

MEAN VALUES OF ARITHMETIC FUNCTIONS ON A SPARSE SET AND APPLICATIONS

BY

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Abstract. Let f be an arithmetic function satisfying some simple conditions. The aim of this paper is to establish some asymptotic estimates for the quantities

$$\psi_f(x) := \sum_{n \leq x} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right), \quad \pi_f(x) := \sum_{p \leq x} f\left(\left[\frac{x}{p}\right]\right)$$

for $x \rightarrow \infty$, where $\Lambda(n)$ is the von Mangoldt function and $[t]$ is the integer part of $t \in \mathbb{R}$. These generalise or sharpen some recent results of Saito–Suzuki–Takeda–Yoshida. As an application, we show that

$$\sum_{p \leq x, [x/p] \text{ is prime}} 1 \underset{x \rightarrow \infty}{\sim} \left(\sum_p \frac{1}{p(p+1)} \right) \frac{x}{\log x}.$$

1. Introduction. A fundamental question in analytic number theory is to study the asymptotic behaviour of the counting summatory function of arithmetic functions. A typical example is the Dirichlet divisor problem: Let $\tau(n)$ be the number of divisors of n . Then for $x \rightarrow \infty$, we have

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}),$$

where γ is the Euler constant. Notice that

$$(1.1) \quad \sum_{n \leq x} \tau(n) = \sum_{n \leq x} \left[\frac{x}{n} \right],$$

where $[t]$ denotes the integer part of $t \in \mathbb{R}$. It seems natural to consider a more general problem

$$S_f(x) := \sum_{n \leq x} f\left(\left[\frac{x}{n}\right]\right),$$

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so that (1.1) corresponds the case of $f = \text{id}$. An interesting aspect of this problem is that the set

$$\mathcal{S}(x) := \left\{ \left[\frac{x}{n} \right] : n \in \mathbb{N} \cap [1, x] \right\}$$

is sparse in $\mathbb{N} \cap [1, x]$, since

$$|\mathcal{S}(x)| \underset{x \rightarrow \infty}{\sim} 2\sqrt{x}.$$

Because the Euler function φ is the convolution $\varphi = \text{id} * \mu$ of the identity function id and the Möbius function μ , we could think that φ is rather close to id . Bordellès, Dai, Heyman, Pan and Shparlinski [2] proposed to investigate the asymptotic behaviour of the summatory function $S_\varphi(x)$ as $x \rightarrow \infty$, and proved the following inequalities:

$$(1.2) \quad \left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + o(1) \right) x \log x \leq S_\varphi(x) \leq \left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + \frac{1380}{4009} + o(1) \right) x \log x.$$

They also conjectured that

$$(1.3) \quad S_\varphi(x) \underset{x \rightarrow \infty}{\sim} \frac{6}{\pi^2} x \log x.$$

The bounds in (1.2) have been sharpened by Wu [15]. Recently Zhai [17] resolved the conjecture (1.3) by the Vinogradov method, and a more general result has been established by Ma, Wu and Zhao [8]. Let r_1, r_2, r_3 be three non-decreasing functions defined on $[1, \infty[$ such that

$$(1.4) \quad 1 \leq r_j(x) \ll x^{\eta_j} \quad (x \geq 1, j = 1, 2, 3), \quad r_3(x) \xrightarrow{x \rightarrow \infty} \infty,$$

where $\eta_j \in [0, 1[$ are constants. Let $f = \text{id} * g$ be an arithmetic function satisfying the following conditions:

$$(1.5) \quad |f(n)| \leq nr_1(n) \quad (n \geq 1),$$

$$(1.6) \quad \sum_{n \leq x} |g(n)| \ll xr_2(x) \quad (x \geq 1),$$

$$(1.7) \quad \sum_{n \leq x} g(n) = D_g x + O(x/r_3(x)) \quad (x \geq 1),$$

where D_g is a constant (possibly 0). Ma, Wu and Zhao [8] proved that for any constant $A > 0$, the asymptotic formula

$$(1.8) \quad S_f(x) = C_f x \log x + O_A(x\mathcal{R}(x, z))$$

holds uniformly for $x \geq 3$ and $1 \leq z \leq \exp\{A^{1/3}(\log x)^{2/3}(\log_2 x)^{1/3}\}$, where

$$(1.9) \quad C_f := \sum_{n=1}^{\infty} \frac{g(n)}{n^2}$$

and

$$\mathcal{R}(x, z) := (\log x)^{2/3} (\log_2 x)^{1/3} r_1(x) + \frac{r_2(x/z)}{(\log x)^A} + \frac{r_2(x/z) \log x}{z} + \frac{z \log x}{r_3(\sqrt{x}/z)}.$$

When $f = \varphi$, we can take $A = 1$ and choose

$$r_1(x) = r_2(x) = 1, \quad r_3(x) = \exp(-c(\log x)^{3/5} (\log_2 x)^{-1/5}), \quad z = \log x,$$

where \log_k is the k -fold iterated logarithm and $c > 0$ is a positive constant. Thus (1.8) gives Zhai's result (1.3).

For small arithmetic functions, i.e., $f(n) \ll n^\theta$ ($n \geq 1$) with some $\theta \in [0, 1[$, many authors also studied the asymptotic behaviour of $S_f(x)$; see [2, 16, 17, 20, 12, 3]. In particular, Wu [16] and Zhai [17] proved independently

$$S_f(x) = c_f x + O(x^{(1+\theta)/2}) \quad (x \rightarrow \infty),$$

where

$$(1.10) \quad c_f := \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$

Very recently Saito, Suzuki, Takeda and Yoshida [11] considered a prime analogue of $S_f(x)$. Let $\Omega(n)$ be the number of all prime factors of n . Then it is well known that

$$\sum_{n \leq x} \Omega(n) = \sum_{p \leq x} \left[\frac{x}{p} \right] = x \log_2 x + O(x) \quad (x \rightarrow \infty).$$

Let $\Lambda(n)$ be the von Mangoldt function and define

$$(1.11) \quad \psi_f(x) := \sum_{n \leq x} \Lambda(n) f\left(\left[\frac{x}{n} \right]\right), \quad \pi_f(x) := \sum_{p \leq x} f\left(\left[\frac{x}{p} \right]\right).$$

In [11, Theorems 7 and 9] it was proved that for $x \rightarrow \infty$,

$$(1.12) \quad \psi_\varphi(x) = \frac{6}{\pi^2} x \log x + O(x(\log x)^{2/3} (\log_2 x)^{1/3})$$

and

$$(1.13) \quad \frac{2}{\pi^2} x \log_2 x + O(x \log_3 x) \leq \pi_\varphi(x) \leq \left(\frac{2}{\pi^2} + \frac{2}{3} \right) x \log_2 x + O(x \log_3 x).$$

In [11], Saito, Suzuki, Takeda and Yoshida also studied the asymptotic behaviour of $\psi_f(x)$ and $\pi_f(x)$ for small arithmetic functions. Suppose that f satisfies the following conditions:

(a) There is a constant $\alpha \geq 0$ such that

$$(1.14) \quad \sum_{n \leq x} \frac{|f(n)|}{n} \ll (\log x)^\alpha \quad (x \geq 2).$$

(b) There is a constant $\theta \in [0, 1/2[$ such that

$$(1.15) \quad f(n) \ll n^\theta \quad (n \geq 1).$$

Then by [11, Theorems 5 and 6] there is a positive constant $c > 0$ such that for $x \rightarrow \infty$,

$$(1.16) \quad \psi_f(x) = c_f x + O(x \exp(-c\sqrt{\log x}))$$

and for any non-negative integer K ,

$$(1.17) \quad \pi_f(x) = \frac{x}{\log x} \sum_{k=0}^K \frac{a_k(f)}{(\log x)^k} + O_K\left(\frac{x}{(\log x)^{K+2}}\right),$$

where c_f is defined as in (1.10) and

$$(1.18) \quad a_k(f) := \int_1^\infty \frac{f([u])(\log u)^k}{u^2} du.$$

The aim of this paper is to generalize and sharpen these results.

THEOREM 1.1.

- (i) *Let f be an arithmetic function satisfying conditions (1.5)–(1.7). Then for any $A \geq 1$, there exists a positive constant $B = B(A)$ such that the asymptotic formula*

$$(1.19) \quad \psi_f(x) = C_f x \log x + O_A(x \mathcal{R}_\psi(x, z))$$

holds uniformly for $x \geq 3$ and $(\log x)^B \leq z \leq x^{1/12}$, where C_f is defined as in (1.9) and

$$(1.20) \quad \mathcal{R}_\psi(x, z) := (\log x)^{2/3} (\log_2 x)^{1/3} r_1(x) + \frac{r_2(x/z)}{(\log x)^A} + \frac{\log x}{r_3(x^{1/4}/z)} + \frac{z \log x}{r_3(x/z)}.$$

Here the implied constant depends on A only.

