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LARGE ORBITS OF NILPOTENT SUBGROUPS OF SOLVABLE LINEAR GROUPS

BY

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Abstract. Suppose that G is a finite solvable group and V is a finite, faithful and completely reducible G-module. Let H be a nilpotent subgroup of G. Then there exists $v \in V$ such that $|\mathbf{C}_H(v)| \leq (|H|/p)^{1/p}$, where $\mathbf{C}_H(v)$ is the centralizer of v in H and p is the smallest prime divisor of $|H|$.

1. Introduction. Let G be a finite group and V a finite, faithful and completely reducible G-module. It is a classical topic to study the orbit structure of G acting on V. One of the most important and natural questions about the orbit structure is to establish the existence of an orbit of a certain size. For a long time, there has been a deep interest and need to examine the largest possible size orbits in linear group actions. The orbit $\{v^g \mid g \in G\}$ is called regular if $\mathbf{C}_G(v) = 1$, or equivalently the size of the orbit v^G is $|G|$.

Isaacs proved the following result [6, Theorem A]. Let N be a nontrivial p-group that acts faithfully on a group H, where $|H|$ is not divisible by p, then there exists an element $x \in H$ such that $|\mathbf{C}_N(x)| \leq (|N|/p)^{1/p}$ where p is the smallest prime divisor of $|N|$. By the Hartley–Turull Lemma [3, Lemma 2.6.2], this could be reduced to the following statement: Let P be a nontrivial p-group that acts faithfully on a vector space V, where $|V|$ is not divisible by p. Then there exists an element $v \in V$ such that $|\mathbf{C}_P(v)| \leq (|P|/p)^{1/p}$. We remark that Isaacs's proof was extremely elegant and he found a way to avoid the detailed analysis of primitive linear groups.

Our result generalizes the case of the linear group actions and shows that the same bound holds for an arbitrary p-subgroup of a solvable linear group where the action is completely reducible. This can be used to strengthen a result of Keller and the second author [7, Theorem 1.2]. The technique of our proof was developed in some recent papers of the second author [9, 14]. The method, in some cases, can provide a good estimation on the size of the orbit.

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2. Notation and lemmas. If V is a finite vector space of dimension n over $\mathrm{GF}(q)$, where q is a prime power, we denote by $\Gamma(q^n) = \Gamma(V)$ the semilinear group of V, i.e.,

$$\Gamma(q^n) = \{x \mapsto ax^\sigma \mid x \in \mathrm{GF}(q^n),\, a \in \mathrm{GF}(q^n)^\times,\, \sigma \in \mathrm{Gal}(\mathrm{GF}(q^n)/\mathrm{GF}(q))\},$$

and we define

$$\Gamma_0(q^n) = \{x \mapsto ax \mid x \in \mathrm{GF}(q^n),\, a \in \mathrm{GF}(q^n)^\times\}.$$

We use $H \wr S$ to denote the wreath product of H with S where H is a group and S is a permutation group.

DEFINITION 2.1. Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V. Let $\mathbf{F}(G)$ be the Fitting subgroup of G and $\mathbf{F}(G) = \prod_i^m P_i$, where P_i are normal p_i-subgroups of G for different primes p_i. Let $Z_i = \Omega_1(\mathbf{Z}(P_i))$. We define

$$E_i = \begin{cases} \Omega_1(P_i) & \text{if } p_i \text{ is odd,} \\ [P_i, G, \ldots, G] & \text{if } p_i = 2 \text{ and } [P_i, G, \ldots, G] \neq 1, \\ Z_i & \text{otherwise.} \end{cases}$$

By proper reordering we may assume that $E_i \neq Z_i$ for $i = 1, \ldots, s$, $0 \leq s \leq m$ and $E_i = Z_i$ for $i = s+1, \ldots, m$. We define $E = \prod_{i=1}^s E_i$, $Z = \prod_{i=1}^s Z_i$ and $\bar{E}_i = E_i/Z_i$, $\bar{E} = E/Z$. Furthermore, we define $e_i = \sqrt{|E_i/Z_i|}$ for $i = 1, \ldots, s$ and $e = \sqrt{|E/Z|}$.

We use the notation of Definition 2.1 in the following theorem.

