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Euler products associated to multivariate rational functions: maximal domain of meromorphy, zeros and poles

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Contents

Abstract

We determine the maximal domain of meromorphy of any Euler product

$$f(s_1,\ldots,s_k)=\prod_p h(p^{-s_1},\ldots,p^{-s_k}),$$

where h is the quotient of two polynomials in k variables, with integer coefficients and with constant term equal to 1. More precisely, we define a domain $\Gamma\subseteq\mathbb{C}^k$, on which f admits a meromorphic extension, and such that for any $\boldsymbol{z}$ in $\partial\Gamma$, there is no neighborhood of $\boldsymbol{z}$ on which f admits a meromorphic extension. This maximal domain Γ is described as $\mathring{K}+\mathrm{i}\mathbb{R}^k$, where $K\in\mathbb{R}^k$ is a rational cone, computed from the set of exponents of the two polynomials defining h. We also describe the divisor of f over Γ, which comes from the local factors $h(p^{-s_1},\ldots,p^{-s_k})$, and from the zeta factors $\zeta(\alpha_1s_1+\cdots+\alpha_ks_k)^{-c_{\boldsymbol{\alpha}}}$, where ζ denotes the Riemann ζ-function, corresponding to terms in the expansion of h as a formal infinite product $\prod_{\boldsymbol{\alpha}}(1-X_1^{\alpha_1}\cdots X_k^{\alpha_k})^{c_{\boldsymbol{\alpha}}}$. We focus our study on the hyperplanes in the divisor, allowing us to use tools developed for the single variable case. We complete our study by giving a geometric and arithmetic description of the set of exponents occurring in the infinite product expansion of h, and by showing a new result on the geometric nature of the set of singular points of a holomorphic function defined over a tubular domain.

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1. Introduction

Zeta functions are analytic objects associated with discrete phenomena that need to be described asymptotically, such as counting of objects or function values. They are named after the Riemann ζ-Function, the function of one complex variable defined for $\Re e(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and extended meromorphically to all the complex plane, with an only pole at $s = 1$ of residue equal to 1. Some of these functions do have special representations related to arithmetic properties, such as Euler products, named after the representation found by Euler in 1737:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

(valid for $\Re(s) > 1$, and with p running over the integer prime numbers), or other symmetries, like the so-called functional equations.

Typically, the asymptotic information of the counting function is extracted through Tauberian techniques, after two preliminary steps. First, the zeta function is analytically continued to the extent that it reveals the underlying data. Next, its growth is analyzed in the extended region. With these in place, Tauberian techniques extract the asymptotic behavior of the counting function from the poles of the zeta function.

While many applications arise from one-variable functions, working with several variables allows the introduction of multiple parameters in the counting. As such, several variable functions appear in the work of Bhowmik, Essouabri and Lichtin [BEL07].

Our goal is to study the Euler product $f(s_1, \ldots, s_k) = \prod_p h(p^{-s_1}, \ldots, p^{-s_k})$ as a meromorphic function, where $h \in \mathbb{Z}[[X_1, \ldots, X_k]]$ is any quotient of two polynomials in k variables with constant term equal to 1. In particular, we determine its domain of meromorphy and its divisor over this domain.

Before giving a historical account on this problem and stating our results, we mention some obvious facts about Euler products.

(1) The process associating a function h to its (somewhere convergent) Euler product f is multiplicative: if a function h_1 (respectively h_2) is associated to its Euler product f_1 (respectively f_2) which admits a meromorphic extension on a domain D_1 (respectively D_2), then the function $h_1 h_2$ is associated to the Euler product $f_1 f_2$ which admits a meromorphic extension on a connected component of $D_1 \cap D_2$. Moreover, h_1/h_2 is associated to f_1/f_2, which is meromorphic over a connected component of $D_1 \cap D_2$.

(2) The polynomial $h = 1 - X^n$ is associated to the Euler product $\prod_p (1 - p^{-ns}) = \zeta(ns)^{-1}$, which is meromorphic over $\mathbb{C}$. More generally, the multivalued polynomial $1 - X_1^{\alpha_1} \cdots X_k^{\alpha_k}$, which we denote by $1 - \boldsymbol{X}^{\boldsymbol{\alpha}}$, with $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, is associated to the Euler product $\zeta(\alpha_1 s_1 + \dots + \alpha_k s_k)^{-1}$, which is meromorphic over $\mathbb{C}^k$.

These are facts from which we can derive some general consequences.

(3) The nth cyclotomic polynomial Φ_n can be written as a quotient $\prod_{d|n}(1-X^d)^{\mu(n/d)}$ where μ is the Möbius function. In consequence, its associated Euler product is $\prod_{d|n} \zeta(ds)^{-\mu(n/d)}$, which is meromorphic on $\mathbb{C}$. More generally, we define a *cyclotomic polynomial in k variables* to be any polynomial of the form $\Phi(\boldsymbol{X}^{\boldsymbol{\alpha}})$ with Φ a univariate cyclotomic polynomial, and $\boldsymbol{\alpha} \in \mathbb{N}^k$ with $\boldsymbol{\alpha} \neq (0, \dots, 0)$ (*). The Euler product associated to any cyclotomic polynomials in k variables is meromorphic on $\mathbb{C}^k$.

(4) The Euler products f and f' associated to the functions h and $h' = \Phi(\boldsymbol{X}^{\boldsymbol{\alpha}})h$ respectively, admit meromorphic extension on the same domains. If h and h' are polynomials, we say that h' admits a *cyclotomic factor*. Without loss of generality, we can assume that a polynomial has no cyclotomic factor, or for a rational function, that its numerator and denominator have no cyclotomic factors. In particular, the meromorphic functions f and f' have the same singularities (†).

1.1. Previous work on the subject. The first study for a generic h goes back to Estermann [Est28], who treated the case where h is a univariate polynomial with integer coefficients. Here is an overview of his argument.

If $h \in \mathbb{Z}[X]$ is a product of cyclotomic polynomials, it can be written as a finite product $\prod_{n<N}(1 - X^n)^{c_n}$ with $c_n \in \mathbb{Z}$ for some $N \in \mathbb{N}$. The associated Euler product equals $\prod_{n<N} \zeta(ns)^{-c_n}$, which is meromorphic on $\mathbb{C}$.

If h is not such a product, then, for any positive integer N, h can still be approximated by a finite product $\prod_{n<N}(1 - X^n)^{c_n}$ with $c_n \in \mathbb{Z}$, up to an arbitrarily small error term:

$$h(z) = (1 - c_N z^N + o(z^N)) \prod_{n<N} (1 - z^n)^{c_n} \qquad \text{when } z \text{ tends to } 0,$$

and f can be meromorphically extended to the half-plane $\Re e(s) > \frac{1}{N}$. This provides us with the meromorphic continuation of f up to the half-plane $\Re e(s) > 0$. More exactly, for any positive integer N there is some large P such that

$$f(s) = \prod_{p<P} h(p^{-s}) \prod_{n<N} \zeta(ns)^{-c_n} G_{N,P}(s)$$

where $G_{N,P}$ is holomorphic and without zero on $\Re e(s) > 1/N$. This proves that the zeros and poles of f on $\Re e(s) > 0$ come either from a factor $h(p^{-s})$ or from a factor $\zeta(ns)^{-c_n}$. Since h is not a product of cyclotomic polynomials, it has at least one root ρ with $|\rho| < 1$. For each prime number p, this root creates an arithmetic progression of zeros of $s \mapsto h(p^{-s})$ on the half-space $\Re e(s) > 0$, namely $\frac{1}{\log p}(-\log \rho + 2\pi \mathrm{i} \mathbb{Z})$. As p

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(*) Montgomery [Mon76, Theorem 1] has shown that this family is a reasonable generalization of the family of univariate cyclotomic polynomials.

(†) Note that in general, we cannot derive the set of singularities of $f_1 f_2$ from the singularities of f_1 and those of f_2. But if a singularity of f_1 is a point of meromorphy of f_2, then it is a singularity of $f_1 f_2$.

grows, these arithmetic progressions cluster at all points of the imaginary line, proving that this line is made up of singularities of f. More precisely, these zeros may cancel with poles coming from ζ factors, but one root of h creates so many zeros of f that they cannot be significantly canceled by the poles arising from the ζ-factors.

Notice that the result and the proof of Estermann can be easily adapted to a rational fraction $h \in \mathbb{Z}(X)$ with $h(0) = 1$ and without cyclotomic factors. By Fatou's Lemma ([Fat04, pp. 369–370], [Pól16, Hilfssatz III]), h is the quotient of two coprime polynomials q^+ and q^- in $\mathbb{Z}[X]$ with $q^+(0) = q^-(0) = 1$. To each polynomial $q^\pm$ is associated its Euler product $g^\pm$ meromorphic on the half-space $\Re e(s) > 0$, therefore $f = g^+/g^-$ is meromorphic on the half-space. If they are not constant, the polynomials q^+ and q^- have at least one root inside the unit disc, giving rise to a large collection of zeros of g^+ or g^- that cluster at any point on the imaginary line. The collection associated with g^+ might cancel the collection associated with g^-, but only when these collections are generated by the same root of q^+ and of q^-, which contradicts the fact that q^+ and q^- do not share a common factor.

Dahlquist [Dah52] extended the result for $h \in \mathbb{Z}\{X\}$ (i.e. when h is a power series with integer coefficients that converges in a neighborhood of 0), obviously with $h(0) = 1$. His convergence theorem states that if h is meromorphic on the open unit disc, then f is meromorphic on the half-plane $\Re e(s) > 0$. The proof roughly follows the lines of Estermann's, with a more careful treatment of the zeta factors. The maximality of this domain of meromorphy is proved when h admits a finite number of zeros and poles in the open unit disc, again with an argument similar to Estermann's. The main novelty is an elaborate argument to prove the maximality of the half-plane when h has neither a zero nor a pole inside the unit disk. In this case, all the zeros and poles of f originate from the ζ-factors and Estermann's argument is no longer useful (‡).

Dahlquist's treatment of the case of h without zero or pole has some drawbacks:

- The argument is convoluted and somewhat unconventional. Up to our knowledge, this argument has never been used in another setting.
- This argument is no longer needed if the Riemann Hypothesis is assumed. On the plus side, only a small amount of knowledge on the zeros of the Riemann ζ-function is required for that proof. This allows Dahlquist to generalize his results to different kinds of Euler products, which can be expressed in terms of Dirichlet L-functions instead of the Riemann ζ-function.
- The functions h concerned by this part of the proof (power series with integer coefficients, with radius of convergence equal to 1, and which are not rational fractions) are pathological: by the Pólya–Carlson Theorem [Rem98, p. 265], any point of the unit circle is a singular point for such functions.

However, this article is our main source of inspiration for focusing on the terms of the divisor of f which comes from the ζ factors instead of those coming from the local

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(‡) Notice that Estermann's argument also fails when there are infinitely many zeros and poles of h inside the unit disc. Dahlquist [Dah52, Section 4.2] remarked that if the number of zeros and poles of h in the disc of radius $r < 1$ does not grow too fast when r tends to 1, it is still possible to conclude with a version of Estermann's argument.

factors (even if, for cancellation reasons, both are to be considered). We will give a new insoght into Dahlquist's argument, and we will show that the absence of zeros or poles is indeed common in the multivariate case, even for elementary examples.

Regarding the multivariate case, Bhowmik, Essouabri and Lichtin [BEL07] gave the first general convergence result when h is a multivariate polynomial. We call the *support* of a polynomial $h \in \mathbb{Z}[\boldsymbol{X}]$ the set of exponents $\boldsymbol{\alpha}$ such that the term $\boldsymbol{X}^{\boldsymbol{\alpha}}$ has a non-zero coefficient in h. When h does not have any cyclotomic factor, Bhowmik et al. proved the meromorphy of f on the tubular polyhedral open convex cone Γ of $\mathbb{C}^k$ defined by the inequalities $\Re\langle \boldsymbol{\alpha}, \boldsymbol{s}\rangle > 0$ for every $\boldsymbol{\alpha} \neq \boldsymbol{0}$ in the support of h, where $\langle \cdot, \cdot \rangle$ is the canonical hermitian inner product of $\mathbb{C}^k$. From a geometric point of view, the closure $\overline{\Gamma}$ is the dual cone of the support of h. On the other hand, they proved a weak maximality result: if h has a non-cyclotomic factor, then the origin $\boldsymbol{0}$ is a singular point for f [BEL07, Theorem 2].

In his PhD thesis, Delabarre [Del13] stated the maximality result in a remarkable way. By duality, the facets of Γ are in relation with the extreme rays of its dual cone $\Gamma^\vee$, which is the conical hull of the support of h in this case. To an extreme ray $\boldsymbol{\nu}\mathbb{R}_+$ of $\Gamma^\vee$ (where $\boldsymbol{\nu}$ can be chosen so that its coordinates are coprime integers), we associate the facet $\mathcal{F}_{\boldsymbol{\nu}} = \overline{\Gamma} \cap \{\boldsymbol{s} : \Re\langle \boldsymbol{\nu}, \boldsymbol{s}\rangle = 0\}$ and the subpolynomial $h_{\boldsymbol{\nu}}$ of h consisting of terms of h whose exponents belong to the ray $\boldsymbol{\nu}\mathbb{R}_+$. The polynomial $h_{\boldsymbol{\nu}}$ is an element of $\mathbb{Z}[\boldsymbol{X}^{\boldsymbol{\nu}}]$ and can be written as a non-constant polynomial in one single variable $T = \boldsymbol{X}^{\boldsymbol{\nu}}$ with constant coefficient equal to 1. Delabarre proved that [Del10, Théorème principal] if h has no cyclotomic factor, then any point of the facet $\mathcal{F}_{\boldsymbol{\nu}}$ is a singular point for f as long as $h_{\boldsymbol{\nu}}$ is square-free. The proof follows the work of Estermann, in the way that Delabarre constructs zeros of h in the neighborhood of zeros of $h_{\boldsymbol{\nu}}$ and those zeros create a large collection of zeros of f near the facet $\mathcal{F}_{\boldsymbol{\nu}}$. The study of possible cancellations between zeros is complicated in this setting, but the more tractable $h_{\boldsymbol{\nu}}$ is square-free.

The results of Bhowmik et al. and Delabarre can also be applied to a rational function $h = q^+/q^-$ where $q^+, q^- \in \mathbb{Z}[\boldsymbol{X}]$ have no cyclotomic factors and $q^+(\boldsymbol{0}) = q^-(\boldsymbol{0}) = 1$. The polynomials q^+ and q^- are associated to their respective Euler products g^+ and g^- which are meromorphic over Γ^+ and Γ^- respectively. Their quotient $f = g^+/g^-$ is a meromorphic extension of the Euler product associated to h over $\Gamma^+ \cap \Gamma^-$, which is the open dual cone of $\operatorname{supp}(q^+) \cup \operatorname{supp}(q^-)$.

On the maximality of this domain, we may conclude that h has a singularity at every point of $\Gamma^+ \cap \partial\Gamma^-$ and $\partial\Gamma^+ \cap \Gamma^-$, at least if the corresponding $h_{\boldsymbol{\nu}}$ is square-free, but we cannot decide this for a point of $\partial\Gamma^+ \cap \partial\Gamma^-$. In particular, it is not possible to get a maximality result for $h(X_1, X_2) = \frac{1+X_1+X_2-X_1X_2}{1+X_1+X_2}$ directly from the previous theorems since the domains Γ^+ and Γ^- are the same.

1.2. Statement of results. In this article, we extend and complete previous results in several ways.

First, we produce a tubular convex cone Γ on which the Euler product f is meromorphic, provided the function h satisfies the following conditions:

- $h(\mathbf{0}) = 1$,
- in a neighborhood of $\mathbf{0}$, h is the sum of a converging power series from $\mathbb{Z}[\![\boldsymbol{X}]\!]$,
- h is meromorphic on the Reinhardt domain $\{(\mathrm{e}^{-s_1}, \dots, \mathrm{e}^{-s_k}) \mid \boldsymbol{s} \in \Gamma\}$.

This domain Γ is not determined by the support of the power series defining h, but from the support of the infinite product representing h. Indeed, we prove that any function h satisfying the first two assumptions can be written as $\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c_{\boldsymbol{\alpha}}}$, where $c_{\boldsymbol{\alpha}} \in \mathbb{Z}$ for any $\boldsymbol{\alpha} \in \mathbb{N}^k$, with $c_{\mathbf{0}} = 0$, and where the product converges on a small neighborhood around $\mathbf{0}$. We introduce the notion of *exponential support* of h, denoted $\operatorname{xsupp}(h)$ as the set of exponents $\boldsymbol{\alpha} \in \mathbb{N}^k$ for which $c_{\boldsymbol{\alpha}} \neq 0$. The exponential support of h is useful to describe not only the domain Γ (as the interior of the inferior limit of the half-spaces $\{\boldsymbol{s} \in \mathbb{C}^k \mid \Re\langle\boldsymbol{\alpha}, \boldsymbol{s}\rangle > 0\}$ for $\boldsymbol{\alpha} \in \operatorname{xsupp}(h)$) but also the divisor of h and especially the divisor of f over Γ.

THEOREM 1.1. *Let h be a power series from $\mathbb{Z}[\![\boldsymbol{X}]\!]$ with $h(\mathbf{0}) = 1$. Let C be its exponential support and $(c_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in C}$ the family of exponents in its product representation $\prod_{\boldsymbol{\alpha}\in C}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c_{\boldsymbol{\alpha}}}$. Define Γ as the interior of the set of $\boldsymbol{s} \in \mathbb{C}^k$ such that $\Re\langle\boldsymbol{\alpha}, \boldsymbol{s}\rangle \geqslant 0$ for all but finitely many $\boldsymbol{\alpha} \in C$. If the function $\boldsymbol{s} \mapsto h(\mathrm{e}^{-s_1}, \dots, \mathrm{e}^{-s_k})$ is meromorphic on Γ, then f is meromorphic on Γ and the divisor of f over Γ is*

$$(f) = \sum_{p\,\text{prime}} \left[(h_p) - \sum_{\boldsymbol{\alpha}\in C} c_{\boldsymbol{\alpha}} \sum_{\kappa\in\mathbb{Z}} H_{\boldsymbol{\alpha}}\left(\frac{2\kappa\pi\mathrm{i}}{\log p}\right) \right] - \sum_{\boldsymbol{\alpha}\in C}\sum_{\gamma\in G} c_{\boldsymbol{\alpha}} m_\gamma \, H_{\boldsymbol{\alpha}}(\gamma),$$

where h_p is the meromorphic function $\boldsymbol{s} \mapsto h(p^{-s_1}, \dots, p^{-s_k})$ defined on Γ, G and m_γ are the support and coefficients of the divisor of the Riemann ζ-function on $\mathbb{C}$, i.e. $(\zeta) = \sum_{\gamma\in G} m_\gamma \, \gamma$, and

$$H_{\boldsymbol{\lambda}}(\gamma) = \{\boldsymbol{s} \in \Gamma \mid \langle\boldsymbol{\lambda}, \boldsymbol{s}\rangle = \gamma\}.$$

Notice that if all the real parts of $s_1, \dots, s_k$ are large enough, the double product $\prod_p \prod_{\boldsymbol{\alpha}\in C}(1-p^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle})^{c_{\boldsymbol{\alpha}}}$ is convergent, and we derive three expressions for f as an infinite product:

$$f(\boldsymbol{s}) = \prod_p h_p(\boldsymbol{s}) = \prod_p \prod_{\boldsymbol{\alpha}\in C} (1 - p^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle})^{c_{\boldsymbol{\alpha}}} = \prod_{\boldsymbol{\alpha}\in C} \zeta(\langle\boldsymbol{\alpha}, \boldsymbol{s}\rangle)^{-c_{\boldsymbol{\alpha}}}.$$

The divisors of the terms of these three products correspond exactly to the terms in the expression of the divisor (f) over Γ of our Theorem 1.1. More exactly, the divisor (f) is the formal sum of the divisors of the terms of the first and third products, corrected by the divisors of the terms of the second one.

Notice also that Γ contains the dual cone of $\operatorname{supp}(h)$. Of course, in the case of a rational function without cyclotomic factors, our domain of meromorphy Γ is exactly the open dual cone of $\operatorname{supp}(q^+) \cup \operatorname{supp}(q^-)$ obtained by Bhowmik et al. and Delabarre, since this domain turns up to be maximal as we will see. In that particular case, the main novelty of Theorem 1.1 is the complete description of the divisor of f on Γ. However, there is no assumption about cyclotomic factors in our statement. This is taken care of by the construction of Γ: the multiplication or division of h by a cyclotomic factor affects its exponential support only by adding or removing finitely many elements, it has no effect on Γ described as a limit inferior.

We now state our theorem of maximality in the case where $h \in \mathbb{Z}[[\boldsymbol{X}]]$ is a rational power series. In that case, the domain of meromorphy Γ is a rational polyhedral convex cone.

Theorem 1.2. *Let $h \in \mathbb{Z}[[\boldsymbol{X}]]$ be a rational power series with constant term* 1. *There exist two coprime $q^+, q^- \in \mathbb{Z}[\boldsymbol{X}]$ such that $q^- h = q^+$. Assume that neither q^+ nor q^- has a cyclotomic factor. Then the Euler product $f(\boldsymbol{s}) = \prod_p h(p^{-s_1}, \dots, p^{-s_k})$ admits a meromorphic extension on the interior of the dual cone of* $\operatorname{supp}(q^+) \cup \operatorname{supp}(q^-)$*, that is, on*

$$\Gamma = \bigcap_{\substack{\boldsymbol{\alpha} \in \operatorname{supp}(q^+) \cup \operatorname{supp}(q^-) \\ \boldsymbol{\alpha} \neq \boldsymbol{0}}} \{\boldsymbol{s} \in \mathbb{C}^k \mid \Re e \langle \boldsymbol{\alpha}, \boldsymbol{s} \rangle > 0\}.$$

This domain is the domain of meromorphy of f, meaning that any point on the boundary of Γ is a singular point of f.

This result completely determines the maximal domain of meromorphy in the polynomial case, but also in the rational case. The methods used may also apply to more general functions h, but some steps in the argumentation are only valid for rational functions. We have not tried to determine the broadest class of functions h for which our methods can prove the maximality of Γ, but it seems to us that there is no simple and practical description of a class of functions h for which our methods are sufficient to conclude.

Our strategy to prove Theorem 1.2 is to use the divisor of f described in Theorem 1.1 and to identify points on the different faces of the polyhedral cone Γ where the local finiteness of (f) is not satisfied. Due to the shape of the expected result (a polyhedral cone) and the terms in the second sum of the divisor f (which all are hyperplanes), we have restricted our study to hyperplanes in the divisor.

For each face $\mathcal{F}_{\boldsymbol{\nu}}$ of Γ, we split the argument into three cases.

First, if q^+ or q^- admits a factor which creates a hyperplane orthogonal to $\boldsymbol{\nu}$ in the divisor (f) (this factor has to appear in $h_{\boldsymbol{\nu}}$ as well), then, similarly to the argument of Estermann [Est28], this factor creates an arithmetic progression of hyperplanes for each factor p, and this double family of hyperplanes clusters everywhere along $\mathcal{F}_{\boldsymbol{\nu}}$.

Second, if no such factor is to be found, we are in a situation very similar to the one detailed by Dahlquist in [Dah52], where only the zeros and poles of the ζ factors appear. Actually, his argument works as is in this situation, unless $h_{\boldsymbol{\nu}}$ is a finite quotient of cyclotomic polynomials, a case in which the hyperplanes in (f) that are parallel to the face $\mathcal{F}_{\boldsymbol{\nu}}$ form a discrete set, and therefore produce no accumulation.

Taking care of this third case has required a completely new set of arguments. To begin with, the fact that the domain Γ described in Theorem 1.1 coincides with the expected dual cone of $\operatorname{supp}(q^+) \cup \operatorname{supp}(q^-)$ is no longer obvious. This fact relies on the following geometric description of the support of a rational power series.

Proposition 1.3. *Let $g \in \mathbb{Z}[[\boldsymbol{X}]]$ be a rational power series, and $p^+, p^- \in \mathbb{Z}[\boldsymbol{X}]$ be the uniquely determined coprime polynomials with $p^-(\boldsymbol{0}) = 1$ and $p^- g = p^+$. We have*

$$\overline{\operatorname{conv}}(\operatorname{supp}(g)) = \operatorname{conv}(\operatorname{supp}(p^+)) + \operatorname{cone}(\operatorname{supp}(p^-)).$$

Moreover, there exists $\boldsymbol{\alpha} \in \mathbb{Z}^k$ *such that the convex hull of* $\operatorname{supp}(g)$ *can be spanned only by elements of* $\operatorname{supp}(g)$ *which do not belong to the affine cone* $\boldsymbol{\alpha} + \operatorname{cone}(\operatorname{supp}(p^-))$.

To prove this result, we establish a multivariate generalization of a result known for Fatou rings. We will provide a proof of this result only for a unique factorization domain R.

Proposition 1.4. *Let* R *be a Fatou ring with fraction field* F. *For any rational power series* $g \in R[[\boldsymbol{X}]]$, *there exists a unique pair of polynomials* $p^+, p^- \in R[\boldsymbol{X}]$, *coprime in* $F[\boldsymbol{X}]$, *with* $p^-(\mathbf{0}) = 1$, *such that* $p^- g = p^+$.

These results help to understand the support of a rational power series, but the objects in Theorem 1.1 are defined in terms of the exponential support. If their geometric properties are related, a proper study of $\operatorname{xsupp}(h)$ is required. A striking result is the following.

Proposition 1.5. *Let* $h \in \mathbb{Z}[[\boldsymbol{X}]]$ *be a rational power series with* $h(\mathbf{0}) = 1$. *Let* $(c_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^k}$ *be the family of exponents in its product representation* $\prod_{\boldsymbol{\alpha} \in \mathbb{N}^k} (1 - \boldsymbol{X}^{\boldsymbol{\alpha}})^{c_{\boldsymbol{\alpha}}}$, *and* K *the cone generated by* $\operatorname{xsupp}(h)$. *For any* $\boldsymbol{\nu} \in \mathbb{N}^k$ *directing an extremal ray of* K *and any* $\boldsymbol{\mu} \in \mathbb{N}^k$ *not collinear with* $\boldsymbol{\nu}$, *the power series* $\sum_{n \in \mathbb{N}} c_{\boldsymbol{\mu} + n\boldsymbol{\nu}} T^n \in \mathbb{Z}[[T]]$ *is rational.*

With this proposition, we can ensure an accumulation of hyperplanes along any face of Γ, but only in a tubular neighborhood of $\mathbf{0}$. In order to conclude this case, we establish the following general result about tubular domains of holomorphy:

Proposition 1.6. *Let* ξ *be a function holomorphic on a tubular domain* $K + \mathrm{i}\mathbb{R}^k$, *with* $K \subseteq \mathbb{R}^k$ *convex, and let* F *be a proper face of* $\overline{K}$ *and let* W *be the direction of the affine hull of* F *in* $\mathbb{R}^k$. *The function* ξ *is holomorphic at* $\boldsymbol{x} + \mathrm{i}\boldsymbol{y} \in F + \mathrm{i}\mathbb{R}^k$ *if and only if* ξ *is holomorphic at any* $\boldsymbol{x}' + \mathrm{i}\boldsymbol{y}' \in F + \mathrm{i}(\boldsymbol{y} + W)$.

This result, based on a theorem of Siciak [Sic69], can be seen as a qualitative local version of Bochner's Theorem for tubular domains. It exhibits an all-or-nothing property which completes our findings perfectly, since we have a polyhedral tubular convex cone as the domain of meromorphy for f with singular points along its faces over a tubular neighborhood of $\mathbf{0}$.

Although we use tools of different areas of mathematics, we make particular use of the geometry of convex cones, a subject that pops up at different points in this article, within various settings. We spend the next part of this section introducing notations and conventions that will be used throughout this paper, and the most basic definitions and results about convex cones that we will need.

1.3. Notations, definitions and basic results.

Out of habit, we will use the Bourbaki brackets notation for open intervals, and we will denote the natural logarithm by log. The set of non-negative integers is denoted by $\mathbb{N}$.

In this article, $k \geqslant 1$ always refers to the dimension of the parameter space, or equivalently, the number of variables of its power series expansion. Any bold letter or symbol denotes a k-tuple whose coordinates are denoted by the same letter or symbol with a regular font weight, and with an index running from 1 to k. For example, $\boldsymbol{s}$ stands for

$(s_1, \ldots, s_k)$ and $\mathbf{0}$ (respectively $\mathbf{1}$) denotes the k-tuple $(0, \ldots, 0)$ (respectively $(1, \ldots, 1)$). The family $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_k\}$ denotes the canonical basis of $\mathbb{R}^k$.

The ring of formal power series in k variables with integer coefficients is denoted by $\mathbb{Z}[[\boldsymbol{X}]]$ where $\boldsymbol{X}$ stands for $X_1, \ldots, X_k$. Members of the monomial basis are denoted $\boldsymbol{X}^{\boldsymbol{\alpha}}$, which stands for the power product $X_1^{\alpha_1} \cdots X_k^{\alpha_k}$. Here, $\boldsymbol{\alpha}$ runs through $\mathbb{N}^k$. An exponent $\boldsymbol{\alpha} \in \mathbb{N}^k$ with coprime coordinates is said to be *primitive.*

Algebraically, a formal series $h \in \mathbb{Z}[[\boldsymbol{X}]]$ is identified with its *coefficient function* $b \in \mathbb{Z}^{\mathbb{N}^k}$, related by the equation $h = \sum_{\boldsymbol{\beta} \in \mathbb{N}^k} b(\boldsymbol{\beta}) \boldsymbol{X}^{\boldsymbol{\beta}}$. The *support* of h, denoted $\operatorname{supp} h$, is defined by $\{\boldsymbol{\beta} \in \mathbb{N}^k \mid b(\boldsymbol{\beta}) \neq 0\}$. For any given primitive $\boldsymbol{\nu} \in \mathbb{N}^k$, we let $h_{\boldsymbol{\nu}}$ be the formal series $\sum_{n \in \mathbb{N}} b(n\boldsymbol{\nu}) \boldsymbol{X}^{n\boldsymbol{\nu}}$, which is a power series in the single variable $T = \boldsymbol{X}^{\boldsymbol{\nu}}$. The map $h \mapsto h_{\boldsymbol{\nu}}$ is generally not a ring homomorphism.

The ring of polynomials in k variables with integer coefficients $\mathbb{Z}[\boldsymbol{X}]$ is seen as a subring of $\mathbb{Z}[[\boldsymbol{X}]]$. It is a unique factorization domain (UFD). For any such domain R, the content and the primitive part of a polynomial in $R[\boldsymbol{X}]$ are defined up to an invertible element of R. If the only common factors between two polynomials are invertible, we say that these polynomials are *coprime* in $R[\boldsymbol{X}]$. They are coprime in $F[\boldsymbol{X}]$, where F is the fraction field of R, if and only if their primitive parts are coprime. A power series $h \in \mathbb{Z}[[\boldsymbol{X}]]$ is *rational* if there is a non-zero polynomial q^- such that $q^- h$ is a polynomial, denoted by q^+. Moreover, there exists a unique choice of q^-, up to the sign, such that q^+ and q^- are coprime. In that case, the polynomials q^+ and q^- are referred to respectively as *the* numerator and denominator of the rational power series.

Analytically, the power series h is identified with the holomorphic function it defines on its *domain of convergence* in $\mathbb{C}^k$. By convergence, we mean absolute convergence and by domain of convergence we mean the largest open set of $\boldsymbol{z} \in \mathbb{C}^k$ for which $\sum_{\boldsymbol{\beta} \in \mathbb{N}^k} |b(\boldsymbol{\beta}) \boldsymbol{z}^{\boldsymbol{\beta}}| < +\infty$. Once again, $\boldsymbol{z}^{\boldsymbol{\beta}}$ denotes the product $z_1^{\beta_1} \cdots z_k^{\beta_k} \in \mathbb{C}$ where as usual $z^0 = 1$ for any value of $z \in \mathbb{C}$. It is the only sensible mode of (pointwise) convergence for this series, since the set of indices (here $\mathbb{N}^k$) is not well-ordered. By Abel's Lemma (see [Sch05, Lemma 1.5.8]), if the family $(b(\boldsymbol{\beta}) \boldsymbol{z}^{\boldsymbol{\beta}})_{\boldsymbol{\beta} \in \mathbb{N}^k}$ is bounded for some $\boldsymbol{z} \in \mathbb{C}^k$, then the power series h converges absolutely on the *open polydisc* $\boldsymbol{D}(\mathbf{0}, \boldsymbol{r})$ of *polyradius* $\boldsymbol{r} = (|z_1|, \ldots, |z_k|)$, that is, on the Cartesian product of complex open discs $D(0, |z_1|) \times \cdots \times D(0, |z_k|) \subseteq \mathbb{C}^k$. Therefore, for any weaker mode of convergence requiring the boundedness of the terms, the possible new points of convergence would be in the closure of the *domain of convergence* or have a zero coordinate.

Actually, we will be more interested in the convergence of the *associated Dirichlet series*, obtained after an exponential change of variables: in that case, there is no chance to get a zero coordinate. We let h_p be the function $\boldsymbol{s} \mapsto h(p^{-s_1}, \ldots, p^{-s_k})$ where $p \in \,]1, +\infty[$ (generally, p is a prime number). Notice that for any $\lambda \in \mathbb{R}$, we have $h_p \circ S_\lambda = h_{p^\lambda}$, where S_λ is the scaling operator $\boldsymbol{s} \mapsto \lambda \boldsymbol{s}$.

The same letter will denote the power series, the holomorphic function defined by it on its domain of convergence, and often, the meromorphic continuation of this function on any admissible domain. To a meromorphic function f defined on a domain Γ of $\mathbb{C}^k$ is associated its *divisor* (f) (the domain is always specified in the context) which is a formal sum $\sum_S n_S S$ where S runs over the irreducible analytic subsets of Γ of codimension 1, and where its coefficient $n_S \in \mathbb{Z}$ is called the *order* of S. An analytic set of non-zero order is said to *appear in* the divisor (f). A critical property is that any compact subset of Γ intersects finitely many analytic sets appearing in (f). To simplify the expression of (f), we will allow some of its terms to be analytic sets not intersecting Γ, and which are not considered as appearing in (f).

The only analytic surfaces that we care about are affine hyperplanes. The canonical hermitian inner product on $\mathbb{C}^k$ is defined as $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{k} \overline{x_i} y_i$, with linearity in the second variable. Any hyperplane can be described as $\{\boldsymbol{s} \in \mathbb{C}^k \mid \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle = \lambda\}$, with $\boldsymbol{\nu} \in \mathbb{C}^k \smallsetminus \{0\}$ and $\lambda \in \mathbb{C}$. We denote this hyperplane by $H_{\boldsymbol{\nu}}(\lambda)$. Two hyperplanes $H_{\boldsymbol{\nu}}(\lambda)$ and $H_{\boldsymbol{\nu}'}(\lambda')$ are parallel if and only if the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ are proportional; they denote the same hyperplane if and only if the vectors $(\boldsymbol{\nu}, \lambda)$ and $(\boldsymbol{\nu}', \lambda')$ are proportional. If Γ is a convex domain of $\mathbb{C}^k$ and f a meromorphic function on Γ, the intersection $H_{\boldsymbol{\nu}}(\lambda) \cap \Gamma$ is either empty or an irreducible analytic subset of Γ. In the latter case, we denote this analytic subset by $H_{\boldsymbol{\nu}}(\lambda)$ as well.

Since convexity is considered, $\mathbb{C}^k$ is also regarded as a real vector space. The canonical inner product is the real part of the hermitian inner product and is denoted by

$$\Re\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{k} (\Re e(x_i)\, \Re e(y_i) + \Im m(x_i)\, \Im m(y_i)).$$

The hyperplane for this structure will also be considered and has to be distinguished from the previous type of hyperplanes (which have real codimension 2), especially since one of our major arguments is that a given family of (complex) hyperplanes clusters along a (real) hyperplane.

The vector space $\mathbb{R}^k$ is canonically regarded as the subspace of $\mathbb{C}^k$ consisting of k-tuples with real coordinates, and $\mathrm{i}\mathbb{R}^k$ the set consisting of k-tuples with imaginary coordinates. As a real vector space, we have $\mathbb{C}^k = \mathbb{R}^k \oplus \mathrm{i}\mathbb{R}^k$. The canonical inner product of $\mathbb{R}^k$ is the restriction of the Hermitian inner product, and it is denoted in the same way. A subset of $\mathbb{C}^k$ is said to be *tubular* when it can be written as $K + \mathrm{i}\mathbb{R}^k$ where $K \subseteq R^k$ and $+$ denotes the Minkowski sum.

The *Minkowski sum* (or simply the sum) of two subsets A, B of $\mathbb{C}^k$, denoted $A + B$, is defined as $\{a + b \mid a \in A, b \in B\}$. The sum of a subset A by a singleton $\{b\}$ is simply denoted $A + b$. We extend this formalism to scalar multiplication: For a subset A of $\mathbb{C}^k$ and a subset L of scalars, we let $L \cdot A$, or simply LA, denote $\{\lambda a \mid \lambda \in L,\, a \in A\}$.

Finally, the norm generally used on $\mathbb{C}^k$, or $\mathbb{R}^k$, is the one associated with the inner product (hermitian or not), and is denoted by $\|\cdot\|$.

1.4. Classical definitions and results in convex cones. We state some basic facts about convex sets in an $\mathbb{R}$-vector space of finite dimension.

The *convex hull* of a set S, denoted by $\operatorname{conv}(S)$, is the set of all linear combinations $\sum_{i \in I} \lambda_i s_i$ of elements $s_i \in S$ with the conditions $\forall i \in I$, $\lambda_i \geqslant 0$ and $\sum_{i \in I} \lambda_i = 1$. Obviously, a set is convex if and only if it is its own convex hull.

The set of linear combinations of elements of S with the condition $\sum_{i \in I} \lambda_i = 1$ only, forms an affine space, named the *affine hull* of S. The condition $\sum_{i \in I} \lambda_i = 0$ instead defines a vector space, which is the direction of the affine hull.

Finally, the set of linear combinations of elements of S with the condition $\forall i \in I$, $\lambda_i \geqslant 0$ only, form a convex cone, named the *conical hull* of S and denoted by $\operatorname{cone}(S)$. As convex cones are often easier to deal with if they are topologically closed, we denote by $\overline{\operatorname{cone}}(S)$ the closed conical hull of S, as a short-hand for $\overline{\operatorname{cone}(S)}$.