- (ii) *Let f be an arithmetic function satisfying condition (1.7) and there is a positive constant $c < 1$ such that one of the two conditions*

$$(1.21) \quad f(p-1) < cf(1)p \quad \text{or} \quad f(p-1) > c^{-1}f(1)p > 0$$

holds for infinitely many prime numbers p . Then the error term of (1.19) is $\Omega(x)$.

As applications of Theorem 1.1, we consider four special arithmetic functions:

- the Euler function $\varphi(n)$,
- the alternating sum-of-divisors function $\beta(n)$,
- the sum-of-divisors function $\sigma(n)$,
- the Dedekind function $\Psi(n)$.

We have the following relations:

$$(1.22) \quad \varphi = \text{id} * \mu, \quad \beta = \text{id} * \lambda, \quad \sigma = \text{id} * \mathbb{1}, \quad \Psi = \text{id} * \mu^2,$$

where $\lambda(n) := (-1)^{\Omega(n)}$ is the Liouville function and $\mathbb{1}(n) = 1$ for all $n \in \mathbb{N}^*$.

It is well known that they satisfy conditions (1.5)–(1.7) and (1.21).

COROLLARY 1.2.

(i) For $x \rightarrow \infty$, we have

$$(1.23) \quad \psi_\varphi(x) = \frac{6}{\pi^2} x \log x + O(x(\log x)^{2/3}(\log_2 x)^{1/3}),$$

$$(1.24) \quad \psi_\beta(x) = \frac{\pi^2}{15} x \log x + O(x(\log x)^{2/3}(\log_2 x)^{1/3}),$$

$$(1.25) \quad \psi_\sigma(x) = \frac{\pi^2}{6} x \log x + O(x(\log x)^{2/3}(\log_2 x)^{4/3}),$$

$$(1.26) \quad \psi_\Psi(x) = \frac{15}{\pi^2} x \log x + O(x(\log x)^{2/3}(\log_2 x)^{4/3}).$$

(ii) The error terms of (1.23)–(1.26) are $\Omega(x)$.

The next theorem generalizes Saito–Suzuki–Takeda–Yoshida’s (1.13).

THEOREM 1.3.

(i) Let f be an arithmetic function satisfying the conditions $0 \leq f(n) \leq Cn$ ($n \geq 1$) and (1.6) and (1.7). Then for any $A \geq 1$, there exists a positive constant $B = B(A)$ such that

$$(1.27) \quad \frac{C_f}{3} x \log_2 x + O_A(x\mathcal{R}_\pi(x, z)) \leq \pi_f(x) \\ \leq \frac{C_f + 2C}{3} x \log_2 x + O_A(x\mathcal{R}_\pi(x, z))$$

uniformly for $x \geq 3$ and $(\log x)^B \leq z \leq x^{1/12}$, where $C_f := \sum_{n=1}^{\infty} \frac{g(n)}{n^2}$ and

$$(1.28) \quad \mathcal{R}_\pi(x, z) := \log_3 x + \frac{r_2(x/z)}{(\log x)^A} + \frac{\log x}{r_3(x^{1/4}/z)} + \frac{z \log x}{r_3(x/z)}.$$

Here the implied constant depends on A only.

(ii) In particular, (1.13) is true and

$$(1.29) \quad \frac{\pi^2}{45} x \log_2 x + O(x \log_3 x) \leq \pi_\beta(x) \leq \frac{\pi^2 + 30}{45} x \log_2 x + O(x \log_3 x).$$

For small arithmetic functions, we have the following results which improve Saito–Suzuki–Takeda–Yoshida’s (1.16) and (1.17).

THEOREM 1.4. Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be an arithmetic function satisfying

$$(1.30) \quad f(n) \ll n^\theta \quad (n \geq 1)$$

with some constant $\theta \in [0, 1[$.

(i) *There exists a positive constant $c_1 > 0$ such that for $x \rightarrow \infty$, we have*

$$(1.31) \quad \psi_f(x) = c_f x + O(x \exp(-c_1(\log x)^{1/3}(\log_2 x)^{-1/3})),$$

where c_f is defined as in (1.10).

(ii) *Furthermore, if f also satisfies condition (1.14), then the error term in (1.31) can be replaced by $O(x \exp(-c_1(\log x)^{3/5}(\log_2 x)^{-1/5}))$.*

THEOREM 1.5. *Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be an arithmetic function satisfying (1.30).*

(i) *There exists a positive constant $c_2 > 0$ such that for $x \rightarrow \infty$, we have*

$$(1.32) \quad \pi_f(x) = x \int_1^{x^{2/7}} \frac{f([u])}{u^2 \log(x/u)} du + O(x \exp(-c_2(\log x)^{1/3}(\log_2 x)^{-1/3})).$$

Furthermore, if f satisfies condition (1.14), then the error term in (1.31) can be replaced by $O(x \exp(-c_2(\log x)^{3/5}(\log_2 x)^{-1/5}))$.

(ii) *For any non-negative integer K , the asymptotic formula (1.17) holds.*

As an application, we have the following result.

COROLLARY 1.6. *For $x \rightarrow \infty$, we have*

$$(1.33) \quad \sum_{p \leq x, [x/p] \text{ is prime}} 1 = x \int_1^{x^{2/7}} \frac{\mathbb{1}_{\mathbb{P}}([u])}{u^2 \log(x/u)} du + O(x \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5})) = \left(\sum_p \frac{1}{p(p+1)} \right) \frac{x}{\log x} \left\{ 1 + O\left(\frac{1}{\log x} \right) \right\},$$

where $\mathbb{1}_{\mathbb{P}}$ is the characteristic function of the set \mathbb{P} of all prime numbers.

In the sum on the right-hand side of (1.33), we count prime numbers p with multiplicity. It seems more interesting to count without multiplicity. For this we introduce

$$\pi_{\mathcal{S}}(x) := \sum_{\substack{p \leq x \\ \exists n \text{ such that } [x/n]=p}} 1 \quad \text{and} \quad \pi_{\mathcal{S},2}(x) := \sum_{\substack{p \leq x \\ \exists p' \text{ such that } [x/p']=p}} 1.$$

According to [6], we have

$$\pi_{\mathcal{S}}(x) \sim 2x^{1/2}/\log x \quad \text{as } x \rightarrow \infty.$$

A stronger result can be found in [9].

THEOREM 1.7.

- (i) For $x \rightarrow \infty$, we have $\pi_{\mathcal{S},2}(x) \gg x^{19/59}/\log x$.
(ii) Let $\pi(x)$ be the number of primes $\leq x$ and let ε be an arbitrarily small positive number. Suppose that for $y = (\log x)^{2+\varepsilon}$, the inequality

$$(1.34) \quad \pi(x+y) - \pi(x) \gg_{\varepsilon} y/\log x$$

holds for $x \rightarrow \infty$. Then $\pi_{\mathcal{S},2}(x) \gg_{\varepsilon} x^{1/2}/(\log x)^{2+\varepsilon}$.

Noticing that n and $[x/n]$ are symmetric with respect to $x^{1/2}$ logarithmically, it seems interesting to compare Corollary 1.6 and Theorem 1.4 with Goldbach's conjecture and the twin prime conjecture:

$$\sum_{p \leq N, N-p \text{ is prime}} 1 \underset{N \rightarrow \infty}{\sim}^{\text{even}} 2 \prod_{2 < p|N} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{(\log N)^2},$$

$$\sum_{p \leq x, p+2 \text{ is prime}} 1 \underset{x \rightarrow \infty}{\sim} 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \frac{x}{(\log x)^2}.$$

We refer the reader to Granville–Lumley's paper [5] for the hypothesis (1.34).

2. Preliminary lemmas. In this section, we recall some lemmas which will be needed later. The first one is the Vaughan identity [14, formula (3)].

LEMMA 2.1. *There are six real arithmetic functions $\alpha_k(n)$ satisfying*

$$|\alpha_k(n)| \leq \tau(n) \log(2n) \quad (n \geq 1, 1 \leq k \leq 6)$$

such that, for all $N \geq 100$, $N < N' \leq 2N$ and any arithmetic function $a(n)$, we have

$$(2.1) \quad \sum_{N < n \leq N'} \Lambda(n) a(n) = S_1 + S_2 + S_3 + S_4,$$

where $\tau(n)$ is the classical divisor function and

$$S_1 := \sum_{\ell \leq N^{1/3}} \alpha_1(\ell) \sum_{N < \ell m \leq N'} a(\ell m),$$

$$S_2 := \sum_{\ell \leq N^{1/3}} \alpha_2(\ell) \sum_{N < \ell m \leq N'} (\log m) a(\ell m),$$

$$S_3 := \sum_{\substack{N^{1/3} < \ell, m \leq N^{2/3} \\ N < \ell m \leq N'}} \alpha_3(\ell) \alpha_4(m) a(\ell m),$$

$$S_4 := \sum_{\substack{N^{1/3} < \ell, m \leq N^{2/3} \\ N < \ell m \leq N'}} \alpha_5(\ell) \alpha_6(m) a(\ell m).$$

The second lemma is [4, Theorem A.6].