THEOREM 2.2. *Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on an n-dimensional finite vector space V over a finite field $\mathbb{F}$ of characteristic r. Then every normal abelian subgroup of G is cyclic and G has normal subgroups $Z \leq U \leq F \leq A \leq G$ such that*

(1) *$F = EU$ is a central product where $Z = E \cap U = \mathbf{Z}(E)$ and $\mathbf{C}_G(F) \leq F$;*
(2) *$F/U \cong E/Z$ is a direct sum of completely reducible G/F-modules;*
(3) *E_i is an extra-special p_i-group for $i = 1, \ldots, s$ and $e_i = p_i^{n_i}$ for some $n_i \geq 1$. Furthermore $(e_i, e_j) = 1$ when $i \neq j$; moreover, $e = e_1 \ldots e_s$ divides n, and $\gcd(r, e) = 1$;*
(4) *$A = \mathbf{C}_G(U)$ and $G/A \lesssim \mathrm{Aut}(U)$, A/F acts faithfully on E/Z;*
(5) *$A/\mathbf{C}_A(E_i/Z_i) \lesssim \mathrm{Sp}(2n_i, p_i)$;*
(6) *U is cyclic and acts fixed-point-freely on W where W is an irreducible submodule of V_U;*
(7) *$|V| = |W|^{eb}$ for some integer b and $|G : A| \mid \dim(W)$.*
(8) *G/A is cyclic.*

Proof. This is [10, Theorem 2.2]. ■

THEOREM 2.3. *Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V, so by Definition 2.1 and Theorem 2.2, G has a unique normal subgroup E which is a direct product of extra-special p-groups for various p. Set $e = \sqrt{|E/\mathbf{Z}(E)|}$. If $e = 5, 6, 7$ or $e \geq 10$ and $e \neq 16$, then G has at least two regular orbits on V.*

Proof. This follows from [10, Theorem 3.1] and [11, Theorem 3.1]. ∎

LEMMA 2.4. *Let G be a finite solvable primitive permutation group of degree n. Denote the number of cycles by $n(g)$ and the number of fixed points by $s(g)$. If $g \in G^{\#}$, then*

$$n(g) \leq (n + s(g))/2 \leq (p + o(g) - 1)n/(o(g)p) \leq 3n/4.$$

Proof. Let V be a minimal normal subgroup of G and let S denote a point stabilizer. Then $n = |\Omega| = |V|$. If $s(g) = 0$, we clearly have $n(g) \leq n/2$. Thus, we may assume that g has fixed points, and without loss of generality $g \in S$. Since the actions of S on V and Ω are permutation isomorphic, it follows that $s(g) = |\mathbf{C}_V(g)|$ and since S acts faithfully on V, we deduce that $s(g)$ divides $|V|/p = n/p$. Therefore,

$$\begin{aligned} n(g) &\leq s(g) + (n - s(g))/o(g) = (n + (o(g) - 1)s(g))/o(g) \\ &\leq (p + o(g) - 1)n/(o(g)p) \leq \frac{3n}{4}. \end{aligned}$$ ∎

Since we need to compare the orbit size and the group order rather than to prove the existence of regular orbits, we need some quantitative results about how a nilpotent subgroup of a solvable primitive permutation group acts on the power set of the base set.

THEOREM 2.5. *Let G be a solvable primitive permutation group of degree n acting on a set Ω, and H be a nontrivial nilpotent subgroup of G. If $k = \lceil \frac{\ln |H|}{\ln 2} \rceil$, then for any subset Λ of size $0 \leq m \leq k$, there exists $\Delta \subseteq \Omega - \Lambda$ such that $|H : \mathrm{stab}_H(\Delta)|^2 \cdot 2^{m-1} \geq |H|$.*

Proof. Let g be an element in G, we denote by $n(g)$ the number of cycles of g on Ω, and by $s(g)$ the number of fixed points.

For a subset $X \subseteq G$, consider the following set:

$$S(X) = \{(g, \Gamma) \mid g \in X,\ \Gamma \subseteq \Omega,\ g \in \mathrm{stab}_G(\Gamma)\}.$$

Note that $g \in G$ stabilizes exactly $2^{n(g)}$ subsets of Ω. Since H is a primitive solvable permutation group on the finite set Ω, by [8, p. 39] G has a unique minimal normal subgroup M with respect to a point stabilizer G_α, where $\alpha \in \Omega$. From this, we also deduce that $G = G_\alpha \cdot M$, $G_\alpha \cap M = 1$, $C_G(M) = M$, and M acts regularly on Ω. Consequently, $|M| = |\Omega|$. Thus, we get

$$|H| \leq |H_\alpha| \cdot |M| = |H_\alpha| \cdot |\Omega|.$$

Combining this with [8, Theorem 3.3], we get

$$|H| \leq \frac{n^{\beta}}{2} \cdot n = \frac{n^{\beta+1}}{2}$$

where $\beta = \frac{\log 32}{\log 9}$.