The *dimension* of a convex set is the dimension of its affine hull. The interior of a convex set as a subset of its affine hull is never empty, and is always convex, this is the *relative interior* of the convex set, denoted by $\operatorname{ri} C$ for a convex set C. The closure of a convex set is also convex. The relative interior and the closure have the same affine hull as the original set.

The intersection of any family of convex sets is convex, the union of a non-decreasing sequence of convex sets is convex. The *limit inferior* of a countably infinite family of sets is the set of elements belonging to all but finitely many sets of this family. In particular, for any enumeration $(A_n)_{n\in\mathbb{N}}$ of this family, the limit inferior can be written as

$$\liminf A_n = \bigcup_{n\geqslant 0} \bigcap_{m>n} A_m.$$

In particular, the limit inferior of a countably infinite family of convex sets is convex.

The *boundary* of a convex set K, denoted ∂K, is $\overline{K} \smallsetminus \operatorname{ri} K$. For any $x \in \partial K$, there is a non-trivial linear form ϕ such that $\forall x' \in \overline{K}$, $\phi(x') \geqslant \phi(x)$. The affine hyperplane $\phi^{-1}(\phi(x))$ is a *supporting hyperplane* of K at x.

A (proper) *face* of a convex set K is a (proper) subset F of K such that for any $x, y \in K$, if $]x, y[$ intersects F then $[x, y] \subseteq F$. A proper face F of K is also a convex set and its dimension is strictly inferior to the dimension of K. Every proper face of F is also a proper face of K. The union of all the proper faces of $\overline{K}$ is ∂K. The open convex set $\operatorname{ri} K$ has no proper face.

Faces of dimension 0 of a convex set K are called *extreme points* of K, and 1-co-dimensional faces are called *facets.* The only possible extreme point of a convex cone is the origin. A face of dimension 1 of a convex cone is either a ray (called *extreme ray*) or a line (in that case, there is no extreme point). We will be using different characterizations of extreme rays. For a ray R in a convex cone K, the following assertions are equivalent: R is extreme in K; the set $K \smallsetminus R$ is convex; for any $x \in R$, if $x = y_1 + y_2$ with $y_1, y_2 \in K$, then $y_1, y_2 \in R$; any set generating the convex cone K contains a non-zero element of R.

For any subset $S \subseteq \mathbb{R}^k$, the set of linear forms ϕ such that $\forall s \in \overline{S}, \phi(s) \geqslant \phi(0)$ is a closed convex cone of the dual space, called the *dual cone* of S and denoted $S^\vee$. The dual cone of the dual cone can be viewed as the closure of the conical hull of S. We will often identify $\mathbb{R}^n$ with its dual by using the euclidean inner product; in particular, the dual cone will often be regarded as a subset of $\mathbb{R}^k$.

By definition, we expect a convex cone to contain the origin. The relative interior of a convex cone K usually does not meet this requirement. However, we will still refer to it as an *open convex cone.* Notice that $\operatorname{ri}(K) + \overline{K} = \operatorname{ri}(K)$ by [Roc97, Theorem 6.1].

Any convex cone K is stable under addition. It has a natural structure of a *commutative monoid.* The group of invertible elements of this monoid K is the vector space $K \cap (-K)$. The *preordering* $\preccurlyeq$ on K defined by $\forall x, y \in K$, $x \preccurlyeq y \Leftrightarrow y - x \in K$ is antisymmetric when $K \cap (-K) = \{0\}$. In that case, the preordering $\preccurlyeq$ is a *partial order* and the origin is the lower bound of K for this order.

A convex cone which is the conical hull of a finite set is said to be *polyhedral.* Polyhedral cones are all closed sets. If moreover all the points of this generating finite set have rational coordinates, the convex cone is said to be *rational.*

1.5. Examples. In this section we give a handful of examples to illustrate some significant points of our claims.

The first example aims to present the algorithmic efficiency of determining Γ from a polynomial $h \in \mathbb{Z}[\boldsymbol{X}]$ with $h(\mathbf{0}) = 1$, by removing the obstructing cyclotomic factors. This can be adapted for any quotient of such polynomials that are relatively coprime: basically, the support B has to be replaced by the union of the supports of the two polynomials.

A possible algorithm is the following one:

1. Find the support B of h.
2. Find the extreme rays of the conical hull of B.
3. For each extreme ray $\boldsymbol{\nu}\mathbb{R}_+$:
    (a) Compute the associated polynomial $h_{\boldsymbol{\nu}} \in \mathbb{Z}[\boldsymbol{X}^{\boldsymbol{\nu}}]$.
    (b) Check if $h_{\boldsymbol{\nu}}$ is a product of cyclotomic polynomials.
    (c) Check if $h_{\boldsymbol{\nu}}$ divides h (and compute the quotient if it does).
    (d) If both conditions (b) and (c) are satisfied, run the algorithm again for $h/h_{\boldsymbol{\nu}}$. Otherwise, mark the ray $\boldsymbol{\nu}\mathbb{R}_+$ as an extreme ray of $\Gamma^\vee$
4. Construct Γ for the resulting extreme rays of $\Gamma^\vee$

From a computational point of view, there are fast algorithms to carry out step 2, based on repeated linear programming problems (see [Cha96]). If this step is repeated for the quotient $h/h_{\boldsymbol{\nu}}$, it is even faster since all extreme rays computed for h other than $\boldsymbol{\nu}\mathbb{R}_+$ are extreme rays for $h/h_{\boldsymbol{\nu}}$. This remark is also valid for step 3, and for an extreme ray different from $\boldsymbol{\nu}\mathbb{R}_+$, substeps 3(a)–(d) give the same answer for the polynomial h or the quotient $h/h_{\boldsymbol{\nu}}$. The test 3(b) is also fast, by obtaining a fixed point after at most $O(\log \deg h_{\boldsymbol{\nu}})$ iterations of Graeffe's transform (see [BD89]). Finally, in 3(c), the divisibility and the quotient of h by $h_{\boldsymbol{\nu}}$ can also be determined by a fast algorithm (like the Fast Fourier Transform) since the divisor is essentially univariate: if we write h as a polynomial in $k-1$ variables with coefficients in $\mathbb{Z}[\boldsymbol{X}^{\boldsymbol{\nu}}]$, the computation is equivalent to the divisibility and the quotient of every coefficient in $\mathbb{Z}[\boldsymbol{X}^{\boldsymbol{\nu}}]$ by $h_{\boldsymbol{\nu}}$.

We give here an example of a simple polynomial where the different steps of the algorithms can be checked by hand.

Consider the polynomial $h = 1 + X_1 + X_2 + X_3 - X_1X_3^2 + X_2X_3$ in three variables.

1. The support of this polynomial is a finite subset of $\mathbb{N}^3$ whose elements are
$$\mathbf{0},\ \boldsymbol{\epsilon}_1 = (1,0,0),\ \boldsymbol{\epsilon}_2 = (0,1,0),\ \boldsymbol{\epsilon}_3 = (0,0,1),\ \boldsymbol{\epsilon}_1 + 2\boldsymbol{\epsilon}_3,\ \boldsymbol{\epsilon}_2 + \boldsymbol{\epsilon}_3.$$
2. The points $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$, $\boldsymbol{\epsilon}_3$ are the directions of the three extreme rays.

3. For $i \in \{1,2,3\}$, we have $h_{\boldsymbol{\epsilon}_i} = 1 + \boldsymbol{X}^{\boldsymbol{\epsilon}_i} = 1 + X_i$, which is a cyclotomic polynomial. We check the divisibility by $h_{\boldsymbol{\epsilon}_3} = 1 + X_3$ first. We write h as a polynomial with coefficients in $\mathbb{Z}[X_3]$: $h = (1 + X_3) + (1 - X_3^2)X_1 + (1 + X_3)X_2$. All the coefficients are divisible by $h_{\boldsymbol{\epsilon}_3} = 1 + X_3$ and the quotient is $1 + X_1 + X_2 - X_1X_3$. We rerun the algorithm for this polynomial, but this new polynomial is clearly irreducible and not a cyclotomic polynomial. The extreme rays are directed by $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ and $\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_3$.
4. By a direct application of Theorem 1.2, we have the following description of the maximal domain of meromorphy:

$$\Gamma = \{(s_1, s_2, s_3) \in \mathbb{C}^3 \mid \Re e(s_1) > 0, \Re e(s_2) > 0, \Re e(s_1 + s_3) > 0\}.$$

We now give other examples to describe and discuss the three different cases we will encounter during the proof of Theorem 1.2, and the arguments we will use.

First consider $h = 1 - 2X_2 + 4X_1X_2^2 - 4X_1^2X_2^2$. The extreme rays are directed by $(1,1)$ and $(0,1)$, and are associated to the polynomials $h_{(1,1)} = 1 - 4(X_1X_2)^2$ and $h_{(0,1)} = 1 - 2X_2$. There is no risk of a relevant cyclotomic factor, and the maximal domain of meromorphy is determined by these two vectors.

Let us nonetheless continue the process of factorization to understand why the face $\mathcal{F}_{(1,1)}$ consists of singular points. Writing h as a polynomial with coefficients in $\mathbb{Z}[X_1X_2]$, we get a new factorization

$$h = (1 - 4(X_1X_2)^2) + (-2 + 4X_1X_2)X_2 = (1 - 2X_1X_2)(1 + 2X_1X_2 - 2X_2).$$

Each factor is associated to an Euler product which is meromorphic over Γ (at least). This gives a factorization of f over Γ:

$$f = \prod_p (1 - 2p^{-(s_1+s_2)}) \prod_p (1 + 2p^{-(s_1+s_2)} - 2p^{-s_2}).$$

Now consider the hyperplanes normal to $(1,1)$ (that is, lines defined by $s_1 + s_2 = \gamma$ with $\gamma \in \mathbb{C}$) which may appear in the divisor of f. They may come from a zeta factor (in that case γ is a zero of ζ divided by a positive integer) or from a local factor. No term $1 + 2p^{-(s_1+s_2)} - 2p^{-s_2}$ has such hyperplane in its divisor, but each term $1 - 2p^{-(s_1+s_2)}$ does (in that case γ is a zero of $1 - 2p^{-s}$). Indeed, the divisor of $\prod_p(1 - 2p^{-(s_1+s_2)})$ is only composed of those hyperplanes, whose constants γ are the zeros and poles of the meromorphic extension of the univariate Euler product $\prod_p(1 - 2p^{-s})$.

This boils down to the study of the zeros and poles of $\prod_p(1-2p^{-s})$, which was done by Estermann [Est28], who proved that the zeros coming from the local factors accumulate along the imaginary line and are more plentiful than the poles coming from the zeta factors. This proves the accumulation of hyperplanes in the divisor of f along the face $\mathcal{F}_{(1,1)}$ supported by the (real) hyperplane $\Re e(s_1 + s_2) = 0$.

The argument we have just given (when h can be factorized by a polynomial whose exponents belong to one extreme ray) will be referred to as *Estermann's argument.*

We now consider the other face $\mathcal{F}_{(0,1)}$ of Γ, and the family of hyperplanes normal to $(0,1)$ (defined by an equation of the form $s_2 = \gamma$ with γ in $\mathbb{C}$) that appears in the divisor of f. It is obvious that no such hyperplane appears in the divisor of $\prod_p(1 - 2p^{-(s_1+s_2)})$, nor in the zero set of a local factor $1 + 2p^{-(s_1+s_2)} - 2p^{-s_2}$. They all arise from the zeta factors of $\prod_p(1 + 2p^{-(s_1+s_2)} - 2p^{-s_2})$.

These zeta factors are completely described by the expression of h as an infinite product $\prod_{\boldsymbol{\alpha}}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c_{\boldsymbol{\alpha}}}$, where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ goes through $\mathbb{N}^2\smallsetminus\{(0,0)\}$, and $c_{\boldsymbol{\alpha}}\in\mathbb{Z}$. Such formal expansion of h is always possible, and is uniquely determined. In particular, a factor $(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c_{\boldsymbol{\alpha}}}$ with $c_{\boldsymbol{\alpha}}\neq 0$ generates an Euler product $\zeta(\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle)^{-c_{\boldsymbol{\alpha}}}$, whose divisor consists of hyperplanes normal to $\boldsymbol{\alpha}$ and appears in the divisor of f as in Theorem 1.1. In particular, the hyperplanes of (f) normal to $(0,1)$ are completely described by the infinite subproduct $\prod_{\alpha_2>0}(1-2X_2^{\alpha_2})^{c(0,\alpha_2)}$, which is equal to the whole product evaluated at $X_1=0$, that is, $h(0,X_2)=h_{(0,1)}=1-2X_2$. Therefore, the hyperplane defined by $s_2=\gamma$ is in the divisor of f if and only if γ is a zero or pole coming from the zeta factors of $\prod_p(1-2p^{-s_2})$.

Once again, the problem boils down to the study of a univariate Euler product. And in this case, Dahlquist [Dah52] has essentially proven the accumulation along the imaginary axis of zeros and poles arising from the zeta factors of such an Euler product, provided that there are infinitely many zeta factors (that is, as long as $h_{(0,1)}$ is not a product of cyclotomic polynomials). This proves the accumulation of hyperplanes in the divisor of (f) that are orthogonal to $(0,1)$ along the face $\mathcal{F}_{(0,1)}$ supported by the (real) hyperplane $\Re e(s_2)=0$.

This kind of argument (when $h_{\boldsymbol{\nu}}$ has no common factor with h and $h_{\boldsymbol{\nu}}$ is not a finite product of cyclotomic factors) will be referred to as *Dahlquist's argument.*

Finally, it remains to consider the following case:

- there is no common factor between $h_{\boldsymbol{\nu}}$ and h;
- the polynomial (respectively the rational function) $h_{\boldsymbol{\nu}}$ is a finite product (respectively a quotient of finite products) of cyclotomic polynomials.

This is precisely the case of our previous example $1+X_1+X_2-X_1X_3$ for its three extreme rays of respective directions $\boldsymbol{\epsilon}_1,\boldsymbol{\epsilon}_2$ and $\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_3$. The corresponding polynomials are $1+X_i$ for the first two directions and $1-X_1X_3$ for the third. Since we are allowed to multiply or divide h by a cyclotomic polynomial, we may as well consider the rational fraction $h^\circ=\frac{1+X_1+X_2-X_1X_3}{(1+X_1)(1+X_2)(1-X_1X_3)}$, for which the Euler product has the same domain of meromorphy.

For that function h°, we have $h^\circ_{\boldsymbol{\epsilon}_1}=h^\circ_{\boldsymbol{\epsilon}_2}=h^\circ_{(1,0,1)}=1$. This third case then boils down to the case $h_{\boldsymbol{\nu}}=1$, thus the divisor of the Euler product f does not have any hyperplane orthogonal to $\boldsymbol{\nu}$. Our previous strategy of accumulation of parallel hyperplanes (as in Estermann's or Dahlquist's arguments) is bound to fail in that case.

We illustrate this case with a simpler expression $h=\frac{1+X_1+X_2-X_1X_2}{1+X_1+X_2}$, to which we have already referred earlier. Both the numerator and denominator are irreducible polynomials, therefore there is no hyperplane in the divisor of f which comes from the local factors. The only hyperplanes in the divisor (f) are then described by the exponents in the infinite product $h=\prod_{\boldsymbol{\alpha}}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c_{\boldsymbol{\alpha}}}$. The two extreme rays are clearly directed by $(1,0)$ and $(0,1)$; by symmetry, we will only consider the direction $(0,1)$. We have $h_{(0,1)}=h(0,X_2)=\prod_\alpha(1-X_2^\alpha)^{c(0,\alpha)}=1$, therefore $c_{(0,\alpha)}=0$ for any $\alpha\geqslant 1$, and there are no hyperplanes in (f) orthogonal to $(0,1)$.

We look for other hyperplanes in (f) by computing more coefficients $c_{\boldsymbol{\alpha}}$. In this case, even if computing $c(1,\alpha)$ is straightforward for this particular polynomial, it is still a fair description of the argument we use for any polynomial.

Consider the value of the infinite product modulo X_1^2 (setting $X_1 = 0$ in the product is equivalent to computing its residue modulo X_1). Taking advantage of the fact that $c(0,\alpha) = 0$ for any $\alpha \geqslant 1$, we have

$$\prod_{\boldsymbol{\alpha}} (1 - \boldsymbol{X}^{\boldsymbol{\alpha}})^{c_{\boldsymbol{\alpha}}} \equiv \prod_{\alpha \geqslant 1} (1 - X_1 X_2^{\alpha})^{c_{(1,\alpha)}} \equiv 1 - \sum_{\alpha \geqslant 1} c_{(1,\alpha)} X_1 X_2^{\alpha} \pmod{X_1^2}.$$

But computing the MacLaurin series of h with coefficients in $\mathbb{Q}(X_2)$ is fairly straightforward:

$$h = 1 - X_1 X_2 \frac{1}{1 + X_1 + X_2} = 1 + \sum_{n \geqslant 1} (-1)^n \frac{X_2}{(1+X_2)^n} X_1^n.$$

Therefore, we have

$$1 - \sum_{\alpha \geqslant 1} c_{(1,\alpha)} X_1 X_2^{\alpha} \equiv h \equiv 1 - \frac{X_2}{1 + X_2} X_1 \equiv 1 - \sum_{\alpha \geqslant 1} (-1)^{\alpha - 1} X_1 X_2^{\alpha} \pmod{X_1^2},$$

and finally, $c_{(1,\alpha)} = (-1)^{\alpha-1}$ for any $\alpha \geqslant 1$.

Therefore, for any $\alpha \geqslant 1$, and any non-trivial zero γ of the Riemann ζ-function with $\Re e(\gamma) \geqslant 1/2$, the hyperplane $s_1 + \alpha s_2 = \gamma$ is a simple zero or a pole of f, which could only be cancelled by another zeta factor, if 2γ (or $k\gamma$ with $k \geqslant 2$) were a zero of ζ. By the prime number theorem, ζ has no zero within the half-plane $\Re e(s) \geqslant 1$.

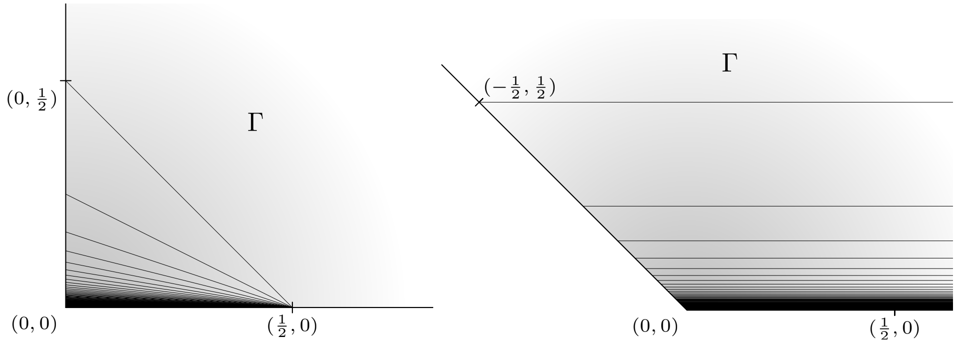

Fig. 1. Real part projections on $\mathbb{R}^2$ of the domain of meromorphy Γ and of an accumulating family of hyperplanes of the divisor (f) for the functions $h = \frac{1+X_1+X_2-X_1X_2}{1+X_1+X_2}$ (left) and $h = 1 + 2X_1X_2 - 2X_2$ (right)

The set of clustering points in $\overline{\Gamma} = \{s \in \mathbb{C} \mid \Re e(s) \geqslant 0\}^2$ of this family of hyperplanes contains the set $[0, 1/2] \times \{0\} + \mathrm{i}\mathbb{R}^2$ as shown in Figure 1. Notice the difference with the previous cases where a family of parallel hyperplanes were exhibited. Therefore, only considering hyperplanes in the divisor is not enough to determine the singularity of f at any point of the face $\mathbb{R}_+ \times \{0\} + \mathrm{i}\mathbb{R}^2$. We will reach this goal thanks to Proposition 1.6.

2. The convergence theorem

In this section, we provide a proof of our main theorem of convergence, which gives an explicit description of a domain on which the Euler product $\prod_p h(p^{-s_1}, \dots, p^{-s_k})$ can be meromorphically extended, as well as an expression of the divisor of this meromorphic function on this domain.

This theorem applies to any power series h with some simple conditions:

- it is convergent in a neighborhood of the origin,
- its constant term is equal to 1,
- its coefficients are integers.

The first two conditions are necessary to ensure the existence of the Euler product. As a matter of fact, these assumptions imply that $h(\boldsymbol{z}) = 1 + O(\max_i |z_i|)$ as $\boldsymbol{z} = (z_1, \dots, z_k)$ tends to $\mathbf{0}$, therefore the Euler product $\prod_p h(p^{-s_1}, \dots, p^{-s_k})$ converges to a limit $f(s_1, \dots, s_k)$ if all the real parts of $s_1, \dots, s_k$ are larger than 1. The convergence being locally uniform, the limit f is meromorphic at $\boldsymbol{s}$ whenever every term of the product is meromorphic at $\boldsymbol{s}$.

The primary reason for the third condition is to simplify the treatment. The presence of a non-integer coefficient roughly corresponds to the factorization of f by a non-integer power of the Riemann ζ-function, which accounts for a family of singularities of f. In specific examples, these singularities are usually easy to keep track of, for example by considering terms with non-integer multiplicities in the expression of the divisor of f.

Actually, we will require a fourth assumption on h, which is difficult to state at this point. Basically, any default of meromorphy of the function $\boldsymbol{s} \mapsto h(\mathrm{e}^{-s_1}, \dots, \mathrm{e}^{-s_k})$ is passed down to every term h_p of the product defining f. Since we want to avoid every foreseeable singularity for f from local factors h_p, we will require the function $\boldsymbol{s} \mapsto h(\mathrm{e}^{-s_1}, \dots, \mathrm{e}^{-s_k})$ to be meromorphic on the domain where f is meromorphic.

However, we will suggest a less precise statement for functions which do not satisfy this assumption.

2.1. Formal development as an infinite product. We show that any formal series

$$\sum_{\boldsymbol{\beta} \in \mathbb{N}^k} b(\boldsymbol{\beta}) \boldsymbol{X}^{\boldsymbol{\beta}} \in \mathbb{Z}[\![\boldsymbol{X}]\!] \quad \text{with } b(\mathbf{0}) = 1 \tag{2.1}$$

has a unique representation as a formal infinite product

$$\prod_{\boldsymbol{\alpha} \in \mathbb{N}^k} (1 - \boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})} \quad \text{with } c(\mathbf{0}) = 0, \tag{2.2}$$

and conversely. Moreover, we give formulas to compute the exponent function $c : \mathbb{N}^k \to \mathbb{Z}$ of (2.2) from the coefficient function $b : \mathbb{N}^k \to \mathbb{Z}$ of (2.1), and the other way around. We will always assume that $c(\mathbf{0}) = 0$ so that the term with $\boldsymbol{\alpha} = \mathbf{0}$ is interpreted as 1, and that $b(\mathbf{0}) = 1$.

Our results generalize the argument of Dahlquist [Dah52, Equation (2.1)] for power series in a single variable and the argument of Delabarre [Del13, Corollary 2.2] for polynomials in multiple variables.

We first show that these infinite products (2.2) describe exactly the set of formal power series in k variables with constant term equal to 1.

Lemma 2.1. *For any integer exponent function* $c : \mathbb{N}^k \to \mathbb{Z}$ *with* $c(\mathbf{0}) = 0$, *there exists an integer coefficient function* $b : \mathbb{N}^k \to \mathbb{Z}$ *with* $b(\mathbf{0}) = 1$ *such that*

$$\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k} (1 - \boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^k} b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}} \tag{2.3}$$

is an identity in $\mathbb{Z}[[\boldsymbol{X}]]$. *Moreover, for any* $\boldsymbol{\beta} \in \mathbb{N}^k$, *we have*

$$b(\boldsymbol{\beta}) = \sum_{\substack{m\colon \mathbb{N}^k\to\mathbb{N} \\ \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} m(\boldsymbol{\alpha})\boldsymbol{\alpha}=\boldsymbol{\beta}}} \prod_{\boldsymbol{\alpha}\in\mathbb{N}^k} (-1)^{m(\boldsymbol{\alpha})}\binom{c(\boldsymbol{\alpha})}{m(\boldsymbol{\alpha})}. \tag{2.4}$$

Note that the outer sum runs over the partitions of $\boldsymbol{\beta}$ with parts in $\mathbb{N}^k$, which are described through the function $m : \mathbb{N}^k \to \mathbb{N}$. Since m is of finite support, almost every term of the inner product is equal to 1. Thus, this product is well-defined.

Since $c(\mathbf{0}) = 0$, the term with $\boldsymbol{\alpha} = \mathbf{0}$ of this product is zero, unless $m(\mathbf{0}) = 0$. Since there are finitely many partitions of $\boldsymbol{\beta}$ in non-zero parts, only finitely many terms of the outer sum are different from zero; thus, the sum is well-defined as well.

Proof. Recall that

$$\binom{c}{m} = \frac{1}{m!}\prod_{i=0}^{m-1}(c-i) \quad\text{and}\quad (1-X)^c = \sum_{m\geqslant 0}(-1)^m\binom{c}{m}X^m$$

for any $c \in \mathbb{Z}$. Therefore, the product $\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1 - \boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}$ is formally equal to

$$\sum_{\boldsymbol{\beta}\in\mathbb{N}^k}\Bigg(\sum_{\substack{m\colon \mathbb{N}^k\to\mathbb{N} \\ \sum_{\boldsymbol{\alpha}} m(\boldsymbol{\alpha})\boldsymbol{\alpha}=\boldsymbol{\beta}}} \prod_{\boldsymbol{\alpha}\in\mathbb{N}^k} (-1)^{m(\boldsymbol{\alpha})}\binom{c(\boldsymbol{\alpha})}{m(\boldsymbol{\alpha})}\Bigg)\boldsymbol{X}^{\boldsymbol{\beta}} \in \mathbb{Z}[[\boldsymbol{X}]], \tag{2.5}$$

where the inner sum runs over the partitions of $\boldsymbol{\beta}$ with parts in $\mathbb{N}^k$. We have already noticed that this expression is well-defined. For the constant term of this series (with $\boldsymbol{\beta} = \mathbf{0}$) the inner sum has only one non-zero term — the empty partition with $m(\boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha}$ – and its value is equal to 1. ■

We now prove the converse of Lemma 2.1, generalizing a formula of Delabarre [Del10, Lemme 25] obtained when the coefficient function $b : \mathbb{N}^k \to \mathbb{Z}$ has finite support.

Lemma 2.2. *For any integer coefficient function* $b : \mathbb{N}^k \to \mathbb{Z}$ *with* $b(\mathbf{0}) = 1$, *there exists an integer exponent function* $c : \mathbb{N}^k \to \mathbb{Z}$ *with* $c(\mathbf{0}) = 0$ *such that* (2.3) *is an identity in* $\mathbb{Z}[[\boldsymbol{X}]]$. *Moreover, for any* $\boldsymbol{\alpha} \in \mathbb{N}^k$, *we have*

$$c(\boldsymbol{\alpha}) = \sum_{r\geqslant 1}\sum_{d\geqslant 1}\frac{(-1)^r\mu(d)}{dr}\sum_{\substack{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_r\neq\mathbf{0} \\ d\sum_{i=1}^r \boldsymbol{\beta}_i=\boldsymbol{\alpha}}}\Big(\prod_{i=1}^r b(\boldsymbol{\beta}_i)\Big). \tag{2.6}$$

The inner sum is over r-tuples of non-zero elements $\boldsymbol{\beta}_i$ of $\mathbb{N}^k$ whose sum is equal to $\frac{1}{d}\boldsymbol{\alpha}$. In particular, the integers d and r and the sum of the coordinates of any $\boldsymbol{\beta}_i$ are all less than or equal to the sum of coordinates of $\boldsymbol{\alpha}$. Thus, every sum is finite in this formula.

Proof of Lemma 2.2. We first show that there exists an exponent function c satisfying the first identity, then give a necessary expression for its values. For the existence, we use inductive reasoning thanks to a non-decreasing (*) sequence $(\boldsymbol{\lambda}_n)_{n\in\mathbb{N}}$ enumerating $\mathbb{N}^k \smallsetminus \{\mathbf{0}\}$.

We recursively define the sequence $(c_n)_{n\in\mathbb{N}}$ of integers such that for any $n \in \mathbb{N}$, c_n is the opposite of the coefficient of $\boldsymbol{X}^{\boldsymbol{\lambda}_n}$ in the finite product of power series

$$P_n = \prod_{i=0}^{n-1} (1 - \boldsymbol{X}^{\boldsymbol{\lambda}_i})^{-c_i} \Big(1 + \sum_{\boldsymbol{\beta}\neq\mathbf{0}} b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}}\Big) \in \mathbb{Z}[[\boldsymbol{X}]]$$

where the sum is defined over non-zero $\boldsymbol{\beta} \in \mathbb{N}^k$.

We prove by induction on $n \in \mathbb{N}$ that for any $m < n$, the coefficient of $\boldsymbol{X}^{\boldsymbol{\lambda}_m}$ in P_n is zero. For $m = n-1$, this follows from the definition of c_{n-1}, for $m < n-1$, it follows from the induction hypothesis for P_{n-1} and the fact that the sequence $(\boldsymbol{\lambda}_n)_{n\in\mathbb{N}}$ is non-decreasing.

Since the sequence $(\boldsymbol{\lambda}_n)_{n\in\mathbb{N}}$ enumerates $\mathbb{N}^k \smallsetminus \{\mathbf{0}\}$, we have the first formal identity by setting $c(\mathbf{0}) = 0$ and $c(\boldsymbol{\alpha})$ as c_n where n is such that $\boldsymbol{\lambda}_n = \boldsymbol{\alpha}$. We derive

$$\log\Big(1 + \sum_{\boldsymbol{\beta}\neq\mathbf{0}} b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}}\Big) = \log\Big(\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k} (1 - \boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}\Big) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha}) \log(1 - \boldsymbol{X}^{\boldsymbol{\alpha}}). \tag{2.7}$$

The left-hand side term may be rewritten as

$$\sum_{r\geqslant 1} \frac{(-1)^{r-1}}{r} \Big(\sum_{\boldsymbol{\beta}\neq\mathbf{0}} b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}}\Big)^r = \sum_{r\geqslant 1} \frac{(-1)^{r-1}}{r} \sum_{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_r\neq\mathbf{0}} \Big(\prod_{i=1}^r b(\boldsymbol{\beta}_i)\Big) \boldsymbol{X}^{\sum_{i=1}^r \boldsymbol{\beta}_i}.$$

By Möbius inversion of the formal identity $-\log(1-X) = \sum_{m\geqslant 1} X^m/m$, we get $X = -\sum_{d\geqslant 1} \frac{\mu(d)}{d} \log(1 - X^d)$, and we derive the formal identity

$$\begin{aligned}\log\Big(1 + \sum_{\boldsymbol{\beta}\neq\mathbf{0}} b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}}\Big)& \\ = \sum_{r\geqslant 1}\sum_{d\geqslant 1} \frac{(-1)^r \mu(d)}{dr} &\sum_{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_r\neq\mathbf{0}} \Big(\prod_{i=1}^r b(\boldsymbol{\beta}_i)\Big) \log(1 - \boldsymbol{X}^{d\sum_{i=1}^r \boldsymbol{\beta}_i}).\end{aligned} \tag{2.8}$$

In (2.7) and (2.8), we get two expressions of the same series as an infinite expansion with respect to the family $(\log(1 - \boldsymbol{X}^{\boldsymbol{\alpha}}))_{\boldsymbol{\alpha}\in\mathbb{N}^k}$. Since $\log(1 - \boldsymbol{X}^{\boldsymbol{\alpha}})$ is the sum of $-\boldsymbol{X}^{\boldsymbol{\alpha}}$ and of monomials of larger exponent vectors, this expansion is unique. By identifying coefficients, we have a necessary expression for $c(\boldsymbol{\alpha})$ for every $\boldsymbol{\alpha} \in \mathbb{N}^k$:

$$c(\boldsymbol{\alpha}) = \sum_{r\geqslant 1}\sum_{d\geqslant 1} \frac{(-1)^r \mu(d)}{dr} \sum_{\substack{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_r\neq\mathbf{0} \\ d\sum_{i=1}^r \boldsymbol{\beta}_i = \boldsymbol{\alpha}}} \Big(\prod_{i=1}^r b(\boldsymbol{\beta}_i)\Big). \quad \blacksquare$$

Incidentally, this proves that this complicated formula always yields an integer (cf. [Est28, p. 448]).

(*) For the usual semi-order of $\mathbb{N}^k$, that is, $\boldsymbol{\lambda}_n - \boldsymbol{\lambda}_m \in \mathbb{N}^k \Rightarrow n > m$. Any sequence such that the sum of coordinates is not decreasing satisfies this condition.

2.2. Analytic convergence. In this section, we give an analytic meaning of the formal identity in $\mathbb{Z}[[\boldsymbol{X}]]$ of Lemmata 2.1 and 2.2. This will allow us to replace the three conditions on h by the equivalent assumption:

For $\boldsymbol{z}$ in the neighborhood of $\mathbf{0}$, the value $h(\boldsymbol{z})$ is defined by an infinite product $\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1-\boldsymbol{z}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}$ with $c(\boldsymbol{\alpha})\in\mathbb{Z}^*$ and $c(\mathbf{0})=0$.

Lemma 2.3. *Let* $b:\mathbb{N}^k\to\mathbb{Z}$ *and* $c:\mathbb{N}^k\to\mathbb{Z}$ *be two functions with* $b(\mathbf{0})=1$ *and* $c(\mathbf{0})=0$ *satisfying the formal relation*

$$\sum_{\boldsymbol{\beta}\in\mathbb{N}^k} b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}} = \prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})} \in \mathbb{Z}[[\boldsymbol{X}]].$$

The LHS-term converges in a neighborhood of $\mathbf{0}$ *if and only if the RHS-term converges in a neighborhood of* $\mathbf{0}$.

We recall that we refer here to the absolute convergence of the series $\sum_{\boldsymbol{\beta}\in\mathbb{N}^k} b(\boldsymbol{\beta})\boldsymbol{z}^{\boldsymbol{\beta}}$, and of the absolute convergence of the infinite product $\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1-\boldsymbol{z}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}$ for some $\boldsymbol{z}\in\mathbb{C}^k$.

It is standard (see [Kno90, Chapter VII]) that the absolute convergence of an infinite product $\prod_{n\in\mathbb{N}}(1+a_n)$ is equivalent to the absolute convergence of the series $\sum_{n\in\mathbb{N}} a_n$. Taking into account the multiplicities among the terms, we derive that the absolute convergence of an infinite product $\prod_{n\in\mathbb{N}}(1+a_n)^{c_n}$ is equivalent to the absolute convergence of the series $\sum_{n\in\mathbb{N}} c_n a_n$, where c_n are positive integers. Treating positive exponents c_n and negative ones separately, this equivalence also holds for any sequence $(c_n)_{n\in\mathbb{N}}$ of integers, as long as $a_n\neq -1$ for every $n\in\mathbb{N}$.

Therefore, if $\boldsymbol{z}$ is in the open unit polydisc $\boldsymbol{D}(\mathbf{0},\mathbf{1})$, the absolute convergence of this latter product $\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1-\boldsymbol{z}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}$ is equivalent to the absolute convergence of the series $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\boldsymbol{z}^{\boldsymbol{\alpha}}$, which is easier to deal with.

Proof of Lemma 2.3. If the series $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\boldsymbol{z}^{\boldsymbol{\alpha}}$ converges absolutely at a non-zero $\boldsymbol{z}\in\boldsymbol{D}(\mathbf{0},\mathbf{1})$, then it converges normally on the open polydisc of polyradius $(|z_1|,\dots,|z_k|)$, and the product defines a holomorphic function on this polydisc, which is the sum of its power series expansion, that is, $\sum_{\boldsymbol{\beta}\in\mathbb{N}^k} b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}}$, on this polydisc. More generally, the domain of convergence of $\sum_{\boldsymbol{\beta}\in\mathbb{N}^k} b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}}$ contains the intersection of the domain of convergence of $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\boldsymbol{X}^{\boldsymbol{\alpha}}$ with the open unit polydisc $\boldsymbol{D}(\mathbf{0},\mathbf{1})$.

We now consider the converse. Assume that $\sum_{\boldsymbol{\beta}\in\mathbb{N}^k} b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}}$ converges on a compact polydisc of polyradius $\boldsymbol{r}\in\,]0,1[^k$. Then there exists $M\geqslant 1$ such that $|b(\boldsymbol{\beta})|\leqslant M\boldsymbol{r}^{-\boldsymbol{\beta}}$ for any $\boldsymbol{\beta}\in\mathbb{N}^k$. By Lemma 2.2 we deduce that for any $\boldsymbol{\alpha}$,

$$|c(\boldsymbol{\alpha})| \leqslant \sum_{d\geqslant 1}\sum_{r\geqslant 1}\frac{1}{dr}M^r\boldsymbol{r}^{-\boldsymbol{\alpha}/d}\left|\left\{(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_r)\in(\mathbb{N}^k\smallsetminus\{\mathbf{0}\})^r \;\middle|\; d\sum_{i=1}^r \boldsymbol{\beta}_i=\boldsymbol{\alpha}\right\}\right|;$$

therefore

$$|c(\boldsymbol{\alpha})| \leqslant M^{\alpha_1+\cdots+\alpha_k}\boldsymbol{r}^{-\boldsymbol{\alpha}}\frac{\sigma(\alpha_1+\cdots+\alpha_k)}{\alpha_1+\cdots+\alpha_k}P(\boldsymbol{\alpha}),$$

where $P(\boldsymbol{\lambda})$ is the number of representations of $\boldsymbol{\lambda}$ as a sum of non-zero elements of $\mathbb{N}^k$, and where $\sigma(n)$ is the sum of the divisors of n. It is well-known that for any $\varepsilon>0$ we have $\sigma(n)/n \ll_\varepsilon (1+\varepsilon)^n$ for $n\in\mathbb{N}$.

