I_2 = 0, \quad c_1 = 0, \quad c_2 = 0,$$

and the values of U_i, $1 \le i \le 3$, I_2, I_3 and c_3 are arbitrary. It is clear that this is the Lagrange case.

Thus a partial integral of type 2, $F(\omega_3, \gamma_3)$, does not exist.

Type 3. Let us consider the existence of a first integral of type 3, i.e. $F(\omega_3, \gamma_2)$. Now we have

$$\frac{dF}{dt} = \frac{d\omega_3}{dt}\frac{\partial F}{\partial \omega_3} + \frac{d\gamma_2}{dt}\frac{\partial F}{\partial \gamma_2} = Y_1(F) = 0, \tag{9.14}$$

where Y_1 is a vector field defined on Ω.

Equation (9.14) should be an identity with respect to all three variables $(\omega_3, \gamma_2, \gamma_3)$. As F does not depend on γ_3, its partial derivatives will not depend on γ_3 either. Thus if we differentiate identity (9.14) with respect to γ_3 we shall obtain again a linear partial differential equation for F:

$$\frac{\partial Y_1(F)}{\partial \gamma_3} = \frac{\partial}{\partial \gamma_3}\left(\frac{d\omega_3}{dt}\right)\frac{\partial F}{\partial \omega_3} + \frac{\partial}{\partial \gamma_3}\left(\frac{d\gamma_2}{dt}\right)\frac{\partial F}{\partial \gamma_2} = Y_2(F) = 0, \tag{9.15}$$

where Y_2 is a vector field defined on Ω.

Equations (9.14) and (9.15) can be considered as a system of two homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_2}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if an integral F exists, system (9.14)–(9.15) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant $\Delta(R)$ composed of the coefficients of this system satisfies $\Delta(R) \equiv 0$ provided that R is a root of (9.4).

We compute this determinant and obtain

$$\Delta(R) = \frac{\widehat{\Delta}(R)}{I_2 I_3 \gamma_2 \Gamma},$$

where

$$\begin{aligned}\widehat{\Delta}(R) = {} & \left[-2I_1(I_1-I_2)\omega_3\Gamma^2\frac{\partial R}{\partial\gamma_3}R - I_2c_1\gamma_2^2\gamma_3\frac{\partial R}{\partial\gamma_3} + (I_1-I_2)U_1R^2 - I_2c_1\gamma_2^2R\right]\Gamma \\ & + I_1(I_1-I_2)\gamma_3\Gamma^2\frac{\partial R}{\partial\gamma_3}R^2 + \Gamma^2[I_3(I_2-I_1)\omega_3^2\gamma_3 + (I_1-I_2)U_1\omega_3 + I_2c_2\gamma_2\gamma_3]\frac{\partial R}{\partial\gamma_3} \\ & + [I_3(I_1-I_2)\omega_3^2\gamma_2^2 - I_3(I_1-I_2)U_2\omega_3^2 + (I_1-I_2)U_1\omega_3\gamma_3 + I_2c_2\gamma_2(-\gamma_2^2+U_2)]R \\ & - I_1(I_1-I_2)(-\gamma_2^2+U_2)R^3 - I_2c_1\omega_3\gamma_2^2\gamma_3.\end{aligned}$$

Thus $\Delta(R)=0$ is equivalent to $\widehat{\Delta}(R)=0$.

$\widehat{\Delta}(R)$ depends on $\frac{\partial R}{\partial\gamma_3}$. To determine this derivative we use the same steps as in the case of the first integral of type 1 and obtain

$$\frac{\partial R}{\partial\gamma_3} = -\frac{\frac{\partial A}{\partial\gamma_3}R^2 + \frac{\partial B}{\partial\gamma_3}R + \frac{\partial C}{\partial\gamma_3}}{2AR+B}.$$

We put it in the expression for $\widehat{\Delta}(R)$. After this substitution the non-zero expression $2AR+B$ appears as a denominator of $\widehat{\Delta}(R)$ and we denote

$$\widehat{\Delta}(R) = \frac{\widetilde{\Delta}(R)}{2AR+B}.$$

The identity $\widehat{\Delta}(R)=0$ is equivalent to $\widetilde{\Delta}(R)=0$ but $\widetilde{\Delta}(R)$ depends on the functions A, B and C from (9.5) and their derivatives with respect to γ_3. We put these functions in the expression for $\widetilde{\Delta}(R)$ and find that $\widetilde{\Delta}(R)$ has a denominator Γ. We denote

$$\widetilde{\Delta}(R) = \frac{\delta(R)}{\Gamma},$$

where $\delta(R)$ is a polynomial in R of degree 4.

The identity $\widetilde{\Delta}(R)=0$ is equivalent to $\delta(R)=0$. We know that if $Q(R)=0$, then if in addition some supplementary first integral $F(\omega_3,\gamma_3)$ of system (9.6) exists, then also $\delta(R)=0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in $\mathbb{K}[x]$, where $\mathbb{K}=\mathbb{A}\mathrm{lg}(\omega_3,\gamma_2,\gamma_3)$, the polynomial $Q(x)$ divides $\delta(x)$.

Using MAPLE we divide δ by Q and obtain a remainder which is a polynomial $\widetilde{r}$ of the form

$$\widetilde{r} = \frac{\widetilde{r}_0x+\widetilde{r}_1}{[(I_2-I_1)\gamma_2^2 - I_1\gamma_3^2 + I_1U_2]^3},$$

where

$$\widetilde{r}_0 = \widetilde{r}_{01}\Gamma + \widetilde{r}_{02} \quad\text{and}\quad \widetilde{r}_1 = \widetilde{r}_{11}\Gamma + \widetilde{r}_{12},$$

Here $\widetilde{r}_{01}$, $\widetilde{r}_{02}$, $\widetilde{r}_{11}$ and $\widetilde{r}_{12}$ are polynomials in the variables ω_3, γ_2, γ_3 and the parameters $\mathcal{I}c$ and U_i, $1\le i\le 3$.