If the group H has a regular orbit on $\Omega \oplus \Omega$, then a similar argument to the one in the proof of [14, Lemma 3.1] implies that there exists $\Delta \in \Omega$ such that $|H : \mathrm{stab}_H(\Delta)|^2 \geq |H|$.

Note that the following inequality

$$(|H| - 1) \cdot (2^{n(g)})^2 < \left(\frac{2^n}{2^k}\right)^2$$

implies that for any subset Λ of size k, there exists $\Delta \subseteq \Omega - \Lambda$ such that $|H : \mathrm{stab}_H(\Delta)|^2 \geq |H|$.

By definition we have $2^k < 2|H|$, and furthermore $2^k = |H|$ when $|H|$ is a power of 2. Thus it suffices to show that

$$|H| \cdot (2^{n(g)})^2 \leq \left(\frac{2^n}{|H|}\right)^2$$

for $|H|$ that is a power of 2, and

$$|H| \cdot (2^{n(g)})^2 \leq \left(\frac{2^{n-1}}{|H|}\right)^2$$

for $|H|$ not a power of 2.

We already know that $|H| \leq n^{\beta+1}/2$. Using Lemma 2.4, we can bound $n(g) \leq \frac{p+o(g)-1}{o(g)p} n \leq \frac{p+1}{2p} n \leq \frac{3n}{4}$ for $g \in G^{\#}$. Simplifying using $n(g) \leq 3n/4$, we get

$$\frac{n^{\beta+1}}{2} \leq \begin{cases} 2^{n/6} & \text{if } |H| \text{ is a power of 2,} \\ 2^{(n-2)/6} & \text{if } |H| \text{ is not a power of 2,} \end{cases}$$

which holds for all $n \geq 113$.

Here we split the remaining $n < 113$ into two cases: where $n = p$ is a prime or $n = p^{\alpha}, \alpha > 1$.

First consider the prime n case. In this case, it is easy to see that $|H| \leq n$. If we now use the finer bound $n(g) \leq \frac{(p+1)n}{2p}$, we get the inequality

$$|H|^3 \cdot (2^{\frac{(p+1)n}{2p}})^2 \leq \begin{cases} 2^{2n} & \text{if } |H| \text{ is a power of 2,} \\ 2^{2n-2} & \text{if } |H| \text{ is not a power of 2,} \end{cases}$$

which simplifies to

$$n^3 \leq \begin{cases} 2^{n-1} & \text{if } |H| \text{ is a power of 2,} \\ 2^{n-3} & \text{if } |H| \text{ is not a power of 2.} \end{cases}$$

This holds for all primes $n > 13$.

Now consider the prime power cases (i.e. $n = p^\alpha, \alpha > 1$). It remains to check all such prime powers less than 113. Using a GAP program, we obtain the following specific bounds for $|H|$:

Table 1

n	Largest $\|H\|$
4	8
8	8
9	27
16	128
25	32
27	81
32	32
49	96
64	1024
81	729

Note that only $n = 4, 8, 9, 16, 27$ fail to satisfy the inequality

$$|H|^3 \cdot (2^{\frac{(p+1)n}{2p}})^2 \leq \begin{cases} 2^{2n} & \text{if } |H| \text{ is a power of 2,} \\ 2^{2n-2} & \text{if } |H| \text{ is not a power of 2.} \end{cases} \tag{2.1}$$

So, to evaluate the smaller n, we run a GAP program to individually check that some subset $\Delta \subseteq \Omega - \Lambda$ satisfies the inequality in every nilpotent subgroup H of all the solvable primitive permutation groups G with degree n. This holds for primes $n = 2, 3, 5, 7, 11$ and prime powers $4, 8, 9, 16$ but takes too long to run for $n = 27$. Instead, we will perform a more detailed analysis for this one remaining case.