LEMMA 2.2. *Let $\psi(t) := t - [t] - 1/2$ for $t \in \mathbb{R}$. For $x, H \geq 1$, we have*

$$(2.2) \quad \psi(x) = - \sum_{1 \leq |h| \leq H} \Phi \left(\frac{h}{H+1} \right) \frac{e(hx)}{2\pi i h} + R_H(x),$$

where

$$e(t) := e^{2\pi i t}, \quad \Phi(t) := \pi t(1 - |t|) \cot(\pi t) + |t|,$$

and the error term $R_H(x)$ satisfies

$$(2.3) \quad |R_H(x)| \leq \frac{1}{2H+2} \sum_{0 \leq |h| \leq H} \left(1 - \frac{|h|}{H+1} \right) e(hx).$$

The third lemma is an estimate of an exponential sum of Vinogradov type.

LEMMA 2.3. *For any $A \geq 1$, there is a positive constant $B = B(A) > 0$ such that*

$$(2.4) \quad \sum_{N \leq n < N'} e(T/n) \ll N(\log T)^{-A}$$

uniformly for

$$(2.5) \quad \exp(B(\log T)^{2/3}(\log_2 T)^{1/3}) \leq N \leq T(\log T)^{-B}, \quad N < N' \leq 2N,$$

where the implied constant depends on A only.

Proof. According to [19, Lemma 2.2], there is a positive constant c_1 such that

$$\sum_{N \leq n < N'} e(T/n) \ll N \exp(-c_1(\log N)^3/(\log T)^2)$$

uniformly for $N \leq T^{2/3}$. This implies (2.4) provided that

$$\exp(B(\log T)^{2/3}(\log_2 T)^{1/3}) \leq N \leq T^{2/3} \quad \text{with} \quad B = (A/c_1)^{1/3}.$$

Next we suppose that $T^{2/3} \leq N \leq T(\log T)^{-B}$ with $B = A$. Applying the exponent pair $(1/6, 4/6)$ (see [4]), we get

$$\begin{aligned} \sum_{N \leq n < N'} e(T/n) &\ll (T/N)^{1/6} N^{4/6} + (T/N)^{-1} N \\ &\ll (TN^3)^{1/6} + T^{-1} N^2 \ll N(\log T)^{-A}. \end{aligned}$$

Combining these, we get the required result with $B = \max(A, \sqrt[3]{A/c_1})$. ■

The fourth lemma is [19, Lemma 2.4].

LEMMA 2.4. *Assume that the arithmetic function f satisfies conditions (1.6) and (1.7). Then*

$$(2.6) \quad \sum_{n \leq x} f(n) = \frac{1}{2} C_f x^2 - D_g x \frac{(z - [z])^2 + [z]}{2z} + O\left(\frac{x r_2(x/z)}{z} + \frac{xz}{r_3(x/z)}\right) - \Delta_g(x, z)$$

uniformly for $x \geq 2$ and $1 \leq z \leq x^{1/3}$, where the constant C_f is defined as in (1.9) and

$$(2.7) \quad \Delta_g(x, z) := x \sum_{d \leq x/z} \frac{g(d)}{d} \psi\left(\frac{x}{d}\right).$$

Furthermore, for $x \geq 2$ we have

$$(2.8) \quad \sum_{n \leq x} f(n) = \frac{1}{2} C_f x^2 + O(x(\log x) r_2(x)).$$

3. Bound on an average of Δ_g . The aim of this section is to prove the following proposition, which will play a key role in the proof of Theorems 1.1 and 1.3.

PROPOSITION 3.1. *Assume that the arithmetic function f satisfies (1.6) and (1.7). Let $\delta \in \{0, 1\}$, $\eta > \frac{12}{11}$, $A \geq 1$ and let B be the constant determined by Lemma 3.2 below. Define*

$$(3.1) \quad N_0 := \exp(B(\log x)^{2/3}(\log_2 x)^{1/3}).$$

Let $\Delta_g(x, z)$ be defined by (2.7). Then

$$(3.2) \quad \Delta_{g,\delta}^\psi(x, z) := \sum_{N_0 < n \leq x^{1/\eta}} \Lambda(n) \Delta_g\left(\frac{x}{n} - \delta, z\right) \ll_{A,\eta} \frac{x r_2(x/z)}{(\log x)^A} + \frac{x \log x}{r_3(x^{1-1/\eta}/z)}$$

and

$$(3.3) \quad \Delta_{g,\delta}^\pi(x, z) := \sum_{N_0 < p \leq x^{1/\eta}} \Delta_g\left(\frac{x}{p} - \delta, z\right) \ll_{A,\eta} \frac{x r_2(x/z)}{(\log x)^A} + \frac{x \log x}{r_3(x^{1-1/\eta}/z)}$$

uniformly for $x \geq 10$ and $(\log x)^B \leq z \leq x^{1/12}$.

In order to prove Proposition 3.1, we need to establish the following lemma. This lemma is essentially due to Liu [7] (see also [13, Lemmas 6, 7, 12])

and [11, Lemma 22]). For convenience, we give a complete (and simpler) proof.

LEMMA 3.2. *Let $\psi(t) := t - [t] - \frac{1}{2}$ for $t \in \mathbb{R}$ and $\iota \in \mathbb{R}$. Then for any $A > 1$, there is a constant $B = B(A) \geq 2A + 8$ such that the inequalities*

$$(3.4) \quad \Xi(x; N, N') := \sum_{N \leq n < N'} \Lambda(n) \psi\left(\frac{x}{n} - \iota\right) \ll_A N(\log x)^{-(A+1)}$$

and

$$(3.5) \quad \sum_{N \leq n < N'} \frac{\Lambda(n)}{n} \psi\left(\frac{x}{n} - \iota\right) \ll_A (\log x)^{-A},$$

$$(3.6) \quad \sum_{N \leq p < N'} \frac{1}{p} \psi\left(\frac{x}{p} - \iota\right) \ll_A (\log x)^{-A}$$

hold uniformly for $x \geq 10$ and

$$(3.7) \quad \exp(B(\log x)^{2/3}(\log_2 x)^{1/3}) \leq N \leq x(\log x)^{-B}, \quad N < N' \leq 2N.$$

The implied constants depend on A and are independent of ι .

Proof. First we prove (3.4). Using (2.2), for any $H \geq 1$ we can write

$$\begin{aligned} \Xi(x; N, N') &= - \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(-h\iota)}{2\pi i h} \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right) \\ &\quad + \sum_{N \leq n < N'} \Lambda(n) R\left(\frac{x}{n} - \iota\right). \end{aligned}$$

In view of (2.3), it follows that

$$\begin{aligned} |\Xi(x; N, N')| &\leq \left| \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(-h\iota)}{2\pi i h} \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right) \right| \\ &\quad + \frac{1}{2H+2} \sum_{0 \leq |h| \leq H} \left(1 - \frac{|h|}{H+1}\right) e(-h\iota) \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right). \end{aligned}$$

Noticing that $0 < \Phi(t) < 1$ ($0 < |t| < 1$) and using the prime number theorem:

$$(3.8) \quad \sum_{n \leq x} \Lambda(n) = x + O(x \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5})) \quad (x \geq 3),$$

where $c > 0$ is a positive constant, we can derive, for any $H \in [1, N]$,

$$(3.9) \quad \Xi(x; N, N') \ll \frac{N}{H} + \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right) \right|.$$

For the last inner sum, we apply the Vaughan identity (2.1) to write

$$(3.10) \quad \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right) = \mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 + \mathfrak{S}_4,$$

where

$$\begin{aligned} \mathfrak{S}_1 &:= \sum_{d \leq N^{1/3}} \alpha_1(d) \sum_{N < d\ell \leq N'} e\left(\frac{hx}{d\ell}\right), \\ \mathfrak{S}_2 &:= \sum_{d \leq N^{1/3}} \alpha_2(d) \sum_{N < d\ell \leq N'} (\log \ell) e\left(\frac{hx}{d\ell}\right), \\ \mathfrak{S}_3 &:= \sum_{\substack{N^{1/3} < d, \ell \leq N^{2/3} \\ N < d\ell \leq N'}} \alpha_3(d) \alpha_4(\ell) e\left(\frac{hx}{d\ell}\right), \\ \mathfrak{S}_4 &:= \sum_{\substack{N^{1/3} < d, \ell \leq N^{2/3} \\ N < d\ell \leq N'}} \alpha_5(d) \alpha_6(\ell) e\left(\frac{hx}{d\ell}\right). \end{aligned}$$

The sums \mathfrak{S}_1 and \mathfrak{S}_2 are said to be of type I, and \mathfrak{S}_3 and \mathfrak{S}_4 of type II.