According to Propositions 4.2 we have $\widetilde{r}_0=\widetilde{r}_1=0$. Then by Propositions 4.3 we conclude that $\widetilde{r}_{01}=\widetilde{r}_{02}=\widetilde{r}_{11}=\widetilde{r}_{12}=0$ because $\Gamma\notin\mathbb{C}(\gamma_2,\gamma_3)$. It turns out that for our aims the equation $\widetilde{r}_{11}=0$ is sufficient. The equation $\widetilde{r}_{11}=0$ will be identically satisfied if and only if all the coefficients of the polynomial $\widetilde{r}_{11}$ are zero. The polynomial $\widetilde{r}_{11}$ has 179 coefficients. We should find all values of the parameters $\mathcal{I}c$ and U_i, $1\le i\le 3$, for which the 179 coefficients are zero. After three consecutive simplifications we obtain the reduced system of seven equations

$$\begin{gathered} I_1 - I_2 = 0, \quad c_2c_3 = 0, \quad (I_2 - I_3)c_2 = 0, \quad c_2U_3 = 0, \\ c_2U_2 = 0, \quad c_2U_1 = 0, \qquad c_1^2 + 2c_2^2 = 0. \end{gathered}$$

We solve this system by using `solve` and obtain two solutions. The first of them gives the Lagrange case $I_1 = I_2$, $c_1 = 0$, $c_2 = 0$, and the second a particular case of the kinetic symmetry case.

Thus a partial integral of type 3, $F(\omega_3, \gamma_2)$, does not exist.

9.3. Elimination of γ_1, γ_2, γ_3. Using `solve` we determine the variables γ_1 and γ_2 from the equations $H_1 = U_1$ and $H_3 = U_3$ (see (9.1)). Then we put the values of γ_1 and γ_2 in the equation $H_2 = U_2$ and we determine γ_3. In this way we obtain the following solution:

$$\begin{aligned} \gamma_1 &= \frac{I_2\omega_2(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 - U_3) + 2c_2U_1 + 2(I_2c_3\omega_2 - I_3c_2\omega_3)R}{2(I_1c_2\omega_1 - I_2c_1\omega_2)}, \\ \gamma_2 &= -\frac{I_1\omega_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 - U_3) + 2c_1U_1 + 2(I_1c_3\omega_1 - I_3c_1\omega_3)R}{2(I_1c_2\omega_1 - I_2c_1\omega_2)}, \\ \gamma_3 &= R, \end{aligned} \tag{9.16}$$

where $R = R(\omega_1, \omega_2, \omega_3)$ is a root of $Q(x) = Ax^2 + Bx + C = 0$, that is,

$$Q(R) = AR^2 + BR + C = 0, \tag{9.17}$$

and $A = A(\omega_1, \omega_2, \omega_3)$, $B = B(\omega_1, \omega_2, \omega_3)$ and $C = C(\omega_1, \omega_2, \omega_3)$ are the following polynomials:

$$\begin{aligned} A = {} & 4I_1^2(c_2^2 + c_3^2)\omega_1^2 - 8I_1I_2c_1c_2\omega_1\omega_2 - 8I_1I_3c_1c_3\omega_1\omega_3 \\ & + 4I_2^2(c_1^2 + c_3^2)\omega_2^2 - 8I_2I_3c_2c_3\omega_2\omega_3 + 4I_3^2(c_1^2 + c_2^2)\omega_3^2, \\ B = {} & 4I_1^3c_3\omega_1^4 - 4I_1^2I_3c_1\omega_1^3\omega_3 + 4I_1I_2(I_1 + I_2)c_3\omega_1^2\omega_2^2 - 4I_1I_2I_3c_2\omega_1^2\omega_2\omega_3 \\ & + 4I_1^2I_3c_3\omega_1^2\omega_3^2 - 4I_1I_2I_3c_1\omega_1\omega_2^2\omega_3 - 4I_1I_3^2c_1\omega_1\omega_3^3 + 4I_2^3c_3\omega_2^4 \\ & - 4I_2^2I_3c_2\omega_2^3\omega_3 + 4I_2^2I_3c_3\omega_2^2\omega_3^2 - 4I_2I_3^2c_2\omega_2\omega_3^3 - 4I_1^2c_3U_3\omega_1^2 \\ & + 4I_1I_3c_1U_3\omega_1\omega_3 - 4I_2^2c_3U_3\omega_2^2 + 4I_2I_3c_2U_3\omega_2\omega_3 \\ & + 8I_1c_1c_3U_1\omega_1 + 8I_2c_2c_3U_1\omega_2 - 8I_3(c_1^2 + c_2^2)U_1\omega_3, \\ C = {} & I_1^4\omega_1^6 + I_1^2I_2(2I_1 + I_2)\omega_1^4\omega_2^2 + 2I_1^3I_3\omega_1^4\omega_3^2 + I_1I_2^2(I_1 + 2I_2)\omega_1^2\omega_2^4 \\ & + 2I_1I_2I_3(I_1 + I_2)\omega_1^2\omega_2^2\omega_3^2 + I_1^2I_3^2\omega_1^2\omega_3^4 + I_2^4\omega_2^6 + 2I_2^3I_3\omega_2^4\omega_3^2 + I_3^2I_2^2\omega_2^2\omega_3^4 \\ & - 2I_1^3U_3\omega_1^4 - 2I_1I_2(I_1 + I_2)U_3\omega_1^2\omega_2^2 - 2I_1^2I_3U_3\omega_1^2\omega_3^2 - 2I_2^3U_3\omega_2^4 \\ & - 2I_2^2I_3U_3\omega_2^2\omega_3^2 + 4I_1^2c_1U_1\omega_1^3 + 4I_1I_2c_2U_1\omega_1^2\omega_2 + 4I_1I_2c_1U_1\omega_1\omega_2^2 \\ & + 4I_1I_3c_1U_1\omega_1\omega_3^2 + 4I_2^2c_2U_1\omega_2^3 + 4I_2I_3c_2U_1\omega_2\omega_3^2 - I_1^2(4c_2^2U_2 - U_3^2)\omega_1^2 \\ & + 8I_1I_2c_1c_2U_2\omega_1\omega_2 - I_2^2(4c_1^2U_2 - U_3^2)\omega_2^2 \\ & - 4I_1c_1U_1U_3\omega_1 - 4I_2c_2U_1U_3\omega_2 + 4(c_1^2 + c_2^2)U_1^2. \end{aligned} \tag{9.18}$$

Putting the values of γ_1, γ_2 and γ_3 from (9.16) in the Euler–Poisson equations (1.1) and removing the last three equations we obtain the following system of three equations in the unknowns ω_1, ω_2 and ω_3:

$$\frac{d\omega_i}{dt} = \frac{M_i}{2I_i(I_1c_2\omega_1 - I_2c_1\omega_2)}, \quad 1 \le i \le 3, \tag{9.19}$$

where M_1, M_2 and M_3 are polynomials in ω_j, γ_j, I_j, c_j, U_j, $1 \le i \le 3$, and R. The system (9.19) is correctly defined only if

$$(c_1, c_2) \neq (0, 0). \tag{9.20}$$

Let us suppose first that condition (9.20) is satisfied.