The series $\sum_{\boldsymbol{\lambda}\in\mathbb{N}^k} P(\boldsymbol{\lambda})\boldsymbol{X}^{\boldsymbol{\lambda}}$ is formally equal to the product $\prod_{\boldsymbol{\mu}\neq\boldsymbol{0}}(1+\boldsymbol{X}^{\boldsymbol{\mu}})$. This product converges absolutely wherever the series $\sum_{\boldsymbol{\mu}\in\mathbb{N}^k}\boldsymbol{X}^{\boldsymbol{\mu}}$ does. But this sum has a formal expansion $\prod_{i=1}^{k}(1-X_i)^{-1}$ which corresponds to a holomorphic function on the open unit polydisc $\boldsymbol{D}(\boldsymbol{0},\boldsymbol{1})$. For $\boldsymbol{z}$ in this polydisc, the power series $\sum_{\boldsymbol{\mu}\in\mathbb{N}^k}\boldsymbol{z}^{\boldsymbol{\mu}}$ converges, the infinite product $\prod_{\boldsymbol{\mu}\neq\boldsymbol{0}}(1+\boldsymbol{z}^{\boldsymbol{\mu}})$ also converges, and so does the power series $\sum_{\boldsymbol{\lambda}\in\mathbb{N}^k} P(\boldsymbol{\lambda})\boldsymbol{z}^{\boldsymbol{\lambda}}$. Therefore, for any $\rho\in\,]0,1[$ we have

$$P(\boldsymbol{\lambda})\ll_\rho \rho^{-(\lambda_1+\cdots+\lambda_k)}\quad\text{for } \boldsymbol{\lambda}\in\mathbb{N}^k.$$

We now set $\boldsymbol{r'}=\frac{\rho}{M(1+\varepsilon)}\boldsymbol{r}$, and we get $|c(\boldsymbol{\alpha})|\ll \boldsymbol{r'}^{-\boldsymbol{\alpha}}$ for any $\boldsymbol{\alpha}\in\mathbb{N}^k$. Therefore, both the power series $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\boldsymbol{z}^{\boldsymbol{\alpha}}$ and the infinite product $\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}$ converge on the open polydisc of polyradius $\boldsymbol{r'}\in\,]0,1[^k$. ■

2.3. Uniform convergence of a power series. In this section, we make precise a classical result on the convergence of power series in several variables.

Let $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\boldsymbol{z}^{\boldsymbol{\alpha}}$ be a power series and define E as the set of $\boldsymbol{z}\in\mathbb{C}^k$ for which there exists a real number γ such that $|c(\boldsymbol{\alpha})\boldsymbol{z}^{\boldsymbol{\alpha}}|\leqslant\gamma$ for all $\boldsymbol{\alpha}\in\mathbb{N}^k$.

A direct generalization of Abel's Lemma (e.g. [Sch05, Lemma 1.5.8]) implies that the interior $\mathring{E}$ is the domain of convergence of the series and, if $\boldsymbol{z}\in\mathring{E}$, then the power series converges normally on the closed polydisc of polyradius $(|z_1|,\ldots,|z_k|)$.

This result is sufficient to deduce that the sum of a power series is a holomorphic function on its domain of convergence. We will show that if the support of the coefficient function c is sufficiently small compared to $\mathbb{N}^k$, then we can replace the closed polydiscs in Abel's Lemma by larger, unbounded closed sets.

In the framework of Dirichlet series, namely, substituting (†) $\boldsymbol{z}$ by $(\mathrm{e}^{-s_1},\ldots,\mathrm{e}^{-s_k})$ with $\boldsymbol{s}\in\mathbb{C}^k$, Abel's Lemma is translated as follows.

LEMMA 2.4. *The domain of convergence of the Dirichlet series* $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}$ *is the interior of the set* D *of* $\boldsymbol{s}\in\mathbb{C}^k$ *for which there exists* $\theta\in\mathbb{R}$ *such that* $\Re\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle\geqslant\theta+\log|c(\boldsymbol{\alpha})|$ *for every* $\boldsymbol{\alpha}\in\mathbb{N}^k$.

For any $\boldsymbol{s}_0\in\mathring{D}$, *the Dirichlet series converges normally on* $\boldsymbol{s}_0+\{\boldsymbol{s}\in\mathbb{C}^k\mid\forall i\in[\![1,k]\!],\ \Re e\,s_i\geqslant 0\}$.

Notice that D is a tubular convex set (‡), and if $c(\boldsymbol{\alpha})=0$, the condition $\Re\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle\geqslant\log|c(\boldsymbol{\alpha})|+\theta$ has to be interpreted as satisfied for all $\boldsymbol{s}\in\mathbb{C}^k$. The parameter θ in D is related to the parameter γ in E by $\gamma=\mathrm{e}^{-\theta}$.

LEMMA 2.5. *Let* $s_0\in\mathbb{C}^k$ *be in the domain of convergence of the Dirichlet series*

$$\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}\,.$$

Let $C\subseteq\mathbb{N}^k$ *be the support of the coefficient function* c *and* $C^\vee$ *its dual cone in* $\mathbb{C}^k$. *Then the Dirichlet series* $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}$ *is normally convergent on* $\boldsymbol{s}_0+C^\vee$.

(†) Only $\boldsymbol{z}$ with a zero coordinate cannot be represented this way.

(‡) This corresponds to the fact that $\mathring{E}$ is a logarithmically convex Reinhardt domain.

Proof. Let $\boldsymbol{s} = \boldsymbol{s}_0 + \boldsymbol{s'}$ with $\boldsymbol{s'} \in C^\vee$. By definition of $C^\vee$, for any $\boldsymbol{\alpha} \in \mathbb{N}^k$ such that $c(\boldsymbol{\alpha}) \neq 0$, we have $\Re\langle \boldsymbol{\alpha}, \boldsymbol{s'} \rangle \geqslant 0$, thus

$$|c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}\rangle}| = |c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}_0\rangle}|\,\mathrm{e}^{-\Re\langle \boldsymbol{\alpha},\boldsymbol{s'}\rangle} \leqslant |c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}_0\rangle}|.$$

Finally, for any $\boldsymbol{\alpha} \in \mathbb{N}^k$, we have

$$\sup_{\boldsymbol{s} \in \boldsymbol{s}_0 + C^\vee} |c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}\rangle}| = |c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}_0\rangle}|,$$

which is the term of a convergent series by assumption. ∎

This result is optimal, in the sense that for any cone K larger than $C^\vee$, there is an $\boldsymbol{\alpha} \in C$ such that the function $c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}\rangle}$ is not bounded on $\boldsymbol{s}_0 + K$.

However, if we remove finitely many terms from the series, corresponding to a finite set $C_0 \subseteq C$ of indices, the lemma applies to the remaining sum but with normal convergence on the larger cone $(C \smallsetminus C_0)^\vee$. The whole series may not converge normally on $\boldsymbol{s}_0 + (C \smallsetminus C_0)^\vee$, since some terms may not be bounded, but it still converges uniformly on this set.

We set

$$D_\infty = \bigcup_{\substack{C_0 \subseteq C \\ C_0 \text{ finite}}} (C \smallsetminus C_0)^\vee = \bigcup_{\substack{C_0 \subseteq C \\ C_0 \text{ finite}}} \bigcap_{\boldsymbol{\alpha} \in C \smallsetminus C_0} \{\boldsymbol{s} \in \mathbb{C}^k \mid \Re\langle \boldsymbol{\alpha}, \boldsymbol{s} \rangle \geqslant 0\}.$$

Notice that D_∞ is the limit inferior of the family of sets $\{\boldsymbol{s} \in \mathbb{C}^k \mid \Re\langle \boldsymbol{\alpha}, \boldsymbol{s} \rangle \geqslant 0\}$ indexed by $\boldsymbol{\alpha} \in C$, which means that $\boldsymbol{s} \in D_\infty$ if and only if $\Re\langle \boldsymbol{\alpha}, \boldsymbol{s} \rangle \geqslant 0$ for all but finitely many $\boldsymbol{\alpha} \in C$. This cone will play a central role in the following. For now, it can be seen as the maximal cone on which the series uniformly converges.

Lemma 2.6. *Let $\boldsymbol{s}_0 \in \mathbb{C}^k$ be in the domain of convergence of the Dirichlet series*

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}\rangle}.$$

Let K be a polyhedral cone. Then the Dirichlet series converges uniformly on $\boldsymbol{s}_0 + K$ if and only if $K \subseteq D_\infty$.

For this reason, we will call D_∞ the *cone of uniform convergence* of the Dirichlet series $\sum_{\boldsymbol{\alpha} \in \mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}\rangle}$.

Proof of Lemma 2.6. Let $\boldsymbol{s}_0 \in \mathbb{C}^k$ be such that the Dirichlet series $\sum_{\boldsymbol{\alpha} \in \mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}_0\rangle}$ converges. Let K be a polyhedral cone, spanned by finitely many vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$.

Assume that $K \subseteq D_\infty$. For every $j \in [\![1, m]\!]$, the vector $\boldsymbol{v}_j$ is in some dual cone $(C \smallsetminus C_j)^\vee$ where C_j is a finite subset of C which may depend on the vector $\boldsymbol{v}_j$. The union $C_0 = \bigcup_{j=1}^m C_j$ is also a finite subset of C, and every generating vector $\boldsymbol{v}_j$ belongs to $(C \smallsetminus C_0)^\vee$, i.e. $K \subseteq (C \smallsetminus C_0)^\vee$. By Lemma 2.5, the Dirichlet series converges uniformly on $\boldsymbol{s}_0 + (C \smallsetminus C_0)^\vee$, therefore on $\boldsymbol{s}_0 + K$.

Assume that $K \nsubseteq D_\infty$. Then there exists a generating vector $\boldsymbol{v}_j$ outside of D_∞. In particular, there exist infinitely many $\boldsymbol{\alpha} \in C$ for which $\Re\langle \boldsymbol{\alpha}, \boldsymbol{v}_j \rangle < 0$. For all these $\boldsymbol{\alpha}$, the term $\boldsymbol{s} \mapsto c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle \boldsymbol{\alpha},\boldsymbol{s}\rangle}$ is not bounded on $\boldsymbol{s}_0 + K$, which contradicts Cauchy's uniform convergence test. ∎

Corollary 2.7. *Let* $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}$ *be a Dirichlet series with integer coefficients, with* $c(\mathbf{0}) = 0$*, and which converges at* $\boldsymbol{s}_0 \in \mathbb{C}^k$*. Let* D_∞ *be the cone of uniform convergence of this series. Then the function* g *defined by the product*

$$g(\boldsymbol{s}) = \prod_{\boldsymbol{\alpha}\in\mathbb{N}^k} (1 - \mathrm{e}^{\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle})^{c(\boldsymbol{\alpha})}$$

is meromorphic on $\boldsymbol{s}_0 + \mathring{D}_\infty$*, and its divisor on this domain is*

$$(g) = \sum_{\kappa\in\mathbb{Z}} \sum_{\boldsymbol{\alpha}\in C} c(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}(2\kappa\pi\mathrm{i}),$$

where $H_{\boldsymbol{\lambda}}(\gamma)$ *is the set of elements* $\boldsymbol{s} \in \boldsymbol{s}_0 + \mathring{D}_\infty$ *such that* $\langle\boldsymbol{\lambda},\boldsymbol{s}\rangle = \gamma$*.*

Proof. Let K be a polyhedral cone inside D_∞. By uniform convergence, the given Dirichlet series has at most finitely many terms taking values larger than $1/2$ on $\boldsymbol{s}_0 + K$; we define C_0 to be the finite set of indices of those terms. Then the subseries $\sum_{\boldsymbol{\alpha}\in C\smallsetminus C_0} c(\boldsymbol{\alpha}) \log(1 - \mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle})$ of analytic functions converges uniformly on $\boldsymbol{s}_0 + K$, the subproduct $\prod_{\boldsymbol{\alpha}\in C\setminus C_0}(1 - \mathrm{e}^{\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle})^{c(\boldsymbol{\alpha})}$ is uniformly convergent on $\boldsymbol{s}_0 + K$ (see [Kno90, §51] for a detailed argument), and its logarithm is bounded over $\boldsymbol{s}_0 + K$. Therefore, the subproduct is an analytic function of $\boldsymbol{s}_0 + \mathring{K}$, with no zero on $\boldsymbol{s}_0 + \mathring{K}$.

The whole product $\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1 - \mathrm{e}^{\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle})^{c(\boldsymbol{\alpha})}$ is therefore meromorphic on $\boldsymbol{s}_0 + K$, and its divisor on $\boldsymbol{s}_0 + \mathring{K}$ comes only from the terms of the infinite product:

$$(g) = \sum_{\kappa\in\mathbb{Z}} \sum_{\boldsymbol{\alpha}\in C_0} c(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}(2\kappa\pi\mathrm{i}).$$

Note that for any $\boldsymbol{\alpha} \in C \smallsetminus C_0$, we have $|\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}| \leqslant 1/2$ for every $\boldsymbol{s} \in \boldsymbol{s}_0 + K$, therefore $\boldsymbol{\alpha} \in K^\vee$, which means that $\Re\langle\boldsymbol{\alpha},\boldsymbol{s}'\rangle \geqslant 0$ for any $\boldsymbol{s}' \in K$, and $|\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}_0\rangle}| \leqslant 1/2$, which means that $\Re\langle\boldsymbol{\alpha},\boldsymbol{s}_0\rangle \geqslant \log 2 > 0$. We deduce that for any $\boldsymbol{s} \in \boldsymbol{s}_0 + K$, we have $\Re\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle > 0$. Thus, $H_{\boldsymbol{\alpha}}(2\kappa\pi\mathrm{i})$ does not intersect $\boldsymbol{s} \in \boldsymbol{s}_0 + K$ if $\boldsymbol{\alpha} \in C \smallsetminus C_0$. We may as well write the divisor of g on $\boldsymbol{s}_0 + \mathring{K}$ as $(g) = \sum_{\kappa\in\mathbb{Z}}\sum_{\boldsymbol{\alpha}\in C} c(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}(2\kappa\pi\mathrm{i})$, even if infinitely many terms of this sum are empty.

Since g is meromorphic on $\boldsymbol{s}_0 + \mathring{K}$ with divisor $(g) = \sum_{\kappa\in\mathbb{Z}}\sum_{\boldsymbol{\alpha}\in C} c(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}(2\kappa\pi\mathrm{i})$ for any polyhedral cone inside D_∞, we deduce that g is meromorphic on $\boldsymbol{s}_0 + \mathring{D}_\infty$ with divisor $(g) = \sum_{\kappa\in\mathbb{Z}}\sum_{\boldsymbol{\alpha}\in C} c(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}(2\kappa\pi\mathrm{i})$. ■

This result gives some legitimacy to the study of the infinite product expansion of a power series: (some of) the terms of the product correspond directly to (some of) the terms of the divisor of the meromorphic extension, which is intrinsic to the power series. The parentheses in the previous statement come from the fact that this correspondence only concerns the terms of the divisor which intersect the domain $\boldsymbol{s}_0 + \mathring{D}_\infty$.

It is natural to ask under which conditions the hyperplane of equation $\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle = \gamma$ meets the domain $\boldsymbol{s}_0 + \mathring{D}_\infty$. Since $\mathring{D}_\infty$ is tubular, we can only consider the real parts. Recall that $\boldsymbol{\alpha} \in \mathbb{N}^k \smallsetminus \{\mathbf{0}\}$ and $\mathbb{R}_+^k \subseteq D_\infty$. Therefore, the image of $\mathring{D}_\infty$ by the real linear form $\boldsymbol{s} \mapsto \Re\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle$ is either $]0,+\infty[$ if $\boldsymbol{\alpha} \in D_\infty^\vee$ or $]-\infty,+\infty[$ if $\boldsymbol{\alpha} \notin D_\infty^\vee$.

We deduce that the hyperplane of equation $\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle = \gamma$ meets the domain $\boldsymbol{s}_0 + \mathring{D}_\infty$ if and only if $\Re\langle\boldsymbol{\alpha},\boldsymbol{s}_0\rangle < \Re e(\gamma)$ or $\boldsymbol{\alpha} \notin D_\infty^\vee$.

With this result in hand, we give two examples. The first one shows how the product representation can help to study the meromorphic expansion.

First example. Consider the power series $\sum_{k,n} p_k(n) X^k Y^n \in \mathbb{Z}[[X,Y]]$, where $p_k(n)$ counts the partitions of n into k parts, and let g be the associated Dirichlet series. The domain of convergence of g is $\mathbb{C}_{\mathrm{r}}^2$, where $\mathbb{C}_{\mathrm{r}}$ denotes $\{s \in \mathbb{C} \mid \Re e(s) > 0\}$.

This power series has the well-known product expansion $\prod_{m\geqslant 1}(1 - XY^m)^{-1}$ [And98, Corollary 2.2]. Therefore,

$$\mathrm{xsupp}(g) = \{(1,m) \mid m \geqslant 1\} \quad \text{and} \quad \mathring{D}_\infty = \mathbb{C} \times \mathbb{C}_{\mathrm{r}}.$$

The dual $D_\infty^\vee = \{0\} \times \mathbb{R}_+$ does not contain any element of $\mathrm{xsupp}(g)$.

Using Corollary 2.7, we deduce that g is meromorphic on $\mathring{D}_\infty$, and its divisor is exactly $(g) = \sum_{\kappa\in\mathbb{Z}} \sum_{m\geqslant 1} (-1) H_{(1,m)}(2\kappa\pi \mathrm{i})$, where every hyperplane actually intersects the domain of meromorphy, which is easy to satisfy in this case.

As an exercise, the reader could prove that the family of poles clusters at any point $(s_1, s_2) \in \partial D_\infty$ with $\Re e(s_1) \leqslant 0$, and with the help of Theorem 4.14, that every point of ∂D_∞ is a singular point for the meromorphic function g. This means that $\mathbb{C} \times \mathbb{C}_{\mathrm{r}}$ is the domain of meromorphy of g.

In the second example, we use our knowledge on the meromorphic extension to deduce a property of the exponential support.

Second example. Consider an algebraic power series $\sum_{\boldsymbol{\beta}\in\mathbb{N}^k} b(\boldsymbol{\beta}) \boldsymbol{X}^{\boldsymbol{\beta}} \in \mathbb{Z}[[\boldsymbol{X}]]$ with minimal polynomial

$$Q(\boldsymbol{X}, Y) = P_0(\boldsymbol{X}) + Y P_1(\boldsymbol{X}) + \cdots + Y^d P_d(\boldsymbol{X}) \in \mathbb{Z}[\boldsymbol{X}, Y],$$

and let g be the associated Dirichlet series. With $d = 1$, we recover the case of rational power series. We assume that $b(\mathbf{0}) = 1$, and that the series converges around $\mathbf{0}$ (§).

If neither P_0 nor P_d has a cyclotomic factor, no meromorphic extension of g can have a zero or a pole written as $H_{\boldsymbol{\alpha}}(2\kappa\pi\mathrm{i})$ (see Lemma 4.1). Therefore, by Corollary 2.7, the divisor of g is empty on $\boldsymbol{s}_0 + \mathring{D}_\infty$. For example, for any $\boldsymbol{\alpha} \in C$, the hyperplane $H_{\boldsymbol{\alpha}}(2\kappa\pi\mathrm{i})$ does not meet $\boldsymbol{s}_0 + \mathring{D}_\infty$, which implies that $\boldsymbol{\alpha} \in D_\infty^\vee$ and $\Re\langle \boldsymbol{\alpha}, \boldsymbol{s}_0 \rangle \geqslant 0$.

The first fact, that $C \subseteq D_\infty^\vee$, can also be written as $(C \setminus C_0)^\vee = C^\vee = \overline{D_\infty}$ for all finite subsets $C_0 \subseteq C$, or again $\overline{\mathrm{cone}}(C \setminus C_0) = \overline{\mathrm{cone}}(C)$. The second fact amounts to saying that the domain of convergence Ω of g is contained in $C^\vee = \overline{D_\infty}$. Notice that by Lemma 2.5, we have $\Omega + C^\vee = C^\vee$.

2.4. Meromorphy of the Euler product. Before stating the main theorem of this section, we establish a result on an unusual kind of product of Dirichlet series.

LEMMA 2.8. *Let $\sum_{n\in\mathbb{N}} b_n \,\mathrm{e}^{-\lambda_n s}$ be a Dirichlet series with $0 < \lambda_0 < \lambda_1 < \cdots$ and $b_0 \neq 0$. Let $\theta \geqslant 0$ be a real value at which this series converges absolutely.*

(§) This is satisfied if $Q(\mathbf{0}, Y) = 1 - Y$. More generally, if $Q(\mathbf{0}, 1) = 0$ and $\frac{\partial Q}{\partial Y}(\mathbf{0}, 1) \neq 0$, we can always construct a solution $y(\boldsymbol{X}) \in \mathbb{Q}[[\boldsymbol{X}]]$ with $y(\mathbf{0}) = 1$ and convergent around $\mathbf{0}$. By the multivariable Eisenstein Theorem [Saf00, Theorem 5], a dilation by a positive integer a is sufficient to get $y(a\boldsymbol{X}) \in \mathbb{Z}[[\boldsymbol{X}]]$.

Let $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}$ be a Dirichlet series and let C be the support of its coefficient function. Let $\boldsymbol{s}_0$ be a point of $C^\vee$ where this series converges. Set $m_0 = \min\{\langle\boldsymbol{\alpha},\boldsymbol{1}\rangle \mid \boldsymbol{\alpha}\in C\}$. Assume that $m_0 \geqslant 1$, that is, $c(\boldsymbol{0}) = 0$.

Then the compound Dirichlet series

$$\sum_{n\in\mathbb{N}}\sum_{\boldsymbol{\alpha}\in C} b_n c(\boldsymbol{\alpha})\,\mathrm{e}^{-\lambda_n\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}$$

converges normally on $\frac{1}{\lambda_0}\boldsymbol{s}_0 + \frac{1}{m_0}\theta\boldsymbol{1} + C^\vee$.

Proof. By Lemma 2.5, the series $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}$ converges normally on $\boldsymbol{s}_0 + C^\vee$, and for any $n\in\mathbb{N}$ we see that $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\lambda_n\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}$ converges on $\frac{1}{\lambda_n}\boldsymbol{s}_0 + C^\vee$, therefore also on $\frac{1}{\lambda_0}\boldsymbol{s}_0 + C^\vee$.

Now we set $\boldsymbol{s} = \frac{1}{\lambda_0}\boldsymbol{s}_0 + \frac{1}{m_0}\theta\boldsymbol{1}$. For any $n\in\mathbb{N}$ and $\boldsymbol{\alpha}\in C$, we have

$$\lambda_n\Re\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle = \frac{\lambda_n}{\lambda_0}\,\Re\langle\boldsymbol{\alpha},\boldsymbol{s}_0\rangle + \theta\lambda_n\frac{1}{m_0}\,\Re\langle\boldsymbol{\alpha},\boldsymbol{1}\rangle \geqslant \Re\langle\boldsymbol{\alpha},\boldsymbol{s}_0\rangle + \theta\lambda_n,$$

since $\boldsymbol{s}_0\in C^\vee$ and $\theta\geqslant 0$. This implies that

$$\sum_{n\in\mathbb{N}}\sum_{\boldsymbol{\alpha}\in C} |b_n c(\boldsymbol{\alpha})\,\mathrm{e}^{-\lambda_n\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}| \leqslant \Big(\sum_{n\in\mathbb{N}} |b_n\,\mathrm{e}^{-\lambda_n\theta}|\Big)\Big(\sum_{\boldsymbol{\alpha}\in C} |c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}_0\rangle}|\Big).$$

This proves that the double series converges in $\boldsymbol{s}$ and normally on $\boldsymbol{s} + C^\vee$. ■

We may now state and prove our convergence theorem.

THEOREM 2.9. *Let $h(\boldsymbol{z})$ be defined by the infinite product $\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1-\boldsymbol{z}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}$ with $c(\boldsymbol{\alpha})\in\mathbb{Z}$ and $c(\boldsymbol{0}) = 0$, for $\boldsymbol{z}$ in a neighborhood of $\boldsymbol{0}$. Let D_∞ be the cone of uniform convergence of the Dirichlet series $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}$, which is defined as*

$$D_\infty = \{\boldsymbol{s}\in\mathbb{C}^k \mid \Re\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle \geqslant 0 \text{ for all but finitely many } \boldsymbol{\alpha}\in\mathbb{N}^k \text{ such that } c(\boldsymbol{\alpha})\neq 0\},$$

and assume that the function $\boldsymbol{s}\mapsto h(\mathrm{e}^{-s_1},\dots,\mathrm{e}^{-s_k})$ is meromorphic on the open convex cone $\Gamma = \overset{\circ}{D}_\infty$.

Then the Euler product

$$\prod_p h(p^{-s_1},\dots,p^{-s_k})$$

defines a function f that admits a meromorphic continuation on Γ, and the divisor of f on Γ is

$$(f) = \sum_p\bigg[(h_p) - \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\sum_{\kappa\in\mathbb{Z}} H_{\boldsymbol{\alpha}}\Big(\frac{2\kappa\pi\mathrm{i}}{\log p}\Big)\bigg] - \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k}\sum_{\gamma\in G} c(\boldsymbol{\alpha})m_\gamma H_{\boldsymbol{\alpha}}(\gamma), \tag{2.9}$$

where h_p is the meromorphic function $\boldsymbol{s}\mapsto h(p^{-s_1},\dots,p^{-s_k})$ defined on Γ, G is the set of zeros and poles of the Riemann ζ-function, m_γ is the multiplicity of γ as a zero or its (negative) order as a pole ($m_1 = -1$) for ζ, and

$$H_{\boldsymbol{\lambda}}(\gamma) = \{\boldsymbol{s}\in\Gamma \mid \langle\boldsymbol{\lambda},\boldsymbol{s}\rangle = \gamma\}.$$

This result requires some conditions on h.

The function h is defined by an infinite product in the neighborhood of $\boldsymbol{0}$: by Lemma 2.3, this assumption is equivalent to the three assumptions presented at the beginning of this section.

The second assumption is that the function $\boldsymbol{s} \mapsto h(\mathrm{e}^{-s_1}, \dots, \mathrm{e}^{-s_k})$ is meromorphic on $\overset{\circ}{D}_\infty$. Actually, this is not really a requirement, but the theorem would be even heavier to state without it. The proof shows that if f is not meromorphic at some point of Γ, then one of the local factors h_p is not meromorphic at that point. The converse may not be true since by default of meromorphy of different local factors at the same point may cancel altogether.

We give an alternate statement. By Corollary 2.7, we know that the function $\boldsymbol{s} \mapsto h(\mathrm{e}^{-s_1}, \dots, \mathrm{e}^{-s_k})$ is meromorphic on $\boldsymbol{s}_0 + \overset{\circ}{D}_\infty$, for some $\boldsymbol{s}_0 \in \overline{D_\infty}$. In this case, the function f is meromorphic on $\frac{1}{\log 2}\boldsymbol{s}_0 + \overset{\circ}{D}_\infty$.

Every term between brackets in (2.9) is the scaled version of the same divisor, which is the divisor of the function $\boldsymbol{s} \mapsto h(\mathrm{e}^{-s_1}, \dots, \mathrm{e}^{-s_k})$ subtracted by the part of this divisor identified in Corollary 2.7. Just as in this corollary, only the terms $\boldsymbol{\alpha} \in C \smallsetminus D_\infty^\vee$ are relevant since the other ones do not intersect $\overset{\circ}{D}_\infty$. Note that if there exists some $\boldsymbol{\alpha}_0 \in C \smallsetminus D_\infty^\vee$, then it is not possible to remove the brackets, since $H_{\boldsymbol{\alpha}_0}(0)$ would have infinite multiplicity in the resulting sum.

On the contrary, if $C \subseteq D_\infty^\vee$, we may replace the term between brackets simply by (h_p). As noticed previously, this is the case when h is a rational power series without cyclotomic factors. In this case, we can also simplify the expression of the domain Γ; this is the aim of a large part of the next section, which is finally attained in Proposition 3.17. In that case, we may write Theorem 2.9 in a simpler form.

Theorem 2.10. *Let $h \in \mathbb{Z}[[\boldsymbol{X}]]$ be a rational power series, with constant term equal to* 1*, and with product expansion $h = \prod_{\boldsymbol{\alpha} \in \mathbb{N}^k} (1 - \boldsymbol{z}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}$. Let q^+ and $q^- \in \mathbb{Z}[\boldsymbol{X}]$ be its numerator and its denominator, respectively. Then the Euler product*

$$\prod_p h(p^{-s_1}, \dots, p^{-s_k})$$

defines a function f that admits a meromorphic continuation on the interior Γ of $(\operatorname{supp} q^+ \cup \operatorname{supp} q^-)^\vee$*, and the divisor of f on Γ is*

$$(f) = \sum_p (h_p) - \sum_{\boldsymbol{\alpha} \in \mathbb{N}^k} \sum_{\gamma \in G} c(\boldsymbol{\alpha}) m_\gamma H_{\boldsymbol{\alpha}}(\gamma),$$

with the same notations as in Theorem 2.9*.*

Proof. Like its associated product $\prod_{\boldsymbol{\alpha} \in \mathbb{N}^k} (1 - \boldsymbol{z}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}$, the power series $\sum_{\boldsymbol{\alpha} \in \mathbb{N}^k} c(\boldsymbol{\alpha}) \boldsymbol{z}^{\boldsymbol{\alpha}}$ is convergent on a neighborhood of $\mathbf{0}$, and there exists some real number $r \in \,]0, 1[$ such that both expressions converge on the open polydisc of polyradius $(r, \dots, r)$. In particular, for $\rho > -\log r > 0$, the product $\prod_{\boldsymbol{\alpha} \in \mathbb{N}^k} (1 - \mathrm{e}^{-\langle \boldsymbol{\alpha}, \boldsymbol{s} \rangle})^{c(\boldsymbol{\alpha})}$ converges at $\boldsymbol{s}_0 = (\rho, \dots, \rho)$.

At least formally, the function f may be represented by the double product

$$\prod_p \prod_{\boldsymbol{\alpha}} (1 - \mathrm{e}^{-\log p \, \langle \boldsymbol{\alpha}, \boldsymbol{s} \rangle})^{c(\boldsymbol{\alpha})}. \tag{2.10}$$

This product converges wherever the double Dirichlet series

$$\sum_p \sum_{\boldsymbol{\alpha} \in \mathbb{N}^k} c(\boldsymbol{\alpha}) \, \mathrm{e}^{-\log p \, \langle \boldsymbol{\alpha}, \boldsymbol{s} \rangle} \tag{2.11}$$

does. We apply Lemma 2.8 to the Dirichlet series $\sum_p p^{-s}$, which converges at θ if $\theta > 1$ and for which $\lambda_0 = \log 2$ since 2 is the smallest prime number, and to the Dirichlet series $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha})\,\mathrm{e}^{-\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle}$, which converges at $\rho\mathbf{1}$ if $\rho > -\log r$ and for which $m_0 \geqslant 1$ since $c(\mathbf{0}) = 0$: we find that the double Dirichlet series (2.11) converges on $\frac{1}{\log 2}\rho\mathbf{1} + \theta\mathbf{1} + C^\vee$.

We deduce that the double product (2.10) converges uniformly on this domain. In particular, f is meromorphic on this domain and can be expressed in two different ways as a convergent infinite product

$$f(\boldsymbol{s}) = \prod_{p\,\text{prime}} h(p^{-s_1},\dots,p^{-s_k}) = \prod_{\boldsymbol{\alpha}\in C} \zeta(\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle)^{-c(\boldsymbol{\alpha})}. \tag{2.12}$$

Notice that on this domain, we have $\Re\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle > (\theta + \frac{1}{\log 2}\rho)\langle\boldsymbol{\alpha},\mathbf{1}\rangle > 1$ for any $\boldsymbol{\alpha} \in C$, therefore using the right-hand side product in (2.12), we deduce that the divisor of f is null on this domain.

If C is finite, then h is a quotient of polynomials, and the divisor of the function $\boldsymbol{s} \mapsto h(\mathrm{e}^{-s_1},\dots,\mathrm{e}^{-s_k})$ on $\mathbb{C}^k$ is exactly

$$\sum_{\boldsymbol{\alpha}\in C} c(\boldsymbol{\alpha}) \sum_{\kappa\in\mathbb{Z}} H_{\boldsymbol{\alpha}}(2\kappa\pi\mathrm{i}).$$

Therefore, the first sum of (2.9) is null in this case. Since C is finite, we have $D_\infty = \mathbb{C}^k$ and the product of zeta functions in (2.12) gives the meromorphic extension of f to the whole space $\mathbb{C}^k$. From this product, the divisor of f over $\mathbb{C}^k$ is

$$-\sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha}) \sum_{\gamma\in G} m_\gamma H_{\boldsymbol{\alpha}}(\gamma),$$

which is the second part of (2.9).

If C is not finite, then it is countable and we can fix a bijection $n \mapsto \boldsymbol{\alpha}_n$ from $\mathbb{N}$ to C. We set $c_n = c(\boldsymbol{\alpha}_n)$ for all $n \in \mathbb{N}$. For any $N \in \mathbb{N}$, we set $C_N = \{\boldsymbol{\alpha}_n \mid n \geqslant N\} \subseteq C$, $D_N = C_N^\vee$ and $m_N = \min\{\langle\boldsymbol{\alpha},\mathbf{1}\rangle \mid \boldsymbol{\alpha} \in C_N\}$. It is clear that the sequence $(m_N)_{N\in\mathbb{N}}$ is non-decreasing and tends to infinity.

We fix two integral parameters $N \geqslant 0$ and $P \geqslant 2$, and set $\mu_{N,P} = \frac{1}{\log P}\rho + \frac{1}{m_N}\theta$. By absolute convergence of the double product (2.10) defining f, we may reorganize the terms to produce three subproducts

$$\prod_{n<N}\Big(\prod_{p}(1-p^{-\langle\boldsymbol{\alpha}_n,\boldsymbol{s}\rangle})^{c_n}\Big)\cdot\prod_{p<P}\Big(\prod_{n\geqslant N}(1-p^{-\langle\boldsymbol{\alpha}_n,\boldsymbol{s}\rangle})^{c_n}\Big)\cdot\prod_{p\geqslant P}\prod_{n\geqslant N}(1-p^{-\langle\boldsymbol{\alpha}_n,\boldsymbol{s}\rangle})^{c_n}$$

for all $\boldsymbol{s}$ in $\mu_{0,2}\mathbf{1} + C^\vee$.

The leftmost term converges to

$$\prod_{n<N} \zeta(\langle\boldsymbol{\alpha}_n,\boldsymbol{s}\rangle)^{-c_n},$$

which admits a meromorphic extension on $\mathbb{C}^k$, and its divisor is

$$-\sum_{n<N} c_n \sum_{\gamma\in G} m_\gamma H_{\boldsymbol{\alpha}_n}(\gamma).$$

In the middle term, each term indexed by $p < P$ converges to

$$h_p(\boldsymbol{s}) \prod_{n<N} (1-p^{-\langle\boldsymbol{\alpha}_n,\boldsymbol{s}\rangle})^{-c_n},$$

which admits a meromorphic extension on $\mathring{D}_\infty$ at least, and its divisor on $\mathring{D}_\infty$ is

$$(h_p) - \sum_{n<N} \sum_{\kappa\in\mathbb{Z}} c_n H_{\boldsymbol{\alpha}_n}\left(\frac{2\kappa\pi\mathrm{i}}{\log p}\right).$$

Finally, by Lemma 2.8, the rightmost product converges uniformly on

$$\frac{1}{\log P}\rho\mathbf{1} + \frac{1}{m_N}\theta\mathbf{1} + C_N^\vee = \mu_{N,P}\mathbf{1} + D_N,$$

and by Corollary 2.7, the product is meromorphic on the interior of this cone with divisor

$$\sum_{p\geqslant P} \sum_{n\geqslant N} \sum_{\kappa\in\mathbb{Z}} c_n H_{\boldsymbol{\alpha}_n}\left(\frac{2\kappa\pi\mathrm{i}}{\log p}\right).$$

Altogether, we obtain a meromorphic extension of f on $\mu_{N,P}\mathbf{1} + \mathring{D}_N$ with divisor

$$\begin{aligned} -\sum_{n<N} \sum_{\gamma\in G} c_n m_\gamma H_{\boldsymbol{\alpha}_n}(\gamma) + \sum_{p<P}\left[(h_p) - \sum_{n<N} \sum_{\kappa\in\mathbb{Z}} c_n H_{\boldsymbol{\alpha}_n}\left(\frac{2\kappa\pi\mathrm{i}}{\log p}\right)\right] \\ + \sum_{p\geqslant P} \sum_{n\geqslant N} \sum_{\kappa\in\mathbb{Z}} c_n H_{\boldsymbol{\alpha}_n}\big(\tfrac{2\kappa\pi\mathrm{i}}{\log p}\big). \end{aligned} \tag{2.13}$$

For any $\boldsymbol{s}$ in this domain and any $n \geqslant N$, we have

$$\Re\langle\boldsymbol{\alpha}_n, \boldsymbol{s}\rangle \geqslant \mu_{N,P}\langle\boldsymbol{\alpha}_n, \mathbf{1}\rangle \geqslant m_n\mu_{N,P} > \theta > 1.$$

Therefore, any hyperplane of equation $\langle\boldsymbol{\alpha}_n, \boldsymbol{s}\rangle = \lambda$ with $n \geqslant N$ and $\Re e(\lambda) \leqslant 1$ has an empty intersection with the domain $\mu_{N,P}\mathbf{1} + \mathring{D}_N$. For this reason, for any $n \geqslant N$, we may add or remove the terms $H_{\boldsymbol{\alpha}_n}(\gamma)$ or $H_{\boldsymbol{\alpha}_n}\big(\frac{2\kappa\pi\mathrm{i}}{\log p}\big)$ from the divisor (2.13). We obtain an expression of (f) independent of N:

$$(f) = -\sum_{n\in\mathbb{N}} c_n \sum_{\gamma\in G} m_\gamma H_{\boldsymbol{\alpha}_n}(\gamma) + \sum_{p<P}\left[(h_p) - \sum_{n\in\mathbb{N}} \sum_{\kappa\in\mathbb{Z}} c_n H_{\boldsymbol{\alpha}_n}\left(\frac{2\kappa\pi\mathrm{i}}{\log p}\right)\right].$$

Finally, applying Corollary 2.7 to the Dirichlet series $\sum_n c_n\, \mathrm{e}^{-\langle\boldsymbol{\alpha}_n, \boldsymbol{s}\rangle}$ which converges at $\rho\mathbf{1}$, we deduce that for any prime p, the divisor of (h_p) on $\frac{1}{\log p}\rho\mathbf{1} + \mathring{D}_\infty$ is equal to

$$(h_p) = \sum_{n\in\mathbb{N}} \sum_{\kappa\in\mathbb{Z}} c_n H_{\boldsymbol{\alpha}_n}\left(\frac{2\kappa\pi\mathrm{i}}{\log p}\right).$$

If $p \geqslant P$, then $\mu_{N,P} > \frac{1}{\log P}\rho > \frac{1}{\log p}\rho$, thus $\mu_{N,P}\mathbf{1} \in \frac{1}{\log p}\rho\mathbf{1} + \mathring{D}_\infty$, and since $D_N \subseteq D_\infty$, we have $\mu_{N,P}\mathbf{1} + D_N \subseteq \frac{1}{\log p}\rho\mathbf{1} + \mathring{D}_\infty$. We may add or remove the term

$$(h_p) - \sum_{n\in\mathbb{N}} \sum_{\kappa\in\mathbb{Z}} c_n H_{\boldsymbol{\alpha}_n}\left(\frac{2\kappa\pi\mathrm{i}}{\log p}\right),$$

which is null, from the divisor of f on $\mu_{N,P}\mathbf{1} + \mathring{D}_N$. We obtain the following expression of (f), which is independent of P:

$$(f) = -\sum_{n\in\mathbb{N}} c_n \sum_{\gamma\in G} m_\gamma H_{\boldsymbol{\alpha}_n}(\gamma) + \sum_{p}\left[(h_p) - \sum_{n\in\mathbb{N}} \sum_{\kappa\in\mathbb{Z}} c_n H_{\boldsymbol{\alpha}_n}\left(\frac{2\kappa\pi\mathrm{i}}{\log p}\right)\right].$$

We deduce that f has a meromorphic extension on the union

$$\Gamma = \bigcup_{N,P} (\mu_{N,P}\mathbf{1} + \mathring{D}_N),$$

and its divisor on this domain is given by the expression above. It remains to prove that $\Gamma = \mathring{D}_\infty$.