A. *Estimates of \mathfrak{S}_1 and \mathfrak{S}_2 .* Since $N \leq x/(\log x)^B$ and $d \leq N^{1/3} \leq x^{1/3}$, we have

$$hx/d \geq x/d \geq (N/d)(\log x)^B \geq (N/d)(\log(x/d))^B \quad \text{and} \quad hx/d \geq x^{2/3}.$$

Thus, we can apply Lemma 2.3 and the inequality $\alpha_1(\ell) \ll \tau(\ell) \log x$ to get

$$(3.11) \quad \begin{aligned} \mathfrak{S}_1 &\ll \sum_{d \leq N^{1/3}} |\alpha_1(d)| (N/d) (\log(hx/d))^{-(A+5)} \\ &\ll N(\log x)^{-(A+4)} \sum_{\ell \leq N^{1/3}} \tau(d)/d \ll N(\log x)^{-(A+2)} \end{aligned}$$

uniformly for $1 \leq h \leq H \leq N$ and $x \geq 3$. The same bound also holds for \mathfrak{S}_2 , since the factor $\log m$ in \mathfrak{S}_2 can be removed by a simple partial integration.

B. *Estimates of \mathfrak{S}_3 and \mathfrak{S}_4 .* We shall only bound \mathfrak{S}_3 , since \mathfrak{S}_4 can be treated in the same way. For this, we shall use two different methods according to the size of N :

$$\exp(B(\log x)^{2/3}(\log_2 x)^{1/3}) \leq N \leq x^{1/2} \quad \text{or} \quad x^{1/2} \leq N \leq x(\log x)^{-B}.$$

First we consider the first case. In view of the symmetry of the variables d and ℓ , we can suppose that $N^{1/2} \leq d \leq N^{2/3}$. Thus using the bound $\alpha_3(\ell) \ll \tau(\ell) \log x$, we can derive

$$\begin{aligned} \mathfrak{G}_3 &\ll (\log x) \sum_{N^{1/2} \leq d \leq N^{2/3}} \tau(d) \left| \sum_{N/d < \ell \leq N'/d} \alpha_4(\ell) e\left(\frac{hx}{d\ell}\right) \right| \\ &\ll (\log x)^2 \max_{N^{1/2} \leq D \leq N^{2/3}} \sum_{d \sim D} \tau(d) \left| \sum_{N/d < \ell \leq N'/d} \alpha_4(\ell) e\left(\frac{hx}{d\ell}\right) \right|, \end{aligned}$$

where $d \sim D$ means that $D < d \leq 2D$. By the Cauchy inequality,

$$\begin{aligned} |\mathfrak{G}_3|^2 &\ll (\log x)^5 \max_{N^{1/2} \leq D \leq N^{2/3}} D \sum_{d \sim D} \tau(d) \left| \sum_{N/d < \ell \leq N'/d} \alpha_4(\ell) e\left(\frac{hx}{d\ell}\right) \right|^2 \\ &\ll (\log x)^7 \max_{N^{1/2} \leq D \leq N^{2/3}} D \sum_{\ell \sim N/D} \sum_{\ell' \sim N/D} \tau(\ell) \tau(\ell') \left| \sum_{d \sim D} \tau(d) e\left(\frac{h(\ell' - \ell)x}{d\ell\ell'}\right) \right|. \end{aligned}$$

Separating the contribution of $\ell = \ell'$ and $\ell \neq \ell'$, we find that

$$(3.12) \quad |\mathfrak{G}_3|^2 \ll (\log x)^{10} \times \max_{N^{1/2} \leq D \leq N^{2/3}} D \left(N + \sum_{\substack{\ell \sim N/D \\ \ell' \sim N/D \\ \ell \neq \ell'}} \tau(\ell) \tau(\ell') |S(D; \ell, \ell')| \right),$$

where

$$S(D; \ell, \ell') := \sum_{d \sim D} \tau(d) e\left(\frac{h(\ell' - \ell)x}{d\ell\ell'}\right).$$

Noticing that $\tau(d) = \sum_{d_1 d_2 = d} 1$ and using the symmetry of d_1 and d_2 , we can suppose that $d_1 \leq d_2$ and write

$$(3.13) \quad \begin{aligned} S(D; \ell, \ell') &= \sum_{D < d_1 d_2 \leq 2D} e\left(\frac{h(\ell' - \ell)x}{d_1 d_2 \ell \ell'}\right) \\ &\ll (\log x) \max_{1 \leq D_1 \leq D^{1/2}} \sum_{d_1 \sim D_1} \left| \sum_{d_2 \sim D/d_1} e\left(\frac{h(\ell' - \ell)x}{d_1 d_2 \ell \ell'}\right) \right|. \end{aligned}$$

Using the hypothesis $N \leq x^{1/2}$ and $\ell \neq \ell'$, it is easy to verify that

$$x^{4/3} \geq Dx \geq \left| \frac{h(\ell' - \ell)x}{d_1 \ell \ell'} \right| \geq \frac{x}{d_1 (N/D)^2} \geq \left(\frac{D}{d_1}\right) x^{1/4}.$$

Thus we can apply Lemma 2.3 to get

$$S(D; \ell, \ell') \ll (\log x) \max_{1 \leq D_1 \leq D^{1/2}} \sum_{d_1 \sim D_1} \frac{D}{d_1} (\log x)^{-(2A+30)} \ll D (\log x)^{-(2A+28)}.$$

Inserting this bound into (3.12), we easily derive

$$(3.14) \quad \mathfrak{G}_3 \ll N (\log x)^{-(A+3)}$$

uniformly for $1 \leq h \leq H \leq N$ and $x \geq 3$. The same bound holds for \mathfrak{S}_4 . Combining (3.11) and (3.14) with (3.10), it follows that

$$\sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right) \ll N(\log x)^{-(A+3)}$$

uniformly for $1 \leq h \leq H \leq N$ and $x \geq 3$. Inserting it into (3.9) and taking $H = (\log x)^{A+1}$, we obtain the required result (3.4).

Next we suppose that $x^{1/2} \leq N \leq x(\log x)^{-B}$. In (3.13), applying the exponent pair (κ, λ) to the inner sum over d and using the fact that $\ell, \ell' \sim N/D$ with $\ell \neq \ell'$, we obtain

$$\begin{aligned} S(D; \ell, \ell') &\ll (\log x) \max_{1 \leq D_1 \leq D^{1/2}} \sum_{d_1 \sim D_1} \left(\left(\frac{h|\ell' - \ell|x}{D(D/d_1)\ell\ell'} \right)^\kappa \left(\frac{D}{d_1} \right)^\lambda + \left(\frac{h|\ell' - \ell|x}{D(D/d_1)\ell\ell'} \right)^{-1} \right) \\ &\ll (\log x)^2 \left(\left(\frac{h|\ell' - \ell|x}{N^2} \right)^\kappa D^{(1+\kappa+\lambda)/2} + \left(\frac{h|\ell' - \ell|x}{N^2} \right)^{-1} \right). \end{aligned}$$

From this, we deduce that

$$\begin{aligned} \max_{N^{1/2} \leq D \leq N^{2/3}} D \sum_{\substack{\ell \sim N/D \\ \ell' \sim N/D \\ \ell \neq \ell'}} \tau(\ell)\tau(\ell') |S(D; \ell, \ell')| \\ \ll (\log x)^6 ((hx)^\kappa N^{(7-5\kappa+\lambda)/4} + (hx)^{-1} N^3). \end{aligned}$$

Inserting it into (3.12) shows that

$$(3.15) \quad \mathfrak{S}_3 \ll (\log x)^8 (N^{5/6} + (hx)^{\kappa/2} N^{(7-5\kappa+\lambda)/8} + (hx)^{-1/2} N^{3/2}).$$

The same bound also holds for \mathfrak{S}_4 . Combining (3.11), (3.15) with (3.9), it follows that

$$\Xi(x; N, N') \ll (\log x)^8 (N^{5/6} + NH^{-1} + (x^{4\kappa} N^{7-5\kappa+\lambda} H^{4\kappa})^{1/8} + (x^{-1} N^3)^{1/2})$$

for all $H \in [1, N]$. Optimizing H over $[1, N]$ and using the hypothesis $x^{1/2} \leq N \leq x(\log x)^{-B}$, we find that

$$\begin{aligned} \Xi(x; N, N') &\ll (\log x)^8 (N^{5/6} + (x^{4\kappa} N^{7-\kappa+\lambda})^{1/(8+4\kappa)} + (x^{4\kappa} N^{7-5\kappa+\lambda})^{1/8} + (x^{-1} N^3)^{1/2}) \\ &\ll (\log x)^8 (N^{5/6} + N^{(7+7\kappa+\lambda)/(8+4\kappa)} + N^{(7+3\kappa+\lambda)/8} + N(\log x)^{-B/2}). \end{aligned}$$

Taking $(\kappa, \lambda) = (\frac{1}{30}, \frac{26}{30})$, we obtain the required inequality (3.4). Then a simple partial integration implies (3.5).