As we are going to study the first integrals of system (9.19) we can multiply its right-hand sides by the non-zero factor $2I_1I_2I_3(I_1c_2\omega_1 - I_2c_1\omega_2)$. In this way we come to the following system:

$$\begin{aligned}
\frac{d\omega_1}{dt} &= -I_2I_3\{2[I_1(c_2^2 + c_3^2)\omega_1 - I_2c_1c_2\omega_2 - I_3c_1c_3\omega_3]R \\
&\quad + I_1^2c_3\omega_1^3 + I_1I_2c_3\omega_1\omega_2^2 - 2I_1(I_2 - I_3)c_2\omega_1\omega_2\omega_3 + I_1I_3c_3\omega_1\omega_3^2 \\
&\quad + 2I_2(I_2 - I_3)c_1\omega_2^2\omega_3 - I_1c_3U_3\omega_1 + 2c_1c_3U_1\}, \\
\frac{d\omega_2}{dt} &= I_1I_3\{2[I_1c_1c_2\omega_1 - I_2(c_1^2 + c_3^2)\omega_2 + I_3c_2c_3\omega_3]R \\
&\quad - I_1I_2c_3\omega_1^2\omega_2 - 2I_1(I_1 - I_3)c_2\omega_1^2\omega_3 + 2I_2(I_1 - I_3)c_1\omega_1\omega_2\omega_3 \\
&\quad - I_2^2c_3\omega_2^3 - I_2I_3c_3\omega_2\omega_3^2 + I_2c_3U_3\omega_2 - 2c_2c_3U_1\}, \\
\frac{d\omega_3}{dt} &= I_1I_2\{2[I_1c_1c_3\omega_1 + I_2c_2c_3\omega_2 - I_3(c_1^2 + c_2^2)\omega_3]R \\
&\quad + I_1^2c_1\omega_1^3 + I_1(2I_1 - I_2)c_2\omega_1^2\omega_2 - I_2(I_1 - 2I_2)c_1\omega_1\omega_2^2 + I_1I_3c_1\omega_1\omega_3^2 \\
&\quad + I_2^2c_2\omega_2^3 + I_2I_3c_2\omega_2\omega_3^2 - I_1c_1U_3\omega_1 - I_2c_2U_3\omega_2 + 2(c_1^2 + c_2^2)U_1\}.
\end{aligned} \tag{9.21}$$

We study the existence of a first integral of system (9.21) that depends on at most two variables among $(\omega_1, \omega_2, \omega_3)$. There are three possible types of such first integrals:

1. $F(\omega_1, \omega_2)$ (case (iv)).
2. $F(\omega_1, \omega_3)$ (case (iv)).
3. $F(\omega_2, \omega_3)$ (case (iv)).

As all the three types of first integrals belong to case (iv) it suffices to study the first type.

Type 1. We consider a first integral F of system (9.21) of type 1, i.e. $F(\omega_1, \omega_2)$. We have

$$\frac{dF}{dt} = \frac{d\omega_1}{dt}\frac{\partial F}{\partial \omega_1} + \frac{d\omega_2}{dt}\frac{\partial F}{\partial \omega_2} = Y_1(F) = 0, \tag{9.22}$$

where $\frac{d\omega_1}{dt}$ and $\frac{d\omega_2}{dt}$ are taken from (9.21) and Y_1 is a vector field defined on $\mathbb{C}^3(\omega_1, \omega_2, \omega_3)$.

This equation should hold identically with respect to the variables ω_1, ω_2 and ω_3. As F does not depend on ω_3, its partial derivatives will not depend on ω_3 either. Thus if we differentiate identity (9.22) with respect to ω_3 we obtain again a linear partial differential equation for F:

$$\frac{\partial Y_1(F)}{\partial \omega_3} = \frac{\partial}{\partial \omega_3}\left(\frac{d\omega_1}{dt}\right)\frac{\partial F}{\partial \omega_1} + \frac{\partial}{\partial \omega_3}\left(\frac{d\omega_2}{dt}\right)\frac{\partial F}{\partial \omega_2} = Y_2(F) = 0, \tag{9.23}$$

where Y_2 is a vector field defined on $\mathbb{C}^3(\omega_1, \omega_2, \omega_3)$.

Equations (9.22) and (9.23) can be considered as a system of two homogeneous linear algebraic equations with the unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}\right)$. This linear system admits a non-zero solution if and only if its determinant $\Delta(R)$ vanishes identically with respect to ω_1, ω_2 and ω_3 provided that R is a root of (9.17).

We compute $\Delta(R)$, remove its non-zero factor $2I_1I_2I_3^2(I_1c_2\omega_1 - I_2c_1\omega_2)$ and obtain

$$\begin{aligned}
\widehat{\Delta}(R) = \frac{\Delta(R)}{2I_1I_2I_3^2(I_1c_2\omega_1 - I_2c_1\omega_2)} &= -2I_3c_3(c_1^2+c_2^2+c_3^2)R^2 \\
&- \big[I_1(2I_3c_2^2 - 2I_1c_2^2 - 2I_1c_3^2 + 3I_3c_3^2)\omega_1^2 + 2(2I_1I_2 - I_1I_3 - I_2I_3)c_1c_2\omega_1\omega_2 \\
&- 2I_1I_3c_1c_3\omega_1\omega_3 + I_2(2I_3c_1^2 - 2I_2c_1^2 - 2I_2c_3^2 + 3I_3c_3^2)\omega_2^2 \\
&- 2I_2I_3c_2c_3\omega_2\omega_3 - I_3^2c_3^2\omega_3^2 - I_3c_3^2U_3\big]R \\
&- \big[I_1^2c_1c_3\omega_1^3 + I_1I_2c_2c_3\omega_1^2\omega_2 + I_1(2I_1c_2^2 - 2I_3c_2^2 + 2I_1c_3^2 - I_3c_3^2)\omega_1^2\omega_3 \\
&+ I_1I_2c_1c_3\omega_1\omega_2^2 - 2(2I_1I_2 - I_1I_3 - I_2I_3)c_1c_2\omega_1\omega_2\omega_3 \\
&- I_3(I_1 - 2I_3)c_1c_3\omega_1\omega_3^2 + I_2^2c_2c_3\omega_2^3 \\
&+ I_2(2I_2c_1^2 - 2I_3c_1^2 + 2I_2c_3^2 - I_3c_3^2)\omega_2^2\omega_3 \\
&+ I_3(2I_3 - I_2)c_2c_3\omega_2\omega_3^2 + I_3^2c_3^2\omega_3^3 - I_1c_1c_3U_3\omega_1 - I_2c_2c_3U_3\omega_2 \\
&- I_3c_3^2U_3\omega_3 + 2c_3(c_1^2+c_2^2+c_3^2)U_1\big]\frac{\partial R}{\partial \omega_3} \\
&+ c_3\big[I_1^2(I_1 - I_3)\omega_1^4 + I_1I_2(I_1 + I_2 - 2I_3)\omega_1^2\omega_2^2 - I_1I_3(I_1 - I_3)\omega_1^2\omega_3^2 \\
&+ I_2^2(I_2 - I_3)\omega_2^4 - I_2I_3(I_2 - I_3)\omega_2^2\omega_3^2 - I_1(I_1 - I_3)U_3\omega_1^2 \\
&- I_2(I_2 - I_3)U_3\omega_2^2 + 2(I_1 - I_3)c_1U_1\omega_1 + 2(I_2 - I_3)c_2U_1\omega_2 - 2I_3c_3U_1\omega_3\big]. \quad (9.24)
\end{aligned}$$