For any $N > N'$ and P, we have $\mu_{N',P} \geqslant \mu_{N,P} > 0$. In particular, $\mu_{N',P}\mathbf{1} + D_N \subseteq \mu_{N,P}\mathbf{1} + D_N \subseteq D_N$, and taking the union over N, we have

$$\mu_{N',P}\mathbf{1} + D_\infty \subseteq \bigcup_{N} \big(\mu_{N,P}\mathbf{1} + D_N\big) \subseteq D_\infty.$$

Since $\mu_{N',P}$ tends to 0 as N' and P both grow to infinity, we get

$$\bigcup_{N',P} \big(\mu_{N',P}\mathbf{1} + D_\infty\big) = \mathring{D}_\infty.$$

Using the two previous formulas, we derive that

$$\mathring{D}_\infty = \bigcup_{N,P} \big(\mu_{N,P}\mathbf{1} + D_N\big),$$

which leads directly to $\Gamma = \mathring{D}_\infty$. ■

Even if it may not seem like it, the roles of h and ζ are relatively symmetric in this theorem. Actually, we may replace ζ by any function ξ, which is meromorphic over $\mathbb{C}$, and which can be expressed as $\prod_{i\in I}(1-{p_i}^{-s})^{-m_i}$ for $\Re e(s) > \sigma$, where the exponents m_i are all integers, the real numbers p_i are all larger than 1 and $\sigma \in \mathbb{R}$. We denote by $\sum_{\gamma\in G_\xi} n_\gamma[\gamma]$ the divisor of ξ over $\mathbb{C}$. In that case, and with the same assumptions on h and the same definition of Γ, the same proof shows that the product $\prod_{i\in I} h(p_i^{-s_1}, \dots, p_i^{-s_k})^{m_i}$ defines a meromorphic function over Γ with divisor

$$\sum_{i\in I} m_i \left[(h_{p_i}) - \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha}) \sum_{\kappa\in\mathbb{Z}} H_{\boldsymbol{\alpha}}\left(\frac{2\kappa\pi \mathrm{i}}{\log p_i}\right)\right] - \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha}) \sum_{\gamma\in G_\xi} n_\gamma H_{\boldsymbol{\alpha}}(\gamma).$$

This idea to replace ζ by other meromorphic functions also appears in [Dah52], or to a larger extent in [Kur86].

To conclude this part, we give a statement which is directly derived from the proof of the previous theorem, and in which the role played by the two functions has been made perfectly symmetric. To make the statement shorter, we fix all exponents $c(\alpha)$ to -1: we have remarked that there is no loss of generality by doing so.

Corollary 2.11. *Let two functions h and h' such that there exist two abscissae $\sigma_a \geqslant \sigma_m \geqslant 0$ and a real number $\lambda > 0$ such that $h(s)$ is defined by the converging product*

$$\prod_{j\in J} (1 - \mathrm{e}^{-\omega_j s})^{-1}$$

if $\Re e(s) > \sigma_a$, where the real parameters ω_j satisfy $\omega_j \geqslant \lambda$, and such that h admits a meromorphic extension on the half-plane $\Re e(s) > \sigma_m$ with divisor (h) (and respectively with σ'_a, σ'_m, λ', J' and ω'_j for the function h').

Then the double product

$$\prod_{j\in J}\prod_{j'\in J'}(1-\mathrm{e}^{-\omega_j\omega'_{j'}s})^{-1}$$

is convergent for $\Re e(s) > \frac{\sigma'_a}{\lambda} + \frac{\sigma_a}{\lambda'}$ *to a function* f *which admits a meromorphic extension on the half-plane* $\Re e(s) > \max(\sigma'_m/\lambda, \sigma_m/\lambda')$ *and whose divisor is*

$$(f) = \sum_{j\in J} S_{1/\omega_j}(h') + \sum_{j'\in J'} S_{1/\omega'_{j'}}(h),$$

where S_λ *is the scaling operator* $s \mapsto \lambda s$ *acting* $\mathbb{Z}$*-linearly of the components on the divisor.*

Notice that if $\Re e(s) > \sigma'_a/\lambda + \sigma_a/\lambda'$, then $f(s)$ may also be defined by the two converging products

$$f(s) = \prod_{j\in J} h'(\omega_j s) = \prod_{j'\in J'} h(\omega'_{j'}s).$$

In the context of additive arithmetical semigroups (see [Kno75]), the function h in the corollary is the zeta function associated to the free abelian semigroup $G = \mathbb{N}^{(J)}$ generated by J associated to the morphism $\mathbb{N}^{(J)} \to \mathbb{R}_+$ defined by $j \mapsto \omega_j$, and similarly for the function h' and the arithmetical semigroup $G' = \mathbb{N}^{(J')}$. The function f is the zeta function associated to the tensor product $G \otimes G'$, isomorphic to $\mathbb{N}^{(J\times J')}$ and associated to the morphism $(j, j') \mapsto \omega_j\omega'_{j'}$.

3. Geometry of C

Theorem 2.9 gives a domain Γ on which the Euler product $\prod_p h_p$ is meromorphic. It gives also a description of the divisor of this Euler product over Γ. However, both objects are mainly described in terms of the exponential support C of h. The exponential support is generally infinite (except in the trivial case when h is a quotient of products of cyclotomic polynomials), and usually, it is not part of the usual description of h, and it has to be computed, which can be rather ineffective. Moreover, in order to determine the domain Γ, or to exhibit accumulation in the divisor and detect singular points, computing infinitely many elements is required.

The goal of this section is twofold. First, we will give a simple description of Γ as a rational convex cone when h is a rational function, only in terms of the (finite) supports of its numerator and denominator. More precisely, if h has no cyclotomic factors, we will show that the dual $\Gamma^\vee$ of Γ, defined by

$$\Gamma^\vee = \{\boldsymbol{\mu} \in \mathbb{R}^k \mid \forall \boldsymbol{s} \in \Gamma,\ \Re\langle\boldsymbol{\mu}, \boldsymbol{s}\rangle \geqslant 0\},$$

is the convex cone generated by the supports of the numerator and denominator of h. Second, we will show that the exponential support C of a rational function h contains infinite subsets of elements arranged in straight lines. More precisely, for any $\boldsymbol{\nu} \in \mathbb{N}^k$ generating an extreme ray $\boldsymbol{\nu}\mathbb{R}_+$ of the closed convex cone $\overline{\mathrm{cone}}(C)$, we will show that there exists a $\boldsymbol{\mu} \in \mathbb{Z}^k$ such that the ray $\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{R}_+$ goes through infinitely many points of C.

Those infinite subsets on straight lines are important in order to prove our maximality result: intuitively, they correspond to infinite families of hyperplanes in the divisor of f, which will generate cluster points, therefore singular points, on the boundary of Γ.

Alignments of this kind naturally appear in the support of h if h is rational, and they play a key role in achieving our first goal.

Our strategy to complete both goals is as follows. We notice that, by definition of Γ as limit inferior, and [Zăl02, Theorem 1.1.9], we have

$$\Gamma^\vee = \bigcap_{\substack{E \subseteq C \\ E \text{ finite}}} \overline{\mathrm{cone}}(C \smallsetminus E),$$

and therefore $\Gamma^\vee \subseteq \overline{\mathrm{cone}}(C)$. For the reciprocal inclusion, it suffices to prove that any extreme ray $\boldsymbol{\nu}\mathbb{R}_+$ of $\overline{\mathrm{cone}}(C)$ is included in $\Gamma^\vee$.

Except in favorable cases when C contains infinitely many elements on the extreme ray $\boldsymbol{\nu}\mathbb{R}_+$, it comes down to proving that $\boldsymbol{\nu} \in \overline{\mathrm{cone}}(C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+)$. It turns out that it is easy to define a function h° with $C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+$ as its exponential support. But in most cases, the exponential support is not explicit enough to reach this conclusion, and we turn to the support B° of h° instead. It happens that for any function, the support and the exponential support generate the same convex cone. Thus, $\overline{\mathrm{cone}}(C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+) = \overline{\mathrm{cone}}(B^\circ)$ and it remains to prove that $\boldsymbol{\nu} \in \overline{\mathrm{cone}}(B^\circ)$.

At this point, we will make additional assumptions on h and use polynomial arithmetic to find a line directed by $\boldsymbol{\nu}$ which goes through infinitely many elements of B°.

If h is a polynomial, arguments are fairly straightforward, but even when h is rational, our arguments are already substantially more involved. The seemingly simple fact of connecting the support of h with the supports of its numerator and denominator is already a bit of a challenge.

To get over this difficulty, we use a special property of $\mathbb{Z}$, and of $\mathbb{Z}[X]$ actually, known as the *Fatou property*, and we generalize this property to multivariate formal series. From here, we deduce that the supports of the denominator and numerator are contained in $\overline{\mathrm{cone}}(B^\circ)$, and that B° has infinitely many elements on a line directed by $\boldsymbol{\nu}$.

This is sufficient to conclude that

$$\Gamma^\vee = \overline{\mathrm{cone}}(C) = \overline{\mathrm{cone}}(B) = \mathrm{cone}(B^+) + \mathrm{cone}(B^-),$$

where B^+ and B^- are the supports of the numerator and denominator of h, when h has no cyclotomic factors.

Finally, the last step to achieve our second goal is to study the rationality of the subseries of h of the form $\sum_{n\in\mathbb{N}} b(\boldsymbol{\mu} + n\boldsymbol{\nu})T^n$, where $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are in $\mathbb{N}^k$, and to carry over the results to similar series involving the exponent function.

3.1. First steps. In this subsection, we gradually reformulate the assertion that a nonzero vector $\boldsymbol{\nu} \in \mathbb{R}^k$ directing an extreme ray of $\overline{\mathrm{cone}}(C)$ belongs to $\Gamma^\vee$, until we are able to give a simple, arithmetic criterion in two particular cases: when h is a polynomial and when h is a rational power series in two variables. The general case will be treated in the next subsection.

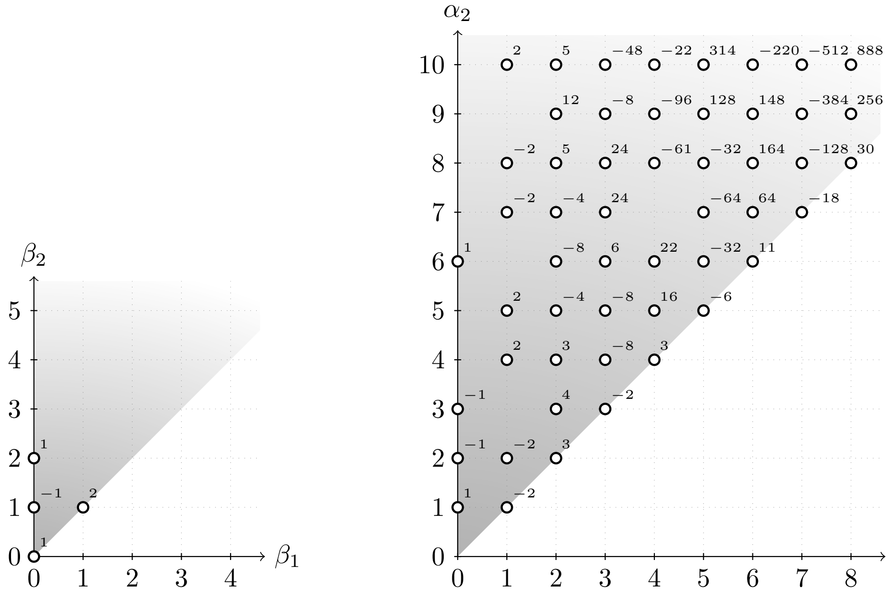

Fig. 2. Support, exponential support and corresponding coefficients for the polynomial $h(X_1, X_2) = 1 - X_2 + X_2^2 + 2X_1X_2$

3.1.1. Step 1: The disjunction. We first state a geometric criterion to decide whether an extreme ray of $\overline{\operatorname{cone}}(C)$ is in $\Gamma^\vee$ or not.

Lemma 3.1. *Let $\boldsymbol{\nu} \in \mathbb{R}^k$ be a non-zero vector such that $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of the closed convex cone $\overline{\operatorname{cone}}(C)$. We have*

$$\boldsymbol{\nu} \in \bigcap_{\substack{E \subseteq C \\ E \text{ finite}}} \overline{\operatorname{cone}}(C \smallsetminus E)$$

if and only if the set $C \cap \boldsymbol{\nu}\mathbb{R}_+$ is infinite or $\boldsymbol{\nu} \in \overline{\operatorname{cone}}(C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+)$.

Proof. The necessity is immediate: if the set $C \cap \boldsymbol{\nu}\mathbb{R}_+$ is a finite subset of C, and $\boldsymbol{\nu}$ is a member of $\overline{\operatorname{cone}}(C \smallsetminus E)$ for any finite subset E of C, then

$$\boldsymbol{\nu} \in \overline{\operatorname{cone}}\big(C \smallsetminus (C \cap \boldsymbol{\nu}\mathbb{R}_+)\big) = \overline{\operatorname{cone}}(C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+).$$

The first term of the disjunction is also immediately sufficient: if the set $C \cap \boldsymbol{\nu}\mathbb{R}_+$ is infinite, so is the set $(C \smallsetminus E) \cap \boldsymbol{\nu}\mathbb{R}_+$ for any finite subset E of C, and $\boldsymbol{\nu} \in \operatorname{cone}(C \smallsetminus E)$ for any such E.

It remains to prove that the second term of the disjunction is also sufficient. Since $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of $\overline{\operatorname{cone}}(C)$, the set $\overline{\operatorname{cone}}(C) \smallsetminus \boldsymbol{\nu}\mathbb{R}_+^*$ is also a convex cone, and it includes the set $C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+$, which we denote C°. Therefore, $\boldsymbol{\nu} \notin \operatorname{cone}(C^\circ)$.

Suppose that $\boldsymbol{\nu} \in \overline{\operatorname{cone}}(C^\circ)$. For any finite subset $E \subseteq C^\circ$, we have $\overline{\operatorname{cone}}(E) = \operatorname{cone}(E) \subseteq \operatorname{cone}(C^\circ)$, thus $\boldsymbol{\nu} \notin \overline{\operatorname{cone}}(E)$. We also find that $\boldsymbol{\nu} \in \overline{\operatorname{cone}}(C^\circ) = \overline{\operatorname{cone}}(E) + \overline{\operatorname{cone}}(C^\circ \smallsetminus E)$. And since the ray $\boldsymbol{\nu}\mathbb{R}_+$ is extremal in $\overline{\operatorname{cone}}(C^\circ)$, it is inside at least one of the terms of the sum $\overline{\mathrm{cone}}(E)+\overline{\mathrm{cone}}(C^\circ\smallsetminus E)$. But since $\boldsymbol{\nu}\notin\overline{\mathrm{cone}}(E)$, we conclude that $\boldsymbol{\nu}\in\overline{\mathrm{cone}}(C^\circ\smallsetminus E)$, and this holds for any finite subset E of C°. This implies that

$$\boldsymbol{\nu}\in\bigcap_{\substack{E\subseteq C^\circ\\ E\text{ finite}}}\overline{\mathrm{cone}}(C^\circ\smallsetminus E)\subseteq\Gamma^\vee. \ \blacksquare$$

Notice that when $\boldsymbol{\nu}$ is not rational, this criterion states that $\boldsymbol{\nu}\in\Gamma^\vee$ if and only if $\boldsymbol{\nu}\in\overline{\mathrm{cone}}(C)$, which means that $\overline{\mathrm{cone}}(C)$ and $\Gamma^\vee$ have the same non-rational extreme rays. However, we will see that $\overline{\mathrm{cone}}(C)$ is a rational cone if h is rational, so the non-rational extreme rays are not particularly of interest.

For these reasons, we will only consider rational extreme rays $\boldsymbol{\nu}\mathbb{R}_+$ from now on, and the direction vectors $\boldsymbol{\nu}$ will be chosen to be primitive, that is, with integral coefficients that are (setwise) coprime.

3.1.2. Step 2: Rewriting in terms of h and of its support. Let $\boldsymbol{\nu}$ be a primitive integer vector such that $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of $\overline{\mathrm{cone}}(C)$. By Lemma 3.1, whether $\boldsymbol{\nu}$ is in $\Gamma^\vee$ or not can be decided by considering the sets $C\cap\boldsymbol{\nu}\mathbb{R}_+$ and $C\smallsetminus\boldsymbol{\nu}\mathbb{R}_+$ separately.

Now, we recall that C is the exponential support of the power series h, defined as the support $\{\boldsymbol{\alpha}\in\mathbb{N}^k\mid c(\boldsymbol{\alpha})\neq 0\}$ of the exponent function $c:\mathbb{N}^k\to\mathbb{Z}$ present in the representation of h as a formal product

$$h=\prod_{\boldsymbol{\alpha}\in\mathbb{N}^k}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}.$$

The partition

$$C=(C\cap\boldsymbol{\nu}\mathbb{R}_+)\cup(C\smallsetminus\boldsymbol{\nu}\mathbb{R}_+)\tag{3.1}$$

from Lemma 3.1 motivates us to write h as a product

$$h=\Big(\prod_{\boldsymbol{\alpha}\in C\cap\boldsymbol{\nu}\mathbb{R}_+}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}\Big)\times\Big(\prod_{\boldsymbol{\alpha}\in C\smallsetminus\boldsymbol{\nu}\mathbb{R}_+}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}\Big).\tag{3.1$'$}$$

As we expect h to be more easily representable as a formal power series, we want to rephrase this decomposition in terms of the coefficient function b of h defined by

$$h=\sum_{\boldsymbol{\beta}\in\mathbb{N}^k}b(\boldsymbol{\beta})\boldsymbol{X}^{\boldsymbol{\beta}}.$$

Lemmata 2.1 and 2.2 provide formulas to derive the coefficient function b from the exponent function c and vice versa. However, instead of computing their values using those formulas, we are tackling the simpler problem of studying their supports, independently of their magnitude.

Lemma 3.2. *Let $h\in\mathbb{Z}[[\boldsymbol{X}]]$ with $h(\mathbf{0})=1$, support B and exponential support C. Then the monoid generated by B is equal to the monoid generated by C.*

Proof. The fact that B is contained in the monoid generated by C is a simple consequence of the conditions given in the sum of (2.4). Conversely, C is contained in the monoid generated by B by formula (2.6). ■

In particular, we have $\mathrm{cone}(C)=\mathrm{cone}(B)$, and $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of $\overline{\mathrm{cone}}(B)$. We now state a useful result about the behavior of the coefficients along such an extreme ray in a product.

Lemma 3.3. *Let $K \subseteq (\mathbb{R}_+)^k$ be a convex cone, and let $\boldsymbol{\nu}$ be a primitive vector of $\mathbb{N}^k$ such that $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of K. For any power series $h \in \mathbb{Z}[[\boldsymbol{X}]]$ with support in K, define*

$$h_{\boldsymbol{\nu}} = \sum_{n \geqslant 0} b(n\boldsymbol{\nu})\boldsymbol{X}^{n\boldsymbol{\nu}} \in \mathbb{Z}[[\boldsymbol{X}]],$$

where b is the coefficient function of h. Then, for any two power series h and h' with support in K, we have $(hh')_{\boldsymbol{\nu}} = h_{\boldsymbol{\nu}}h'_{\boldsymbol{\nu}}$.

Proof. Every element in the support of hh' has to be written as the sum of an element in $\operatorname{supp}(h) \subset K$ and an element in $\operatorname{supp}(h') \subset K$. Since K is closed under addition, $\operatorname{supp}(hh') \subset K$.

Let $n \in \mathbb{N}$. The coefficient of index $n\boldsymbol{\nu}$ of hh' is

$$\sum_{\boldsymbol{\alpha}+\boldsymbol{\alpha}'=n\boldsymbol{\nu}} b(\boldsymbol{\alpha})b'(\boldsymbol{\alpha}'),$$

where b (respectively b') is the coefficient function of h (respectively h'). Since the supports of h and h' are included in K, we may restrict the sum to pairs $(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$ of elements of $K \cap \mathbb{N}^k$. If the sum of two elements of K is in the extreme ray $\boldsymbol{\nu}\mathbb{R}_+$, then each of these elements is in $\boldsymbol{\nu}\mathbb{R}_+$. Thus, we may restrict the sum to $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ of the form $m\boldsymbol{\nu}$ and $m'\boldsymbol{\nu}$ respectively, and we get the following expression for the coefficient of index $n\boldsymbol{\nu}$ of hh':

$$\sum_{m+m'=n} b(m\boldsymbol{\nu})b(m'\boldsymbol{\nu}),$$

which is the coefficient of index $n\boldsymbol{\nu}$ of $h_{\boldsymbol{\nu}}h'_{\boldsymbol{\nu}}$. ■

We are now in a position to identify the left-hand side term of the product (3.1′).

Lemma 3.4. *Let $K \subseteq (\mathbb{R}_+)^k$ be a convex cone, and let $\boldsymbol{\nu}$ be a primitive vector of $\mathbb{N}^k$ such that $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of K. For any power series $h \in \mathbb{Z}[[\boldsymbol{X}]]$ with $h(\boldsymbol{0}) = 1$ and with support in K, we have*

$$\prod_{n \geqslant 1} (1 - \boldsymbol{X}^{n\boldsymbol{\nu}})^{c(n\boldsymbol{\nu})} = \sum_{n \geqslant 0} b(n\boldsymbol{\nu})\boldsymbol{X}^{n\boldsymbol{\nu}} \in \mathbb{Z}[[\boldsymbol{X}]],$$

where b (respectively c) is the coefficient (respectively exponent) function of h.

Proof. Notice that the right-hand side of the formula is precisely the power series $h_{\boldsymbol{\nu}}$ of Lemma 3.3.

By Lemma 2.2, h can be represented as an infinite product with c as its associated exponent function. By Lemma 3.2, the exponential support C of h is inside the cone K. Consider the decomposition (3.1′) of h as a product of two terms. By Lemma 2.2, both terms can be expressed as power series, and by Lemma 3.2, their support is contained in $\boldsymbol{\nu}\mathbb{R}_+$ for the left-hand side, and in $K \smallsetminus \boldsymbol{\nu}\mathbb{R}_+^*$ for the right-hand side. In particular, both supports are inside K, so we may apply Lemma 3.3 to this product to get

$$h_{\boldsymbol{\nu}} = \Big(\prod_{\boldsymbol{\alpha} \in C \cap \boldsymbol{\nu}\mathbb{R}_+} (1 - \boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}\Big)_{\boldsymbol{\nu}} \times \Big(\prod_{\boldsymbol{\alpha} \in C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+} (1 - \boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}\Big)_{\boldsymbol{\nu}}.$$

Consider first the right-hand side: since its support is inside $K \smallsetminus \boldsymbol{\nu}\mathbb{R}_+^*$, the last product is trivially equal to 1. On the contrary, the support of the left-hand side is inside $\boldsymbol{\nu}\mathbb{R}_+$, thus

$$\Big(\prod_{\boldsymbol{\alpha}\in C\cap\boldsymbol{\nu}\mathbb{R}_+}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}\Big)_{\boldsymbol{\nu}} = \prod_{\boldsymbol{\alpha}\in C\cap\boldsymbol{\nu}\mathbb{R}_+}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}.$$

We conclude that

$$\sum_{n\geqslant 0} b(n\boldsymbol{\nu})\boldsymbol{X}^{n\boldsymbol{\nu}} = h_{\boldsymbol{\nu}} = \prod_{\boldsymbol{\alpha}\in C\cap\boldsymbol{\nu}\mathbb{R}_+}(1-\boldsymbol{X}^{\boldsymbol{\alpha}})^{c(\boldsymbol{\alpha})}. \ \blacksquare$$

In other words, the subseries $h_{\boldsymbol{\nu}}$ of h is also its subproduct. To emphasize that $h_{\boldsymbol{\nu}}$ is actually a formal series in one variable $\boldsymbol{X}^{\boldsymbol{\nu}}$, we often write $h_{\boldsymbol{\nu}}$ as an element of $\mathbb{Z}[[T]]$ where $T = \boldsymbol{X}^{\boldsymbol{\nu}}$.

We set $h^\circ = h/h_{\boldsymbol{\nu}}$. This is the right-hand side of the decomposition (3.1′) of h. It is a subproduct of h, and its exponential support C° is exactly $C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+$. However, it is generally not a subseries of h. Its support B° still satisfies $\operatorname{cone}(B^\circ) = \operatorname{cone}(C^\circ)$ by Lemma 3.2.

We are now able to rewrite Lemma 3.1 without any reference to the exponential support of the function h.

Lemma 3.1′. *Let $h \in \mathbb{Z}[[\boldsymbol{X}]]$ with $h(\boldsymbol{0}) = 1$, and let Γ be the domain described in Theorem 2.9. Let $\boldsymbol{\nu} \in \mathbb{N}^k$ be a primitive vector such that $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of $\overline{\operatorname{cone}}(B)$, where B is the support of h. Set*

$$h_{\boldsymbol{\nu}} = \sum_{n\geqslant 0} b(n\boldsymbol{\nu})\boldsymbol{X}^{n\boldsymbol{\nu}} \quad \text{and} \quad h^\circ = h/h_{\boldsymbol{\nu}} \in \mathbb{Z}[[\boldsymbol{X}]].$$

We have $\boldsymbol{\nu} \in \Gamma^\vee$ if and only if $h_{\boldsymbol{\nu}}$ is not a quotient of products of cyclotomic polynomials, or $\boldsymbol{\nu}$ is an element of $\overline{\operatorname{cone}}(B^\circ)$, where B° is the support of h°.

Proof. In Lemma 3.1, $\boldsymbol{\nu}\mathbb{R}_+$ is chosen as an extreme ray of $\overline{\operatorname{cone}}(C)$, where C is the exponential support h. By Lemma 3.2, this assumption is equivalent to being an extreme ray of $\overline{\operatorname{cone}}(B)$.

In Lemma 3.1, the necessary and sufficient condition for $\boldsymbol{\nu} \in \Gamma^\vee$ is that the set $C\cap\boldsymbol{\nu}\mathbb{R}_+$ is infinite or $\boldsymbol{\nu} \in \overline{\operatorname{cone}}(C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+)$.

Any finite product of the form $\prod_n (1-T^n)^{a_n}$ may be written as a finite product of the form $\prod_n \Phi_n(T)^{a'_n}$, and conversely. By contraposition, the first part of the disjunction is equivalent to the fact that $h_{\boldsymbol{\nu}}$ is not a quotient of products of cyclotomic polynomials.

For the latter part, we only observe that

$$\overline{\operatorname{cone}}(C \smallsetminus \boldsymbol{\nu}\mathbb{R}_+) = \overline{\operatorname{cone}}(C^\circ) = \overline{\operatorname{cone}}(B^\circ),$$

where C° is the exponential support of h° and B° is its support. ■

3.1.3. Step 3: Finding alignments.

Lemma 3.1′ takes advantage of the decomposition of h as $h_{\boldsymbol{\nu}}h^\circ$, which corresponds to the partition (3.1) of the exponential support C of h.

Exponential support behaves particularly well with respect to products and quotients: the product (respectively quotient) of two functions corresponds to the sum (respectively difference) of their exponent functions. Therefore, for two functions h and h' with exponential support C and C' respectively, we have

$$C \mathbin{\triangle} C' \subseteq \operatorname{xsupp}(hh') \subseteq C \cup C',$$

(respectively $C \triangle C' \subseteq \operatorname{xsupp}(h/h') \subseteq C \cup C'$) where $\triangle$ denotes symmetric difference. As an example, the exponential supports C of h and C° of h° are quite similar.

On the contrary, the support of a formal series does not behave as well as the exponential support with respect to products and quotients, with the exception of results like Lemma 3.3, which can be generalized for any face of the cone K. Actually, the supports B of h and B° of h° may differ considerably (see Figures 2 and 4).

In general, the previous relations obtained for the exponential support of a product have weaker counterparts in terms of supports: if h and h' are formal series of respective supports B and B', we have

$$B \dotplus B' \subseteq \operatorname{supp}(hh') \subseteq B + B', \tag{3.2}$$

where $A \dotplus A'$ denotes the set of elements with a unique representation as a sum of an element $a \in A$ and an element $a' \in A'$.

This relation satisfied by $\operatorname{supp}(hh')$ is generally not sufficient to determine its geometry, unless we are dealing with finite supports.

A basic result in this setting is a theorem of Ostrowski regarding Newton polytopes. For a polynomial h, we define its *Newton polytope* $\Delta(h)$ as the convex hull of its support. Ostrowski's Theorem (cf. [Ost75]) states that for two polynomials h and h', we have

$$\Delta(hh') = \Delta(h) + \Delta(h'). \tag{3.3}$$

Unfortunately, for the product $h = h_{\boldsymbol{\nu}} h^\circ$, this result is helpful only in trivial cases.

However, we state a corollary of Ostrowski's Theorem which will be extremely useful.

Corollary 3.6. *Let $K \subseteq \mathbb{R}^k$ be a convex cone, and $q, q' \in \mathbb{Z}[\boldsymbol{X}]$. If $\operatorname{supp}(qq') \subset K$ and $q'(\mathbf{0}) \neq 0$, then $\operatorname{supp}(q) \subset K$.*

Proof. We have $0 \in \Delta(q')$. Using Ostrowski's Theorem (3.3), we get

$$\operatorname{supp}(q) \subseteq \Delta(q) \subseteq \Delta(q) + \{\mathbf{0}\} \subseteq \Delta(q) + \Delta(q') = \Delta(qq') \subset K. \ \blacksquare$$

Assuming h is a polynomial. Before handling the general case with h rational, we deal with two easier cases, which are particularly revealing: when h is a polynomial, and when h is a rational function in two variables.

Lemma 3.7. *Let $h \in \mathbb{Z}[\boldsymbol{X}]$, $\boldsymbol{\nu} \in \mathbb{N}^k$ be a non-zero vector, and q be a polynomial in the single variable $\boldsymbol{X}^{\boldsymbol{\nu}}$ and with constant term 1. The rational function $h' = h/q$ has a representation as a formal series, whose support is denoted by B'.*

Then we have the following exclusive disjunction: either

- *q divides h in $\mathbb{Z}[\boldsymbol{X}]$, or*
- *there exists an integer vector $\boldsymbol{\mu}$ such that $B' \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N})$ is infinite.*

Proof. First notice that if q divides h, then h' is a polynomial and its support B' is finite. We now show the converse.

Since q has constant term 1, $1/q$ can be represented as a power series in $\boldsymbol{X}^{\boldsymbol{\nu}}$, and so can the rational function h/q.

Let $\boldsymbol{\mu} \in B'$. Assume that $B' \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N})$ is finite. In particular, there is a maximal integer n such that $\boldsymbol{\mu} + n\boldsymbol{\nu} \in B'$: we denote this maximum by n^+ and we write $\boldsymbol{\mu}^+ = \boldsymbol{\mu} + n^+\boldsymbol{\nu}$ for the corresponding element in B'.

Let d be the degree of q as a polynomial in $\boldsymbol{X}^{\boldsymbol{\nu}}$. Any element of the support of q is written as $(d-m)\boldsymbol{\nu}$ for some integer $m \in [0,d]$. The vector $\boldsymbol{\alpha} = \boldsymbol{\mu}^+ + d\boldsymbol{\nu}$ belongs to $B' + \mathrm{supp}(h)$ and any representation of $\boldsymbol{\alpha}$ as the sum of an element of B' and an element of $\mathrm{supp}(h)$ has the form $\big(\boldsymbol{\mu}^+ + m\boldsymbol{\nu}\big) + \big((d-m)\boldsymbol{\nu}\big)$ for some $m \in [0,d]$. By the maximality of n^+, only the choice $m = 0$ provides such a representation. By (3.2), this uniqueness implies that $\boldsymbol{\alpha}$ belongs to the support of $h'q = h$, that is, $\boldsymbol{\alpha} \in B$.

We deduce that there exists an element $\boldsymbol{\alpha} \in B$ such that $\boldsymbol{\mu} \in \mathbb{N}^k \cap (\boldsymbol{\alpha} - \boldsymbol{\nu}\mathbb{N})$ since $\boldsymbol{\mu} = \boldsymbol{\alpha} - (d + n^+)\boldsymbol{\nu}$. Since $\boldsymbol{\nu}$ is a non-zero vector with non-negative coefficients, the intersection $\mathbb{N}^k \cap (\boldsymbol{\alpha} - \boldsymbol{\nu}\mathbb{N})$ is a finite set.

We have shown that any $\boldsymbol{\mu} \in B'$ such that the set $B' \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N})$ is finite belongs to the set

$$\bigcup_{\boldsymbol{\alpha} \in B} \mathbb{N}^k \cap (\boldsymbol{\alpha} - \boldsymbol{\nu}\mathbb{N}),$$

which is a finite union of finite sets. Therefore, unless B' contains infinitely many elements on a line $\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N}$, B' is a subset of the above set, which is finite, thus h' is a polynomial, which satisfies $h = h'q$, and q is a factor of h. ∎

Lemma 3.7 is sufficient to retrieve the description of the domain Γ on which the Euler product f has a meromorphic extension, when h is a non-trivial polynomial without cyclotomic factor.

Proposition 3.8. *Let $h \in \mathbb{Z}[\boldsymbol{X}]$ with $h(\mathbf{0}) = 1$ without cyclotomic factor and with support B. Then* $\mathrm{cone}(B) = \Gamma^\vee$.

Proof. Let $\boldsymbol{\nu} \in \mathbb{N}^k$ be the primitive vector directing an extreme ray of $\mathrm{cone}(B) = \overline{\mathrm{cone}}(B)$, and consider h° and $h_{\boldsymbol{\nu}}$ defined as above; the latter can be considered as a non-constant polynomial in the variable $\boldsymbol{X}^{\boldsymbol{\nu}}$. If $h_{\boldsymbol{\nu}}$ divides h, then it cannot be a product of cyclotomic polynomials, and we conclude that $\boldsymbol{\nu} \in \Gamma^\vee$, by the first part of the disjunction in Lemma 3.1′.

If $h_{\boldsymbol{\nu}}$ does not divide h, then, by Lemma 3.7, there is a ray $\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{R}_+$ containing infinitely many elements of the support B° of h°, therefore $\boldsymbol{\nu} \in \overline{\mathrm{cone}}(B^\circ)$, and we also conclude that $\boldsymbol{\nu} \in \Gamma^\vee$, by the second part of the disjunction in Lemma 3.1′.

Since this is true for any extreme ray of $\mathrm{cone}(B)$, we deduce that $\mathrm{cone}(B) = \Gamma^\vee$. ∎

As obtained by Delabarre [Del13], when h is a polynomial without cyclotomic factor, the associated Euler product f has a meromorphic extension on the dual cone of the support of h.

Assuming h is rational in two variables. We will now turn our attention to the case of a rational function $h \in \mathbb{Q}(X_1, X_2)$ in two variables, which can be represented as a power series with integer coefficients, and such that $h(0,0) = 1$.

Moreover, we assume that h can be written as the quotient of two coprime polynomials q^+ and q^- in $\mathbb{Z}[X_1, X_2]$ such that $q^+(0,0) = q^-(0,0) = 1$. The existence of such a representation will be obtained later in a broader setting by generalizing the property of Fatou in Proposition 3.12.

Each polynomial q^+ and q^- has a support denoted B^+ and B^- respectively, each generating a cone in $\mathbb{R}^2$. Convex cones in $\mathbb{R}^2$ are particularly easy to describe: they have at most two extreme rays. Since $q^-(0,0) = 1$, the inverse $1/q^-$ is represented as a power series in $\mathbb{Z}[[X_1, X_2]]$, and the convex hull of the support of $1/q^-$ is exactly the cone generated by B^-. Therefore, the support B of h has to be inside $\operatorname{conv}(B^+) + \operatorname{cone}(B^-)$, thus

$$\operatorname{cone}(B) \subseteq \operatorname{cone}(B^+) + \operatorname{cone}(B^-) = \operatorname{cone}(B^+ \cup B^-). \tag{3.4}$$

However, as shown by the example $h(X_1, X_2) = \frac{1+X_1+X_2-X_1X_2}{1+X_1+X_2}$ developed in the introduction, the previous inclusion is not always an equality, and the extreme rays of $\overline{\operatorname{cone}}(B)$ are not at our disposal *a priori*. We will use the extreme rays of $\operatorname{cone}(B^+ \cup B^-)$ instead.

In this case, we are able to prove a result similar to Lemma 3.7.

Lemma 3.9. *Let q^+ and q^- be two coprime polynomials in $\mathbb{Z}[X_1, X_2]$ with constant terms equal to 1, and B^+ and B^- their respective supports. Their quotient $h = q^+/q^-$ has a representation as a power series in $\mathbb{Z}[[X_1, X_2]]$.*

Let $\boldsymbol{\nu} \in \mathbb{N}^2$ be a primitive vector such that $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of $\operatorname{cone}(B^+ \cup B^-)$. Define $h_{\boldsymbol{\nu}}$, $q^+_{\boldsymbol{\nu}}$ and $q^-_{\boldsymbol{\nu}}$ as in Lemma 3.4. The rational function $h^\circ = h/h_{\boldsymbol{\nu}}$ has a representation as a formal series, with B° as its support.