Finally, we prove (3.6). Under the condition (3.7), we have

$$\begin{aligned} \left| \sum_{N \leq p^\nu < N', \nu \geq 2} \frac{\log p}{p^\nu} \psi\left(\frac{x}{p^\nu} - \iota\right) \right| &\leq \sum_{N \leq p^\nu < N', \nu \geq 2} \frac{\log p}{p^\nu} \ll \frac{(\log N)^2}{N} \\ &\ll \exp(-(B/2)(\log x)^{2/3}). \end{aligned}$$

Thus, under the same condition, we have

$$\begin{aligned} &\sum_{N \leq p < N'} \frac{\log p}{p} \psi\left(\frac{x}{p} - \iota\right) \\ &= \sum_{N \leq n < N'} \frac{\Lambda(n)}{n} \psi\left(\frac{x}{n} - \iota\right) + O(\exp(-(B/2)(\log x)^{2/3})) \ll_A (\log x)^{-A}. \end{aligned}$$

From this, a simple partial integration gives us (3.6). ■

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. By (2.7), we can write

$$\begin{aligned} (3.16) \quad \Delta_{g,\delta}^\psi(x, z) &= \sum_{N_0 < n \leq x^{1/\eta}} \Lambda(n) \left(\frac{x}{n} - \delta\right) \sum_{d \leq (x/n-\delta)/z} \frac{g(d)}{d} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right) \\ &= x \sum_{N_0 < n \leq x^{1/\eta}} \frac{\Lambda(n)}{n} \sum_{d \leq x/(nz)} \frac{g(d)}{d} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right) \\ &\quad + O(|E_1| + |E_2|), \end{aligned}$$

where the implied constant is absolute and

$$\begin{aligned} E_1 &:= \sum_{N_0 < n \leq x^{1/\eta}} \Lambda(n) \sum_{d \leq (x/n-\delta)/z} \frac{g(d)}{d} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right), \\ E_2 &:= x \sum_{N_0 < n \leq x^{1/\eta}} \frac{\Lambda(n)}{n} \sum_{(x/n-\delta)/z < d \leq x/(nz)} \frac{g(d)}{d} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right). \end{aligned}$$

In view of the prime number theorem and the hypothesis (1.6), we have

$$E_1 \ll \sum_{n \leq x^{1/\eta}} \Lambda(n) \sum_{d \leq x/z} \frac{|g(d)|}{d} \ll x^{1/\eta} r_2(x/z) \log x.$$

Since $\delta/z < 1$, the hypothesis (1.7) and the prime number theorem allow us to deduce that

$$\begin{aligned} E_2 &\leq x \sum_{n \leq x^{1/\eta}} \frac{\Lambda(n)}{n} \frac{|g([x/(nz)])|}{[x/(nz)]} \ll x \sum_{n \leq x^{1/\eta}} \frac{\Lambda(n)}{n} \frac{1}{r_3([x/(nz)])} \\ &\ll \frac{x \log x}{r_3(x^{1-1/\eta}/z)}. \end{aligned}$$

Inserting these into (3.16) and inverting the order of summations, we get

$$(3.17) \quad \Delta_{g,\delta}^\psi(x, z) = \Delta_{g,\delta}^{\psi,\dagger}(x, z) + O\left(x^{1/\eta} r_2(x/z) \log x + \frac{x \log x}{r_3(x^{1-1/\eta}/z)}\right).$$

where

$$\Delta_{g,\delta}^{\psi,\dagger}(x, z) := x \sum_{d \leq x/(N_0 z)} \frac{g(d)}{d} \sum_{N_0 < n \leq \min\{x^{1/\eta}, x/(dz)\}} \frac{\Lambda(n)}{n} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right).$$

For $0 \leq k \leq (\log(\min\{x^{1/\eta}, x/(dz)\}/N_0))/\log 2$, let $N_k := 2^k N_0$ and define

$$\mathfrak{S}_k(d) := \sum_{N_k < n \leq 2N_k} \frac{\Lambda(n)}{n} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right).$$

Noticing that $N_0 \leq N_k \leq x/(dz) \leq (x/d)(\log x)^{-B} \leq (x/d)(\log(x/d))^{-B}$, we can apply (3.5) with $\iota = \delta/d$ to deduce that

$$\mathfrak{S}_k(d) \ll (\log(x/d))^{-B} \ll (\log x)^{-A-2},$$

since $\log(x/d) \geq \log N_0 \geq B(\log x)^{2/3}$. Note that the implied constant is independent of δ/d . Inserting this into the expression of $\Delta_{g,\delta}^{\psi,\dagger}(x, z)$ and using the condition (1.6), we can deduce that

$$(3.18) \quad \begin{aligned} \Delta_{g,\delta}^{\psi,\dagger}(x, z) &\ll x \sum_{d \leq x/(N_0 z)} \frac{|g(d)|}{d} \sum_{2^k N_0 \leq x/(dz)} |\mathfrak{S}_k(d)| \\ &\ll \frac{x}{(\log x)^{A+1}} \sum_{d \leq x/z} \frac{|g(d)|}{d} \ll \frac{x r_2(x/z)}{(\log x)^A}. \end{aligned}$$

Inserting (3.18) into (3.17), we obtain (3.2).

The inequality (3.3) can be proved in the same way (the only difference is to apply (3.6) in place of (3.5)). ■

4. Proof of Theorem 1.1. First, we divide the sum over n into three parts:

$$(4.1) \quad \psi_f(x) = \psi_{f,1}(x) + \psi_{f,2}(x) + \psi_{f,3}(x),$$

where

$$\begin{aligned} \psi_{f,1}(x) &:= \sum_{n \leq N_0} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right), \\ \psi_{f,2}(x) &:= \sum_{N_0 < n \leq x^{3/4}} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right), \\ \psi_{f,3}(x) &:= \sum_{x^{3/4} < n \leq x} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right). \end{aligned}$$

Here $N_0 := \exp(B(\log x)^{2/3}(\log_2 x)^{1/3})$ is defined as in (3.1).

In view of the hypothesis (1.5) and the prime number theorem (3.8), we have

$$(4.2) \quad \psi_{f,1}(x) \leq \sum_{n \leq N_0} \Lambda(n) \frac{x}{n} r_1(x) \ll x r_1(x) \log N_0.$$

Noticing that $d = [x/n]$ is equivalent to $x/n - 1 < d \leq x/n$ and using Lemma 2.4, for any $(\log x)^B \leq z \leq x^{1/12}$ ($\leq (x/n)^{1/3}$) we can write

$$\begin{aligned} \psi_{f,2}(x) &= \sum_{N_0 < n \leq x^{3/4}} \Lambda(n) \sum_{x/n-1 < d \leq x/n} f(d) \\ &= \sum_{N_0 < n \leq x^{3/4}} \Lambda(n) \left\{ \frac{1}{2} C_f \left(\frac{2x}{n} - 1 \right) - D_g \frac{(z - [z])^2 + [z]}{2z} \right. \\ &\quad \left. + O \left(\frac{xz}{nr_3(x/z)} + \frac{xr_2(x/z)}{nz} \right) - \Delta_g \left(\frac{x}{n}, z \right) + \Delta_g \left(\frac{x}{n} - 1, z \right) \right\}, \end{aligned}$$

where $\Delta_g(x, z)$ is defined by (2.7). With the help of (3.8) and (3.2) with $\eta = \frac{4}{3}$, we easily see that

$$(4.3) \quad \psi_{f,2}(x) = \frac{3}{4} C_f x \log x + O \left(\frac{xr_2(x/z)}{(\log x)^A} + \frac{xz \log x}{r_3(x/z)} + \frac{x \log x}{r_3(x^{1/4}/z)} \right),$$

where we have removed the term $xr_2(x/z)(\log x)/z$ (since it can be bounded by $xr_2(x/z)/(\log x)^A$).