As in Sec. 9.2 we should obtain $\widehat{\Delta}(R)$ as a polynomial of R, that is, we should determine $\frac{\partial R}{\partial \omega_3}$ as a function of R. For this purpose we use equation (9.17) where the polynomials $A(\omega_1,\omega_2,\omega_3)$, $B(\omega_1,\omega_2,\omega_3)$ and $C(\omega_1,\omega_2,\omega_3)$ are taken from (9.18). We differentiate (9.17) with respect to ω_3 and obtain

$$\frac{\partial Q(R)}{\partial \omega_3} = \frac{\partial A}{\partial \omega_3}R^2 + \frac{\partial B}{\partial \omega_3}R + \frac{\partial C}{\partial \omega_3} + \frac{dQ}{dR}\frac{\partial R}{\partial \omega_3} = 0. \quad (9.25)$$

Determining $\frac{\partial R}{\partial \omega_3}$ from (9.25) is possible if and only if $\frac{dQ}{dR} = 2AR + B$ is not zero when R is a root of Q. Then we obtain

$$\frac{\partial R}{\partial \omega_3} = -\frac{\frac{\partial A}{\partial \omega_3}R^2 + \frac{\partial B}{\partial \omega_3}R + \frac{\partial C}{\partial \omega_3}}{2AR + B}.$$

Let us prove that $\frac{dQ}{dR}$ is not zero. We use Proposition 4.1. Let R be a root of $Q(R) = 0$. We consider the resultant ρ of Q and $\frac{dQ}{dR}$ and prove that it can never be identically zero with respect to ω_1, ω_2 and ω_3. We have

$$\rho = A(4AC - B^2)$$

and as A never vanishes identically we do not consider ρ but $\widehat{\rho} = 4AC - B^2$ instead. Putting in $\widehat{\rho}$ the expressions for A, B and C from (9.18) we obtain

$$\widehat{\rho} = 16(I_1c_2\omega_1 - I_2c_1\omega_2)^2\widetilde{\rho}.$$

As we consider the case (9.20), the first factor never vanishes identically. The second one, i.e. $\widetilde{\rho}$, is a polynomial of ω_1, ω_2 and ω_3 with 35 monomials. Among them is the monomial $I_1^4\omega_1^6$ and therefore $\widetilde{\rho}$ never vanishes identically. Consequently, ρ never vanishes identically either.

We put the value of $\frac{\partial R}{\partial \omega_3}$ obtained from (9.25) in (9.24) and find $\widehat{\Delta}(R)$. After this substitution the non-zero expression $2AR+B$ appears as a denominator of $\widehat{\Delta}(R)$ and we denote

$$\widehat{\Delta}(R) = \frac{\delta(R)}{2AR+B},$$

where $\delta(R)$ is a polynomial in R of degree 3.

It is clear that $\widehat{\Delta}(R)=0$ is equivalent to $\delta(R)=0$. We know that if $Q(R)=0$, then if in addition some supplementary first integral $F(\omega_1,\omega_2)$ of system (9.21) exists, then also $\delta(R)=0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in $\mathbb{K}[x]$, where $\mathbb{K}=\mathbb{A}\mathrm{lg}(\omega_1,\omega_2,\omega_3)$, the polynomial $Q(x)$ divides $\delta(x)$.

Using the MAPLE command `rem` we compute the remainder of dividing δ by Q and obtain

$$\widehat{r}(x) = \frac{4(I_1c_2\omega_1 - I_2c_1\omega_2)}{(I_1c_2\omega_1 - I_2c_1\omega_2)^2 + (I_1c_3\omega_1 - I_3c_1\omega_3)^2 + (I_2c_3\omega_2 - I_3c_2\omega_3)^2}(\widehat{r}_0 x + \widehat{r}_1),$$

where $\widehat{r}_0$ and $\widehat{r}_1$ are polynomials in ω_1, ω_2 and ω_3.

It is easily seen that when $(c_1,c_2)\neq(0,0)$ the fraction in the above equality is non-zero on an open dense subset of $\mathbb{C}^3(\omega_1,\omega_2,\omega_3)$.

Thus $\widehat{r}_0=\widehat{r}_1=0$ identically with respect to ω_1, ω_2 and ω_3. Below we consider only $\widehat{r}_0=0$, which turns out to be sufficient for our needs.

As $\widehat{r}_0$ has a non-zero factor I_3 we remove it. The polynomial obtained has 74 coefficients. To find all values of the parameters $\mathcal{I}c$ and U_i, $1\leq i\leq 3$, for which these coefficients are zero we apply simplification and after four consecutive simplifications we obtain the reduced system of five equations

$$\begin{gathered} c_2c_3=0,\quad c_1c_3=0,\\ (I_1-I_2)c_3=0,\quad (I_1-I_3)c_2=0,\quad (I_2-I_3)c_1=0. \end{gathered}$$

Solving it by using `solve` we obtain the following five solutions:

$$\begin{array}{ll} c_1=0,\ c_2=0,\ c_3=0 & \text{with arbitrary } I_1,\ I_2,\ I_3,\ U_1,\ U_2,\ U_3,\\ I_1=I_2,\ c_1=0,\ c_2=0 & \text{with arbitrary } I_2,\ I_3,\ c_3,\ U_1,\ U_2,\ U_3,\\ I_2=I_3,\ c_2=0,\ c_3=0 & \text{with arbitrary } I_1,\ I_3,\ c_1,\ U_1,\ U_2,\ U_3,\\ I_1=I_3,\ c_1=0,\ c_3=0 & \text{with arbitrary } I_2,\ I_3,\ c_2,\ U_1,\ U_2,\ U_3,\\ I_1=I_3,\ I_2=I_3,\ c_3=0 & \text{with arbitrary } I_3,\ c_1,\ c_2,\ U_1,\ U_2,\ U_3. \end{array}$$

As $(c_1,c_2)\neq(0,0)$ we discard the first and second solutions. The third and fourth solutions give the Lagrange case, and the fifth the kinetic symmetry case. Thus a partial integral of type 1, $F(\omega_1,\omega_2)$, does not exist when $(c_1,c_2)\neq(0,0)$.