A quick GAP program lists out all potential nilpotent $H \leq G$ for $\deg(G) = n = 27$. We find that the largest subgroup $H = (C_3 \times C_3 \times C_3) \rtimes C_3$ is the only subgroup of order 81, and is the only subgroup that violates inequality 2.1. All other H satisfy $|H| \leq 27$ and (2.1). Now, looking at $(C_3 \times C_3 \times C_3) \rtimes C_3$, we see that there is 1 element of order 1, 44 elements of order 3, and 36 elements of order 9.

Now we use the finer bound $n(g) \leq \frac{p+o(g)-1}{o(g)p} n$, where $n = p^\alpha = 3^3 = 27$. For elements of order $o(g) = 3$, we have $n(g) \leq 15$. Clearly $n(g)$ for a higher order element will have a lower bound in this case. Thus, it suffices to show (2.1) assuming $o(g) = 3$. Plugging this into the inequality, we get

$$(81)^3 \cdot (2^{15})^2 \leq 2^{2 \cdot 26}.$$

Thus, we have verified the last remaining case of $n = 27$, proving the theorem in its full generality. ∎

3. Main theorems

THEOREM 3.1. *Suppose that G is a finite solvable group and V is a finite, faithful and completely reducible G-module. Assume H is a nilpotent subgroup of G and $2 \mid |H|$. Then there exists $v \in V$ such that $2 \cdot |H| \leq |v^H|^2$.*

Proof. We prove by induction on $|G| + \dim V$.

STEP 1. We first show that V is an irreducible G-module. Assume otherwise, then $V = V_1 \oplus V_2$ where V_1, V_2 are nontrivial G-submodules of V. Let m_1 be the largest orbit size of H on V_1, and let m_2 be the largest orbit size in the action of $\mathbf{C}_H(V_1)$ on V_2. Moreover, let $v_i \in V_i$ $(i = 1, 2)$ be representatives of these orbits. Put $v = v_1 + v_2$. Then $\mathbf{C}_H(v) = \mathbf{C}_H(v_1) \cap \mathbf{C}_H(v_2)$ and hence

$$\begin{aligned} M \geq \left|v^H\right| &= |H : (\mathbf{C}_H(v_1) \cap \mathbf{C}_H(v_2))| \\ &= |H : \mathbf{C}_H(v_1)| \cdot |\mathbf{C}_H(v_1) : (\mathbf{C}_H(v_1) \cap \mathbf{C}_H(v_2))| \\ &= m_1 \cdot |\mathbf{C}_H(v_1) : \mathbf{C}_{\mathbf{C}_H(v_1)}(v_2)| \\ &= m_1 \cdot \left|v_2^{\mathbf{C}_H(v_1)}\right| \geq \ m_1 \left|v_2^{\mathbf{C}_H(V_1)}\right| = m_1 m_2. \end{aligned}$$

Since $2 \mid |H|$, either $2 \mid |H/\mathbf{C}_H(V_1)|$ or $2 \mid |\mathbf{C}_H(V_1)|$. By induction, we conclude that

$$M^2 \geq m_1^2 m_2^2 \geq 2 \cdot |H/\mathbf{C}_H(V_1)| \cdot |\mathbf{C}_H(V_1)| = 2 \cdot |H|.$$

So from now on we assume the action of G on V is irreducible.

STEP 2. We now assume that the action of G on V is not primitive. Thus, there exists a proper subgroup L_1 of G and an irreducible L_1-submodule V_1 of V such that $V = {V_1}^G$. By transitivity of induction, we can choose L_1 to be a maximal subgroup of G. In particular, $S \cong G/N$ is a primitive permutation group on a right transversal of L_1 in G, where N is the normal core of L_1 in G. Let $V_N = V_1 \oplus \cdots \oplus V_m$, where each V_i is an irreducible L_i-module with $L_i = \mathbf{N}_G(V_i)$ and $m > 1$. We know G/N primitively permutes the elements of $\{V_1, \ldots, V_m\}$. Clearly, $|H| = |HN/N| \cdot |H \cap N|$.