We have the following exclusive disjunction: either

- *$q^+_{\boldsymbol{\nu}}$ divides q^+ and $q^-_{\boldsymbol{\nu}}$ divides q^-, or*
- *there exists an integer vector $\boldsymbol{\mu} \in \mathbb{N}^2$ such that $B^\circ \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N})$ is infinite.*

The outline of the proof contains two main steps: apply a suitable change of indeterminates and use Fatou's Lemma. This will serve as template for the general case.

Fatou proved (cf. [Fat04, pp. 368–371]) that if P and Q are two coprime polynomials in $\mathbb{Q}[X]$ with $Q(0) = 1$ such that the power series expansion of P/Q has coefficients in $\mathbb{Z}$, then the polynomials P and Q are in $\mathbb{Z}[X]$. This property has been generalized to other integral domains than $\mathbb{Z}$, in particular to any unique factorization domain such as $\mathbb{Z}[Y]$: if R is the integral domain $\mathbb{Z}[Y]$, and F its fraction field $\mathbb{Q}(Y)$, then any pair of coprime polynomials P and Q in $F[X]$ with $Q(0) = 1$ and such that the power series expansion of P/Q has coefficients in R, are indeed polynomials in $R[X]$.

Proof of Lemma 3.9. If $q^+_{\boldsymbol{\nu}}$ divides q^+ and $q^-_{\boldsymbol{\nu}}$ divides q^-, we define $q^\oplus = q^+/q^+_{\boldsymbol{\nu}}$ and $q^\ominus = q^-/q^-_{\boldsymbol{\nu}}$, both polynomials in $\mathbb{Z}[X_1, X_2]$; we denote their supports by $B^\oplus$ and $B^\ominus$ respectively. The supports of q^+, q^-, h, $q^\oplus$ and $q^\ominus$ are all inside the cone $K = \operatorname{cone}(B^+ \cup B^-)$ of extreme ray $\boldsymbol{\nu}\mathbb{R}_+$. By Lemma 3.3, we have

$$h_{\boldsymbol{\nu}} = \frac{q^+_{\boldsymbol{\nu}}}{q^-_{\boldsymbol{\nu}}}, \qquad \text{thus } h^\circ = \frac{q^\oplus}{q^\ominus}.$$

Also, $B^\oplus$ and $B^\ominus$ do not contain any non-trivial multiple of $\boldsymbol{\nu}$. Thus, they are inside $K \smallsetminus \boldsymbol{\nu}\mathbb{R}^*_+$, which is a convex cone by the extremality of the ray $\boldsymbol{\nu}\mathbb{R}_+$ in K. Consequently, $\operatorname{cone}(B^\oplus \cup B^\ominus) \subseteq K \smallsetminus \boldsymbol{\nu}\mathbb{R}^*_+$.

Applying (3.4) to h°, we get $\operatorname{cone}(B^\circ) \subseteq \operatorname{cone}(B^\oplus) + \operatorname{cone}(B^\ominus)$ (thus $B^\circ \subseteq K$). Since both cones in the right-hand term are finitely generated and therefore closed, we also have $\overline{\operatorname{cone}}(B^\circ) \subseteq \operatorname{cone}(B^\oplus) + \operatorname{cone}(B^\ominus)$. Therefore, $\overline{\operatorname{cone}}(B^\circ) \subseteq K \smallsetminus \boldsymbol{\nu}\mathbb{R}^*_+$, which means that $\boldsymbol{\nu}$ does not belong to $\overline{\operatorname{cone}}(B^\circ)$. As a consequence, the set $B^\circ \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N})$ cannot be infinite for any $\boldsymbol{\mu} \in \mathbb{N}^2$.

Now, let us assume that for any $\boldsymbol{\mu} \in \mathbb{N}^2$, the set $B^\circ \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N})$ is finite. We first change our indeterminates.

Since $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of the cone $K = \operatorname{cone}(\operatorname{supp} q^+ \cup \operatorname{supp} q^-)$, which includes B°, we find that for any $\boldsymbol{\beta} = (\beta_1, \beta_2) \in B^\circ$ the sign of $\nu_1\beta_2 - \nu_2\beta_1$ does not change (it may be zero though); we set $\epsilon = 1$ or $\epsilon = -1$ according to this sign.

Since the coordinates of $\boldsymbol{\nu}$ are coprime, there exists a vector $\boldsymbol{\mu} \in \mathbb{Z}^2$ such that $\det(\boldsymbol{\nu}, \boldsymbol{\mu}) = \epsilon$. Consequently, the family $\{\boldsymbol{\nu}, \boldsymbol{\mu}\}$ forms a basis of the lattice $\mathbb{Z}^2$, in which every element of $K \cap \mathbb{Z}^2$ has a non-negative coordinate with respect to the basis vector $\boldsymbol{\mu}$. Since $K \subseteq (\mathbb{R}_+)^2$, there exists a negative integer $m \in \mathbb{Z}$ such that $\boldsymbol{\mu} + m\boldsymbol{\nu}$ is not in K: set $\boldsymbol{\nu}' = \boldsymbol{\mu} + m\boldsymbol{\nu}$, so that $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ forms a basis of the lattice $\mathbb{Z}^2$, and every element of $K \cap \mathbb{Z}^2$ has non-negative integer coordinates in this basis. We set $Y_1 = \boldsymbol{X}^{\boldsymbol{\nu}}$ and $Y_2 = \boldsymbol{X}^{\boldsymbol{\nu}'}$, so that any power series with support inside K may be regarded as a power series in the indeterminates Y_1 and Y_2: this change of variables is indeed a $\mathbb{Z}$-algebra homomorphism.

In this setting, we have $h_{\boldsymbol{\nu}}(Y_1, Y_2) = h(Y_1, 0)$, therefore $h^\circ(Y_1, 0) = 1$, and by the fact that $B^\circ \cap (m\boldsymbol{\nu}' + \boldsymbol{\nu}\mathbb{N})$ is finite for any $m \in \mathbb{N}$, the coefficient in Y_2^m of $h^\circ(Y_1, Y_2)$ when regarded as a power series in $(\mathbb{Z}[[Y_1]])[[Y_2]]$ is a polynomial in Y_1. As a result, we may regard $h^\circ(Y_1, Y_2)$ as a power series in Y_2 with coefficients in $\mathbb{Z}[Y_1]$ and with constant term equal to 1.

We set $R = \mathbb{Z}[Y_1]$ and $F = \mathbb{Q}(Y_1)$ its quotient field. We now consider every power series of $\mathbb{Z}[[X_1, X_2]]$ with support inside K as a power series in $R[[Y_2]]$, and make explicit the variable. For example, the polynomial $q^+ \in \mathbb{Z}[X_1, X_2]$ is now considered as $q^+(Y_2) \in R[Y_2]$, and $q^+_{\boldsymbol{\nu}} \in \mathbb{Z}[X_1, X_2]$ as its constant term $q^+(0) \in R$. The power series h° is considered as $h^\circ(Y_2)$ which can be seen as the element $h(Y_2)/h(0)$ of $F(Y_2)$ with constant term equal to 1, and as an element of $R[[Y_2]]$. Since the integral domain $R = \mathbb{Z}[Y_1]$ satisfies Fatou's property, there are two polynomials $p^+(Y_2)$ and $p^-(Y_2)$ in $R[Y_2]$, coprime in $F[Y_2]$ with $p^+(0) = 1$ and $p^-(0) = 1$, such that $h^\circ(Y_2) = p^+(Y_2)/p^-(Y_2)$. On the other hand, we have

$$h^\circ(Y_2) = \frac{h(Y_2)}{h(0)} = \frac{q^+(Y_2)/q^+(0)}{q^-(Y_2)/q^-(0)}.$$

Since $F[Y_2]$ has unique factorization, there exists a polynomial $r(Y_2) \in F[Y_2]$ such that

$$q^+(Y_2) = q^+(0)r(Y_2)p^+(Y_2) \quad \text{and} \quad q^-(Y_2) = q^-(0)r(Y_2)p^-(Y_2).$$

The polynomial $r(Y_2)$ divides q^+ and q^- in $F[Y_2]$. Since $R = \mathbb{Z}[Y_1]$ is a UFD, the primitive part $\tilde{r}(Y_2)$ of $r(Y_2)$ is in $R[Y_2]$ and, by Gauss's Lemma [DF04, §9.3], divides $q^+(Y_2)$ and $q^-(Y_2)$ in $R[Y_2]$. Back to $\mathbb{Z}[Y_1, Y_2]$, $\tilde{r}(Y_1, Y_2)$ divides $q^+(Y_1, Y_2)$ and $q^-(Y_1, Y_2)$. By Corollary 3.6, the support of $\tilde{r}(Y_1, Y_2)$ is in the cone generated by the supports of $q^+(Y_1, Y_2)$ and $q^-(Y_1, Y_2)$. In particular, every monomial of $\tilde{r}(\boldsymbol{X}^{\boldsymbol{\nu}}, \boldsymbol{X}^{\boldsymbol{\nu}'})$ is written as $\boldsymbol{X}^{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} \in K \cap \mathbb{Z}^2$. We conclude that $\tilde{r}(\boldsymbol{X}^{\boldsymbol{\nu}}, \boldsymbol{X}^{\boldsymbol{\nu}'})$ is an element of $\mathbb{Z}[X_1, X_2]$ which divides q^+ and q^-, which are coprime in $\mathbb{Z}[Y_1, Y_2]$. We deduce that the primitive part of $r(Y_2)$ is equal to 1, which means that $r(Y_2)$ is of degree 0 as a polynomial in $F[Y_2]$. Since we have

$r(0) = 1$, we deduce that $r = 1$ and

$$q^+(Y_2) = q^+(0)p^+(Y_2) \quad \text{and} \quad q^-(Y_2) = q^-(0)p^-(Y_2).$$

Each term appearing in these identities belongs to $\mathbb{Z}[Y_1, Y_2]$, which implies that q_ν^+ divides q^+, and q_ν^- divides q^-, by substituting the original indeterminates back. ■

Using this lemma, along with Lemma 3.1′, we may reason as for Proposition 3.8 and prove that the dual cone of the domain Γ defined in Theorem 2.9 is the cone generated by the supports of q^+ and q^-, when h is the quotient of two coprime polynomials q^+ and q^- of $\mathbb{Z}[X_1, X_2]$ with constant terms equal to 1 and without cyclotomic factors.

3.2. The general case. In this section, we generalize the two main ingredients in the proof of Lemma 3.9 to higher dimensions: the change of coordinates and the property of Fatou. We conclude by giving a proof of the generalization of Lemmata 3.7 and 3.9.

3.2.1. Change of coordinates. To correctly convey this idea, we need to generalize the notions of polynomials and power series, mainly by generalizing the notion of monomials.

For K a convex cone in $\mathbb{R}^k$, we define the algebra $\mathbb{Z}[\boldsymbol{X}; K]$ of generalized polynomials in k variables with support in K to consist of finite sums

$$\sum_{\boldsymbol{\alpha} \in \mathbb{Z}^k \cap K} b_{\boldsymbol{\alpha}} \boldsymbol{X}^{\boldsymbol{\alpha}} \tag{3.5}$$

with coefficients in $\mathbb{Z}$. This algebra is the standard *semigroup ring* associated to the monoid $\mathbb{Z}^k \cap K$ over the domain $\mathbb{Z}$, as defined in [Gil84, §7] for example. To avoid confusion between the different occurrences of the ring of integers, we extend this definition to the algebra $R[\boldsymbol{X}; K]$ of generalized polynomials with coefficients in an integral domain R.

The set of infinite sums of the form (3.5) admits a natural algebra structure, provided that K does not contain any affine line (see [AMK13] for an expository account): we name it the algebra of *generalized power series in k variables*, and denote it $R[[\boldsymbol{X}; K]]$.

If $K = (\mathbb{R}_+)^k$, we retrieve the usual ring of power series in k variables, $R[[\boldsymbol{X}; K]] = R[[\boldsymbol{X}]]$.

More generally, when the cone K is generated by a basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ of $\mathbb{Z}^k$, the associated power series ring is isomorphic to the usual ring of power series in k variables.

Let us consider the change-of-basis linear map associated to this basis,

$$\phi(x_1, \dots, x_k) = \sum_{i=1}^{k} x_i \boldsymbol{v}_i,$$

where $\phi\big((\mathbb{R}_+)^k\big) = K$, and set $Y_i = \boldsymbol{X}^{\boldsymbol{v}_i}$ as the new indeterminates for our polynomial/power series ring. For any $\boldsymbol{\beta} \in \mathbb{Z}^k$, we have $\boldsymbol{X}^{\phi(\boldsymbol{\beta})} = \boldsymbol{Y}^{\boldsymbol{\beta}}$. Since ϕ is an isomorphism from $\mathbb{Z}^k$ to itself, every monomial may be rewritten as $\boldsymbol{X}^{\boldsymbol{\alpha}} = \boldsymbol{Y}^{\phi^{-1}(\boldsymbol{\alpha})}$. This defines the isomorphism

$$\begin{aligned} \phi_* : R[[\boldsymbol{X}; K]] &\to R[[\boldsymbol{Y}; \phi^{-1}(K)]], \\ \sum_{\boldsymbol{\alpha} \in K} b_{\boldsymbol{\alpha}} \boldsymbol{X}^{\boldsymbol{\alpha}} &\mapsto \sum_{\boldsymbol{\alpha} \in K} b_{\boldsymbol{\alpha}} \boldsymbol{Y}^{\phi^{-1}(\boldsymbol{\alpha})}, \end{aligned} \tag{3.6}$$

where $\phi^{-1}(K)$ is actually $(\mathbb{R}_+)^k$. Hence, we have an isomorphism between $R[\![\boldsymbol{X};K]\!]$ and $R[\![\boldsymbol{Y}]\!]$.

Changing the name of the indeterminates from $\boldsymbol{X}$ to $\boldsymbol{Y}$ is purely cosmetic here, but it is useful for keeping track of which algebra we are operating in.

If a cone K lies inside another cone K', we have a trivial injective algebra morphism

$$R[\![\boldsymbol{X};K]\!] \to R[\![\boldsymbol{X};K']\!].$$

It is well-known (*) that a cone is line-free if and only if it is contained in a simplicial cone K'. By slightly tweaking this statement, we can even ask for this simplicial cone to be generated by a basis of $\mathbb{Z}^k$. Using the corresponding isomorphism ϕ_*, the algebra $R[\![\boldsymbol{X};K']\!]$ is isomorphic to $R[\![\boldsymbol{Y}]\!]$. This means that for any line-free cone K, the algebra $R[\![\boldsymbol{X};K]\!]$ can be interpreted as a subalgebra of the usual power series algebra $R[\![\boldsymbol{Y}]\!]$.

In our setting, the cone K is generated by the support of two polynomials in $R[\boldsymbol{X}]$, it is therefore rational and line-free. When a cone K is rational (or, more generally, when it is closed), K is line-free if and only if it does not contain any non-trivial vector subspace, which is equivalent to $K \cap (-K) = \{\boldsymbol{0}\}$, or to the fact that $\{\boldsymbol{0}\}$ is a face of K.

We also want the chosen extreme ray $\boldsymbol{\nu}\mathbb{R}_+$ of the cone K to be an extreme ray of the containing cone K'. The point of this assumption is the following: given a power series $g \in \mathbb{Z}[\![\boldsymbol{X};K]\!]$, and assuming that K' is generated by the basis $\boldsymbol{v}_1,\dots,\boldsymbol{v}_k$ of $\mathbb{Z}^k$ with $\boldsymbol{v}_1 = \boldsymbol{\nu}$, the image of g and $g_{\boldsymbol{\nu}}$ by ϕ_* satisfies $\phi_* g_{\boldsymbol{\nu}}(\boldsymbol{Y}) = \phi_* g(Y_1, 0, \dots, 0)$.

We will actually prove a slightly more precise result.

Lemma 3.10. *Let K be a full-dimensional line-free rational convex cone in $\mathbb{R}^k$. Let*

$$F_0 = \{\boldsymbol{0}\} \subset F_1 \subset \cdots \subset F_k = K$$

be a maximal chain of faces of K. Then there exists a basis $\boldsymbol{v}_1,\dots,\boldsymbol{v}_k$ of $\mathbb{Z}^k$ such that $F_i \subseteq \operatorname{cone}(\boldsymbol{v}_1,\dots,\boldsymbol{v}_i)$ for every i between 0 and k.

Proof. We will prove this result by induction on k. The statement is trivial for $k = 1$. Assume that $k \geqslant 2$. We want to find a linear projection π onto the hyperspace H spanned by F_{k-1} such that the projected cone $K' = \pi(K)$ satisfies the assumptions of the lemma. In particular, we want F_i to be a face of K' for any $i \leqslant k-2$ in order to apply the induction hypothesis, and $\pi(\mathbb{Z}^k) = H \cap \mathbb{Z}^k$ to exploit its conclusion.

As a face of codimension 2, F_{k-2} is the intersection of two facets of K (see [Ful93, Section 1.2]), one of them being F_{k-1}. Let H' be the hyperplane spanned by the other facet. Both H and H' are rational hyperplanes. In particular, $H \cap \mathbb{Z}^k$ is a free abelian group of rank $k-1$.

Let $\phi : \mathbb{Q}^k \to \mathbb{Q}$ (respectively ϕ') be a linear form with kernel H (respectively H') and which is non-negative on K. The image of $\mathbb{Z}^k$ by ϕ is a subgroup of $\mathbb{Q}$ generated by at most k rational numbers. In particular, $\phi(\mathbb{Z}^k)$ is a discrete subgroup of $\mathbb{Q}$, and it has a minimal positive element. Let $\boldsymbol{x} \in \mathbb{Z}^k$ with $\phi(\boldsymbol{x}) = \min \phi(\mathbb{Z}^k) \cap \mathbb{Q}_+^*$.

(*) See [Grü03, §2.5, exercise 9]. Basically, a cone K is line-free if and only if its closure $\overline{K}$ does not contain any non-trivial vector subspace, which is equivalent to its dual cone $\overline{K}^\vee$ containing a basis of the ambient vector space.

The free abelian group $H \cap \mathbb{Z}^k$ is of rank $k-1$, and is not contained in H'. Let $\boldsymbol{u} \in H \cap \mathbb{Z}^k$ such that $\phi'(\boldsymbol{u}) > 0$. There is $n \in \mathbb{N}$ such that $n\phi'(\boldsymbol{u}) > \phi'(\boldsymbol{x})$. Set $\boldsymbol{v}_k = \boldsymbol{x} - n\boldsymbol{u}$. We have $\phi(\boldsymbol{v}_k) = \phi(\boldsymbol{x})$ so $\phi(\mathbb{Z}^k)$ is the subgroup of $\mathbb{Q}$ generated by $\phi(\boldsymbol{v}_k)$, and $\phi'(\boldsymbol{v}_k) = \phi'(\boldsymbol{x}) - n\phi'(\boldsymbol{u}) < 0$.

Let H'' be the hyperplane generated by $\boldsymbol{v}_k$ and $H \cap H'$. For any $\boldsymbol{y} \in H''$ there is a unique $t \in \mathbb{R}$ with $\boldsymbol{y} - t\boldsymbol{v}_k \in H \cap H'$. Assume also that $\boldsymbol{y} \in K$. We deduce that $\phi(\boldsymbol{y}) = t\phi(\boldsymbol{v}_k) \geqslant 0$, so $t \geqslant 0$, and $\phi'(\boldsymbol{y}) = t\phi'(\boldsymbol{v}_k) \geqslant 0$, so $t \leqslant 0$. This means that $K \cap H'' \subseteq K \cap H \cap H' = F_{k-2}$.

Let K' be the image of K by projection along $\boldsymbol{v}_k\mathbb{R}$ onto H; K' is also rational, since it is generated by the finite family of projected images of the generators of K. Since $\boldsymbol{v}_k \in H''$, the projection of H'' is $H \cap H'$. In particular, $H \cap H'$ is a supporting hyperplane of K' within H, and the intersection $H \cap H' \cap K'$ is the projection of $H'' \cap K$, which equals F_{k-2}. Finally, since $F_{k-1} = K \cap H$, we have $F_{k-1} \subseteq K'$. Therefore, K' is of dimension $k-1$ and F_{k-2} is a facet of K'. Since every face of a face of a cone is a face of this cone, every F_i with $i \leqslant k-2$ is a face of K'. In particular, $F_0 = \{0\}$ is face of K' which means that K' is line-free.

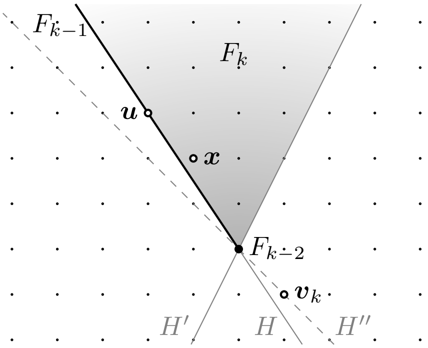

Fig. 3. Construction of the basis vector $\boldsymbol{v}_k$ and of its corresponding projection. Everything has been projected along the affine hull of F_{k-2} for readability.

Since K' is a rational cone of dimension $k-1$ in H, containing no line, we can apply the lemma for the following chain of faces

$$F_0 = \{\boldsymbol{0}\} \subset F_1 \subset \cdots \subset F_{k-2} \subset F'_{k-1} = K'.$$

Therefore, there exists a basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k-1}$ of $H \cap \mathbb{Z}^k$ such that $F_i \subseteq \operatorname{cone}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_i)$ for every i between 0 and $k-2$, and $K' \subseteq \operatorname{cone}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k-1})$. Since $F_{k-1} \subseteq K'$, we also have $F_{k-1} \subseteq \operatorname{cone}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k-1})$

It remains to prove that the family $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$ is a basis of $\mathbb{Z}^k$, and that the cone K is contained in $\operatorname{cone}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$.

First, let $\boldsymbol{v} \in \mathbb{Z}^k$. Recall that $\phi(\mathbb{Z}^k)$ is a discrete subgroup of $\mathbb{Q}$ generated by $\phi(\boldsymbol{v}_k)$, thus $\phi(\boldsymbol{v})$ is a multiple of this generator, and there exists a unique $n_k \in \mathbb{Z}$ such that $\phi(\boldsymbol{v}) = n_k\phi(\boldsymbol{v}_k)$. We deduce that $\boldsymbol{v} - n_k\boldsymbol{v}_k \in \ker\phi = H$. Since $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k-1}$ is a basis of $H \cap \mathbb{Z}^k$, there exist $k-1$ integers $n_1, \dots, n_{k-1}$ such that $\boldsymbol{v} - n_k\boldsymbol{v}_k = n_1\boldsymbol{v}_1 + \cdots + n_{k-1}\boldsymbol{v}_{k-1}$.

We conclude that the family $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$ spans $\mathbb{Z}^k$, and since $\mathbb{Z}^k$ is of rank k, $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$ is a basis of $\mathbb{Z}^k$.

Now, let $\boldsymbol{x} \in K$. Since K' is the projected image of K along $\boldsymbol{v}_k$, $\boldsymbol{x}$ can be represented as $\boldsymbol{x}' + t_k\boldsymbol{v}_k$ with $\boldsymbol{x}' \in K'$ and $t_k \in \mathbb{R}$. Since $K' \subseteq \operatorname{cone}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k-1})$, there are non-negative real numbers $t_1, \dots, t_{k-1}$ such that $\boldsymbol{x}' = t_1\boldsymbol{v}_1 + \dots + t_{k-1}\boldsymbol{v}_{k-1}$. The form ϕ has non-negative value on K, positive at $\boldsymbol{v}_k$ and zero on $K' \subseteq H$, thus $0 \leqslant \phi(\boldsymbol{x}) = t_k\phi(\boldsymbol{v}_k)$ which implies that $t_k \geqslant 0$. Altogether, we have $\boldsymbol{x} = t_1\boldsymbol{v}_1 + \dots + t_k\boldsymbol{v}_k$ with non-negative scalars $t_1, \dots, t_k$. This proves that K is contained in $\operatorname{cone}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$. ■

We remark that the dual statement asserts that there also exists a basis $\boldsymbol{v}'_1, \dots, \boldsymbol{v}'_k$ of $\mathbb{Z}^k$ such that $\boldsymbol{v}'_i \in F_i$ for any i between 1 and k.

With the goal of applying this lemma to a change of indeterminates, we may rewrite Lemma 3.10 in a more suitable way.

COROLLARY 3.11. *Let K be a full-dimensional line-free rational convex cone in $\mathbb{R}^k$. Let*

$$F_0 = \{\boldsymbol{0}\} \subset F_1 \subset \dots \subset F_k = K$$

be a maximal chain of faces of K. There exists a change-of-basis isomorphism $\phi : \mathbb{Z}^k \to \mathbb{Z}^k$ such that ϕ_* injectively maps $\mathbb{Z}[[\boldsymbol{X}; K]]$ into $\mathbb{Z}[[\boldsymbol{Y}]]$ and every subalgebra $\mathbb{Z}[[\boldsymbol{X}; F_i]]$ into $\mathbb{Z}[[Y_1, \dots, Y_i]]$.

3.2.2. The property of Fatou. In this section we want to generalize the following characteristic property of Fatou rings to multivariable power series.

DEFINITION 3.1. A power series g of $R[[X]]$ is said to be *rational* if there exists two polynomials $P, Q \in R[X]$ satisfying $Qg = P$. The power series g can be identified with the quotient $P/Q \in F(X)$, where F is the quotient field of F.

An integral domain R with quotient field F is said to be a *Fatou ring* if any rational element of $R[[X]]$ can be identified as the quotient of two polynomials in $R[X]$ that are coprime in $F[X]$ and such that the constant term of the denominator is equal to 1.

For more context on this notion, see [Eil74, Chapter XVI].

We are able to prove that, if R is a Fatou ring, and K a line-free cone of $\mathbb{R}^k$, then any rational power series of $R[[\boldsymbol{X}; K]]$ is the quotient of two polynomials $P, Q \in R[\boldsymbol{X}; K]$ which are coprime in $F[\boldsymbol{X}; K']$ for any line-free convex cone K' containing K, and with $Q(\boldsymbol{0}) = 1$.

Even if this proof relies on some arguments already developed in this article, presenting it entirely would make this discourse even longer. We shall restrict ourselves to proving the case with R being a UFD, and $K = K' = (\mathbb{R}_+)^k$:

PROPOSITION 3.12. *Let R be a UFD, and F its fraction field. For any rational power series $g \in R[[\boldsymbol{X}]]$, there is a unique pair of coprime polynomials $P, Q \in R[\boldsymbol{X}]$ with $Q(\boldsymbol{0}) = 1$ such that $Qg = P$.*

Consider the special case of a field, that is, with $R = F$. Proving that a field is a Fatou ring is straightforward, because with one indeterminate, a zero constant term implies a factorization by the indeterminate. However, the corresponding statement of

Proposition 3.12 in this case is surprisingly difficult to derive. We refer to [KKT14] for the only proof we know of; it is equivalent to the following statement.

Proposition 3.13 ([KKT14]). *Let R be an integral domain, and F its fraction field. For any rational power series $g \in R[[\boldsymbol{X}]]$, there is a pair of polynomials $P, Q \in R[\boldsymbol{X}]$, coprime in $F[\boldsymbol{X}]$, with $Q(\mathbf{0}) \neq 0$, such that $Qg = P$.*

Their proof relies on the fact that a unique factorization domain is a Fatou ring. A proof of this fact may also be found in [Dre68, Lemme 3].

Proof of Proposition 3.12. Let g be a rational power series in $R[[\boldsymbol{X}]]$. By Proposition 3.13, there are polynomials $P'', Q'' \in R[\boldsymbol{X}]$, coprime in $F[\boldsymbol{X}]$, with $Q''(\mathbf{0}) \neq 0$, such that $Q''g = P''$.

Consider the change of indeterminates $Y_i = X_i X_k^{-1}$ for any $i \neq k$ and $Y_k = X_k$. It corresponds to the R-algebra isomorphism

$$\begin{aligned}\psi_* &: R[[\boldsymbol{X}]] \to R[[\boldsymbol{Y}; K]],\\ &\sum_{\boldsymbol{\alpha} \in \mathbb{N}^k} b_{\boldsymbol{\alpha}} \boldsymbol{X}^{\boldsymbol{\alpha}} \mapsto \sum_{\boldsymbol{\alpha} \in \mathbb{N}^k} b_{\boldsymbol{\alpha}} Y_1^{\alpha_1} \cdots Y_{k-1}^{\alpha_{k-1}} Y_k^{\alpha_1 + \cdots + \alpha_k},\end{aligned}$$

where K is the cone generated by the vectors $\boldsymbol{e}_k$ and all the $\boldsymbol{e}_i + \boldsymbol{e}_k$ for i between 1 and $k-1$, with $(\boldsymbol{e}_i)_{1 \leqslant i \leqslant k}$ being the canonical basis of $\mathbb{R}^k$.

Each term of $\psi_* g$ can be written as $a_n(Y_1, \ldots, Y_{k-1})(Y_k)^n$ where a_n is a monomial in $R[Y_1, \ldots, Y_{k-1}]$ of total degree less than or equal to n. We may as well regard $\psi_* g$ as a power series of $R'[[Y_k]]$, with $R' = R[Y_1, \ldots, Y_{k-1}]$.

On the other hand, since ψ_* is an algebra homomorphism, we have

$$\psi_* P'' = \psi_*(Q''g) = \psi_* Q'' \cdot \psi_* g,$$

where $\psi_* P''$ and $\psi_* Q''$ are elements in $R[[\boldsymbol{Y}]]$ with finite supports, and can be seen as elements of $R'[Y_k]$. Therefore, $\psi_* g$ is a rational power series of $R'[[Y_k]]$.

Since R' is a UFD, it is a Fatou ring, and there are two polynomials P' and Q' in $R'[Y_k]$, coprime in $F'[Y_k]$ where F' is the quotient field of R', with $Q'(0) = 1 \in R'$, such that $Q' \cdot \psi_* g = P'$.

We now make full use of the fact that R' is a unique factorization domain. Since $Q' \in R'[Y_k]$ has constant term 1, its content is 1 and the contents of P' and Q' are coprime in R'. Since P' and Q' are coprime in $F'[Y_k]$, their primitive parts are coprime in $R'[Y_k]$. Finally, we deduce that P' and Q' are coprime in $R'[Y_k]$, that is, in $R[\boldsymbol{Y}]$. From now on, P' and Q' are considered as polynomials in $R[\boldsymbol{Y}]$.

From the relations $\psi_* Q \cdot \psi_* g = \psi_* P$ and $Q' \cdot \psi_* g = P'$, we deduce that $P' \cdot \psi_* Q = Q' \cdot \psi_* P$. Since $R[\boldsymbol{Y}]$ is a UFD, and since P' and Q' are coprime in this domain, there exists a polynomial $S \in R[\boldsymbol{Y}]$ such that $\psi_* Q'' = SQ'$ and $\psi_* P'' = SP'$.

Since $Q'(\mathbf{0}) = 1$, we have $S(\mathbf{0}) = \psi_* Q''(\mathbf{0}) = Q''(\mathbf{0}) \neq 0$. By Corollary 3.6, we deduce that $\operatorname{supp} Q' \subset K$ from the fact that $\operatorname{supp}(\psi_* Q'') \subset K$. Moreover, we have

$$\operatorname{supp} P' = \operatorname{supp}(Q' \psi_* g) \subseteq \operatorname{supp}(Q') + \operatorname{supp}(\psi_* g) \subseteq K + K = K.$$

We deduce that P' and Q' are elements of $R[\boldsymbol{Y}; K]$, thus of $R[[\boldsymbol{Y}; K]]$ also. Using the inverse ψ_*^{-1} of the algebra isomorphism ψ_*, we can define $P = \psi_*^{-1} P'$ and $Q = \psi_*^{-1} Q'$ which are polynomials in $R[[\boldsymbol{X}]]$, therefore in $R[\boldsymbol{X}]$.

Finally, we show that P and Q are the polynomials we are looking for.

Since ψ_*^{-1} is an algebra homomorphism, we have

$$P = \psi_*^{-1} P' = \psi_*^{-1}(Q' \cdot \psi_* g) = (\psi_*)^{-1} Q' \cdot g = Qg.$$

Every common divisor of P and Q is sent by ψ_* to a common divisor of P' and Q', which can only be a unit. Therefore, P and Q are coprime in $R[\boldsymbol{X}]$.

The constant term of Q is $Q'(\mathbf{0})$, that is, 1.

Finally, the uniqueness of these polynomials P and Q relies on the fact that R is a UFD. Let $P_0, Q_0 \in R[\boldsymbol{X}]$ be coprime in $F[\boldsymbol{X}]$, with $Q_0(\mathbf{0}) = 1$ and $Q_0 g = P_0$. We have $QP_0 = QQ_0 g = PQ_0$, and since P and Q are coprime in $R[\boldsymbol{X}]$, there is a polynomial $S_0 \in R[\boldsymbol{X}]$ such that $P_0 = S_0 P$ and $Q_0 = S_0 Q$. Since P_0 and Q_0 are coprime in $R[\boldsymbol{X}]$, S_0 is a constant polynomial, and by evaluating the relation $Q_0 = S_0 Q$ at $\mathbf{0}$, we get $S_0 \equiv 1$. Therefore, $P_0 = P$ and $Q_0 = Q$. ■

When $R = \mathbb{Z}$, this result solves our problem of representing the rational power series $h \in \mathbb{Z}[[\boldsymbol{X}]]$ satisfying $h(\mathbf{0}) = 1$ as a quotient of two coprime polynomials in $\mathbb{Z}[\boldsymbol{X}]$ with constant term 1.

3.2.3. The main result. Before stating our main result concerning the support B°, we recall the different objects and properties we have considered so far.

Let $h \in \mathbb{Z}[[\boldsymbol{X}]]$ be a rational power series.

By Proposition 3.12, there exist two unique coprime polynomials q^+ and q^- in $\mathbb{Z}[\boldsymbol{X}]$ with $q^-(\mathbf{0}) = 1$ and such that $q^- h = q^+$. From now on, we shall refer to these polynomials, respectively, as *the* numerator and *the* denominator of h.

We set $K = \overline{\operatorname{cone}}(\operatorname{supp} h)$, and let $\boldsymbol{\nu} \in \mathbb{N}^k$ be the primitive vector directing an extreme ray of K. From Lemma 3.1, we are only interested in rational extreme rays of this cone. But by Theorem 1.1 of [KKT14], we have

$$\overline{\operatorname{cone}}(\operatorname{supp} h) = \operatorname{cone}(\operatorname{supp} q^+) + \operatorname{cone}(\operatorname{supp} q^-). \tag{3.7}$$

In particular, each extreme ray of K is rational; it is even generated by an element of the support of q^+ or of q^-.

Once h and $\boldsymbol{\nu}$ are fixed, we choose a convenient change of coordinates ϕ_*.

Since K is rational and line-free (it is contained in $(\mathbb{R}_+)^k$), and $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of this cone (that is, a face of dimension 1), we can apply Corollary 3.11 with a maximal chain of faces of $\overline{\operatorname{cone}}(\operatorname{supp} h)$ starting from $F_1 = \boldsymbol{\nu}\mathbb{R}_+$. If K is not full-dimensional, say of dimension $d < k$, we can build a larger cone $\tilde{K} \subset (\mathbb{R}_+)^k$ of dimension k by appending $k - d$ vectors, such that $\tilde{K}$ admits K as a face, and apply Corollary 3.11 to it. In any case, we have $h \in \mathbb{Z}[[\boldsymbol{X}; K]]$.

By Corollary 3.11, there exists an isomorphism $\phi : \mathbb{Z}^k \to \mathbb{Z}^k$, with $\phi(1, 0, \ldots, 0) = \boldsymbol{\nu}$, such that the mapping ϕ_* defined by (3.6) is an injective morphism of algebras from $\mathbb{Z}[[\boldsymbol{X}; K]]$ to $\mathbb{Z}[[\boldsymbol{Y}]]$, which maps $\mathbb{Z}[[\boldsymbol{X}; F_1]]$ to $\mathbb{Z}[[Y_1]]$.

Since the supports of h, q^+ and q^- are inside K, their images $\phi_* h$, $\phi_* q^+$ and $\phi_* q^-$ are elements of $\mathbb{Z}[[\boldsymbol{Y}]]$. Since the supports of q^+ and q^- are finite, so are the supports of $\phi_* q^+$ and $\phi_* q^-$, which means that $\phi_* q^+$ and $\phi_* q^-$ are two polynomials in $\mathbb{Z}[\boldsymbol{Y}]$. Under the morphism of algebras, the relation $q^- h = q^+$ is sent to $\phi_* q^- \cdot \phi_* h = \phi_* q^+$. Therefore, $\phi_* h$ is a rational power series in $\mathbb{Z}[[\boldsymbol{Y}]]$.

Finally, we identify $\mathbb{Z}[[\boldsymbol{Y}]]$ with the isomorphic algebra $\mathbb{Z}[[Y_1]][[Y_2,\dots,Y_k]]$, and we define the coefficient function $\mathbb{N}^{k-1} \ni \boldsymbol{\beta'} \mapsto a_{\boldsymbol{\beta'}} \in \mathbb{Z}[[Y_1]]$ of $\phi_* h$ so that

$$\phi_* h = \sum_{(\beta_2,\dots,\beta_k)\in\mathbb{N}^{k-1}} a_{(\beta_2,\dots,\beta_k)} Y_2^{\beta_2}\cdots Y_k^{\beta_k}.$$

By formula (3.6), we deduce that $\phi_* h = \sum_{\boldsymbol{\beta}\in\mathbb{N}^k} b_{\phi(\boldsymbol{\beta})}\boldsymbol{Y}^{\boldsymbol{\beta}}$, where the coefficient function $\boldsymbol{\alpha} \mapsto b_{\boldsymbol{\alpha}}$ of h is extended on all of $\mathbb{Z}^k$ by taking the zero value. By equating the coefficients, we find that for any $(\beta_2,\dots,\beta_k) \in \mathbb{N}^{k-1}$ we have

$$a_{(\beta_2,\dots,\beta_k)} = \sum_{\beta_1\in\mathbb{N}} b_{\phi(\beta_1,\beta_2,\dots,\beta_k)} Y_1^{\beta_1} \in \mathbb{Z}[[Y_1]].$$

By writing $\phi(0,\beta_2,\dots,\beta_k)$ as $\boldsymbol{\mu}$, and since $\phi(1,0,\dots,0) = \boldsymbol{\nu}$, we can rewrite this power series as $\sum_{n\in\mathbb{N}} b_{\boldsymbol{\mu}+n\boldsymbol{\nu}} Y_1^n$. In particular, the support of $a_{(\beta_2,\dots,\beta_k)}$ as an element of $\mathbb{Z}[[Y_1]]$ is the set $\{n \in \mathbb{N} \mid \boldsymbol{\mu} + n\boldsymbol{\nu} \in \operatorname{supp}(h)\}$.