Let us suppose now that $(c_1, c_2) = (0, 0)$. To avoid the Euler case we suppose that $c_3 \neq 0$. Solving (9.1) with respect to γ_1, γ_2 and γ_3 by using `solve` we obtain

$$\begin{aligned} \gamma_1 &= \frac{I_1 I_3 \omega_1^2 \omega_3 + I_2 I_3 \omega_2^2 \omega_3 + I_3^2 \omega_2^3 - I_3 U_3 \omega_3 + 2c_3 U_1 - 2I_2 \omega_2 R}{2I_1 c_3 \omega_1}, \\ \gamma_2 &= \frac{R}{c_3}, \qquad \gamma_3 = -\frac{I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 - U_3}{2c_3}, \end{aligned} \tag{9.26}$$

where $R = R(\omega_1, \omega_2, \omega_3)$ satisfies

$$Q(R) = AR^2 + BR + C = 0. \tag{9.27}$$

Here $A = A(\omega_1, \omega_2)$, $B = B(\omega_1, \omega_2, \omega_3)$ and $C = C(\omega_1, \omega_2, \omega_3)$ are the following polynomials:

$$\begin{aligned} A &= 4(I_1^2 \omega_1^2 + I_2^2 \omega_2^2), \\ B &= -4I_2 \omega_2 (I_1 I_3 \omega_1^2 \omega_3 + I_2 I_3 \omega_2^2 \omega_3 + I_3^3 \omega_3^3 - I_3 U_3 \omega_3 + 2c_3 U_1), \\ C &= I_1^4 \omega_1^6 + 2I_1^3 I_2 \omega_1^4 \omega_2^2 + I_1^2 I_3 (2I_1 + I_3) \omega_1^4 \omega_3^2 + I_1^2 I_2^2 \omega_1^2 \omega_2^4 \\ &\quad + 2I_1 I_2 I_3 (I_1 + I_3) \omega_1^2 \omega_2^2 \omega_3^2 + I_1 I_3^2 (I_1 + 2I_3) \omega_1^2 \omega_3^4 + I_2^2 I_3^2 \omega_2^4 \omega_3^2 \\ &\quad + 2I_2 I_3^3 \omega_2^2 \omega_3^4 + I_3^4 \omega_3^6 - 2I_1^3 U_3 \omega_1^4 - 2I_1^2 I_2 U_3 \omega_1^2 \omega_2^2 - 2I_1 I_3 (I_1 + I_3) U_3 \omega_1^2 \omega_3^2 \\ &\quad - 2I_2 I_3^2 U_3 \omega_2^2 \omega_3^2 - 2I_3^3 U_3 \omega_3^4 + 4I_1 I_3 c_3 U_1 \omega_1^2 \omega_3 + 4I_2 I_3 c_3 U_1 \omega_1^2 \omega_3 \\ &\quad + 4I_3^2 c_3 U_1 \omega_3^3 - I_1^2 (4c_3^2 U_2 - U_3^2) \omega_1^2 + I_3^2 U_3^2 \omega_3^2 - 4I_3 c_3 U_1 U_3 \omega_3 + 4U_1^2 c_3^2. \end{aligned} \tag{9.28}$$

After substitution of γ_1, γ_2 and γ_3 from (9.26) in the first three Euler–Poisson equations (1.1) we obtain the following system for ω_1, ω_2 and ω_3:

$$\begin{aligned} \frac{d\omega_1}{dt} &= \frac{R + (I_2 - I_3)\omega_2 \omega_3}{I_1}, \\ \frac{d\omega_2}{dt} &= \frac{2I_2 \omega_2 R - I_1 (2I_1 - I_3) \omega_1^2 \omega_3 - I_2 I_3 \omega_2^2 \omega_3 - I_3^2 \omega_3^3 + I_3 U_3 \omega_3 - 2c_3 U_1}{2I_1 I_2 \omega_1}, \\ \frac{d\omega_3}{dt} &= \frac{(I_1 - I_2)\omega_1 \omega_2}{I_3}. \end{aligned} \tag{9.29}$$

As in case (9.20), we examine only type 1 first integrals of system (9.29).

Type 1. As when $(c_1, c_2) \neq (0, 0)$ we define the vector fields Y_1 and Y_2 by $Y_1(F) = \frac{dF}{dt}$ (see (9.22)) and $Y_2(F) = \frac{\partial Y_1(F)}{\partial \omega_3}$ (see (9.23)) but now $\frac{d\omega_1}{dt}$ and $\frac{d\omega_2}{dt}$ are taken from (9.29).

The determinant $\Delta(R)$ of the linear system (9.22)–(9.23) should vanish identically with respect to ω_1, ω_2 and ω_3 provided that R is a root of (9.27).

We compute $\Delta(R)$. It has a non-zero denominator $2I_1^2 I_2 \omega_1$. We denote

$$\begin{aligned} \widehat{\Delta}(R) = 2I_1^2 I_2 \omega_1 \Delta(R) &= -[I_1 (2I_1 - I_3) \omega_1^2 + I_2 (2I_2 - I_3) \omega_2^2 + 3I_3^2 \omega_3^2 - I_3 U_3] R \\ &\quad + [I_1 (2I_1 - I_3) \omega_1^2 \omega_3 + I_2 (2I_2 - I_3) \omega_2^2 \omega_3 + I_3^2 \omega_3^3 - I_3 U_3 \omega_3 + 2U_1 c_3] \frac{\partial R}{\partial \omega_3} \\ &\quad - 2(I_2 - I_3)(I_3^2 \omega_3^3 - U_1 c_3) \omega_2. \end{aligned} \tag{9.30}$$

In order to obtain $\widehat{\Delta}(R)$ as a polynomial in R we determine $\frac{\partial R}{\partial \omega_3}$ using (9.27) where $A(\omega_1, \omega_2)$, $B(\omega_1, \omega_2, \omega_3)$ and $C(\omega_1, \omega_2, \omega_3)$ are taken from (9.28). After differentiating

(9.27) with respect to ω_3 we obtain

$$\frac{\partial Q}{\partial \omega_3} = \frac{\partial B}{\partial \omega_3} R + \frac{\partial C}{\partial \omega_3} + \frac{dQ}{dR}\frac{\partial R}{\partial \omega_3} = 0. \tag{9.31}$$

In the same way as in the case $(c_1, c_2) \neq (0,0)$ we deduce by Proposition 4.1 that $\frac{dQ}{dR}$ is not zero and determine $\frac{\partial R}{\partial \omega_3}$ from (9.31). Then we put it in (9.30) and find $\widehat{\Delta}(R)$. After this substitution the non-zero expression $2AR + B$ appears as the denominator of $\widehat{\Delta}(R)$ and we denote

$$\widehat{\Delta}(R) = \frac{\delta(R)}{2AR + B},$$

where $\delta(R)$ is a polynomial in R of degree 2.