Define

$$N_i = \mathbf{C}_N\Big(\sum_{j=1}^{i-1} V_j\Big) / \mathbf{C}_N\Big(\sum_{j=1}^{i} V_j\Big)$$

for $i = 1, \ldots, m$, and note that $N_1 = N/\mathbf{C}_N(V_1)$ and $N_m = \mathbf{C}_N(\sum_{j=1}^{m-1} V_j)$. We define H_i to be the image of $H \cap N$ in N_i. Then $|H \cap N| \leq \prod_{i=1}^{m} |H_i|$. Clearly N_i acts completely reducibly on V_i for $i = 1, \ldots, m$. Let M_i be the largest orbit size of the action of H_i on V_i $(i = 1, \ldots, m)$, and let $v_i \in V_i$ be representatives of the corresponding orbits for all i. Thus,

$$M_{H\cap N} \;\geq\; \Big|\Big(\sum_{i=1}^{m} v_i\Big)^{H\cap N}\Big| \;\geq\; \prod_{i=1}^{m} |v_i^{H_i}| \;=\; \prod_{i=1}^{m} M_i.$$

We define $\bar{H} = HN/N$ and $\Omega = \{1, \dots, m\}$. Let Λ be the subset of Ω that contains indices i such that $H_i > 1$.

If $|\bar{H}| \leq 2^{|\Lambda|-1}$, then we set $w = w_1 + \cdots + w_m$ where $w_i = v_i$ if $H_i > 0$ and $w_i = 0$ if $H_i = 1$.

If $|\bar{H}| > 2^{|\Lambda|-1}$, then by Theorem 2.5, we may find $\Delta \subseteq \Omega - \Lambda$ such that $|\bar{H} : \mathrm{stab}_{\bar{H}}(\Delta)|^2 \cdot 2^{|\Lambda|-1} \geq |\bar{H}|$. We set $w = w_1 + \cdots + w_m$ where $w_i = v_i$ if $i \notin \Delta$ and $w_i = 0$ if $i \in \Delta$.

We have

$$M \;\geq\; |\bar{H} : \mathrm{stab}_{\bar{H}}(\Delta)| \cdot \prod_{i\in\Lambda} M_i.$$

We conclude

$$\begin{aligned} M^2 &\geq |\bar{H} : \mathrm{stab}_{\bar{H}}(\Delta)|^2 \cdot \prod_{i\in\Lambda} M_i^2 \geq |\bar{H} : \mathrm{stab}_{\bar{H}}(\Delta)|^2 \cdot 2^{|\Lambda|} \cdot \prod_{i\in\Lambda} |H_i| \\ &\geq 2 \cdot |\bar{H}| \cdot |H \cap N| = 2|H|. \end{aligned} \tag{3.1}$$

Step 3. Hence, we may assume that the action of G on V is primitive. By the main results of [10, 11, 15, 4], we know that either $e = 1$ or G has a regular orbit on V except for a finite number of cases listed in [4]. For all those exceptional cases listed in [4], we may use GAP to verify that the result holds.

Thus, we may assume $e = 1$ and G is a subgroup of $\Gamma(V)$. Since H is a subgroup of G, we may set $|V| = p^n$ and $|H| = |H/C||C|$ where $|H/C| \mid n$ and $|C| \mid |V|-1$. For $g \in H\backslash C$, $|\mathbf{C}_V(g)| \leq |V|^{1/2}$, and for $g \in C$, $|\mathbf{C}_V(g)| \leq |V|^{1/2}$. If $|C| < |V|^{1/2}$, then H has a regular orbit. Thus, we may assume that $|C| \geq |V|^{1/2}$. This shows that $|C|$ is pretty large. It suffices to show that $|C|^2 \geq 2|H|$, which is equivalent to showing $|C| \geq 2|H/C|$.

Assume otherwise; then by plugging in the bounds we have $|V|^{1/2} < 2(\log_p |V|)$, which holds only when $|V| = 2^2, 2^3, 3^2, 2^4, 3^3, 2^5, 2^6, 2^7$. For all those small cases, we can construct possible representations explicitly in GAP and check that the result holds. ∎

Isaacs proved the following result [6, Theorem A]. Let P be a nontrivial p-group that acts faithfully on a group H, where $|H|$ is not divisible by p. Then there exists an element $x \in H$ such that $|\mathbf{C}_P(x)| \leq (|P|/p)^{1/p}$. By the Hartley–Turull's Lemma [3, Lemma 2.6.2], this is equivalent to the following statement: Let P be a nontrivial p-group that acts faithfully on a vector space V, where $|V|$ is not divisible by p. Then there exists an element

$x \in V$ such that $|\mathbf{C}_P(x)| \leq (|P|/p)^{1/p}$. This compares well with the following theorem.