For $(\beta_2,\dots,\beta_k) = (0,\dots,0)$, the constant term of $\phi_* h \in \mathbb{Z}[[Y_1]][[Y_2,\dots,Y_K]]$ is

$$\phi_* h(Y_1, 0,\dots,0) = a_{(0,\dots,0)} = \sum_{n\in\mathbb{N}} b_{n\boldsymbol{\nu}} Y_1^n.$$

This is also the image by ϕ_* of $h_{\boldsymbol{\nu}}$, defined by $\sum_{n\in\mathbb{N}} b_{n\boldsymbol{\nu}} \boldsymbol{X}^{n\boldsymbol{\nu}}$.

Proposition 3.14. *Let* $h = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} b_{\boldsymbol{\alpha}} \boldsymbol{X}^{\boldsymbol{\alpha}} \in \mathbb{Z}[[\boldsymbol{X}]]$ *be a rational power series with denominator* q^-*, and let* $\boldsymbol{\nu} \in \mathbb{N}^k$ *be the primitive vector directing an extreme ray of the* $\overline{\operatorname{cone}}(\operatorname{supp} h)$*. Then there exists a vector* $\boldsymbol{\mu} \in \mathbb{Z}^k$ *such that there are infinitely many points of* $\operatorname{supp}(h)$ *on the half-line* $\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{R}_+$ *if and only if* $q^-_{\boldsymbol{\nu}} \neq 1$.

Proof. Let K^- denote $\operatorname{cone}(\operatorname{supp} q^-)$. This convex cone is finitely generated and therefore closed. Recall that the cone $K = \overline{\operatorname{cone}}(\operatorname{supp} h)$ contains K^-.

Since $q^-(\mathbf{0}) = 1$, q^- is invertible in $\mathbb{Z}[[\boldsymbol{X}]]$, and $1/q^- = 1 + \sum_{n\geqslant 1}(1-q^-)^n$. Since the convex cone K^- contains the support of $1 - q^-$, it also contains the support of its powers, therefore it contains the support of $1/q^-$ too. Consequently,

$$\operatorname{supp}(h) \subseteq \operatorname{supp}(q^+) + \operatorname{supp}(1/q^-) \subseteq \operatorname{supp}(q^+) + K^-.$$

Let us first assume that there exists a vector $\boldsymbol{\mu} \in \mathbb{Z}^k$ such that the set $\{n \in \mathbb{N} \mid \boldsymbol{\mu} + n\boldsymbol{\nu} \in \operatorname{supp}(h)\}$ is infinite. Since $\operatorname{supp}(h) \subseteq \operatorname{supp}(q^+) + K^-$, for each of these n, there exists an element $\boldsymbol{\kappa}_n \in \operatorname{supp}(q^+)$ such that $\boldsymbol{\mu} + n\boldsymbol{\nu} - \boldsymbol{\kappa}_n \in K^-$. Therefore, we have $\boldsymbol{\nu} + \frac{1}{n}(\boldsymbol{\mu} - \boldsymbol{\kappa}_n) \in K^-$ for infinitely many n. Since the sequence $\boldsymbol{\mu} - \kappa_n$ is bounded, the sequence $\boldsymbol{\nu} + \frac{1}{n}(\boldsymbol{\mu} - \kappa_n)$ converges to $\boldsymbol{\nu}$, and since K^- is closed, we conclude that $\boldsymbol{\nu} \in K^-$.

Thus, we have the following inclusions of rational cones

$$\boldsymbol{\nu}\mathbb{R}_+ \subseteq K^- \subseteq K.$$

Since $\boldsymbol{\nu}\mathbb{R}_+$ is a face of K, there is a hyperplane H such that $\boldsymbol{\nu}\mathbb{R}_+ = H \cap K$. Thus, we have $\boldsymbol{\nu}\mathbb{R}_+ = H \cap K^-$, and we deduce that $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of K^-. In particular, $\operatorname{supp}(q^-)$, like any other set generating the convex cone K^-, contains a non-zero multiple of $\boldsymbol{\nu}$, which means that $q^-_{\boldsymbol{\nu}} \neq 1$.

Now we assume that for any $\boldsymbol{\mu} \in \mathbb{Z}^k$, the set $\{n \in \mathbb{N} \mid \boldsymbol{\mu} + n\boldsymbol{\nu} \in \operatorname{supp}(h)\}$ is finite. Using the change of indeterminates described above, this means that for any $(\beta_2,\dots,\beta_k)$

$\in \mathbb{N}^{k-1}$, the power series $a_{(\beta_2,\dots,\beta_k)} = \sum_{n\in\mathbb{N}} b_{\boldsymbol{\mu}+n\boldsymbol{\nu}} Y_1^n$ has a finite support, thus it is in $\mathbb{Z}[Y_1]$. Consequently, the power series

$$\phi_* h = \sum_{(\beta_2,\dots,\beta_k)\in\mathbb{N}^{k-1}} a_{(\beta_2,\dots,\beta_k)} Y_2^{\beta_2}\cdots Y_k^{\beta_k}$$

is an element of $\mathbb{Z}[Y_1][\![Y_2,\dots,Y_k]\!]$. It also satisfies $\phi_* q^- \cdot \phi_* h = \phi_* q^+$, where $\phi_* q^+$ and $\phi_* q^-$ are elements of $\mathbb{Z}[Y_1][Y_2,\dots,Y_k]$, that is, polynomials over the ring of coefficients $\mathbb{Z}[Y_1]$. Since $\mathbb{Z}[Y_1]$ is a UFD, we apply Proposition 3.12 and get a unique pair of coprime polynomials (P,Q) in $\mathbb{Z}[Y_1][Y_2,\dots,Y_k]$, with Q of constant term equal to 1 and such that $Q\cdot\phi_* h = P$. We deduce that $Q\cdot\phi_* q^+ = P\cdot\phi_* q^-$. By unique factorization and coprimality of P and Q, there exists $R\in\mathbb{Z}[\boldsymbol{Y}]$ such that $PR = \phi_* q^+$ and $QR = \phi_* q^-$. Since Q and $\phi_* q^-$ have their constant terms equal to 1, so does R.

Notice that $\operatorname{supp}(\phi_* q^+) = \phi^{-1}(\operatorname{supp} q^+) \subseteq \phi^{-1}(K)$. Using Corollary 3.6, we find that $\operatorname{supp}(P) \subseteq \phi^{-1}(K)$, and similarly the supports of Q and R are contained in $\phi^{-1}(K)$. Therefore, the image of P, Q and R under the inverse isomorphism ϕ_*^{-1} (which stands for $(\phi_*)^{-1}$ as well as $(\phi^{-1})_*$) are elements of $\mathbb{Z}[\boldsymbol{X};K] \subseteq \mathbb{Z}[\boldsymbol{X}]$. Since ϕ_*^{-1} is a morphism of algebras, the relations $PR = \phi_* q^+$ and $QR = \phi_* q^-$ can be written as relations $\phi_*^{-1}P\cdot\phi_*^{-1}R = q^+$ and $\phi_*^{-1}Q\cdot\phi_*^{-1}R = q^-$ in $\mathbb{Z}[\boldsymbol{X}]$. Since q^+ and q^- are coprime, we have $\phi_*^{-1}R = 1$, thus $R = 1$, hence $P = \phi_* q^+$ and $Q = \phi_* q^-$. But Q has constant term 1 as element of $\mathbb{Z}[Y_1][Y_2,\dots,Y_k]$, that is, $Q(Y_1,0,\dots,0) = 1$. We conclude that

$$\phi_* q_{\boldsymbol{\nu}}^- = \phi_* q^-(Y_1,0,\dots,0) = Q(Y_1,0,\dots,0) = 1,$$

and therefore $q_{\boldsymbol{\nu}}^- = 1$. ∎

We deduce directly a substantial improvement of the relation (3.7), which extends Ostrowski's Theorem (3.3) in some way.

Corollary 3.15. *Let $h\in\mathbb{Z}[\![\boldsymbol{X}]\!]$ be a rational power series with denominator q^- and numerator q^+. Then we have*

$$\overline{\operatorname{conv}}(\operatorname{supp} h) = \operatorname{conv}(\operatorname{supp} q^+) + \operatorname{cone}(\operatorname{supp} q^-).$$

Proof. We first notice that $\operatorname{supp}(h) \subseteq \operatorname{supp}(q^+) + \operatorname{supp}(1/q^-)$ by application of (3.2), and that $\operatorname{supp}(1/q^-) \subseteq \operatorname{cone}(\operatorname{supp} q^-)$. We deduce that

$$\overline{\operatorname{conv}}(\operatorname{supp} h) \subseteq \operatorname{conv}(\operatorname{supp} q^+) + \operatorname{cone}(\operatorname{supp} q^-)$$

where the right-hand term is closed because the supports of q^+ and q^- are finite. We denote $\overline{\operatorname{conv}}(\operatorname{supp} h)$ by Δ, and $\operatorname{cone}(\operatorname{supp} q^-)$ by K^-. It suffices to show $\operatorname{supp}(q^+) + K^- \subseteq \Delta$.

We show that $\boldsymbol{x} + K^- \subset \Delta$ for any $\boldsymbol{x}\in\Delta$, with extra assumptions at first.

We give a proof assuming that $\operatorname{supp}(q^+) \subseteq K^-$. In this case, the formula (3.7) yields

$$K = \overline{\operatorname{cone}}(\operatorname{supp} h) = \operatorname{cone}(\operatorname{supp} q^+) + \operatorname{cone}(\operatorname{supp} q^-) = K^-.$$

Let $\boldsymbol{\nu}$ be a primitive integer vector such that the ray $\boldsymbol{\nu}\mathbb{R}_+$ is extreme in K^-. In particular, we know that $q_{\boldsymbol{\nu}}^- \neq 1$ and that $\boldsymbol{\nu}\mathbb{R}_+$ is also an extreme ray in K. By Proposition 3.14, there exist $\boldsymbol{\mu}\in\mathbb{Z}^k$ and an infinite subset $S\subseteq\mathbb{N}$ such that $\boldsymbol{\mu}+n\boldsymbol{\nu}\in\operatorname{supp}(h)\subseteq\Delta$ for any $n \in S$. Thus, for any $\boldsymbol{x} \in \Delta$, any $\lambda \in \mathbb{R}_+$ and any $n \in S$ such that $n > \lambda$, we have

$$\boldsymbol{x} + \lambda\boldsymbol{\nu} + \frac{\lambda}{n}(\boldsymbol{\mu} - \boldsymbol{x}) = \left(1 - \frac{\lambda}{n}\right)\boldsymbol{x} + \frac{\lambda}{n}(\boldsymbol{\mu} + n\boldsymbol{\nu}) \in \Delta.$$

Since Δ is closed, the limit $\boldsymbol{x} + \lambda\boldsymbol{\nu}$ is also in Δ. By the convexity of K^-, we deduce that $\boldsymbol{x} + K^- \subseteq \Delta$ for any $\boldsymbol{x} \in \Delta$.

Now, we only assume that there exists a $\boldsymbol{\mu} \in \mathbb{Z}^k$ such that $\operatorname{supp}(q^+) \subseteq \boldsymbol{\mu} + \operatorname{span}(K^-)$. Therefore, any $\boldsymbol{y} \in \operatorname{supp} q^+$ can be written as a linear combination $\boldsymbol{\mu} + \sum_{\boldsymbol{x}} \lambda_{\boldsymbol{y}\boldsymbol{x}}\boldsymbol{x}$ with $\lambda_{\boldsymbol{y}\boldsymbol{x}} \in \mathbb{R}$ and where $\boldsymbol{x}$ runs through $\operatorname{supp}(q^-)$. Choosing $n_{\boldsymbol{x}} \in \mathbb{N}$ such that $n_{\boldsymbol{x}} \geqslant |\lambda_{\boldsymbol{y}\boldsymbol{x}}|$ for all $\boldsymbol{y} \in \operatorname{supp} q^+$, the vector $\boldsymbol{\mu}' = -\boldsymbol{\mu} + \sum_{\boldsymbol{x}} n_{\boldsymbol{x}}\boldsymbol{x} \in \mathbb{Z}^k$ satisfies $\boldsymbol{\mu}' + \operatorname{supp}(q^+) \subseteq K^-$. We deduce that $\boldsymbol{X}^{\boldsymbol{\mu}'} q^+$ has its support in $K^- \cap \mathbb{Z}^k \subseteq \mathbb{N}^k$ and therefore is a polynomial. It is the numerator of the rational power series $h' = \boldsymbol{X}^{\boldsymbol{\mu}'} h$. By the previous argument, for any $\boldsymbol{x}' \in \overline{\operatorname{conv}}(\operatorname{supp} h')$, we have that $\boldsymbol{x}' + K^- \subseteq \overline{\operatorname{conv}}(\operatorname{supp} h')$. Shifting back by $\boldsymbol{\mu}'$, we obtain the desired conclusion.

Finally, we treat the general case. We decompose q^+ as $q_1^+ + \cdots + q_r^+$ such that each subpolynomial q_i^+ is made of the terms of q^+ whose exponents lie in an affine subspace of direction $\operatorname{span}(K^-)$. In particular, for each i, there exists a $\boldsymbol{\mu}_i \in \mathbb{Z}^k$ such that $\operatorname{supp}(q_i^+) = \operatorname{supp}(q^+) \cap (\boldsymbol{\mu}_i + \operatorname{span} K^-)$. For each i we define h_i as the power series expansion of q_i^+/q^- such that $h = h_1 + \cdots + h_r$ and $\operatorname{supp}(h_i) = \operatorname{supp}(h) \cap (\boldsymbol{\mu}_i + \operatorname{span} K^-)$ for all i. Notice that $\operatorname{supp}(h)$ is the union of every $\operatorname{supp}(h_i)$, and that $\Delta = \operatorname{conv}(\bigcup_i \Delta_i)$ where $\Delta_i = \overline{\operatorname{conv}}(\operatorname{supp} h_i)$. Let $\boldsymbol{y} \in K^-$ and $\boldsymbol{x} \in \Delta$. The element $\boldsymbol{x}$ can be written as $\boldsymbol{x} = \lambda_1\boldsymbol{x}_1 + \cdots + \lambda_r\boldsymbol{x}_r$ with $\boldsymbol{x}_i \in \Delta_i$ and $\lambda_i \in [0,1]$ such that $\lambda_1 + \cdots + \lambda_r = 1$. Since $\operatorname{supp} q_i^+ \subseteq \boldsymbol{\mu}_i + \operatorname{span} K^-$ and $\boldsymbol{y} \in K^-$, we have $\boldsymbol{x}_i + \boldsymbol{y} \in \Delta_i$ for any i and

$$\boldsymbol{x} + \boldsymbol{y} = \sum_i \lambda_i(\boldsymbol{x}_i + \boldsymbol{y}) \in \operatorname{conv}\Big(\bigcup_i \Delta_i\Big) = \Delta.$$

This concludes our proof that $\Delta + K^- = \Delta$ for any rational power series h.

We now show that $\operatorname{supp}(q^+) \subseteq \Delta$. If $\boldsymbol{y}$ does not belong to any $\boldsymbol{y}' + K^-$ for any other $\boldsymbol{y}' \in \operatorname{supp}(q^+)$, the only decomposition of $\boldsymbol{y}$ as the sum of an element $\boldsymbol{y}'$ in $\operatorname{supp}(q^+)$ and an element in $\operatorname{supp}(1/q^-) \subseteq K^-$ is the sum $\boldsymbol{y} + \boldsymbol{0}$. The uniqueness of this decomposition proves that $\boldsymbol{y} \in \Delta$. Now, if $\boldsymbol{y} \in \boldsymbol{y}' + K^-$ for some other $\boldsymbol{y}' \in \operatorname{supp}(q^+)$, we let $\boldsymbol{y}_0$ be the element with minimal sum of coordinates among the elements $\boldsymbol{y}' \in \operatorname{supp}(q^+)$ satisfying $\boldsymbol{y} \in \boldsymbol{y}' + K^-$. If there is some other $\boldsymbol{y}_1 \in \operatorname{supp}(q^+)$ such that $\boldsymbol{y}_0 \in \boldsymbol{y}_1 + K^-$, we would have $\boldsymbol{y} \in \boldsymbol{y}_0 + K^- \subseteq \boldsymbol{y}_1 + K^-$, and since $K^- \subseteq (\mathbb{R}^+)^k$, every coordinate of $\boldsymbol{y}_1$ would be smaller than the corresponding coordinate of $\boldsymbol{y}_0$. This would contradict the minimality of $\boldsymbol{y}_0$. Therefore, $\boldsymbol{y}_0$ does not belong to any $\boldsymbol{y}' + K^-$ for any other $\boldsymbol{y}' \in \operatorname{supp}(q^+)$, and by the preceding argument, $\boldsymbol{y}_0 \in \Delta$. This implies that $\boldsymbol{y} \in \boldsymbol{y}_0 + K^- \subseteq \Delta + K^- = \Delta$.

We deduce that $\operatorname{supp}(q^+) \subseteq \Delta$ which implies that $\operatorname{supp}(q^+) + K^- \subseteq \Delta + K^- = \Delta$ and concludes our argument. ■

We are now in a position to prove the generalization of Lemma 3.9 to rational power series with more than two variables, by applying Proposition 3.14 to h°.

Corollary 3.16. *Let $h \in \mathbb{Z}[[\boldsymbol{X}]]$ be a rational power series with constant term equal to 1. Let $\boldsymbol{\nu} \in \mathbb{N}^k$ be the primitive vector directing an extreme ray of $\overline{\operatorname{cone}}(\operatorname{supp} h)$, and* *define* $h_{\boldsymbol{\nu}}$ *as* $\sum_{n\in\mathbb{N}} b(n\boldsymbol{\nu})\boldsymbol{X}^{n\boldsymbol{\nu}}$, *where* b *is the coefficient function of* h, *and* $h^\circ \in \mathbb{Z}[[\boldsymbol{X}]]$ *as* $h/h_{\boldsymbol{\nu}}$.

We have the following exclusive disjunction: either

(i) *either in* $\mathbb{Z}[\boldsymbol{X}]$, *the polynomial* $q_{\boldsymbol{\nu}}^+$ *divides* q^+, *and* $q_{\boldsymbol{\nu}}^-$ *divides* q^-, *or*
(ii) *there exists* $\boldsymbol{\mu} \in \mathbb{N}^k$ *such that the half-line* $\boldsymbol{\mu}+\boldsymbol{\nu}\mathbb{R}_+$ *contains infinitely many elements of* $\operatorname{supp}(h^\circ)$.

These two options correspond to the cases $\boldsymbol{\nu} \notin \overline{\operatorname{cone}}(\operatorname{supp} h^\circ)$ *and* $\boldsymbol{\nu} \in \overline{\operatorname{cone}}(\operatorname{supp} h^\circ)$ *respectively.*

Note that since $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of the rational cone $\overline{\operatorname{cone}}(\operatorname{supp} h)$ generated by the supports of q^+ and q^-, we have $q_{\boldsymbol{\nu}}^+ q_{\boldsymbol{\nu}}^- \neq 1$ and the condition (i) is never trivial.

Proof. By Lemma 3.3, we have $q_{\boldsymbol{\nu}}^- h_{\boldsymbol{\nu}} = q_{\boldsymbol{\nu}}^+$ which implies that the power series $h_{\boldsymbol{\nu}}$ is rational. Therefore, h° is rational and satisfies $q_{\boldsymbol{\nu}}^+ q^- h^\circ = q_{\boldsymbol{\nu}}^- q^+$.

Let $q^\oplus$ and $q^\ominus$ be the numerator and denominator of h°, respectively. By unique factorization, there exists a polynomial $r \in \mathbb{Z}[\boldsymbol{X}]$ such that $q_{\boldsymbol{\nu}}^+ q^- = rq^\ominus$ and $q_{\boldsymbol{\nu}}^- q^+ = rq^\oplus$.

By Proposition 3.14 applied to h°, either $q_{\boldsymbol{\nu}}^\ominus = 1$ or there exists $\boldsymbol{\mu} \in \mathbb{N}^k$ such that the half-line $\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{R}_+$ contains infinitely many elements of $\operatorname{supp}(h^\circ)$, which is exactly the option (ii).

Thus, it remains to prove that (i) is equivalent to $q_{\boldsymbol{\nu}}^\ominus = 1$.

Notice that q^- and $q^\ominus$ have constant terms equal to 1 for being the denominators of rational power series. Since $h(\boldsymbol{0}) = 1$ we also have $q_{\boldsymbol{\nu}}^+(\boldsymbol{0}) = q^+(\boldsymbol{0}) = 1$. Thus, r has constant term equal to 1.

Since the supports of $q_{\boldsymbol{\nu}}^+ q^-$ and of $q_{\boldsymbol{\nu}}^- q^+$ are inside the cone K, Corollary 3.6 implies that the supports of r, of $q^\ominus$ and of $q^\oplus$ are all inside K. We then apply Lemma 3.3 and deduce that $q_{\boldsymbol{\nu}}^+ q_{\boldsymbol{\nu}}^- = r_{\boldsymbol{\nu}} q_{\boldsymbol{\nu}}^\ominus = r_{\boldsymbol{\nu}} q_{\boldsymbol{\nu}}^\oplus$.

If r has an irreducible factor which does not divide $q_{\boldsymbol{\nu}}^+ q_{\boldsymbol{\nu}}^-$, then it has to divide q^+ and q^-, which are coprime polynomials. Therefore, every factor of r has support in $\boldsymbol{\nu}\mathbb{N}$, and $r = r_{\boldsymbol{\nu}}$.

If $q_{\boldsymbol{\nu}}^+$ divides q^+ and $q_{\boldsymbol{\nu}}^-$ divides q^-, then $q_{\boldsymbol{\nu}}^+ q_{\boldsymbol{\nu}}^-$ divides $rq^\oplus$ and $rq^\ominus$. Since $q^\oplus$ and $q^\ominus$ are coprime, we deduce that $q_{\boldsymbol{\nu}}^+ q_{\boldsymbol{\nu}}^-$ divides $r = r_{\boldsymbol{\nu}}$. And from the relation $q_{\boldsymbol{\nu}}^+ q_{\boldsymbol{\nu}}^- = r_{\boldsymbol{\nu}} q_{\boldsymbol{\nu}}^\ominus$ we deduce that $q_{\boldsymbol{\nu}}^\ominus = 1$.

Conversely, if $q_{\boldsymbol{\nu}}^\ominus = 1$, then $r = r_{\boldsymbol{\nu}} = q_{\boldsymbol{\nu}}^+ q_{\boldsymbol{\nu}}^-$. We deduce that $q^- = q_{\boldsymbol{\nu}}^- q^\ominus$ and $q^+ = q_{\boldsymbol{\nu}}^+ q^\oplus$, which implies that $q_{\boldsymbol{\nu}}^-$ divides q^-, and $q_{\boldsymbol{\nu}}^+$ divides q^+. ■

As already noticed, this corollary is actually sufficient to prove the characterization of Γ we are aiming for.

PROPOSITION 3.17. *Let* $h \in \mathbb{Z}[[\boldsymbol{X}]]$ *be a rational power series with constant term equal to* 1. *Let* $q^+, q^- \in \mathbb{Z}[\boldsymbol{X}]$ *be its numerator and its denominator, respectively.*

If h *has no cyclotomic factor* (*meaning that the product* q^+q^- *has no cyclotomic factor*), *then the domain* Γ *described in Theorem* 2.9, *on which the Euler product is* *meromorphic, is the interior of the dual of the rational cone spanned by the supports* B^+ *of* q^+ *and* B^- *of* q^-*, that is,*

$$\Gamma = \bigcap_{\substack{\boldsymbol{\alpha}\in B^+\cup B^- \\ \boldsymbol{\alpha}\neq 0}} \{\boldsymbol{s}\in\mathbb{C}^k \mid \Re\langle\boldsymbol{\alpha},\boldsymbol{s}\rangle > 0\}.$$

Proof. We have already noticed that $\Gamma^\vee \subseteq \overline{\mathrm{cone}}(C)$ where C is the exponential support of h. By Lemma 3.2 we know that $\overline{\mathrm{cone}}(C) = \overline{\mathrm{cone}}(B)$ where B is the support of h, and by (3.7) we see that $\overline{\mathrm{cone}}(B) = \mathrm{cone}(B^+ \cup B^-)$, which is a rational cone. It remains to prove that every extreme ray $\boldsymbol{\nu}\mathbb{R}_+$ of $\mathrm{cone}(B^+ \cup B^-)$ is in $\Gamma^\vee$.

Assume that $\boldsymbol{\nu} \notin \Gamma^\vee$. By Lemma 3.1′, $h_{\boldsymbol{\nu}}$ is a quotient of products of cyclotomic polynomials, and $\boldsymbol{\nu}$ is not in $\overline{\mathrm{cone}}(\mathrm{supp}(h^\circ))$. By Corollary 3.16, this latter condition implies that $q^+_{\boldsymbol{\nu}}$ divides q^+ and $q^-_{\boldsymbol{\nu}}$ divides q^-. Since q^+ and q^- are coprime, so are $q^+_{\boldsymbol{\nu}}$ and $q^-_{\boldsymbol{\nu}}$, thus $q^+_{\boldsymbol{\nu}}$ and $q^-_{\boldsymbol{\nu}}$ are the numerator and denominator of $h_{\boldsymbol{\nu}}$. We conclude that $q^+_{\boldsymbol{\nu}}$ and $q^-_{\boldsymbol{\nu}}$ are products of cyclotomic polynomials, not both reduced to 1, and that they divide q^+ and q^- respectively. This contradicts the fact that h has no cyclotomic factor. Therefore, we have $\boldsymbol{\nu} \in \Gamma^\vee$ for every extreme ray of $\mathrm{cone}(B^+ \cup B^-)$, which proves that $\Gamma^\vee = \mathrm{cone}(B^+ \cup B^-)$. ■

3.3. Rationality in the support and the exponential support. In the previous section we described the domain Γ in terms of $\mathrm{supp}(q^+)$ and $\mathrm{supp}(q^-)$, by proving that the intersection of $\mathrm{supp}(h^\circ)$ with some half-line $\boldsymbol{\mu}+\boldsymbol{\nu}\mathbb{R}_+$ is an infinite set. In this section, we improve the statement about this intersection, by showing that it is a *syndetic* set, which means that the differences between pairs of consecutive elements are bounded. From that, we deduce the same property for the intersection of the exponential support C of h with some half-line $\boldsymbol{\mu}+\boldsymbol{\nu}\mathbb{R}_+$.

In the one variable setting, Proposition 3.14 reduces to the obvious statement that a rational power series from $\mathbb{Z}[[X]]$ is a polynomial if and only if its denominator is equal to 1. However, in this case, we have two results describing the support of a non-polynomial rational power series $\sum_{n\in\mathbb{N}} a_n X^n$:

- Since the sequence $(a_n)_{n>\deg q^+}$ satisfies a linear recurrence relation of order $\deg q^-$, the sequence has a non-zero term among any $\deg q^-$ consecutive terms.
- By the celebrated Skolem–Mahler–Lech Theorem (†), we know that there exist two numbers $N \in \mathbb{N}$ and $d \geqslant 1$ such that the set $\{n \geqslant N \mid a_n = 0\}$ is a union of arithmetic progressions of common difference d. The same statement is also true for its complement $\{n \geqslant N \mid a_n \neq 0\}$.

We will show that both statements apply to the subsequence $(b_{\boldsymbol{\mu}+n\boldsymbol{\nu}})_{n\in\mathbb{N}}$ of the coefficients of h, for any $\boldsymbol{\mu} \in \mathbb{Z}^k$, provided that its support is not finite. Actually, the first one, which is almost trivial, would be sufficient for our needs, but we will also refer to the second, much deeper one, only because it will slightly simplify some arguments.

The reason that these statements are valid for the sequences $(b_{\boldsymbol{\mu}+n\boldsymbol{\nu}})_{n\in\mathbb{N}}$ is the rationality of the corresponding power series.

(†) For example, see [Eil74, Theorem 3.4].

Proposition 3.18. *Let* $h = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} b_{\boldsymbol{\alpha}} \boldsymbol{X}^{\boldsymbol{\alpha}} \in \mathbb{Z}[[\boldsymbol{X}]]$ *be a rational power series, and let* $\boldsymbol{\nu} \in \mathbb{N}^k$ *be the primitive vector directing an extreme ray of* $\overline{\mathrm{cone}}(\operatorname{supp} h)$. *Then for any* $\boldsymbol{\mu} \in \mathbb{Z}^k$, *the power series*

$$\sum_{n\in\mathbb{N}} b_{\boldsymbol{\mu}+n\boldsymbol{\nu}} T^n \in \mathbb{Z}[[T]]$$

is rational.

Proof. If the statement is satisfied for a particular $\boldsymbol{\mu} \in \mathbb{Z}^k$, so that we get a rational power series in $\mathbb{Z}[[T]]$ of numerator p and denominator q, then it is also true for $\boldsymbol{\mu}-\boldsymbol{\nu}$ by considering $Tp(T)/q(T)$, and for $\boldsymbol{\mu}+\boldsymbol{\nu}$ by considering p'/q with $p'(T) = (p(T) - p(0))/T \in \mathbb{Z}[T]$. Therefore, this is true for any $\boldsymbol{\mu} + m\boldsymbol{\nu}$ with $m \in \mathbb{Z}$. The image under ϕ, as described in the previous section, of the canonical basis of $\mathbb{Z}^k$ is a basis of $\mathbb{Z}^k$ whose first vector is $\boldsymbol{\nu}$. We may restrict ourselves to proving the assertion for $\boldsymbol{\mu}$ in the span of the other vectors of this basis, that is, for $\boldsymbol{\mu}$ written as $\phi(0, \beta_2, \ldots, \beta_k)$ for $(\beta_2, \ldots, \beta_k) \in \mathbb{Z}^{k-1}$. We may even restrict ourselves to $(\beta_2, \ldots, \beta_k) \in \mathbb{N}^{k-1}$: otherwise, for any $\beta_1 \in \mathbb{Z}$, we will have $(\beta_1, \beta_2, \ldots, \beta_k) \notin \mathbb{N}^k$, therefore $\phi(\beta_1, \beta_2, \ldots, \beta_k) \notin \operatorname{supp} h$ since $\operatorname{supp} h \subseteq K \cap \mathbb{Z}^k \subseteq \phi(\mathbb{N}^k)$, and finally, the power series under consideration will be equal to zero, therefore rational.

We denote by R the ring of rational power series in $\mathbb{Z}[[Y_1]]$, or equivalently, the intersection $\mathbb{Q}(Y_1) \cap \mathbb{Z}[[Y_1]]$ of subrings of $\mathbb{Q}((Y_1)) = \mathbb{Q}[[Y_1]][Y_1^{-1}]$. Note that $\mathbb{Z}[Y_1] \subset R$. The polynomials $\phi_* q^+$ and $\phi_* q^-$ may be regarded as elements of $\mathbb{Z}[Y_1][Y_2, \ldots, Y_k]$, therefore also as elements of $R[Y_2, \ldots, Y_k]$. The constant term of $\phi_* q^-$ in this ring is $\phi_* q^-(Y_1, 0, \ldots, 0)$. This constant term is a polynomial in $\mathbb{Z}[Y_1]$, and since $q^-(\mathbf{0}) = 1$, it has the constant term equal to 1. In particular, this constant term is an invertible element of R, thus $\phi_* q^-$ is an invertible element of $R[[Y_2, \ldots, Y_k]]$. Therefore, the quotient $\phi_* q^+/\phi_* q^-$, which is equal to $\phi_* h$, is an element of $R[[Y_2, \ldots, Y_k]]$.

For any $(\beta_2, \ldots, \beta_k) \in \mathbb{N}^{k-1}$, we define the coefficient function

$$a_{(\beta_2,\ldots,\beta_k)} = \sum_{\beta_1\in\mathbb{N}} b_{\phi(\beta_1,\beta_2,\ldots,\beta_k)} Y_1^{\beta_1} \in \mathbb{Z}[[Y_1]],$$

so that

$$\phi_* h = \sum_{(\beta_2,\ldots,\beta_k)\in\mathbb{N}^{k-1}} a_{(\beta_2,\ldots,\beta_k)} Y_2^{\beta_2} \cdots Y_k^{\beta_k}.$$

Since $\phi_* h$ belongs to $R[[Y_2, \ldots, Y_k]]$, we can identify its coefficients as members of $\mathbb{Z}[[Y_1]]$, and we deduce that $a_{(\beta_2,\ldots,\beta_k)}$ belongs to R for any $(\beta_2, \ldots, \beta_k) \in \mathbb{N}^{k-1}$. This means that for any $\boldsymbol{\mu}$ of the form $\phi(0, \beta_2, \ldots, \beta_k)$ with $(\beta_2, \ldots, \beta_k) \in \mathbb{N}^{k-1}$, the power series in one variable

$$a_{(\beta_2,\ldots,\beta_k)} = \sum_{\beta_1\in\mathbb{N}} b_{\phi(\beta_1,\beta_2,\ldots,\beta_k)} Y_1^{\beta_1} = \sum_{n\in\mathbb{N}} b_{\boldsymbol{\mu}+n\boldsymbol{\nu}} Y_1^n$$

is rational. ■

We can also provide another proof of the rationality of $a_{(\beta_2,\ldots,\beta_k)}$ for $(\beta_2, \ldots, \beta_k) \in \mathbb{N}^{k-1}$, by using the partial derivatives ∂_i with respect to the variables Y_i. As a matter of fact, we have

$$\frac{\partial_2^{\beta_2}}{\beta_2!} \cdots \frac{\partial_k^{\beta_k}}{\beta_k!} \phi_* h = \sum_{(\beta_2',\ldots,\beta_k')\in\mathbb{N}^{k-1}} a_{(\beta_2+\beta_2',\ldots,\beta_k+\beta_k')} Y_2^{\beta_2'} \cdots Y_k^{\beta_k'},$$

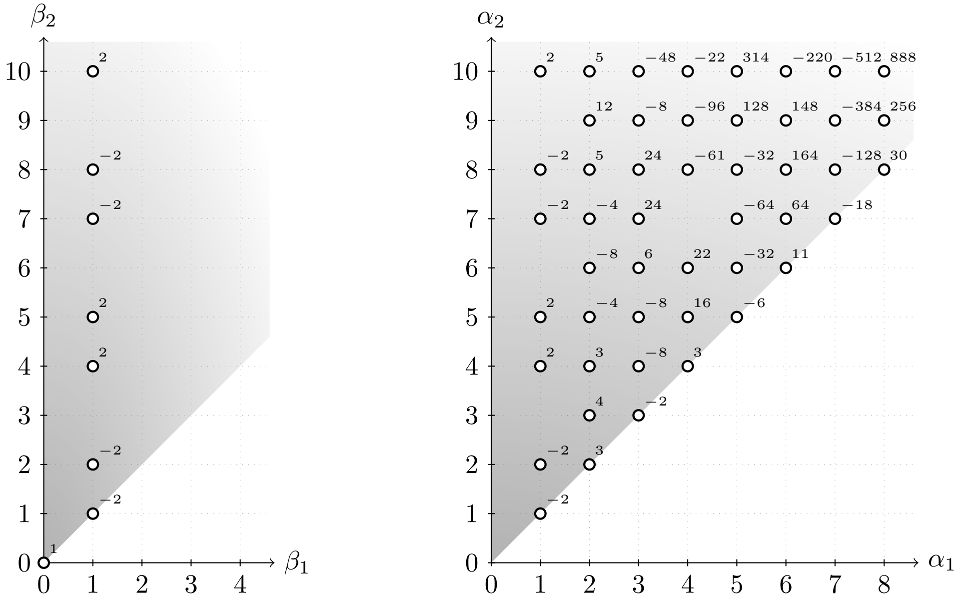

Fig. 4. Support, exponential support and corresponding coefficients for the rational function $h^\circ(X_1,X_2)=\frac{1-X_2+X_2^2+2X_1X_2}{1-X_2+X_2^2}$ associated to the polynomial $h(X_1,X_2)=1-X_2+X_2^2+2X_1X_2$ of Figure 2 and to the direction $\boldsymbol{\nu}=(0,1)$

therefore $a_{(\beta_2,\ldots,\beta_k)}=\frac{\partial_2^{\beta_2}}{\beta_2!}\cdots\frac{\partial_k^{\beta_k}}{\beta_k!}\phi_*h(Y_1,0,\ldots,0)$. Since $\phi_*h=\phi_*q^+/\phi_*q^-$, we have

$$(\phi_*q^-)^{1+\beta_2+\ldots+\beta_k}\frac{\partial_2^{\beta_2}}{\beta_2!}\cdots\frac{\partial_k^{\beta_k}}{\beta_k!}\phi_*h\in\mathbb{Z}[\boldsymbol{Y}].$$

By specializing to $Y_2=\cdots=Y_k=0$ and recalling that $\phi_*q^-(Y_1,0,\ldots,0)=\phi_*q^-_{\boldsymbol{\nu}}$, we find that $(\phi_*q^-_{\boldsymbol{\nu}})^{1+\beta_2+\cdots+\beta_k}a_{(\beta_2,\ldots,\beta_k)}$ is a polynomial in $\mathbb{Z}[Y_1]$. This proves not only that $a_{(\beta_2,\ldots,\beta_k)}$ is rational, but also that its denominator divides $(\phi_*q^-_{\boldsymbol{\nu}})^{1+\beta_2+\cdots+\beta_k}$. In particular, we set $\boldsymbol{\mu}=\phi(0,\beta_2,\ldots,\beta_k)$, and if the sequence $(b_{\boldsymbol{\mu}+n\boldsymbol{\nu}})_{n\in\mathbb{N}}$ has infinite support, it may admit only finitely many subsequences of $(1+\beta_2+\cdots+\beta_k)\deg\phi_*q^-_{\boldsymbol{\nu}}$ consecutive zero terms.

3.3.1. Rationality in the exponential support. Let h be a rational power series with constant term equal to 1, B its support, and K the closed convex cone generated by B. We have seen that K is rational and line-free.

Let $\boldsymbol{\nu}$ be a primitive integer vector such that $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of K. Let $(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k)$ be a basis of the lattice $\mathbb{Z}^k$ such that $\boldsymbol{v}_1=\boldsymbol{\nu}$ and K is contained in the cone spanned by $(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k)$.