It is clear that $\widehat{\Delta}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_1, \omega_2)$ of system (9.29) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently, in $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_1, \omega_2, \omega_3)$, the polynomial $Q(x)$ divides $\delta(x)$.

The remainder $\overline{r}(x)$ of dividing $\delta(x)$ by $Q(x)$ is a polynomial in x of degree 1,

$$\overline{r}(x) = \overline{r}_0 x + \overline{r}_1,$$

where $\overline{r}_0$ and $\overline{r}_1$ are polynomials in ω_1, ω_2 and ω_3 which, by Proposition 4.2, should be identically zero. We consider only the leading coefficient of $\overline{r}(x)$,

$$\overline{r}_0 = -8I_1 I_3 (I_1 - I_2)\omega_1^2 \omega_2 (I_3^2 \omega_3^3 - c_3 U_1) = 0.$$

It is easily seen that $\overline{r}_0$ vanishes identically if and only if $I_1 = I_2$, which together with the restriction $c_1 = c_2 = 0$ considered now leads to the Lagrange case. Thus a partial integral of type 1, $F(\omega_1, \omega_2)$, does not exist.

This concludes our study.

References

[1] *Algebraic functions*, in: Encyclopedia of Mathematics, 1(A-B), Kluwer, 1987; http://encyclopediaofmath.org/index.php?title=Algebraic_function&oldid=45136.

[2] T. Archibald, *Differential equations and algebraic transcendents: French efforts at the creation of a Galois theory of differential equations 1880–1910*, Rev. Histoire Math. 17 (2011), 373–401.

[3] Yu. A. Arkhangelskii, *Analytical Dynamics of a Rigid Body*, Nauka, Moscow, 1977 (in Russian).

[4] Yu. A. Arkhangelskii, *On a new property of Euler–Poisson equations*, Dokl. Akad. Nauk SSSR 258 (1981), 810-811 (in Russian).

[5] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed., Grad. Texts in Math. 60, Springer, Berlin, 1989.

[6] M. Audin, *Spinning Tops: A Course on Integrable Systems*, Cambridge Stud. Adv. Math. 51, Cambridge Univ. Press, 1996.

[7] M. Audin, *Les systèmes hamiltoniens et leur intégrabilité*, Cours Spéc. 8, Soc. Math. France et EDP Sciences, 2001; English transl.: *Hamiltonian Systems and Their Integrability*, SMF/AMS Texts Monogr. 15, Amer. Math. Soc., 2008.

[8] J.-E. Björk, *Rigid bodies and the Bordin prize*, in: Oper. Theory Adv. Appl. 132, Birkhäuser, 2002, 55–60.

[9] V. N. Bogaevskii, *On general integrals of the equations of motion of a rigid body about a fixed point*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1965, 92–95 (in Russian).

[10] O. I. Bogoyavlensky, *Integrable Euler equations on six-dimensional Lie algebras*, Dokl. Akad. Nauk SSSR 268 (1983), 11-15 (in Russian); English transl.: Soviet Math. Dokl. 27 (1983), 1–5.

[11] A. V. Borisov and I. S. Mamaev, *Poisson Structures and Lie Algebras in Hamiltonian Mechanics*, Udmurtskiĭ Univ. Izhevsk, 1999 (in Russian).

[12] A. V. Borisov and I. S. Mamaev, *Modern Methods of the Theory of Integrable Systems*, Institute of Computer Science, Izhevsk, 2003 (in Russian).

[13] A. V. Borisov and I. S. Mamaev, *Rigid Body Dynamics: Hamiltonian Methods, Integrability, Chaos*, Institute of Computer Science, Izhevsk, 2005 (in Russian); http://ics.org.ru/publications/index.php?cat=103&author=23; English transl.: De Gruyter Stud. Math. Phys. 52, De Gruyter, 2019.

[14] N. Bourbaki, *Éléments d'histoire des mathématiques*, Springer, 2007.

[15] S. A. Chaplygin, *A new case of rotation of a heavy rigid body supported at one point*, Trudy Otdel. Fiz. Nauk Obshchestva Lyubit. Estestvozn. 10 (1901), no. 2, 31–34 (in Russian). Reproduced in:

a) Complete works of S. A. Chaplygin, Vol. 1, Akad. Nauk SSSR, Leningrad, 1933, 241–245.

b) S. A. Chaplygin, Collected Works, Vol. 1, Moscow, Gostekhizdat, 1948, 118–124.

c) S. A. Chaplygin, Selected Works on Mechanics and Mathematics, Gostekhizdat, Moscow, 1954, 472–476.

[16] A. Clebsch, *Ueber die simultane Integration linearer partieller Differentialgleichungen*, J. Reine Angew. Math. 65 (1866), 257–268.

[17] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 4th ed., Springer, 2015.

[18] F. Deahna, *Ueber die Bedingungen der Integrabilität lineärer Differentialgleichungen erster Ordnung zwischen einer beliebigen Anzahl veränderlicher Größen*, J. Reine Angew. Math. 20 (1840), 340–349.

[19] A. I. Dokshevich, *On the fourth integral in the problem of the rotation of a heavy rigid body around a fixed point*, in: Integration of Some Differential Equations of Mathematical Physics, Nauka, Tashkent, 1964, 104–116 (in Russian).

[20] A. I. Dokshevich, *Solutions in Finite Terms of the Euler–Poisson Equations*, Naukova Dumka, Kiev, 1992 (in Russian).

[21] D. S. Dummit and R. M. Foote, *Abstract Algebra*, 3rd ed., Wiley, 2004.

[22] A. T. Fomenko, *Integrability and Nonintegrability in Geometry and Mechanics*, Kluwer, Dordrecht, 1988.