Finally, we evaluate $\psi_{f,3}(x)$. Write

$$\psi_{f,3}(x) = \sum_{d \leq x^{1/4}} f(d) \sum_{\max\{x/(d+1), x^{3/4}\} < n \leq x/d} \Lambda(n) = \psi_{f,3}^\dagger(x) + \psi_{f,3}^\sharp(x),$$

where

$$\begin{aligned} \psi_{f,3}^\dagger(x) &:= \sum_{d \leq x^{1/4}} f(d) \sum_{x/(d+1) < n \leq x/d} \Lambda(n), \\ \psi_{f,3}^\sharp(x) &:= \sum_{x^{1/4}-1 < d \leq x^{1/4}} f(d) \sum_{x/(d+1) < n \leq x/d} \Lambda(n). \end{aligned}$$

It is easy to bound $\psi_{f,3}^\sharp(x)$. In view of (1.5), we have

$$\begin{aligned} \psi_{f,3}^\sharp(x) &\ll x(\log x) \sum_{x^{1/4}-1 < d \leq x^{1/4}} \frac{f(d)}{d^2} \\ &\ll x(\log x) \frac{f([x^{1/4}])}{[x^{1/4}]^2} \ll x^{3/4} r_1(x) \log x. \end{aligned}$$

In order to estimate $\psi_{f,3}^\dagger(x)$, first we recall Huxley's prime number theorem in short intervals (see [10]): For any $\varepsilon > 0$,

$$(4.4) \quad \sum_{x < n \leq x+y} \Lambda(n) = y \{1 + O(\exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3}))\},$$

$$(4.5) \quad \sum_{x < p \leq x+y} 1 = \int_x^{x+y} \frac{dt}{\log t} \{1 + O(\exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3}))\}$$

hold uniformly for $x \geq 3$ and $x^{7/12+\varepsilon} \leq y \leq x$. When $d \leq x^{1/4}$, we have $x/d \geq x^{3/4}$ and $x/(d(d+1)) \geq (x/(d+1))^{3/5}$. Thus we can apply (4.4) to deduce that

$$\psi_{f,3}^\dagger(x) = \sum_{d \leq x^{1/4}} f(d) \frac{x}{d(d+1)} \{1 + O(\exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3}))\}.$$

On the other hand, by (2.8) it follows that

$$\begin{aligned} \sum_{d \leq x^{1/4}} \frac{f(d)}{d(d+1)} &= \sum_{d \leq x^{1/4}} \frac{f(d)}{d^2} - \sum_{d \leq x^{1/4}} \frac{f(d)}{d^2(d+1)} \\ &= \int_{1-}^{x^{1/4}} t^{-2} d\left(\frac{1}{2}C_f t^2 + O(tr_2(t) \log t)\right) + O(1) \\ &= \frac{1}{4}C_f \log x + O(1). \end{aligned}$$

Combining these estimates, we find that

$$(4.6) \quad \psi_{f,3}(x) = \frac{1}{4}C_f x \log x + O(x + x^{3/4}r_1(x) \log x).$$

Now the required result (1.19) follows from (4.1), (4.2), (4.3) and (4.6).

Finally, we prove the second assertion. Let $E(x)$ be the error term of (1.19), i.e.

$$\psi_f(x) = C_f x \log x + E(x),$$

and define $E^*(x) := \max\{|E(x)|, |E(x-1)|\}$.

Putting $d = [x/n]$, we have $x/n - 1 < d \leq x/n$ and $x/(d+1) < n \leq x/d$. Thus,

$$\psi_f(x) = \sum_{d \leq x} f(d) \sum_{x/(d+1) < n \leq x/d} \Lambda(n) = \sum_{dn \leq x} f(d)\Lambda(n) - \sum_{(d+1)n \leq x} f(d)\Lambda(n).$$

Setting $f(0) = 0$, we can write

$$(4.7) \quad \psi_f(x) = \sum_{dn \leq x} (f(d) - f(d-1))\Lambda(n) = \sum_{m \leq x} \sum_{d|m} (f(d) - f(d-1))\Lambda\left(\frac{m}{d}\right).$$

First, we suppose that

$$(4.8) \quad f(p-1) < cf(1)p$$

for infinitely many primes p . In view of (4.7), for each prime p we can write

$$\begin{aligned}
\sum_{d|2p} (f(d) - f(d-1))\Lambda(2p/d) &= \psi_f(2p) - \psi_f(2p-1) \\
&= C_f(2p \log(2p) - (2p-1) \log(2p-1)) + E(2p) - E(2p-1) \\
&\leq 2E^*(2p) + O(\log p).
\end{aligned}$$

On the other hand, our hypothesis (4.8) and (1.7) allow us to deduce that

$$\begin{aligned}
\sum_{d|2p} (f(d) - f(d-1))\Lambda(2p/d) \\
&= (f(p) - f(p-1)) \log 2 + (f(2) - f(1)) \log p \\
&= (g(1)p + g(p)f(1) - f(p-1)) \log 2 + (f(2) - f(1)) \log p \\
&\geq (1-c)f(1)p \log 2 + O(p/r_3(p)) \geq \frac{1}{2}(1-c)f(1)p
\end{aligned}$$

for infinitely many primes p . Combining these, we find that

$$E^*(2p) \geq \frac{1}{5}(1-c)f(1)p$$

for infinitely many primes p .

Next we suppose that

$$(4.9) \quad f(p-1) > c^{-1}f(1)p > 0$$

for infinitely many primes p . As before, for each prime p we can write

$$\sum_{d|2p} (f(d) - f(d-1)) \geq -2E^*(p) + O(\log p).$$

On the other hand, our hypothesis (4.9) and (1.7) allow us to deduce that

$$\sum_{d|2p} (f(d) - f(d-1)) \leq -\frac{1}{2}(c^{-1} - 1)f(1)p$$

for infinitely many primes p . Combining these, we find that

$$E^*(p) \geq \frac{1}{5}(c^{-1} - 1)f(1)p > 0$$

for infinitely many primes p . ■

5. Proof of Corollary 1.2

5.1. Proof of (1.23) and (1.24). Since $\varphi = \text{id} * \mu$ and $\beta = \text{id} * \lambda$, we have $g = \mu$ or $g = \lambda$ and the well-known bound

$$\sum_{n \leq x} g(n) \ll x \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5}) \quad (x \geq 2).$$

Thus φ and β satisfy the conditions (1.5)–(1.7) with $D_g = 0$ and

$$r_1(x) = 1, \quad r_2(x) = 1, \quad r_3(x) = \exp(-c_4(\log x)^{3/5}(\log_2 x)^{-1/5}).$$

The asymptotic formulas (1.23)–(1.24) follow from Theorem 1.1 with $A = 1$ and $z = \log^{B(1)}(3x)$.

5.2. Proof of (1.25) and (1.26). In this case, we have $g = \mathbb{1}$ or μ^2 and

$$\sum_{n \leq x} g(n) = \begin{cases} x + O(1) & \text{if } g = \mathbb{1}, \\ (6/\pi^2)x + O(\sqrt{x}) & \text{if } g = \mu^2. \end{cases}$$

Thus the functions $\sigma(n)$ and $\Psi(n)$ satisfy (1.5)–(1.7) with $D_{\mathbb{1}} = 1$ or $D_{\Psi} = 6/\pi^2$ and

$$r_1(x) = C \log_2(3x), \quad r_2(x) = 1, \quad r_3(x) = \sqrt{x},$$

where $C > 0$ is a positive constant. The asymptotic formulas (1.25) and (1.26) follow from Theorem 1.1 with $A = 1$ and $z = \log^{B(1)}(3x)$.

5.3. Proof of the second assertion. First we verify that the functions φ , σ , β and Ψ satisfy the condition (1.21) of Theorem 1.1.

For each odd prime p , we can write $p - 1 = 2^\nu m$ with $2 \nmid m$. Thus

$$\varphi(p - 1) = \varphi(2^\nu)\varphi(m) \leq 2^{\nu-1}m < \frac{1}{2}p$$

for all odd primes p .

For all primes p and integers $\nu \geq 1$, we have

$$\beta(p^\nu) = \sum_{0 \leq j \leq \nu} (-1)^{\nu-j} p^j = \frac{p^{\nu+1} + (-1)^\nu}{p + 1} \leq \begin{cases} \frac{3}{4}2^\nu & \text{if } p = 2, \\ p^\nu & \text{otherwise.} \end{cases}$$

For each odd prime p , we can write $p - 1 = 2^\nu m$ with $2 \nmid m$. Thus

$$\beta(p - 1) = \beta(2^\nu)\beta(m) \leq \frac{3}{4}(p - 1) \leq \frac{3}{4}p$$

for all odd primes p .

For all odd primes p , we have

$$\sigma(p - 1) \geq (p - 1) + \frac{1}{2}(p - 1) + 1 > \frac{5}{4}p.$$

For all primes p and integers $\nu \geq 1$, we have

$$\Psi(p^\nu) = p^\nu + p^{\nu-1} \geq \begin{cases} \frac{3}{2}2^\nu & \text{if } p = 2, \\ p^\nu & \text{otherwise.} \end{cases}$$

For each odd prime p , we can again write $p - 1 = 2^\nu m$ with $2 \nmid m$. Thus

$$\Psi(p - 1) = \Psi(2^\nu)\Psi(m) \geq \frac{3}{2}(p - 1) \geq \frac{5}{4}p$$

for all odd primes $p \geq 7$.