Let M' be the monoid generated by $(\boldsymbol{v}_2,\ldots,\boldsymbol{v}_k)$; it is equipped with a canonical partial order $\preceq$ defined as $\boldsymbol{\mu}\preceq\boldsymbol{\mu}'$ if and only if $\boldsymbol{\mu}'-\boldsymbol{\mu}\in M'$, and $\prec$ its corresponding strict partial order (its irreflexive kernel). For any $(n_2,\ldots,n_k)$ and $(n'_2,\ldots,n'_k)$ in $\mathbb{N}^{k-1}$, we have $n_2\boldsymbol{v}_2+\cdots+n_k\boldsymbol{v}_k\preceq n'_2\boldsymbol{v}_2+\cdots+n'_k\boldsymbol{v}_k$ if and only if $n_i\leqslant n'_i$ for all $i\in[\![2,k]\!]$.

In particular, for any $\boldsymbol{\mu} \in M'$, there are only finitely many elements $\boldsymbol{\mu}' \in M'$ such that $\boldsymbol{\mu}' \preceq \boldsymbol{\mu}$.

Every $\boldsymbol{\alpha} \in B$ is an element of $\mathbb{Z}^k \cap K$ and can be uniquely written as $n_1\boldsymbol{v}_1 + \cdots + n_k\boldsymbol{v}_k$ with $(n_1, \ldots, n_k) \in \mathbb{N}^k$, or again as $n\boldsymbol{\nu} + \boldsymbol{\mu}$, with $n \in \mathbb{N}$ and $\boldsymbol{\mu} \in M'$.

Lemma 3.19. *Let c be the exponent function associated to h. Then the support of c is contained in $\mathbb{Z}^k \cap K$ and for all $\boldsymbol{\mu} \in M' \smallsetminus \{\mathbf{0}\}$, we have*

$$\sum_{n\in\mathbb{N}} c_{\boldsymbol{\mu}+n\boldsymbol{\nu}} T^n = \sum_{d\geqslant 1}\sum_{r\geqslant 1} \frac{\mu(d)(-1)^r}{dr} \sum_{\substack{\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_r \in M'\smallsetminus\{\mathbf{0}\}\\ d\sum \boldsymbol{\mu}_i = \boldsymbol{\mu}}} \prod_{i=1}^{r} \Big(\sum_{m\in\mathbb{N}} b^{\circ}_{\boldsymbol{\mu}_i + m\boldsymbol{\nu}} (T^d)^m \Big),$$

where b° is the coefficient function of $h^{\circ} = h/h_{\boldsymbol{\nu}}$.

Proof. As noted in Lemma 3.2, the exponential support of h generates the same monoid as its support B, therefore any $\boldsymbol{\beta}$ in the exponential support of h is in $\mathbb{Z}^k \cap K$, and it can be uniquely written as $\boldsymbol{\beta} = n\boldsymbol{\nu} + \boldsymbol{\mu}$, with $n \in \mathbb{N}$ and $\boldsymbol{\mu} \in M'$.

if $\boldsymbol{\mu} \neq \mathbf{0}$, and since $\boldsymbol{\nu}\mathbb{N} \cap M' = \{\mathbf{0}\}$, then $\boldsymbol{\beta} \notin \boldsymbol{\nu}\mathbb{R}_+$ and the exponent $c(\boldsymbol{\beta})$ is also equal to the exponent of h°, which is defined by Equation (3.1′). We denote by b° the coefficient function of the power series h°, whose support is also in $\mathbb{Z}^k \cap K$. By Lemma 2.2, we have

$$c_{\boldsymbol{\beta}} = \sum_{d\geqslant 1}\sum_{r\geqslant 1} \frac{\mu(d)(-1)^r}{dr} \sum_{\substack{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r \in \mathbb{Z}^k\cap K\smallsetminus\{\mathbf{0}\}\\ d\sum \boldsymbol{\alpha}_i = \boldsymbol{\beta}}} \prod_{i=1}^{r} b^{\circ}_{\boldsymbol{\alpha}_i}.$$

Note that we may consider only $\boldsymbol{\alpha}_i$ which are not collinear with $\boldsymbol{\nu}$, since $b^{\circ}(n\boldsymbol{\nu}) = 0$ if $n > 0$. We write $\boldsymbol{\beta}$ as $n\boldsymbol{\nu} + \boldsymbol{\mu}$ and $\boldsymbol{\alpha}_i = m_i\boldsymbol{\nu} + \boldsymbol{\mu}_i$, with $\boldsymbol{\mu}_i \in M' \smallsetminus \{\mathbf{0}\}$. The condition $d\sum_{i=1}^{r} \boldsymbol{\alpha}_i = \boldsymbol{\beta}$ is then translated into two equations: $d\sum_{i=1}^{r} \boldsymbol{\mu}_i = \boldsymbol{\mu}$ and $d\sum_{i=1}^{r} m_i = n$, so that

$$c_{\boldsymbol{\mu}+n\boldsymbol{\nu}} T^n = \sum_{d\geqslant 1}\sum_{r\geqslant 1} \frac{\mu(d)(-1)^r}{dr} \sum_{\substack{\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_r \in M'\smallsetminus\{\mathbf{0}\}\\ d\sum \boldsymbol{\mu}_i = \boldsymbol{\mu}}} \sum_{\substack{m_1,\ldots,m_r \in \mathbb{N}\\ d\sum m_i = n}} \prod_{i=1}^{r} \big(b^{\circ}_{\boldsymbol{\mu}_i + m_i\boldsymbol{\nu}} (T^d)^{m_i} \big).$$

By summing over $n \in \mathbb{N}$, we notice that

$$\sum_{n\in\mathbb{N}} \sum_{\substack{m_1,\ldots,m_r \in \mathbb{N}\\ d\sum m_i = n}} \prod_{i=1}^{r} \big(b^{\circ}_{\boldsymbol{\mu}_i + m_i\boldsymbol{\nu}} (T^d)^{m_i} \big) = \prod_{i=1}^{r} \Big(\sum_{m\in\mathbb{N}} b^{\circ}_{\boldsymbol{\mu}_i + m\boldsymbol{\nu}} (T^d)^m \Big),$$

which gives the desired formula. ■

Corollary 3.20. *Let c be the exponent function associated to h. For all $\boldsymbol{\mu} \in \mathbb{Z}^k$ which is non-collinear with $\boldsymbol{\nu}$, the power series*

$$\sum_{n\in\mathbb{N}} c_{\boldsymbol{\mu}+n\boldsymbol{\nu}} T^n \in \mathbb{Z}[[T]]$$

is rational.

Proof. Let $(m_1, \ldots, m_k) \in \mathbb{N}^k$ be the coordinates of $\boldsymbol{\mu}$ in the basis $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$ of $\mathbb{Z}^k$. As for Proposition 3.18, the validity of the statement does not change by adding a multiple of $\boldsymbol{\nu}$ to $\boldsymbol{\mu}$, so we may as well suppose that $m_1 = 0$.

Furthermore, the statement is obviously satisfied if $\boldsymbol{\mu} \notin M'$, so we may assume that $\boldsymbol{\mu} \in M'$, that is, $m_i \geqslant 0$ for all $i \in [\![2, k]\!]$.

Since $\boldsymbol{\mu} \neq \mathbf{0}$, we can use the formula of Lemma 3.19. The relation $d \sum_{i=1}^{r} \boldsymbol{\mu}_i = \boldsymbol{\mu}$ with $\boldsymbol{\mu}_i \neq \mathbf{0}$ implies that d is bounded by the lcm of the m_i, r is bounded by the sum $m_2 + \cdots + m_k$ and each $\boldsymbol{\mu}_i$ has to be chosen among at most $(m_2 + 1) \cdots (m_k + 1)$ values.

By Proposition 3.18, for any $\boldsymbol{\mu}' \in \mathbb{Z}^k$, the power series $\sum_{m \in \mathbb{N}} b^{\circ}_{\boldsymbol{\mu}' + m\boldsymbol{\nu}} (T^d)^m$ is rational. Therefore, the power series $\sum_{n \in \mathbb{N}} c_{\boldsymbol{\mu} + n\boldsymbol{\nu}} T^n$ is a finite sum of finite products of rational power series, hence is also rational. ∎

Notice that when $\boldsymbol{\mu} = \mathbf{0}$, we do not expect the power series $\sum_{n \in \mathbb{N}} c_{n\boldsymbol{\nu}} T^n$ to be rational.

COROLLARY 3.21. *Let $\boldsymbol{\mu} \in M' \setminus \{\mathbf{0}\}$. If for every $\boldsymbol{\mu}' \in M' \setminus \{\mathbf{0}\}$ with $\boldsymbol{\mu}' \prec \boldsymbol{\mu}$, the series $\sum_{n \in \mathbb{N}} b^{\circ}_{\boldsymbol{\mu}' + n\boldsymbol{\nu}} T^n$ is a polynomial, then the difference of series*

$$\sum_{n \in \mathbb{N}} c_{\boldsymbol{\mu} + n\boldsymbol{\nu}} T^n - \sum_{n \in \mathbb{N}} b^{\circ}_{\boldsymbol{\mu} + n\boldsymbol{\nu}} T^n$$

is a polynomial. In particular, there exists $N \in \mathbb{N}$ such that $c_{\boldsymbol{\mu} + n\boldsymbol{\nu}} = b^{\circ}_{\boldsymbol{\mu} + n\boldsymbol{\nu}}$ for every $n \geqslant N$.

Proof. In the formula of Lemma 3.19, the term with $r = d = 1$ is exactly $\sum_{n \in \mathbb{N}} b^{\circ}_{\boldsymbol{\mu} + n\boldsymbol{\nu}} T^n$. The condition $d \sum_{i=1}^{r} \boldsymbol{\mu}_i = \boldsymbol{\mu}$ implies that, if $d > 1$, then for any i we have

$$\boldsymbol{\mu}_i \preceq \sum_{j=1}^{r} \boldsymbol{\mu}_j \prec d \sum_{j=1}^{r} \boldsymbol{\mu}_j = \boldsymbol{\mu},$$

and if $r > 1$, then for any i we have

$$\boldsymbol{\mu}_i \prec \sum_{j=1}^{r} \boldsymbol{\mu}_j \preceq d \sum_{j=1}^{r} \boldsymbol{\mu}_j = \boldsymbol{\mu}.$$

In any case, all the series occurring in the other term are polynomials. By the same analysis as in the proof of the previous corollary, we conclude that the power series $\sum_{n \in \mathbb{N}} c_{\boldsymbol{\mu} + n\boldsymbol{\nu}} T^n - \sum_{n \in \mathbb{N}} b^{\circ}_{\boldsymbol{\mu} + n\boldsymbol{\nu}} T^n$ is a finite sum of finite products of polynomials, therefore it is a polynomial. ∎

We can finally conclude the existence of infinitely many elements in C on a half-line directed by $\boldsymbol{\nu}$, with bounded difference between consecutive terms.

PROPOSITION 3.22. *Let $h \in \mathbb{Z}[\![\boldsymbol{X}]\!]$ be a rational power series with constant term equal to 1, let q^+ and $q^- \in \mathbb{Z}[\boldsymbol{X}]$ be its numerator and denominator, respectively. Assume that h has no cyclotomic factors. Let C be the exponential support of h, and let $\boldsymbol{\nu} \in \mathbb{N}^k$ be the primitive vector directing an extreme ray of $\overline{\mathrm{cone}}(C)$.*

Then exactly one of the following two statements is satisfied: either

(i) *$\boldsymbol{\nu}\mathbb{N}$ contains infinitely many elements of C, or*

(ii) *there exists $\boldsymbol{\mu}_0 \in \mathbb{N}^k$ such that $\boldsymbol{\mu}_0 + \boldsymbol{\nu}\mathbb{N}$ contains infinitely many elements of C, with the following additional properties:*

- *There exist two positive integers n_0 and d_0 such that for every $n \geqslant n_0$, at least one of the d_0 values $\boldsymbol{\mu}_0 + (n+1)\boldsymbol{\nu}, \ldots, \boldsymbol{\mu}_0 + (n+d_0)\boldsymbol{\nu}$ belongs to C.*
- *There are only finitely many elements of C of the form $t\boldsymbol{\mu}_0 + x\boldsymbol{\nu}$, with $t \in [0, 1[$ and $x \in \mathbb{R}_+$.*

The two options correspond to whether $h_{\boldsymbol{\nu}}$ is a quotient of products of cyclotomic polynomials or not. The two technical additional properties in option (ii) will be of great use in the next section.

Proof. Assertions (i) and (ii) cannot hold together since the second property of (ii) (with $t = 0$) contradicts (i). It remains to prove that (ii) holds if (i) does not. As remarked in the proof of Lemma 3.1′, (i) is equivalent to the fact that $h_{\boldsymbol{\nu}}$ is not a quotient of products of cyclotomic polynomials. Thus, we assume that $C \cap \boldsymbol{\nu}\mathbb{R}_+$ is finite, i.e., that $h_{\boldsymbol{\nu}}$ is a quotient of products of cyclotomic polynomials.

The validity of (ii) does not change if we modify C by adding or removing finitely many elements. Since $C = (C \cap \boldsymbol{\nu}\mathbb{R}_+) \cup C^\circ$, where $C^\circ = \operatorname{xsupp}(h^\circ)$, we will prove the statement (ii) for C° instead of C. We first show that it is impossible that q_{ν}^{+} divides q^+ and $q_{\boldsymbol{\nu}}^{-}$ divides q^-, in order to use Corollary 3.16.

Assume that q_{ν}^{+} divides q^+ and $q_{\boldsymbol{\nu}}^{-}$ divides q^-. Since q^+ and q^- are coprime polynomials, q_{ν}^{+} and $q_{\boldsymbol{\nu}}^{-}$ do not share a non-trivial common factor. Thus, they are the numerator and the denominator of $h_{\boldsymbol{\nu}}$, which is a quotient of products of cyclotomic polynomials. This implies that q_{ν}^{+} and $q_{\boldsymbol{\nu}}^{-}$ are products of cyclotomic polynomials. However, q^+ and q^- do not have any cyclotomic factors, therefore $q^+ = q^- = 1$.

On the other hand, by Lemma 3.2 and formula (3.7), we have

$$\overline{\operatorname{cone}}(C) = \overline{\operatorname{cone}}(B) = \operatorname{cone}(B^+ \cup B^-).$$

Since $\boldsymbol{\nu}\mathbb{R}_+$ is an extreme ray of this cone, the generating set $B^+ \cup B^-$ contains a non-zero multiple of $\boldsymbol{\nu}$, which means that not both polynomials q_{ν}^{+} or $q_{\boldsymbol{\nu}}^{-}$ are constant, which contradicts the fact that $q^+ = q^- = 1$.

This contradiction proves that assertion (i) in Corollary 3.16 is not satisfied, so assertion (ii) holds: there exists $\boldsymbol{\mu} \in \mathbb{N}^k$ such that the half-line $\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{R}_+$ contains infinitely many elements of the support B° of h°.

We may choose $\boldsymbol{\mu}$ in $B^\circ \subseteq \overline{\operatorname{cone}}(C)$, and by removing from $\boldsymbol{\mu}$ a suitable multiple of $\boldsymbol{\nu}$, we may choose $\boldsymbol{\mu} \in M'$. In that case, we have $\boldsymbol{\mu} \neq \mathbf{0}$, because $\boldsymbol{\nu}\mathbb{N} \cap B^\circ = \{\mathbf{0}\}$. The set of $\boldsymbol{\mu}' \in M'$ with $\boldsymbol{\mu}' \preceq \boldsymbol{\mu}$ is finite, so we can find $\boldsymbol{\mu}_0 \in M' \setminus \{\mathbf{0}\}$ which is minimal for the partial order relation $\preceq$ in the set of $\boldsymbol{\mu} \in M'$ for which $(\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{R}_+) \cap B^\circ$ is not finite.

We verify the conditions of Corollary 3.21 for any $\boldsymbol{\mu} \preceq \boldsymbol{\mu}_0$: if $\boldsymbol{\mu} \prec \boldsymbol{\mu}_0$, the symmetric difference between the sets $B^\circ \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N})$ and $C \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N})$ is finite, and since the former is also finite, so is $C \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{N})$. The symmetric difference of $B^\circ \cap (\boldsymbol{\mu}_0 + \boldsymbol{\nu}\mathbb{N})$ and $C \cap (\boldsymbol{\mu}_0 + \boldsymbol{\nu}\mathbb{N})$ is also finite, but $B^\circ \cap (\boldsymbol{\mu}_0 + \boldsymbol{\nu}\mathbb{N})$ is not finite, and neither is $C \cap (\boldsymbol{\mu}_0 + \boldsymbol{\nu}\mathbb{N})$.

The series $\sum_{n \in \mathbb{N}} c_{\boldsymbol{\mu} + n\boldsymbol{\nu}} T^n$ is not a polynomial, and is rational by Corollary 3.20. By the Skolem–Mahler–Lech Theorem, we conclude that $C \cap (\boldsymbol{\mu}_0 + \boldsymbol{\nu}\mathbb{N})$ contains an infinite arithmetic progression, which proves the first part of (ii).

Finally, there are finitely many $t \in [0, 1[$ such that $t\boldsymbol{\mu}_0$ is an element of M', and for each of them, we have $t\boldsymbol{\mu}_0 \prec \boldsymbol{\mu}_0$, thus $C \cap (t\boldsymbol{\mu}_0 + \boldsymbol{\nu}\mathbb{N})$ is finite. If $t\boldsymbol{\mu}_0 \notin M'$, the previous intersection is empty. A finite union of finite sets is still finite, therefore there are finitely many elements of C of the form $t\boldsymbol{\mu}_0 + n\boldsymbol{\nu}$, with $t \in [0, 1[$ and $n \in \mathbb{N}$. If $x \notin \mathbb{Z}$ then $t\boldsymbol{\mu}_0 + x\boldsymbol{\nu} \notin \mathbb{Z}^k$, and the last part of (ii) is satisfied. ■

4. Hyperplanes in the divisor of f

In this last section, we study hyperplanes in the divisor of the Euler product f over Γ, in order to detect cluster points, and therefore singularities of f on $\partial\Gamma$.

This divisor is described in Theorem 2.9:

$$(f) = \sum_p \left[(h_p) - \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} c(\boldsymbol{\alpha}) \sum_{\kappa\in\mathbb{Z}} H_{\boldsymbol{\alpha}}\left(\frac{2\kappa\pi \mathrm{i}}{\log p}\right)\right] - \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} \sum_{\gamma\in G} m_\gamma c(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}(\gamma),$$

where the divisor of the Riemann ζ-function over $\mathbb{C}$ is $\sum_{\gamma\in G} m_\gamma\, \gamma$. Most of this expression already involves hyperplanes with rational direction. Only the terms (h_p) are not of this form. Lemma 4.1 below describes the hyperplanes that appear in those terms.

But since this point is crucial, we give some more details on divisors and hypersurfaces. We limit the exposition to holomorphic functions, but it can be extended to meromorphic functions by means of local quotients. We refer the reader to [Huy05, §2.1] for the basics on complex manifolds and affine hypersurfaces.

Let Γ be a domain of $\mathbb{C}^k$, and f be a holomorphic function defined on Γ. The (analytic) *order of vanishing* of f at $\boldsymbol{z}$ is the minimal order $m = m_1 + \cdots + m_k$ of a non-zero partial derivative of f at $\boldsymbol{z}$, that is, such that $\frac{\partial^m f}{\partial^{m_1} z_1 \cdots \partial^{m_k} z_k}(\boldsymbol{z}) \neq 0$. The work of Weierstraß gives a geometric meaning of this order, through the divisor of f as a formal sum of hypersurfaces: if $(f) = \sum_S m_S S$, then the order of f at $\boldsymbol{z}$ is $\sum_{S\ni\boldsymbol{z}} m_S$, where the sum is restricted to hypersurfaces S containing $\boldsymbol{z}$.

These (analytic) hypersurfaces are defined locally as zeros of holomorphic functions, that is, there exists an open cover $(U_i)_{i\in I}$ of Γ and a family of holomorphic functions $g_i : U_i \to \mathbb{C}$ such that $S \cap U_i$ is the set of zeros of g_i for any $i \in I$. A hypersurface is said to be *irreducible* if it cannot be written as a union of two proper hypersurfaces. It can easily be shown that an irreducible hypersurface is connected, and it admits a representation by a family $g_i : U_i \to \mathbb{C}$ where the order of vanishing of g_i is at most 1 for every $\boldsymbol{z} \in U_i$.

If f is a polynomial function, each irreducible hypersurface appearing in (f) corresponds to an irreducible factor of f that vanishes at some point on Γ. In that case, the order m_S of this hypersurface is equal to the algebraic multiplicity of the corresponding factor in f.

Our final tool is the pullback by a non-degenerate holomorphic mapping (see [Huy05, §2.3]). A mapping $\Phi : \Omega \to \Gamma$ between two domains of $\mathbb{C}^k$ is *holomorphic* if every coordinate function $\Omega \to \mathbb{C}$ is holomorphic. We define the *pullback* of f by Φ as the function $\Phi^* f = f \circ \Phi : \Omega \to \mathbb{C}$. We also define the pullback of a hypersurface S defined by the local data $(U_i, g_i)_{i\in I}$ as the hypersurface $\Phi^* S$ defined by the local data $\left(\Phi^{-1}(U_i), \Phi^* g_i\right)_{i\in I}$, so that $\boldsymbol{s} \in \Phi^* S \Leftrightarrow \Phi(\boldsymbol{s}) \in S$.

If $(f) = \sum_S m_S S$, we cannot yet assert that $(\Phi^* f) = \sum_S m_S \Phi^* S$. However, if the Jacobian matrix of Φ at $\boldsymbol{s}$ is invertible, then the order of vanishing of $\Phi^* f$ at $\boldsymbol{s}$ is equal to the order of vanishing of f at $\Phi(\boldsymbol{s})$. Therefore, $\sum_{S\ni\Phi(\boldsymbol{s})} m_S = \sum_{\Phi^* S\ni\boldsymbol{s}} m_S$ is the order of vanishing of $\Phi^* f$ at $\boldsymbol{s}$. If the Jacobian matrix of Φ at $\boldsymbol{s}$ is invertible for any $\boldsymbol{s} \in \Omega$ (Φ is then said to be non-degenerate), we do have $(\Phi^* f) = \sum_S m_S \Phi^* S$, even if $\phi^* S$ is not necessarily an irreducible hypersurface. Since the functions g_i defining S may be chosen with vanishing order at most 1 at any point, so are the functions $\Phi^* g_i$ defining $\Phi^* S$.

We deduce that if S is irreducible and Φ non-degenerate, then Φ^*S is a disjoint union of irreducible hypersurfaces, and each connected component of Φ^*S has order m_S in the divisor (Φ^*f).

LEMMA 4.1. *Let* $\boldsymbol{\nu} \in \mathbb{N}^k$ *have coprime coordinates,* $\rho \in \mathbb{C}$ *be such that* $H_{\boldsymbol{\nu}}(\rho) \cap \Gamma \neq \emptyset$ *and* p *be a prime number. There is equality between:*

(i) *the order of* $H_{\boldsymbol{\nu}}(\rho)$ *in the divisor of* h_p *over* Γ*;*

(ii) *the order of the zero set of the irreducible polynomial* $\boldsymbol{X}^{\boldsymbol{\nu}} - p^{-\rho}$ *in the divisor of* h *over* $\Omega = \{\boldsymbol{s} \in \mathbb{C}^{*k} \mid (-\log|s_1|, \dots, -\log|s_k|) \in \Gamma\}$*;*

(iii) *the order of* $H_{\boldsymbol{\nu}}\big(\rho + m\frac{2\pi \mathrm{i}}{\log p}\big)$ *in the divisor of* h_p *over* Γ *for any* $m \in \mathbb{Z}$.

Moverover, if h *is rational, those numbers are also equal to:*

(iv) *the multiplicity of* $\boldsymbol{X}^{\boldsymbol{\nu}} - p^{-\rho}$ *in the factorization of* h.

Proof. Set $P = \boldsymbol{X}^{\boldsymbol{\nu}} - p^{-\rho}$. Its Newton polytope $\Delta(P)$ is the line segment $[\boldsymbol{\nu}, \boldsymbol{0}]$. By Ostrowski's Theorem, any factor of P has a Newton polytope of the form $[\lambda\boldsymbol{\nu}, \boldsymbol{0}]$, with $\lambda \in [0,1]$. Since $\boldsymbol{\nu}$ has coprime coordinates, the fact that $\lambda\boldsymbol{\nu} \in \mathbb{N}^k$ implies that $\lambda \in \{0,1\}$, thus this factor of P is either a non-zero constant or a multiple of P. We conclude that P is an irreducible polynomial in $\mathbb{C}[X]$; its divisor (P) over Ω is an irreducible hypersurface, which we denote by S.

The holomorphic mapping $\Phi : \boldsymbol{s} \mapsto (p^{-s_1}, \dots, p^{-s_k})$ maps Γ onto Ω, its Jacobian matrix is diagonal and invertible at any point. And by definition, $h_p = \Phi^*h$. Since S is an irreducible hypersurface and Φ is non-degenerate, Φ^*S is the disjoint union of irreducible hypersurfaces. Actually,

$$\Phi^*S = \left\{\boldsymbol{s} \in \Omega \mid \langle \boldsymbol{\nu}, \boldsymbol{s}\rangle \in \rho + \frac{2\pi \mathrm{i}}{\log p}\mathbb{Z}\right\} = \bigcup_{m\in\mathbb{Z}} H_{\boldsymbol{\nu}}\left(\rho + \frac{2\pi \mathrm{i}}{\log p} m\right),$$

and the order of each hyperplane $H_{\boldsymbol{\nu}}\big(\rho + \frac{2\pi \mathrm{i}}{\log p} m\big)$ in the divisor $(\Phi^*h) = (h_p)$ is equal to the order of S in (h), proving the equality between (ii) and (iii), and thus (i).

If h is rational, the order of S in (h) is equal to the algebraic multiplicity of the polynomial P in h, proving the equality between (ii) and (iv). ■

Note that by a hyperplane we actually mean the intersection of an affine $\mathbb{C}$-hyperplane of $\mathbb{C}^k$ with Γ, so $H_{\boldsymbol{\alpha}}(\rho) = \{\boldsymbol{s} \in \Gamma \mid \langle \boldsymbol{\alpha}, \boldsymbol{s}\rangle = \rho\}$. As a matter of fact, if the parts of hyperplanes which are outside Γ create cluster points on $\partial\Gamma$, we cannot conclude that these cluster points are singularities of f (see Figure 1, left panel).

For this reason, any term corresponding to a hyperplane which does not intersect Γ should be removed from our expression of (f). As noticed in Section 2.3, no hyperplane $H_{\boldsymbol{\alpha}}(\rho)$ with $\boldsymbol{\alpha} \in \Gamma^\vee$ and $\Re e(\rho) \leqslant 0$ intersects Γ.

The situation becomes clearer when the power series h is assumed to be rational, and neither its numerator q^+ nor its denominator q^- has a cyclotomic factor. In this case, the exponential support C of h is included in $\Gamma^\vee$, and by Theorem 2.10, the expression of the divisor is simplified to

$$(f) = \sum_p (h_p) - \sum_{\boldsymbol{\alpha}\in C} \sum_{\gamma \in G'} m_\gamma c(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}(\gamma), \tag{4.1}$$

where $G' = \{1\} \cup \{\rho \in \mathbb{R}_+^* + \mathrm{i}\mathbb{R} \mid \zeta(\rho) = 0\}$. Every hyperplane $H_{\boldsymbol{\alpha}}(\gamma)$ in the rightmost sum has non-zero order and intersects the domain Γ.

By Lemma 4.1, for a prime p and a primitive $\boldsymbol{\nu} \in \mathbb{N}^k$, the hyperplanes $H_{\log p\,\boldsymbol{\nu}}(\rho)$ in (h_p) correspond to the factors $\boldsymbol{X}^{\boldsymbol{\nu}} - \mathrm{e}^{-\rho}$ of q^+ or q^-. By Ostrowski's Theorem (3.3), the support $\{\mathbf{0}, \boldsymbol{\nu}\}$ of this factor is included in $\Delta(q^+)$ (or $\Delta(q^-)$). Therefore, by Proposition 3.17, $\boldsymbol{\nu}$ is contained in $\operatorname{cone}(\operatorname{supp} q^+ \cup \operatorname{supp} q^-) = \Gamma^\vee$. This means that the hyperplane $H_{\boldsymbol{\nu}}(\rho)$ intersects Γ if and only if $\Re e(\rho) > 0$, or equivalently, $|\,\mathrm{e}^{-\rho}\,| < 1$.

From now on, we assume that the power series $h \in \mathbb{Z}[[\boldsymbol{X}]]$ is rational and that no factor of its numerator q^+ or its denominator q^- is cyclotomic. We recall that q^+ and q^- are two coprime polynomials in $\mathbb{Z}[\boldsymbol{X}]$ with $q^+(\mathbf{0}) = q^-(\mathbf{0}) = 1$, by Proposition 3.12.

We have identified every hyperplane $H_{\boldsymbol{\alpha}}(\rho)$ that appears in the expression (4.1) of (f), and we parameterize these hyperplanes by using the *parameter pair* $(\boldsymbol{\alpha}, \rho) \in \mathbb{R}^k \times \mathbb{C}$, even if it would be more canonical to consider the image of $(\boldsymbol{\alpha}, \rho)$ in $\mathbb{P}^k(\mathbb{C})$. The set of normal vectors $\boldsymbol{\alpha}$ as well as the set of affine constants ρ have a special structure which we want to take advantage of. For this reason, we will consider a set of parameter pairs rather than a set of hyperplanes. The ambiguity between different parameterizations of a given hyperplane will be taken care of by introducing the notion of *non-recurring hyperplanes* later on.

For any primitive vector $\boldsymbol{\nu} \in \mathbb{N}^k$, we introduce $\Theta_{\boldsymbol{\nu}}$ as the set of complex numbers θ inside the punctured unit disc ($0 < |\theta| < 1$) such that $\boldsymbol{X}^{\boldsymbol{\nu}} - \theta$ divides q^+ or q^-. Notice that this set is finite, and the set C' of primitive vectors $\boldsymbol{\nu} \in \mathbb{N}^k \cap \Gamma^\vee$ such that $\Theta_{\boldsymbol{\nu}}$ is non-empty is also finite. We also define $\Theta = \bigcup_{\boldsymbol{\nu} \in C'} \Theta_{\boldsymbol{\nu}}$.

The hyperplanes in the left-hand side sum in (4.1) are parameterized by the parameter pairs $(\log p\,\boldsymbol{\nu}, -\log\theta + 2\pi\mathrm{i}\kappa)$, where p is a prime, $\boldsymbol{\nu}$ is a primitive vector in C', θ is in $\Theta_{\boldsymbol{\nu}}$, and κ is in $\mathbb{Z}$. Notice that this set of parameter pairs does not depend on the determination of the complex logarithm, and that the parameter pairs are all distinct, even if they may describe the same hyperplane.

The hyperplanes in the right-hand side sum in (4.1) are parameterized by the parameter pairs $(\boldsymbol{\alpha}, \gamma) \in C \times G'$, where C is the exponential support of h and G' is the set of zeros and poles of the ζ-function in the half-plane $\{\Re e(s) > 0\}$. Notice that these parameter pairs are distinct from the previous ones, simply because the chosen normal vector has integer coefficients.

We denote by $\mathcal{P}$ the set of all these parameter pairs. The projection of $\mathcal{P}$ onto the second coordinate lies inside $G' \cup \bigcup_{\theta \in \Theta} G_\theta$, where

$$G_\theta = \{\rho \in \mathbb{C} \mid \mathrm{e}^{-\rho} = \theta\} = \{-\log\theta + 2\pi\mathrm{i}\kappa \mid \kappa \in \mathbb{Z}\}.$$

The elements of this set have bounded positive real parts (because of the finiteness and range of $-\log|\theta|$ for $\theta \in \Theta$) and this set is discrete since Θ is finite. The projection of $\mathcal{P}$ onto the first coordinate lies inside $C \cup \bigcup_p \log p\, C'$ and is also a discrete set.

Every hyperplane in the divisor (f) is parameterized by one parameter pair $(\boldsymbol{\alpha}, \rho) \in \mathcal{P}$. However the converse may be false. Two distinct parameter pairs $(\boldsymbol{\alpha}, \rho)$ and $(\boldsymbol{\alpha}', \rho') \in \mathcal{P}$ generate the same hyperplane if and only if $\rho'\boldsymbol{\alpha} = \rho\boldsymbol{\alpha}'$. In that case, the coefficients of the corresponding term in (4.1) may cancel, and it is not possible to infer that the hyperplane $H_{\boldsymbol{\alpha}}(\rho) = H_{\boldsymbol{\alpha}'}(\rho')$ has a non-zero order in (f) without knowing these coefficients.

We say that a hyperplane is *non-recurring* if it is parameterized by exactly one parameter pair $(\boldsymbol{\alpha}, \rho) \in \mathcal{P}$. Any non-recurring hyperplane has a non-zero order in (f), since no cancellation between coefficients is possible. These hyperplanes are unequivocally parameterized by the set of pairs

$$\mathcal{P}^* = \{(\boldsymbol{\alpha}, \rho) \in \mathcal{P} \mid \forall(\boldsymbol{\alpha}', \rho') \in \mathcal{P},\ \rho'\boldsymbol{\alpha} = \rho\boldsymbol{\alpha}' \Rightarrow (\boldsymbol{\alpha}, \rho) = (\boldsymbol{\alpha}', \rho')\}.$$

Actually, we will study only subsets of $\mathcal{P}^*$, for which the normal vector lies on a half-line directed by a primitive vector $\boldsymbol{\nu} \in \mathbb{N}^k$. We define

$$\mathcal{P}_{\boldsymbol{\nu}} = \{(a, g) \in \mathbb{R}_+^* \times \mathbb{C} \mid (a\boldsymbol{\nu}, g) \in \mathcal{P}\}$$

and $\mathcal{P}_{\boldsymbol{\nu}}^*$ similarly. By extension, we also call elements of $\mathcal{P}_{\boldsymbol{\nu}}$ parameter pairs. By setting $A' = \{\log p \mid p \text{ prime}\}$ and $A_{\boldsymbol{\nu}} = \{a \in \mathbb{N} \mid a\boldsymbol{\nu} \in C\}$, we can represent $\mathcal{P}_{\boldsymbol{\nu}}$ as a disjoint union

$$\mathcal{P}_{\boldsymbol{\nu}} = \Big(\bigcup_{\theta \in \Theta_{\boldsymbol{\nu}}} A' \times G_\theta\Big) \cup A_{\boldsymbol{\nu}} \times G',$$

where the left-hand side term of this union corresponds to the left-hand side term in the expression (4.1) of (f), and similarly for the right-hand-side terms.

In some cases, we will have to work with a different subset, for which the origin of the half-line directed by $\boldsymbol{\nu}$ is not necessarily the origin. For $\boldsymbol{\mu} \in \mathbb{Z}^k$, we define

$$\mathcal{P}_{\boldsymbol{\nu},\boldsymbol{\mu}} = \{(a, g) \in \mathbb{R}_+^* \times \mathbb{C} \mid (\boldsymbol{\mu} + a\boldsymbol{\nu}, g) \in \mathcal{P}\},$$

and $\mathcal{P}_{\boldsymbol{\nu},\boldsymbol{\mu}}^*$ similarly. For $\boldsymbol{\mu} = \mathbf{0}$, we retrieve the former sets of parameter pairs $\mathcal{P}_{\boldsymbol{\nu}}$ and $\mathcal{P}_{\boldsymbol{\nu}}^*$. If $\boldsymbol{\mu} \neq \mathbf{0}$ is collinear with $\boldsymbol{\nu}$, then $\boldsymbol{\mu} = d\boldsymbol{\nu}$ for some $d \in \mathbb{Z}$, thus $P_{\boldsymbol{\nu},\boldsymbol{\mu}}^*$ is the translation by $(-d, 0)$ of $P_{\boldsymbol{\nu}}^* \cap (]d, +\infty[\times\mathbb{C})$, and this case does not add anything new.

If $\boldsymbol{\mu}$ is non-collinear with $\boldsymbol{\nu}$, any parameter pair $(a, g) \in \mathcal{P}_{\boldsymbol{\nu},\boldsymbol{\mu}}$ is such that $(\boldsymbol{\mu} + a\boldsymbol{\nu}, g) \in \mathcal{P}$, thus $\boldsymbol{\mu} + a\boldsymbol{\nu}$ is written as $a'\boldsymbol{\nu}'$ for such primitive vector $\boldsymbol{\nu}' \in \mathbb{N}^k$. The relation $\boldsymbol{\mu} + a\boldsymbol{\nu} = a'\boldsymbol{\nu}'$ ensures that a and a' are rational. Therefore, $a' \notin A'$ since $\log p$ is irrational for any prime p, which means that (a', g) belongs to $A_{\boldsymbol{\nu}'} \times G' \subseteq \mathbb{N} \times G'$. In particular $a\boldsymbol{\nu} = a'\boldsymbol{\nu}' - \boldsymbol{\mu} \in \mathbb{Z}^k$, and since the coordinates of $\boldsymbol{\nu}$ are coprime, we have $a \in \mathbb{N}$ also. Therefore, $\mathcal{P}_{\boldsymbol{\nu},\boldsymbol{\mu}}$ is contained in $\mathbb{N} \times G'$ and satisfies that both of its projections onto the coordinate axes are discrete sets. Notice that $\mathcal{P}_{\boldsymbol{\nu},\boldsymbol{\mu}}^*$ may parameterize far fewer hyperplanes than the family of all non-recurring hyperplanes which admit a normal vector of the form $\boldsymbol{\mu} + a\boldsymbol{\nu}$. To compare, a similar parameterization of this superfamily would be $\{(a, a'g'/b) \mid \boldsymbol{\mu} + a\boldsymbol{\nu} = a'\boldsymbol{\nu}',\ (b, g') \in \mathcal{P}_{\boldsymbol{\nu}'}^*\}$.

In order to prove the existence of singular points for f, we want to show that the family of non-recurring hyperplanes has many cluster points. We state this property through the set of parameter pairs $\mathcal{P}_{\boldsymbol{\nu}}^*$: we say that the set $\mathcal{P}_{\boldsymbol{\nu}}^*$ *accumulates vertically* if every purely imaginary number is a cluster point of the family of g/a for $(g, a) \in \mathcal{P}_{\boldsymbol{\nu}}^*$.