[23] F. G. Frobenius, *Ueber das Pfaffsche Problem*, J. Reine Angew. Math. 82 (1876), 220–315.

[24] I. N. Gashenenko, G. V. Gorr and A. M. Kovalev, *Classical Problems of Rigid Body Dynamics*, Naukova Dumka, Kiev, 2012 (in Russian).

[25] L. Gavrilov, *Non-integrability of the equations of heavy gyrostat*, Compos. Math. 82 (1992), 275–291.

[26] V. V. Golubev, *Lectures on Integration of the Equations of Motion of a Rigid Body about a Fixed Point*, Gostekhizdat, Moscow, 1953 (in Russian); English transl.: Israel Program for Scientific Translations, Haifa, 1960.

[27] A. Goriely, *A brief history of Kovalevskaya exponents and modern developments*, Regul. Chaotic Dynam. 5 (2000), 3–15.

[28] G. V. Gorr, L. V. Kudriashova and L. A. Stepanova, *Classical Problems in Dynamics of a Rigid Body: Development and State of the Art*, Naukova Dumka, Kiev, 1978 (in Russian).

[29] D. N. Goryachev, *On the motion of a heavy rigid body about a fixed point in the case* $A = B = 4C$, Mat. Sb. 21(3) (1900), 431–438 (in Russian).

[30] B. Grammaticos, J. Moulin-Ollagnier, A. Ramani, J.-M. Strelcyn and S. Wojciechowski, *Integrals of quadratic ordinary differential equations in* $\mathbb{R}^3$*: the Lotka–Volterra system*, Phys. A 163 (1990), 683–722.

[31] A. T. Grigorian and B. N. Fradlin, *History of Mechanics of a Rigid Body*, Nauka, Moscow, 1982 (in Russian).

[32] F. Haas, *Frobenius method and invariants for one-dimensional time-dependent Hamiltonian systems*, J. Phys. A 34 (2001), 1005–1017.

[33] T. Hawkins, *Jacobi and the birth of Lie's theory of groups*, Arch. Hist. Exact Sci. 42 (1991), 187–278.

[34] T. Hawkins, *Emergence of the Theory of Lie Groups: An Essay in the History of Mathematics 1869–1926*, Sources and Studies in the History of Mathematics and Physical Sciences, Springer, 2000.

[35] T. Hawkins, *Frobenius, Cartan, and the problem of Pfaff*, Arch. Hist. Exact Sci. 59 (2005), 381–436.

[36] T. Hawkins, *The Mathematics of Frobenius in Context: A Journey Through 18th to 20th Century Mathematics*, Sources and Studies in the History of Mathematics and Physical Sciences, Springer, 2013.

[37] D. D. Holm, *Geometric Mechanics. Part I: Dynamics and Symmetry. Part II: Rotating, Translating and Rolling*, 2nd ed., Imperial College Press, 2011.

[38] D. D. Holm, T. Schmah and C. Stoica, *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions*, Cambridge Univ. Press, 2009.

[39] E. Husson, *Recherches des intégrales algébriques dans le mouvement d'un corps pesant autour d'un point fixe*, Ann. Fac. Sci. Toulouse 8 (1906), 73–152.

[40] E. Husson, *Sur un théorème de M. Poincaré, relativement au mouvement d'un solide pesant*, Acta Math. 31 (1908), 71–88.

[41] A. A. Iliukhin, *Spatial Problems of Nonlinear Theory of Elastic Rods*, Naukova Dumka, Kiev, 1979 (in Russian).

[42] Yu. Ilyashenko and S. Yakovenko, *Lectures on Analytical Differential Equations*, Grad. Stud. Math. 86, Amer. Math. Soc., 2007.

[43] A. V. Kavinov, *On calculation of first integrals for three-dimensional ODE systems*, Math. and Math. Modeling 2018, no. 6, 11–21 (in Russian).

[44] V. V. Kozlov, *Symmetries, Topology and Resonances in Hamiltonian Mechanics*, Springer, 1996.

[45] E. Leimanis, *The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point*, Springer, 1965.

[46] T. Levi Civita e U. Amaldi, *Lezioni di meccanica razionale*, Nicola Zanichelli, Bologna, 1927.

[47] Y. Z. Liu and Y. Xue, *Formulation of Kirchhoff rod based on quasi-coordinates*, Tech. Mechanik 24 (2004), 206–210.

[48] J. Lützen, *Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics*, Springer, New York, 1990.

[49] A. J. Maciejewski and S. I. Popov, *Invariants of homogeneous ordinary differential equations*, Rep. Math. Phys. 41 (1998), 287–310.

[50] A. J. Maciejewski, S. I. Popov and J.-M. Strelcyn, *The Euler equations on Lie algebra so(4); an elementary approach to integrability condition*, J. Math. Phys. 42 (2001), 2701–2717.

[51] A. J. Maciejewski and M. Przybylska, *Differential Galois approach to the non-integrability of the heavy top problem*, Ann. Fac. Sci. Toulouse Math. 6 (2005), 123–160.

[52] K. Magnus, *Kreisel: Theorie und Anwendungen*, Springer, 1971.

[53] J. E. Marsden, *Lectures on Mechanics*, London Math. Soc. Lecture Note Ser. 174, Cambridge Univ. Press, 1992.

[54] J. Montaldi and T. Ratiu (eds), *Geometric Mechanics and Symmetry: The Peyresq Lectures*, London Math. Soc. Lecture Note Ser. 306, Cambridge Univ. Press, 2005.

[55] J. J. Morales-Ruiz, *Kovalevskaya, Lyapunov, Painlevé, Ziglin and the differential Galois theory*, Regul. Chaotic Dynam. 5 (2000), 251–272.

[56] J. J. Morales-Ruiz, *Picard–Vessiot theory and integrability*, J. Geom. Phys. 87 (2015), 314–343.

[57] J. J. Morales-Ruiz and C. Simó, *Picard–Vessiot theory and Ziglin's theorem*, J. Differential Equations 107 (1994), 140–162.

[58] J. Moulin-Ollagnier and J.-M. Strelcyn, *On first integrals of linear systems, Frobenius integrability theorem and linear representations of Lie algebras*, in: Bifurcations of Planar Vector Fields (Luminy, 1989), Lecture Notes in Math. 1455, Springer, Berlin, 1990, 243–271.

[59] R. Narasimhan, *Analysis on Real and Complex Manifolds*, 3rd printing, North-Holland, Amsterdam, 1985.

[60] P. J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd ed., Grad. Texts in Math. 107, Springer, Berlin, 1986.