Thus Theorem 1.1(ii) implies that the error terms of (1.23)–(1.26) are $\Omega(x)$. ■

6. Proof of Theorem 1.3. First we divide the sum over n into three parts:

$$(6.1) \quad \pi_f(x) = \pi_{f,1}(x) + \pi_{f,2}(x) + \pi_{f,3}(x),$$

where

$$\begin{aligned}\pi_{f,1}(x) &:= \sum_{p \leq N_0} f\left(\left[\frac{x}{p}\right]\right), \\ \pi_{f,2}(x) &:= \sum_{N_0 < p \leq x^{3/4}} f\left(\left[\frac{x}{p}\right]\right), \\ \pi_{f,3}(x) &:= \sum_{x^{3/4} < p \leq x} f\left(\left[\frac{x}{p}\right]\right).\end{aligned}$$

Since $0 \leq f(n) \leq Cn$ ($n \geq 1$), the prime number theorem allows us to deduce that

$$(6.2) \quad 0 \leq \pi_{f,1}(x) \leq C \sum_{p \leq N_0} \frac{x}{p} = \frac{2}{3}Cx \log_2 x + O(x \log_3 x),$$

$$(6.3) \quad \pi_{f,3}(x) \leq C \sum_{x^{3/4} < p \leq x} \frac{x}{p} \ll x.$$

Noticing that $d = [x/p]$ is equivalent to $x/p - 1 < d \leq x/p$, for $(\log x)^B \leq z \leq x^{1/12}$ ($\leq (x/p)^{1/3}$) we can apply Lemma 2.4 to write

$$\begin{aligned}\pi_{f,2}(x) &= \sum_{N_0 < p \leq x^{3/4}} \sum_{x/p-1 < d \leq x/p} f(d) \\ &= \sum_{N_0 < p \leq x^{3/4}} \left\{ \frac{1}{2}C_f \left(\frac{2x}{p} - 1 \right) - D_g \frac{(z - [z])^2 + [z]}{2z} \right. \\ &\quad \left. + O\left(\frac{xz}{pr_3(x/z)} + \frac{xr_2(x/z)}{pz} \right) - \Delta_g \left(\frac{x}{p}, z \right) + \Delta_g \left(\frac{x}{p} - 1, z \right) \right\},\end{aligned}$$

where $\Delta_g(x, z)$ is defined by (2.7). By the prime number theorem and (3.3) with $\eta = \frac{4}{3}$, we easily derive

$$(6.4) \quad \pi_{f,2}(x) = \frac{1}{3}C_f x \log_2 x + O\left(\frac{xr_2(x/z)}{(\log x)^A} + \frac{xz \log x}{r_3(x/z)} + \frac{x \log x}{r_3(x^{1/4}/z)} + x \right),$$

where we have removed the term $\frac{xr_2(x/z) \log x}{z}$ (since it can be bounded by $\frac{xr_2(x/z)}{(\log x)^A}$). Now the required result (1.27) follows from (6.1)–(6.4).

Next we prove the second assertion. In the proof of Corollary 1.2, we have seen that φ and β satisfy the conditions (1.6) and (1.7) with $D_g = 0$, $r_2(x) = 1$ and $r_3(x) = \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5})$. On the other hand, we have trivially $\varphi(n) \leq n$ and $\beta(n) \leq n$ for all $n \geq 1$. Thus the first assertion immediately gives the second. ■

7. Proof of Theorem 1.4

7.1. Proof of the first assertion. Write

$$(7.1) \quad \psi_f(x) = \psi_f^\dagger(x) + \psi_f^\sharp(x),$$

where

$$\psi_f^\dagger(x) := \sum_{n \leq x^{5/7}} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right), \quad \psi_f^\sharp(x) := \sum_{x^{5/7} < n \leq x} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right).$$

In view of the hypothesis (1.30) and the prime number theorem (3.8), we have

$$(7.2) \quad \psi_f^\dagger(x) \ll x^\theta \sum_{n \leq x^{5/7}} \frac{\Lambda(n)}{n^\theta} \ll x^{(5+2\theta)/7}.$$

On the other hand, noticing that $m = [x/n] \Leftrightarrow x/(m+1) < n \leq x/m$, we can write

$$\begin{aligned} \psi_f^\sharp(x) &= \sum_{m \leq x^{2/7}} f(m) \sum_{\max\{x^{5/7}, x/(m+1)\} < n \leq x/m} \Lambda(n) \\ &= \sum_{m \leq x^{2/7}} f(m) \sum_{x/(m+1) < n \leq x/m} \Lambda(n) + O(x^{(5+2\theta)/7}), \end{aligned}$$

where we have used the bound

$$\begin{aligned} \sum_{x^{2/7}-1 < m \leq x^{2/7}} f(m) \sum_{x^{5/7} < n \leq x/m} \Lambda(n) &\ll \sum_{n \leq 2x^{5/7}} \Lambda(n) \sum_{x^{2/7}-1 < m \leq x^{2/7}} f(m) \\ &\ll x^{2\theta/7} \sum_{n \leq 2x^{5/7}} \Lambda(n) \ll x^{(5+2\theta)/7}. \end{aligned}$$

On the other hand, it is easy to verify that for $m \leq x^{2/7}$ we have $x/m^2 \geq (x/m)^{3/5}$. Applying Huxley's prime number theorem in short intervals of the form (4.4), we can deduce that

$$(7.3) \quad \begin{aligned} \psi_f^\sharp(x) &= \sum_{m \leq x^{2/7}} \frac{xf(m)}{m(m+1)} \{1 + O(\exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3}))\} \\ &\quad + O(x^{(5+2\theta)/7}). \end{aligned}$$

Using the hypothesis (1.30), we have

$$\begin{aligned} \sum_{m \leq x^{2/7}} \frac{|f(m)|}{m(m+1)} &\ll \sum_{m=1}^{\infty} \frac{1}{m^{2-\theta}} \ll 1, \\ x \sum_{m \leq x^{2/7}} \frac{f(m)}{m(m+1)} &= c_f x + O(x^{(5+2\theta)/7}). \end{aligned}$$

Inserting these into (7.3) yields

$$(7.4) \quad \psi_f^\sharp(x) = c_f x + O(x \exp(-c_1(\log x)^{1/3}(\log_2 x)^{-1/3})).$$

Now the required result follows from (7.1), (7.2) and (7.4).

7.2. Proof of the second assertion. Now suppose that f also satisfies the hypothesis (1.30). In place of (7.3), we have, thanks to the prime number theorem (3.8) and (1.30),

$$\begin{aligned} \psi_f^\sharp(x) &= \sum_{m \leq x^{2/7}} f(m) \left\{ \frac{x}{m(m+1)} + O\left(\frac{x}{m} \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5})\right) \right\} \\ &\quad + O(x^{(5+2\theta)/7}) \\ &= c_f x + O(x \exp(-c_1(\log x)^{3/5}(\log_2 x)^{-1/5})). \end{aligned}$$

Now the required result follows from this, (7.1) and (7.2). ■

8. Proof of Theorem 1.5

8.1. Proof of the first assertion. In view of the hypothesis (1.30), we have

$$\sum_{p \leq x^{5/7}} f\left(\left[\frac{x}{p}\right]\right) \ll x^\theta \sum_{p \leq x^{5/7}} \frac{1}{p^\theta} \ll \frac{x^{(5+2\theta)/7}}{\log x}.$$

Thus, we can write

$$(8.1) \quad \pi_f(x) = \pi_f^\sharp(x) + O(x^{(5+2\theta)/7}/\log x),$$

where

$$\pi_f^\sharp(x) := \sum_{x^{5/7} < p \leq x} f\left(\left[\frac{x}{p}\right]\right).$$

Using Huxley's prime number theorem in short intervals of the form (4.5), we have

$$\begin{aligned} (8.2) \quad \pi_f^\sharp(x) &= \sum_{m \leq x^{2/7}} f(m) \sum_{\max\{x^{5/7}, x/(m+1)\} < p \leq x/m} 1 \\ &= \sum_{m \leq x^{2/7}} f(m) \sum_{x/(m+1) < p \leq x/m} 1 + O\left(\frac{x^{(5+2\theta)/7}}{\log x}\right) \\ &= \sum_{m \leq x^{2/7}} f(m) \int_{x/(m+1)}^{x/m} \frac{dt}{\log t} \{1 + O(e^{-c(\log x)^{1/3}(\log_2 x)^{-1/3}})\} \\ &\quad + O\left(\frac{x^{(5+2\theta)/7}}{\log x}\right). \end{aligned}$$

Inverting the order of summation, it follows that

$$\begin{aligned} \sum_{m \leq x^{2/7}} f(m) \int_{x/(m+1)}^{x/m} \frac{dt}{\log t} &= \int_{x/(x^{2/7}+1)}^x \sum_{x/t-1 < m \leq \min\{x^{2/7}, x/t\}} f(m) \frac{dt}{\log t} \\ &= \int_{x^{5/7}}^x \sum_{x/t-1 < m \leq x/t} f(m) \frac{dt}{\log t} + O\left(\frac{x^{(3+2\theta)/7}}{\log x}\right) \\ &= \int_{x^{5/7}}^x \frac{f([x/t])}{\log t} dt + O\left(\frac{x^{(3+2\theta)/7}}{\log x}\right). \end{aligned}$$

Inserting this into (8.2) and making the change of variables $u = x/t$, it follows that

$$(8.3) \quad \pi_f^\sharp(x) = x \int_1^{x^{2/7}} \frac{f([u])}{u^2 \log(x/u)} du + O(x \exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3})).$$

Now the required result (1.32) follows from (8.1) and (8.3).