THEOREM 3.2. *Suppose that G is a finite solvable group and V is a finite, faithful and completely reducible G-module. Let H be a nilpotent subgroup of G. Then there exists $v \in V$ such that $|\mathbf{C}_H(v)| \leq (|H|/p)^{1/p}$, where p is the smallest prime divisor of $|H|$.*

Proof. If $2 \mid |H|$, then the result follows by Theorem 3.1.

We now assume that $2 \nmid |H|$. Suppose $\mathrm{char}(\mathbb{F}) = 2$, then H acts coprimely on V, and the result follows by the main result of [6]. Suppose that $\mathrm{char}(\mathbb{F}) \neq 2$. Then H has a regular orbit on V by the main result of [12]. ∎

LEMMA 3.3. *Let G be a finite group, and suppose $N \trianglelefteq G$ such that G/N is nilpotent. Then there exists a nilpotent subgroup $U \leq G$ such that $G = NU$.*

Proof. See, for example, [5, III, Satz 3.10]. ∎

The following result strengthens [7, Theorem 1.2].

THEOREM 3.4. *Let G be a finite solvable group and V a finite faithful completely reducible G-module, possibly of mixed characteristic. Let M be the largest orbit size in the action of G on V. Denote by $G^{\mathcal{N}}$ the normal subgroup of G such that $G/G^{\mathcal{N}}$ is the maximum nilpotent quotient. Let p be the smallest prime dividing $|G/G^{\mathcal{N}}|$. Then*

$$p \cdot |G/G^{\mathcal{N}}| \leq M^{\frac{p}{p-1}}.$$

Proof. It is easy to reduce this inequality to the situation that G acts irreducibly on V where V is of characteristic r. By Lemma 3.3, there exists a nilpotent $L \leq G$ such that $G^{\mathcal{N}}L = G$. Thus, there is a subgroup H of L such that $|G/G^{\mathcal{N}}|$ divides $|H|$ and $\pi(|H|) = \pi(|G/G^{\mathcal{N}}|)$.

By Theorem 3.2 it follows that if M_H is the largest orbit size of H on V, then $p|H| \leq M_H^{p/(p-1)}$. Hence, altogether $p|G/G^{\mathcal{N}}| \leq p|H| \leq M_H^{p/(p-1)} \leq M^{p/(p-1)}$, as desired. ∎

THEOREM 3.5. *Suppose that G is a finite solvable group and let H be a nilpotent Hall π-subgroup. Then $p|G : O_{\pi'\pi}(G)|_\pi \leq b(H)^{p/(p-1)}$ where p is the smallest number in π.*

Proof. We may assume that $O_{\pi'}(G) = 1$. Let $N = O_\pi(G)$. Then fairly standard arguments show that $C = \mathbf{C}_G(\mathbf{F}(N)/\Phi(N)) \subseteq N$. Let $U = \mathbf{F}(N)/\Phi(N)$, and let $V = \hat{U}$ be the dual group of U, i.e. the group of the irreducible characters of the abelian group U. Set $\bar{G} = G/C$. It is easy to see that V is a faithful and completely reducible $\bar{G}$-module, possibly of mixed characteristic.

Let H be a nilpotent Hall π-subgroup of G and let $\bar{H} = H/C$. By Theorem 3.2, there exists $\lambda \in V$ such that $|\mathbf{C}_{\bar{H}}(\lambda)| \leq (|\bar{H}|/p)^{1/p}$. Let

$\xi \in \mathrm{Irr}(\mathbf{C}_H(\lambda)|\lambda)$ and $\alpha = \xi^H \in \mathrm{Irr}(H)$. Thus, $p|G : N|_\pi \leq p|\bar{H}| \leq \alpha(1)^{\frac{p}{p-1}} \leq b(H)^{\frac{p}{p-1}}$, as desired. ■

REMARK. This strengthens [13, Theorem 4.2].

COROLLARY 3.6. Suppose that G is a finite solvable group and let $P \in \mathrm{Syl}_2(G)$. Then $2\,|G : O_{2'2}(G)|_2 \leq b(P)^2$.

Proof. This follows from Theorem 3.5 by choosing H to be a Sylow 2-subgroup of G. ■

As another application, we get the following result.

THEOREM 3.7. *Let G be a finite solvable group and $b(G)$ be the largest character degree of G. Let H be a nilpotent subgroup of G. Then*

$$p\,|HF(G)/F(G)| \leq b^{\frac{p}{p-1}}(G)$$

where p is the smallest prime divisor of $|HF(G)/F(G)|$.

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