[61] A. M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Vol. I, Birkhäuser, Basel, 1990.

[62] P. Ya. Polubarinova-Kochina, *On single-valued solutions and algebraic integrals of the problem about rotation of the heavy rigid body problem around a fixed point*, in: Motion of a Rigid Body around a Fixed Point: S. V. Kovalevskaya Memorial Collection, Acad. Sci. USSR, Moscow, 1940, 157–186 (in Russian).

[63] S. I. Popov, *On the existence of a first integral depending on* p, q, r, γ *of the Euler–Poisson system*, Theoretical and Applied Mechanics (Bulgarian Academy of Sciences) 2 (1981), 28–33 (in Bulgarian).

[64] S. I. Popov, *On the nonexistence of a new first integral* $F(p, q, r, \gamma, \gamma') = const$ *of the problem of a heavy rigid body motion about a fixed point*, C. R. Acad. Bulgare Sci. 38 (1985), 583–586 (in Russian).

[65] S. I. Popov, *On the nonexistence of a new first integral* $F(p, q, r, \gamma, \gamma') = const$ *of the problem of a heavy rigid body motion about a fixed point*, Theoretical and Applied Mechanics (Bulgarian Academy of Sciences) 4 (1988), 17–23 (in Russian).

[66] S. I. Popov and J.-M. Strelcyn, *An elementary approach to integrability condition for the Euler equations on Lie algebra so(4)*, in: Hamiltonian Mechanics (Toruń, 1993), NATO Adv. Sci. Inst. Ser. B Phys. 331, Plenum, New York, 1994, 371–376.

[67] S. I. Popov and J.-M. Strelcyn, *The Euler–Poisson equations: an elementary approach to integrability conditions*, J. Geom. Mechanics 10 (2018), 293–329.

[68] V. V. Prasolov, *Polynomials*, Springer, 2004.

[69] B. L. Reinhart, *Differential Geometry of Foliations: The Fundamental Integrability Problem*, Ergeb. Math. Grenzgeb. 99, Springer, 1983.

[70] S. Spodzieja, *The field of Nash functions and factorization of polynomials*, Ann. Polon. Math. 65 (1996), 81–94.

[71] L. N. Sretenskii, *On some cases of integration of the gyrostat motion equations*, Dokl. Akad. Nauk SSSR 149 (1963), 292–294 (in Russian).

[72] L. N. Sretenskii, *On certain cases of motion of a heavy rigid body with a gyroscope*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1963, no. 3, 60–71 (in Russian).

[73] J.-M. Strelcyn and S. Wojciechowski, *A method of finding integrals for three-dimensional dynamical systems*, Phys. Lett. A 133 (1988), 207–212.

[74] V. V. Trofimov, *Introduction to the Geometry of Manifolds with Symmetry*, Math. Appl. 270, Kluwer, Dordrecht, 1994.

[75] E. Vessiot, *Gewöhnliche Differentialgleichungen; elementare Integrationsmethoden*, in: Encyklopädie der mathematischen Wissenschaften, Band II, 1. Teil, 1. Hälfte, Teubner, Leipzig, 1900, 232–293.

[76] E. Vessiot, *Méthodes d'intégration élémentaires. Étude des équations différentielles ordinaires au point de vue formel*, in: Encyclopédie des sciences mathématiques pures et appliquées, Tome II (Vol. 3), Équations différentielles ordinaires, 1910, Gauthier-Villars, Paris et Teubner, Leipzig, 65–179; reprint: Éditions Jacques Gabay, Sceaux, 1992.

[77] G. D. Villa Salvador, *Topics in the Theory of Algebraic Function Fields*, Math. Theory Appl., Birkhäuser Boston, Boston, MA, 2006.

[78] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Math. Library, Cambridge Univ. Press, Cambridge, 1995.

[79] E. von Weber, *Partielle Differentialgleichungen. Allgemeine Eigenschaften der Differentialsysteme. Die linearen partiellen Differentialgleichungen erster Ordnung mit einer Umbekanten*, in: Encyklopädie der mathematischen Wissenschaften, Band II, 1. Teil, 1. Hälfte, Teubner, Leipzig, 1900, 294–322.

[80] E. von Weber, *Propriétés générales des systèmes d'équations aux dérivées partielles. Equations linéaires du premier ordre*, in: Encyclopédie des sciences mathématiques pures et appliquées, Tome II (Vol. 4), Équations aux dérivées partielles, Gauthier-Villars, Paris et Teubner, Lepzig, 1913, 1–55; reprint: Éditions Jacques Gabay, Sceaux, 1991.

[81] J. Wittenburg, *Dynamics of Systems of Rigid Bodies*, Teubner, 1972.

[82] S. Wojciechowski, *Construction of integrable Riccati systems by the use of low-dimensional Lie algebras*, in: Structure, Coherence and Chaos in Dynamical Systems (Lyngby, 1986), Manchester Univ. Press, Manchester, 1989, 523–527.

[83] S. Wojciechowski, communication at the conference Euromech 216, Integrable Systems in Nonlinear Analytical Mechanics Leeds, 1986.

[84] S. Wojciechowski, *A new test for integrability of Riccati systems of equations*, in: Finite-Dimensional Integrable Nonlinear Dynamical Systems (Johannesburg, 1988), Singapore, 1988, 46–59.

[85] S. Wojciechowski, *A method of studying integrals of dynamical systems based on Frobenius' integrability theorem*, in: Symmetries in Science, III (Vorarlberg, 1988), Plenum Press, New York, 1989, 493–503.

[86] S. Wojciechowski, *Method of compatible vector field in studying integrability of 3-dimensional dynamical systems*, in: Complexity, Chaos, and Biological Evolution, Plenum Press, New York, 1991, 383–388.

[87] H. M. Yehia, *New integrable case in the dynamics of rigid bodies*, Mech. Res. Comm. 13 (1986), 169–172.

[88] H. M. Yehia, *Rigid Body Dynamics: A Lagrangian Approach*, Birkhäuser, 2022.

[89] S. L. Ziglin, *Bifurcation of solutions and the nonexistence of first integrals in Hamiltonian mechanics. I*, Funktsional. Anal. i Prilozh. 16 (1982), no. 3, 30–41 (in Russian); English transl.: Funct. Anal. Appl. 16 (1982), 181–189.

[90] S. L. Ziglin, *Bifurcation of solutions and the nonexistence of first integrals in Hamiltonian mechanics, II*, Funktsional. Anal. i Prilozhen. 17 (1983), no. 1, 8–23 (in Russian); English transl.: Funct. Anal. Appl. 17 (1983), 8–23.