Next we suppose that f also satisfies the hypothesis (1.14). As before, in place of (8.2) and (8.3), we have

$$\begin{aligned} (8.4) \quad \pi_f^\sharp(x) &= \sum_{m \leq x^{2/7}} f(m) \left\{ \int_{x/(m+1)}^{x/m} \frac{dt}{\log t} + O\left(\frac{x}{m} e^{-c(\log x)^{3/5}(\log_2 x)^{-1/5}}\right) \right\} \\ &\quad + O\left(\frac{x^{(5+2\theta)/7}}{\log x}\right) \\ &= x \int_1^{x^{2/7}} \frac{f([u])}{u^2 \log(x/u)} du + O(x e^{-c_2(\log x)^{3/5}(\log_2 x)^{-1/5}}). \end{aligned}$$

From (8.1) and (8.4), we can obtain the required result.

8.2. Proof of the second assertion. We have

$$\begin{aligned} x \int_1^{x^{2/7}} \frac{f([u])}{u^2 \log(x/u)} du &= \frac{x}{\log x} \int_1^{x^{2/7}} \frac{f([u]) du}{u^2(1 - (\log u)/\log x)} \\ &= \frac{x}{\log x} \int_1^{x^{2/7}} \left\{ \sum_{k=0}^K \frac{(\log u)^k}{(\log x)^k} + O\left(\frac{(\log u)^{K+1}}{(\log x)^{K+1}}\right) \right\} \frac{f([u])}{u^2} du \\ &= \frac{x}{\log x} \left\{ \sum_{k=0}^K \frac{a_k(f)}{(\log x)^k} + O\left(\frac{1}{(\log x)^{K+1}}\right) \right\}, \end{aligned}$$

where we have used the hypothesis (1.30) to get

$$\int_{x^{2/7}}^{\infty} \frac{f([u])(\log u)^k}{u^2} du \ll \int_{x^{2/7}}^{\infty} \frac{(\log u)^k}{u^{2-\theta}} du \ll \frac{(\log x)^k}{x^{2(1-\theta)/7}},$$

$$\int_1^{x^{2/7}} \frac{f([u])(\log u)^{K+1}}{u^2} du \ll \int_1^{\infty} \frac{(\log u)^{K+1}}{u^{2-\theta}} du \ll 1.$$

Combining these we can obtain

$$(8.5) \quad \pi_f^\sharp(x) = \frac{x}{\log x} \left\{ \sum_{k=0}^K \frac{a_k(f)}{(\log x)^k} + O\left(\frac{1}{(\log x)^{K+1}}\right) \right\}.$$

Inserting (8.5) into (8.1), we obtain the desired result. ■

9. Proof of Corollary 1.6. Clearly the characteristic function of \mathbb{P} satisfies the hypothesis (1.30) with $\theta = 0$ and (1.14) with $\alpha = 1$. Thus Theorem 1.4 gives us the required result by noticing that

$$a_0 = \int_1^{\infty} \frac{\mathbb{1}_{\mathbb{P}}([u])}{u^2} du = \sum_p^{p+1} \int_p \frac{1}{u^2} du = \sum_p \frac{1}{p(p+1)}.$$

10. Proof of Theorem 1.4. First we note that

$$\mathcal{S}(x) = \{p \in \mathbb{P} : \exists n \in [1, x] \text{ such that } [x/n] = p\}.$$

Further, if $[x/p'] = p \in \mathbb{P}$, then $x/(p+1) < p' \leq x/p$. Thus we can write

$$\pi_{\mathcal{S},2}(x) = \sum_{p \leq x} \mathbb{1}([x/p] - [x/(p+1)]) \cap \mathbb{P} \neq \emptyset.$$

For $p \leq x^{19/59}$, we have

$$\left[\frac{x}{p} \right] - \left[\frac{x}{p+1} \right] > \frac{x}{p(p+1)} - 1 \geq \frac{1}{2} \left(\frac{x}{p} \right)^{21/40}.$$

According to Baker–Harman [1, Theorem 1], for $p \leq x^{19/59}$ we have

$$([x/p] - [x/(p+1)]) \cap \mathbb{P} \neq \emptyset.$$

Thus

$$\pi_{\mathcal{S},2}(x) \geq \pi(x^{19/59}).$$

For $p \leq x^{1/2}/(\log x)^{1+\varepsilon}$, we have $x/(p(p+1)) \geq (\log(x/p))^{2+\varepsilon}$. Thus under the hypothesis (1.34), we have

$$\pi_{\mathcal{S},2}(x) \geq \pi(x^{1/2}/(\log x)^{1+\varepsilon}). \quad \blacksquare$$

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REFERENCES

- [1] R. C. Baker, G. Harman and J. Pintz, *The difference between consecutive primes. II*, Proc. London Math. Soc. (3) 83 (2001), 532–562.
- [2] O. Bordellès, L. Dai, R. Heyman, H. Pan and I. E. Shparlinski, *On a sum involving the Euler function*, J. Number Theory 202 (2019), 278–297.
- [3] B. Chen, J. Wu and F. Zhao, *Note on a paper by Bordellès, Dai, Heyman, Pan and Shparlinski, 3*, Publ. Math. Debrecen 102 (2023), 495–505.
- [4] S. W. Graham and G. Kolesnik, *Van der Corput’s Method of Exponential Sums*, Cambridge Univ. Press, Cambridge, 1991.
- [5] A. Granville and A. Lumley, *Primes in short intervals: heuristics and calculations*, Exp. Math. 32 (2023), 378–404.
- [6] R. Heyman, *Primes in floor function sets*, Integers 22 (2022), art. A59, 10 pp.
- [7] H.-Q. Liu, *On Euler’s function*, Proc. Roy. Soc. Edinburgh 146A (2016), 769–775.
- [8] J. Ma, J. Wu and F. Zhao, *On a generalisation of Bordellès–Dai–Heyman–Pan–Shparlinski’s conjecture*, J. Number Theory 236 (2022), 334–348.
- [9] R. Ma and J. Wu, *On the primes in floor function sets*, Bull. Austral. Math. Soc. 108 (2023), 236–243.
- [10] K. Ramachandra, *Some problems of analytic number theory*, Acta Arith. 31 (1976), 313–324.
- [11] K. Saito, Y. Suzuki, W. Takeda and Y. Yoshida, *Some remarks on the $[x/n]$ -sequence*, Res. Number Theory 11 (2025), art. 50, 28 pp.
- [12] J. Stucky, *The fractional sum of small arithmetic functions*, J. Number Theory 238 (2022), 731–739.
- [13] Y. Suzuki, *On error term estimates à la Walfisz for mean values of arithmetic functions*, arXiv:1811.02556v2 (2018).
- [14] R. C. Vaughan, *An elementary method in prime number theory*, Acta Arith. 37 (1980), 111–115.
- [15] J. Wu, *On a sum involving the Euler totient function*, Indag. Math. 30 (2019), 536–541.
- [16] J. Wu, *Note on a paper by Bordellès, Dai, Heyman, Pan and Shparlinski*, Period. Math. Hungar. 80 (2020), 95–102.
- [17] W. Zhai, *On a sum involving the Euler function*, J. Number Theory 211 (2020), 199–219.
- [18] W. Zhai, *On a sum involving small arithmetic functions*, Int. J. Number Theory 18 (2022), 2029–2052.
- [19] F. Zhao and J. Wu, *On the sum involving the sum-of-divisors function*, J. Math. 2021, art. 5574465, 7 pp.
- [20] F. Zhao and J. Wu, *Note on a paper by Bordellès, Dai, Heyman, Pan and Shparlinski, 2*, Acta Arith. 202 (2022), 185–194.

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