More generally, any set $\mathcal{S}$ of parameters $(a, g) \in \mathbb{R}_+^* \times \mathbb{C}$ with $\Re e(g) > 0$ is said to satisfy the *vertical accumulation property* if $\mathrm{i}\mathbb{R} \subset \overline{\{g/a \mid (a, g) \in \mathcal{S}\}}$. Notice that if $\mathcal{S}$ satisfies the vertical accumulation property, then so does any superset $\mathcal{S}' \supseteq \mathcal{S}$.

Every set $\mathcal{S}$ considered is such that the two projections of $\mathcal{S}$ along one coordinate (respectively $A_{\mathcal{S}} \subset \mathbb{R}_+$ and $G_{\mathcal{S}} \subset \mathbb{C}$) are discrete sets. In this case, the vertical accumulation property requires $A_{\mathcal{S}}$ to be unbounded, and the imaginary parts of elements of $G_{\mathcal{S}}$ to be unbounded above and below. In particular, for any $M \in \mathbb{R}_+$, the set $\{(a,g) \in \mathcal{S} \mid a \leqslant M\}$ can be removed from $\mathcal{S}$ while maintaining the vertical accumulation property, because $\{g/a \mid (a,g) \in \mathcal{S}, a \leqslant M\}$ is discrete.

We now assert a general result for subsets of $\mathcal{P}^*_{\boldsymbol{\nu},\boldsymbol{\mu}}$ with the vertical accumulation property.

Proposition 4.2. *Let $\boldsymbol{\nu} \in \mathbb{N}^k$ be a primitive vector such that Γ is inside the half-space $\{\boldsymbol{s} \in \mathbb{C}^k \mid \Re\langle\boldsymbol{\nu}, \boldsymbol{s}\rangle > 0\}$, and $\boldsymbol{\mu} \in \mathbb{N}^k$. Let $\mathcal{S} \subseteq \mathcal{P}^*_{\boldsymbol{\nu},\boldsymbol{\mu}}$ be a set of parameter pairs with the vertical accumulation property, and let r and $r' \in \mathbb{R}_+$ be such that $0 \leqslant r < \Re e(g) \leqslant r'$ for every $(a,g) \in \mathcal{S}$. Then every point $\boldsymbol{s} \in \overline{\Gamma}$ such that $\Re\langle\boldsymbol{\nu}, \boldsymbol{s}\rangle = 0$ and $\Re\langle\boldsymbol{\mu}, \boldsymbol{s}\rangle \leqslant r$ is a singular point for f.*

The condition $\Re e(g) \leqslant r'$ is actually always satisfied, because the second coordinate of any element of $\mathcal{S}$ belongs to $G' \cup \bigcup_{\theta\in\Theta} G_\theta$, and its real part is in $]0, r']$ where r' is 1 or $\max_{\theta\in\Theta}(-\log|\theta|)$.

If $\boldsymbol{\mu}$ is collinear with $\boldsymbol{\nu}$ (in particular when $\boldsymbol{\mu} = \boldsymbol{0}$), the restriction $\Re\langle\boldsymbol{\mu}, \boldsymbol{s}\rangle \leqslant r$ is insignificant, even for $r = 0$. If $\boldsymbol{\mu}$ is non-collinear with $\boldsymbol{\nu}$ and $r > 0$, this restriction may be removed as well by using our result on the structure of singular points in faces of a tubular domain of meromorphy (cf. Theorem 4.14). Since its proof is independent and requires a different setting, we postpone it to the end of this section and state, without proof, the immediate corollary.

Corollary 4.3. *Let $\boldsymbol{\nu} \in \mathbb{N}^k$ be a primitive vector such that Γ is inside the half-space $\{\boldsymbol{s} \in \mathbb{C}^k \mid \Re\langle\boldsymbol{\nu}, \boldsymbol{s}\rangle > 0\}$, and $\boldsymbol{\mu} \in \mathbb{N}^k$. Let $\mathcal{S} \subseteq \mathcal{P}^*_{\boldsymbol{\nu},\boldsymbol{\mu}}$ be a set of parameter pairs with the vertical accumulation property, and let r and $r' \in \mathbb{R}_+$ be such that $0 \leqslant r < \Re e(g) \leqslant r'$ for every $(a,g) \in \mathcal{S}$. If $\boldsymbol{\mu}$ is collinear with $\boldsymbol{\nu}$ or if $r > 0$, then every point $\boldsymbol{s} \in \overline{\Gamma}$ such that $\Re\langle\boldsymbol{\nu}, \boldsymbol{s}\rangle = 0$ is a singular point for f.*

Proof of Proposition 4.2 and its Corollary. Since the normal vector of every hyperplane under consideration has integer (therefore real) coefficients, we can write any hyperplane $H_{\boldsymbol{\alpha}}(\rho)$ as $H^R_{\boldsymbol{\alpha}}(\rho) \oplus \mathrm{i}H^I_{\boldsymbol{\alpha}}(\rho)$, where

$$H^R_{\boldsymbol{\alpha}}(\rho) = \{\boldsymbol{\sigma} \in \Gamma \cap \mathbb{R}^k \mid \langle\boldsymbol{\alpha}, \boldsymbol{\sigma}\rangle = \Re e(\rho)\} \quad \text{and} \quad H^I_{\boldsymbol{\alpha}}(\rho) = \{\boldsymbol{\tau} \in \mathbb{R}^k \mid \langle\boldsymbol{\alpha}, \boldsymbol{\tau}\rangle = \Im m(\rho)\}.$$

Let $\boldsymbol{s} \in \overline{\Gamma}$ be such that $\Re\langle\boldsymbol{\nu}, \boldsymbol{s}\rangle = 0$ and $\Re\langle\boldsymbol{\mu}, \boldsymbol{s}\rangle \leqslant r$. Notice that $\boldsymbol{s} \notin \Gamma$.

Let us start with the real parts. We set $\boldsymbol{\sigma} = (\Re e(s_1), \ldots, \Re e(s_k))$ and $U = \Gamma \cap \mathbb{R}^k$ so that $\Gamma = U + \mathrm{i}\mathbb{R}^k$. We shall show that for any (convex) neighborhood V_R of $\boldsymbol{\sigma}$, there is a positive number M_1 such that for any $(a,g) \in \mathcal{S}$ with $a \geqslant M_1$, the intersection $U \cap V_R \cap H^R_{\boldsymbol{\mu}+a\boldsymbol{\nu}}(\rho)$ is non-empty (*).

Let V_R a convex neighborhood of $\boldsymbol{\sigma}$ in $\mathbb{R}^k$. Since $\boldsymbol{\sigma} \in \overline{U} = \overline{\Gamma} \cap \mathbb{R}^k$, the intersection $U \cap V_R$ is non-empty and we fix an element $\boldsymbol{\sigma}'$ in this set. Since $\boldsymbol{\sigma}' \in U \subseteq \Gamma$, we have $\langle\boldsymbol{\nu}, \boldsymbol{\sigma}'\rangle > 0$, hence there exists $M_1 > 0$ with $\langle\boldsymbol{\mu} + M_1\boldsymbol{\nu}, \boldsymbol{\sigma}'\rangle > r'$. By our assumptions on $\boldsymbol{s}$, we have $\langle\boldsymbol{\mu} + a\boldsymbol{\nu}, \boldsymbol{\sigma}\rangle = \langle\boldsymbol{\mu}, \boldsymbol{\sigma}\rangle \leqslant r$ for any $a \in \mathbb{R}$.

Therefore, for any $(a,g) \in \mathcal{S}$ with $a \geqslant M_1$, we have

$$\langle\boldsymbol{\mu} + a\boldsymbol{\nu}, \boldsymbol{\sigma}\rangle \leqslant r < \Re e(g) \leqslant r' < \langle\boldsymbol{\mu} + a\boldsymbol{\nu}, \boldsymbol{\sigma}'\rangle,$$

(*) The redundant intersection $U \cap H^R_{\boldsymbol{\mu}+a\boldsymbol{\nu}}(\rho)$ is to emphasize that intersection of the neighborhood and the hyperplane has to be over U.

which implies that there exists $\lambda \in]0,1[$ such that $\langle \boldsymbol{\mu}+a\boldsymbol{\nu}, \lambda\boldsymbol{\sigma}'+(1-\lambda)\boldsymbol{\sigma}\rangle = \Re e(g)$. By the convexity of U and V_R, the element $\lambda\boldsymbol{\sigma}'+(1-\lambda)\boldsymbol{\sigma}$ is in the intersection $U \cap V_R \cap H^R_{\boldsymbol{\mu}+a\boldsymbol{\nu}}(\rho)$.

We now consider the imaginary parts. We set $\boldsymbol{\tau} = (\Im m(s_1), \dots, \Im m(s_k))$. We shall show that any neighborhood of $\boldsymbol{\tau}$ in $\mathbb{R}^k$ intersects the hyperplane $H^I_{\boldsymbol{\mu}+a\boldsymbol{\nu}}(g)$ for infinitely many $(a,g) \in \mathcal{S}$.

Let $\varepsilon > 0$. Choose $M_2 \geqslant M_1$ such that $2|\langle \boldsymbol{\mu}, \boldsymbol{\tau}\rangle| < \varepsilon M_2$ and $2\|\boldsymbol{\mu}\| < M_2\|\boldsymbol{\nu}\|$. The set $\mathcal{S}' = \{(a,g) \in \mathcal{S} \mid a \geqslant M_2\}$ also has the vertical accumulation property, and there are infinitely many $(a,g) \in \mathcal{S}'$ with $\Im m(g)/a$ in the interval $[\langle \boldsymbol{\nu}, \boldsymbol{\tau}\rangle - \varepsilon/2, \langle \boldsymbol{\nu}, \boldsymbol{\tau}\rangle + \varepsilon/2]$. For any one of those couples (a,g), we have

$$\frac{|\langle \boldsymbol{\mu}+a\boldsymbol{\nu}, \boldsymbol{\tau}\rangle - \Im m(g)|}{a} = \left|\langle \boldsymbol{\nu}, \boldsymbol{\tau}\rangle - \frac{\Im m(g)}{a} + \frac{\langle \boldsymbol{\mu}, \boldsymbol{\tau}\rangle}{a}\right| \leqslant \left|\langle \boldsymbol{\nu}, \boldsymbol{\tau}\rangle - \frac{\Im m(g)}{a}\right| + \frac{|\langle \boldsymbol{\mu}, \boldsymbol{\tau}\rangle|}{M_2} \leqslant \varepsilon.$$

Therefore, for any of those (a,g), and $\boldsymbol{\eta} \in \mathbb{R}^k$ defined by

$$\boldsymbol{\eta} = (\Im m(g) - \langle \boldsymbol{\mu}+a\boldsymbol{\nu}, \boldsymbol{\tau}\rangle)\frac{\boldsymbol{\mu}+a\boldsymbol{\nu}}{\|\boldsymbol{\mu}+a\boldsymbol{\nu}\|^2}$$

we obtain $\langle \boldsymbol{\mu}+a\boldsymbol{\nu}, \boldsymbol{\tau}+\boldsymbol{\eta}\rangle = \Im m(g)$ and $\|\boldsymbol{\eta}\| \leqslant \big|\langle \boldsymbol{\mu}+a\boldsymbol{\nu}, \boldsymbol{\tau}\rangle - \Im m(g)\big|/\|a\boldsymbol{\nu}+\boldsymbol{\mu}\| \leqslant 2\varepsilon/\|\boldsymbol{\nu}\|$ since $\left\|\boldsymbol{\nu} + \frac{1}{a}\boldsymbol{\mu}\right\| \geqslant \frac{1}{2}\|\boldsymbol{\nu}\|$.

Finally, for any neighborhood of the form $V_R + \mathrm{i}V_I$ of $\boldsymbol{s}$, there are infinitely many $(a,g) \in \mathcal{S}'$ such that $V_I \cap H^I_{\boldsymbol{\mu}+a\boldsymbol{\nu}}(g)$ is non-empty, and for all of them $V_R \cap H^R_{\boldsymbol{\mu}+a\boldsymbol{\nu}}(g)$ is non-empty either. Therefore, the neighborhood intersects infinitely many non-recurring hyperplanes, that is, infinitely many primitive hypersurfaces of the divisor of f, and since this is true for any neighborhood, $\boldsymbol{s}$ is singular for f.

Now, assume that $r > 0$, and that $\boldsymbol{\mu}$ is non-collinear with $\boldsymbol{\nu}$. Let $\mathscr{F}_{\boldsymbol{\nu}}$ be the intersection of $\overline{\Gamma}$ and the hyperplane defined by $\Re\langle \boldsymbol{\nu}, \boldsymbol{s}\rangle = 0$. When possible, consider an element $\boldsymbol{s}$ in the relative interior of $\mathscr{F}_{\boldsymbol{\nu}}$ that satisfies $\Re\langle \boldsymbol{\nu}, \boldsymbol{s}\rangle = 0$ and $\Re\langle \boldsymbol{\mu}, \boldsymbol{s}\rangle > r > 0$. Write $\boldsymbol{s}$ as $\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$. Since Γ is convex and tubular, we know that for any $\lambda \in]0,1]$ the element $\boldsymbol{s}_\lambda$ defined by $\lambda\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$ is also in the relative interior of $\mathscr{F}_{\boldsymbol{\nu}}$ by [Roc97, Theorem 6.1]. We have $\Re\langle \boldsymbol{\mu}, \boldsymbol{s}_\lambda\rangle = \lambda\Re\langle \boldsymbol{\mu}, \boldsymbol{s}\rangle$, and we can choose $\lambda \in]0,1]$ such that $\Re\langle \boldsymbol{\mu}, \boldsymbol{s}_\lambda\rangle \leqslant r$. We have just proven that $\boldsymbol{s}_\lambda$ is a singular point of f. By Theorem 4.14 (with the "meromorphy" wording), the point $\boldsymbol{s}$ is also singular for f. And since the set of singular points of f is closed, any point in $\mathscr{F}_{\boldsymbol{\nu}}$ is a singular point of f. ∎

A direct application of this proposition yields the following maximality result.

THEOREM 4.4. *Let $h \in \mathbb{Z}[[\boldsymbol{X}]]$ be a rational power series with $h(\mathbf{0}) = 1$. The domain Γ is the maximal domain of meromorphy of f, in the sense that f is meromorphic on Γ and any point on the boundary of Γ is a singular point of f.*

Let us recall that h is the quotient q^+/q^- of two coprime elements of $\mathbb{Z}[\boldsymbol{X}]$ with $q^+(\mathbf{0}) = q^-(\mathbf{0}) = 1$. We assume without loss of generality that neither q^+ nor q^- admits any cyclotomic factors. The domain Γ is described as

$$\Gamma = \bigcap_{\substack{\boldsymbol{\beta} \in B^+ \cup B^- \\ \boldsymbol{\beta} \neq \mathbf{0}}} \{\boldsymbol{s} \in \mathbb{C}^k \mid \Re\langle \boldsymbol{\beta}, \boldsymbol{s}\rangle > 0\},$$

where B^+ and B^- are the supports of q^+ and q^-, respectively. Every supporting hyperplane of $\overline{\Gamma}$ is defined by $\Re\langle \boldsymbol{\nu}, \boldsymbol{s}\rangle = 0$, where $\boldsymbol{\nu}$ is a primitive vector of $\mathbb{N}^k$ directing an extreme ray of the closed convex cone generated by $B^+ \cup B^-$.

The theorem derives from Corollary 4.3 if for any such $\boldsymbol{\nu}$ we are able to find a vector $\boldsymbol{\mu} \in \mathbb{N}^k$ and a subset $\mathcal{S} \subseteq \mathcal{P}^*_{\boldsymbol{\nu},\boldsymbol{\mu}}$ that satisfies the condition of the corollary. The construction of such a subset $\mathcal{S}$ of parameter pairs changes greatly according to the arithmetic properties of $q^+_{\boldsymbol{\nu}}$ and $q^-_{\boldsymbol{\nu}}$ and each case will be treated in a different subsection.

In Section 4.1, we will assume that q^+ and $q^+_{\boldsymbol{\nu}}$ (or q^- and $q^-_{\boldsymbol{\nu}}$) have a non-trivial common factor. This common factor will help to describe the part of $(f)_{\boldsymbol{\nu}}$ stemming from $(h_p)_{\boldsymbol{\nu}}$. Using the argument of Estermann, we will show in Lemma 4.6 that some subset $\mathcal{S} \subseteq \mathcal{P}^*_{\boldsymbol{\nu}}$ satisfies the conditions of Corollary 4.3 (with $\boldsymbol{\mu} = \boldsymbol{0}$ and $r > 0$).

In Section 4.2, we assume that q^+ and $q^+_{\boldsymbol{\nu}}$ (as well as q^- and $q^-_{\boldsymbol{\nu}}$) are coprime, and that $h_{\boldsymbol{\nu}} = q^+_{\boldsymbol{\nu}}/q^-_{\boldsymbol{\nu}}$ cannot be written as a quotient of products of cyclotomic polynomials. Using the argument of Dahlquist, we will show in Lemma 4.7 that some subset $\mathcal{S} \subseteq \mathcal{P}^*_{\boldsymbol{\nu}}$ satisfies the conditions of Corollary 4.3 (with $\boldsymbol{\mu} = \boldsymbol{0}$ and $r = 0$).

In Section 4.3, we assume that q^+ and $q^+_{\boldsymbol{\nu}}$ are coprime, so are q^- and $q^-_{\boldsymbol{\nu}}$, and that $h_{\boldsymbol{\nu}} = q^+_{\boldsymbol{\nu}}/q^-_{\boldsymbol{\nu}}$ can be written as a quotient of products of cyclotomic polynomials. Using Proposition 3.22, we will show in Lemma 4.7 that some subset $\mathcal{S} \subset \mathcal{P}^*_{\boldsymbol{\nu},\boldsymbol{\mu}}$ with $\boldsymbol{\mu} \in \mathbb{N}^k$ satisfies the conditions of Corollary 4.3 (with $\boldsymbol{\mu}$ non-collinear with $\boldsymbol{\nu}$ and $r = 1/2$).

The Section 4.4 is dedicated to the proof of Theorem 4.14, which describes the structure of singular points in the boundary of a tubular domain of holomorphy.

4.1. Estermann's argument. In the original case [Est28, Theorem 4] of a univariate polynomial h, Estermann's argument consists in estimating the rate of accumulation of zeros/poles of f associated to each root of h, as well as the rate of accumulation of zeros associated to the zeta factors. A direct comparison of their rates shows that these zeros/poles of f cannot all cancel each other as long as h has a root inside the unit circle.

In our setting, we have seen that $\mathcal{P}_{\boldsymbol{\nu}}$ is the disjoint union of several families of parameter pairs $A_{\boldsymbol{\nu}} \times G' \cup \bigcup_{\theta \in \Theta_{\boldsymbol{\nu}}} A' \times G_\theta$. The rate of accumulation associated to each family is obtained by computing the size of the intersection of the family with

$$B_{M,I} = \{(a, g) \in \mathbb{R}^*_+ \times \mathbb{C} \mid g/a \in [1/M, +\infty[+ \mathrm{i}I\},$$

where M is a large real number and I is an interval of $\mathbb{R}$ of length $|I|$. But this argument will work as long as $\Theta_{\boldsymbol{\nu}}$ is finite and non-empty. The following lemma translates this condition in terms of factors of q^+ and q^-.

Lemma 4.5. *For both choices of the sign ±, we define $s^\pm \in \mathbb{Z}[T]$, where $T = \boldsymbol{X}^{\boldsymbol{\nu}}$, as the greatest common divisor of $q^\pm$ and $q^\pm_{\boldsymbol{\nu}}$. Then the elements of the set $\Theta_{\boldsymbol{\nu}}$ are the roots $\theta \in \mathbb{C}$ of $s^+(T)$ and those of $s^-(T)$ which lie inside the unit circle.*

In particular, $\Theta_{\boldsymbol{\nu}}$ is always finite, and it is empty if and only if both s^+ and s^- are eual to the constant polynomial 1.

Note that when $k \geqslant 2$, it is not a generic property for a polynomial to have a factor of the form $\boldsymbol{X}^{\boldsymbol{\nu}} - \theta$ for some fixed $\boldsymbol{\nu} \in \mathbb{N}^k$. Dahlquist's argument is much more needed in the multivariate setting.

Proof. Let $\theta \in \Theta_{\boldsymbol{\nu}}$. Therefore, $\boldsymbol{X}^{\boldsymbol{\nu}} - \theta$ divides either q^+ or q^-. Without loss of generality, we can assume that $\boldsymbol{X}^{\boldsymbol{\nu}} - \theta$ divides q^+. By Lemma 3.3, it also divides $q^+_{\boldsymbol{\nu}}$. We deduce that $T - \theta$ divides s^+. Conversely, if θ is a root of s^+ with $|\theta| < 1$, then $\boldsymbol{X}^{\boldsymbol{\nu}} - \theta$ divides s^+, which is a divisor of q^+. Therefore, $\theta \in \Theta_{\boldsymbol{\nu}}$.

If $s^+ = 1$ and $s^- = 1$, then $\Theta_{\boldsymbol{\nu}}$ is empty. Otherwise, s^+s^- is a non-constant polynomial of $\mathbb{Z}[T]$ with $s^+(0)s^-(0) = 1$, and without cyclotomic factors (since s^+s^- divides q^+q^-). Therefore, s^+s^- has at least a root θ with $|\theta| < 1$ and $\Theta_{\boldsymbol{\nu}}$ is non-empty. The set $\Theta_{\boldsymbol{\nu}}$ is always finite as the zero set of the polynomial s^+s^-. ∎

For $\theta \in \Theta_{\boldsymbol{\nu}}$, the family $A' \times G_\theta$ consists of all pairs $(\log p, -\log\theta + 2\pi i\kappa)$ where p is a prime number and κ is an integer.

FACT 1. *No parameter pair of $A' \times G_\theta$ can cancel another one.*

That is, for two pairs (a, g) and (a', g') in $A' \times G_\theta$, if $g/a = g'/a'$ then $(a, g) = (a', g')$. In fact, since all g have the same real part, the equality $g/a = g'/a'$ implies that $a = a'$ and therefore $g = g'$.

The same argument works as long as the real parts of the second coordinates are all the same.

FACT 1′. *Let $\theta \neq \theta'$ with $|\theta| = |\theta'|$. No parameter pair of $A' \times G_\theta$ can cancel any parameter pair of $A' \times G_{\theta'}$.*

FACT 2. *Let $\theta \neq 0$ with $|\theta| < 1$. We have* $\operatorname{card}((A' \times G_\theta) \cap B_{M,I}) \sim \frac{|I|}{2\pi}|\theta|^{-M}$ *as $M \to +\infty$.*

The condition $\Re e(g/a) \geqslant 1/M$ translates to $p \leqslant |\theta|^{-M}$. And for each p, there are $\frac{|I|}{2\pi}\log p + O(1)$ possible choices for the imaginary part of g. The prime number theorem states that $\sum_{p \leqslant x} \log p \sim x$ as $x \to +\infty$.

FACT 3. *We have* $\operatorname{card}((A_{\boldsymbol{\nu}} \times G') \cap B_{M,I}) \ll M^2 \log M$ *as $M \to +\infty$.*

Since no zero or pole of the Riemann ζ-function has real part greater than 1, the condition $\Re e(g/a) \geqslant 1/M$ implies that $a \leqslant M$. For such a, the Riemann–von Mangoldt formula indicates that there are at most $\frac{|I|}{2\pi}a\big(\log a + O(1)\big)$ possible choices for g as zero of ζ with an imaginary part in aI.

These simple facts are sufficient to carry out the argument of Estermann in the multivariate setting.

LEMMA 4.6. *If q^+ and $q^+_{\boldsymbol{\nu}}$ are not coprime, or if q^- and $q^-_{\boldsymbol{\nu}}$ are not coprime, then there exist $\theta \in \Theta_{\boldsymbol{\nu}}$ and $\mathcal{S} \subset A' \times G_\theta$ such that $\mathcal{S} \subseteq \mathcal{P}^*_{\boldsymbol{\nu}}$, $\mathcal{S}$ satisfies the vertical accumulation property, and for any $(a, g) \in \mathcal{S}$, $\Re e(g) = -\log|\theta| \in \mathbb{R}^*_+$.*

Proof. By Lemma 4.5, the set $\Theta_{\boldsymbol{\nu}}$ is finite and non-empty. Since $q^+(\mathbf{0}) = q^-(\mathbf{0}) = 1$, this set does not contain $\mathbf{0}$, and by its finiteness, we choose θ in $\Theta_{\boldsymbol{\nu}}$ with minimal modulus.

The third part of the assertion derives directly from $G_\theta = \{-\log\theta + 2\pi i\kappa \mid \kappa \in \mathbb{Z}\}$.

We define $\mathcal{S} \subseteq A' \times G_\theta \subseteq \mathcal{P}_{\boldsymbol{\nu}}$ by

$$\mathcal{S} = A' \times G_\theta \setminus \Big(A_{\boldsymbol{\nu}} \times G' \cup \bigcup_{\substack{\theta' \in \Theta_{\boldsymbol{\nu}} \\ |\theta'| > |\theta|}} A' \times G_{\theta'}\Big).$$

No element of this set cancels with other parameter pairs of $A'\times G_\theta$ by Fact 1, with other parameter pairs of $A'\times G_{\theta'}$ with $|\theta'|=|\theta|$ by Fact $1'$, with other parameter pairs of $A'\times G_{\theta'}$ with $|\theta'|\neq|\theta|$ or of $A_{\boldsymbol{\nu}}\times G'$ by construction. Therefore, it is a subset of $\mathcal{P}^*_{\boldsymbol{\nu}}$.

It remains to prove the second part of the assertion: any point of the imaginary line $\mathrm{i}\mathbb{R}$ is a cluster point of the family $\{g/a \mid (a,g)\in\mathcal{S}\}$. We first show that for any bounded interval I, $\operatorname{card}(\mathcal{S}\cap B_{M,I})\to+\infty$ as $M\to+\infty$.

A crude estimate shows that $\operatorname{card}(\mathcal{S}\cap B_{M,I})$ is larger than

$$\operatorname{card}((A'\times G_\theta)\cap B_{M,I})-\operatorname{card}((A_{\boldsymbol{\nu}}\times G')\cap B_{M,I})-\sum_{\substack{\theta'\in\Theta_{\boldsymbol{\nu}}\\ |\theta'|>|\theta|}}\operatorname{card}((A'\times G_{\theta'})\cap B_{M,I}).$$

The first term is $\sim\frac{|I|}{2\pi}|\theta|^{-M}$ as $M\to+\infty$ by Fact 2. The second and third terms are negligible by Facts 2 and 3, and by the finiteness of $\Theta_{\boldsymbol{\nu}}$. Altogether, we have proven that $\operatorname{card}(\mathcal{S}\cap B_{M,I})\sim\frac{|I|}{2\pi}|\theta|^{-M}$ as $M\to+\infty$.

Therefore, any interval $\mathrm{i}I$ of $\mathrm{i}\mathbb{R}$ contains at least one cluster point of $\{g/a \mid (a,g)\in\mathcal{S}\}$. This means that the set of cluster points is dense in $\mathrm{i}\mathbb{R}$, and since it is a closed set, we obtain the vertical accumulation property for $\mathcal{S}$. ∎

By Corollary 4.3, any point of the facet $\mathcal{F}_{\boldsymbol{\nu}}$ of $\overline{\Gamma}$ is a singular point of f, at least if q^+ and $q^+_{\boldsymbol{\nu}}$ are not coprime, or if q^- and $q^-_{\boldsymbol{\nu}}$ are not coprime.

4.2. Dahlquist's argument. Estermann's argument is efficient as long as $\Theta_{\boldsymbol{\nu}}$ is not empty. For a rational function h of one variable, this emptiness is equivalent to the fact that h is a quotient of products of cyclotomic polynomials. But in several variables, this is no longer the case: it is frequent that $\Theta_{\boldsymbol{\nu}}$ is empty but $h_{\boldsymbol{\nu}}$ is not a quotient of products of cyclotomic polynomials, which is exactly when Dahlquist's argument is useful.

Let us assume in this subsection that q^+ and $q^+_{\boldsymbol{\nu}}$ (as well as q^- and $q^-_{\boldsymbol{\nu}}$) are coprime, and that $h_{\boldsymbol{\nu}}$ is not a quotient of products of cyclotomic polynomials. We deduce that $\mathcal{P}_{\boldsymbol{\nu}}=A_{\boldsymbol{\nu}}\times G'$ by Lemma 4.5, and that $A_{\boldsymbol{\nu}}$ is an infinite subset of $\mathbb{N}$ by Lemma $3.1'$. In that case, the vertical accumulation property is easily satisfied for $\mathcal{P}_{\boldsymbol{\nu}}$, but $\mathcal{P}^*_{\boldsymbol{\nu}}$ may differ from $\mathcal{P}_{\boldsymbol{\nu}}$. If the elements of $G'\smallsetminus\{1\}$ have the same real part (which is the Riemann Hypothesis), then $\mathcal{P}^*_{\boldsymbol{\nu}}=\mathcal{P}_{\boldsymbol{\nu}}$. The linear independence (over $\mathbb{Q}$) of the positive ordinates of the non-trivial zeros of the Riemann ζ-function would also provide this equality.

Without this knowledge about the zeros of the Riemann ζ-function, Dahlquist constructs a subset $A^*\subseteq A_{\boldsymbol{\nu}}$ and for each $a\in A^*$ a subset $G_a\subseteq G'$ such that:

(i) for any $a\in A^*$ and any $g\in G_a$, the parameter pair (a,g) is in $\mathcal{P}^*_{\boldsymbol{\nu}}$;
(ii) for any $\varepsilon>0$, and for any T larger than some T_0 depending on ε, each set G_a with $a\in A^*$ has at least one element in the interval $[(1-\varepsilon)T,T]$ (and one in the interval $[-T,-T(1-\varepsilon)]$);
(iii) the set A^* is infinite.

We revisit Dahlquist's argument in more modern, geometric terms.

Lemma 4.7. *If q^+ and $q^+_{\boldsymbol{\nu}}$ (as well as q^- and $q^-_{\boldsymbol{\nu}}$) are coprime, and $h_{\boldsymbol{\nu}}$ is not a quotient of products of cyclotomic polynomials, then there exists $\mathcal{S}\subset A_{\boldsymbol{\nu}}\times G'$ such that* (a) *$\mathcal{S}\subseteq\mathcal{P}^*_{\boldsymbol{\nu}}$,* (b) *$\mathcal{S}$ satisfies the vertical accumulation property, and* (c) *for any $(a,g)\in\mathcal{S}$, $\Re e(g)\in\,]0,1]$.*

Proof. Assertion (c) comes directly from the fact that non-trivial zeros of ζ are located in the critical strip. To prove assertions (a) and (b), it suffices to prove that $\mathcal{S} \subset \mathbb{N} \times G'$ satisfies the three properties (i)–(iii) above. Indeed, property (i) asserts that $\mathcal{S} \subseteq \mathcal{P}^*_{\boldsymbol{\nu}}$, which is assertion (a). We now show that such $\mathcal{S}$ has the vertical accumulation property.

Let $x \in \mathbb{R}^*$ and $\varepsilon > 0$ and consider the intersection of $\{g/a \mid a \in A^*,\, g \in G_a\}$ and $\{s \in \mathbb{C} \mid \Im m(s) \in [(1-\varepsilon)x, x]\}$ (or $[x,(1-\varepsilon)x]$ if $x < 0$). By property (i), all the ratios g/a are distinct, and this intersection has the same cardinality as $\bigsqcup_{a\in A^*}\{g \in G_a \mid \Im m(g) \in [(1-\varepsilon)ax, ax]\}$. Since $A^* \subseteq \mathbb{N}$ is infinite by property (iii), there are infinitely many $a \in A^*$ with $a|x| > T_0$ and for each one of them the set $\{g \in G_a \mid \Im m(g) \in [(1-\varepsilon)ax, ax]\}$ is non-empty, by property (ii). Therefore,

$$\{g/a \mid a \in A^*,\, g \in G_a\} \cap \{s \in \mathbb{C} \mid \Im m(s) \in [(1-\varepsilon)x, x]\}$$

is an infinite, bounded set, thus it has at least one cluster point, which can only be on the imaginary line with imaginary part in $[(1-\varepsilon)x, x]$. Since the result is true for every $x \in \mathbb{R}^*$ and any $\varepsilon > 0$, we conclude that each point of the imaginary line $\mathrm{i}\mathbb{R}$ is a cluster point of $\{g/a \mid a \in A^*,\, g \in G_a\}$. Therefore, $\mathcal{S}$ satisfies the vertical accumulation property.

It remains to prove that such a set $\mathcal{S}$ of parameters can be constructed. The details are presented in the lemmata below.

We define

$$A^* = \Big\{a \in A_{\boldsymbol{\nu}} \mid \forall r \geqslant 2,\, \forall (a_1,\dots,a_r) \in (A_{\boldsymbol{\nu}} \smallsetminus \{a\})^r,\, \prod_{i=1}^r a_i = a^r \Rightarrow \max_{1\leqslant i\leqslant r} a_i \geqslant 2a\Big\}$$

and we let $\mathcal{L}$ be the set of lines $L = \gamma\mathbb{R} \subset \mathbb{C}$ with $\gamma \in \mathbb{C}^*$ such that the intersection $L \cap G'$ is non-empty and is a subset of a segment $\big]\frac{1}{2}\gamma t, \gamma t\big]$ with $t \in \mathbb{R}$.

Lemma 4.9 proves that for each $a \in A^*$ there is a choice function of $\mathcal{L}$ of image $G_a \subseteq G'$ such that property (i) is satisfied for $\mathcal{S} = \bigcup_{a\in A^*}\{a\}\times G_a$. Property (ii) is a direct consequence of Lemma 4.8. To prove property (iii), we apply Lemma 4.10 to the set $A_{\boldsymbol{\nu}}$, which is infinite by Lemma 3.1′. ∎

We recall [Dah52, Lemma 3.1], which only relies on basic properties of the zeros of the Riemann ζ-function.

LEMMA 4.8. *For every $\varepsilon > 0$ there is a T_0 such that for each $T > T_0$ there is a line $L = \gamma\mathbb{R} \subset \mathbb{C}$ with $\gamma \in \mathbb{C}^*$ such that $L \cap G'$ is non-empty and every point of $L \cap G'$ has its imaginary part in $[(1-\varepsilon)T, T]$.*

LEMMA 4.9. *Consider a line $L = \gamma\mathbb{R} \subset \mathbb{C}$ with $\gamma \in \mathbb{C}^*$ such that $L \cap G'$ is non-empty and every point of $L \cap G'$ has its imaginary part in $]T/2, T]$. Let $a \in A_{\boldsymbol{\nu}}$ be such that any representation of a^r as a product of r elements of $A_{\boldsymbol{\nu}} \smallsetminus \{a\}$ involves an element of $A_{\boldsymbol{\nu}}$ greater than or equal to $2a$, for any $r \geqslant 2$.*

Then there is at least one element $g \in L \cap G'$ such that the hyperplane $H_{a\boldsymbol{\nu}}(g)$ is non-recurring.

Proof. Since G' is discrete and the real parts of its elements are bounded, the set $L \cap G'$ is finite.

Assume that for $g \in L \cap G'$, the hyperplane $H_{a\boldsymbol{\nu}}(g)$ is recurring. Then there exists $(a', g') \in A_{\boldsymbol{\nu}}\times G'$ with $(a', g') \neq (a, g)$ but $g'/a' = g/a$. Of course $g' \neq g$ and $g' = \frac{a'}{a}g \in L$; therefore $g' \in L \cap G'$.

If every hyperplane $H_{a\boldsymbol{\nu}}(g)$ with $g \in L \cap G'$ is recurring, then there exist two functions $\sigma : L \cap G' \to L \cap G'$ and $\tau : L \cap G' \to A_{\boldsymbol{\nu}} \smallsetminus \{a\}$ such that for any $g \in L \cap G'$, we have $g/a = \sigma(g)/\tau(g)$. Notice that σ has no fixed point. Since $L \cap G'$ is finite, there is $g_0 \in L \cap G'$ and an integer $r \geqslant 2$ such that $\sigma^r(g_0) = g_0$. For any i between 1 and r we set $g_i = \sigma(g_{i-1})$ and $a_i = \tau(g_{i-1})$ so that $g_{i-1}/a = g_i/a_i$. In particular, $g_r = g_0$ but also $a^r g_r = \left(\prod_{i=1}^r a_i\right) g_0$.

We deduce that a should satisfy an equation of the form $a^r = \prod_{i=1}^r a_i$ with $r \geqslant 2$ and $a_i \neq a$. According to the assumption on a, any such identity should involve an integer $a_i \geqslant 2a$ for some i. But since g_{i-1} and g_i are elements of $L \cap G'$, their imaginary parts are both in $]T/2, T]$. We obtain a contradiction

$$2 \leqslant \frac{a_i}{a} = \frac{\Im m(g_i)}{\Im m(g_{i-1})} < \frac{T}{T/2} = 2.$$

We conclude that at least one $g \in L \cap G'$ is such that the hyperplane $H_{a\boldsymbol{\nu}}(g)$ is non-recurring. ∎

Lemma 4.10. *Let $A \subset \mathbb{N}$ be an infinite set of positive integers. There exists an infinite subset A^* of A such that any relation $a^k = b_1 \cdots b_k$ with $a \in A^*$ and $b_1, \ldots, b_k \in A$ implies that either $b_1 = \cdots = b_k = a$, or there is some $b_i \geqslant 2a$.*

In [Dah52, Lemma 3.2], a variation of the following statement (related to the concept of *vertex number*, developed therein) is given. We give a shorter proof using convex geometry, but based on the same principles as the original proof. Doing so, we are able to generalize the previous lemma. Hereafter we denote by $\nu_p(a)$ the p-adic valuation of the integer a.

Lemma 4.11. *Let $A \subset \mathbb{N}$ be an infinite set of positive integers and p_0 a prime number such that $\prod_{p \geqslant p_0} p^{\nu_p(a)}$ is not bounded for $a \in A$. There exists an infinite subset A^* of A such that any relation $a^k = b_1 \cdots b_k$ with $a \in A^*$ and $b_1, \ldots, b_k \in A$ implies that either $b_1 = \cdots = b_k = a$, or there is some $b_i \geqslant p_0 a$.*

The *valuation map* $a \mapsto (\nu_p(a))_{p \in \mathscr{P}}$ associates to each positive integer the sequence of its p-adic valuations (here $\mathscr{P}$ denotes the set of all prime numbers). This is a semigroup isomorphism between the multiplicative semigroup $\mathbb{N}^*$ and the additive semigroup $\mathbb{N}^{(\mathscr{P})}$ consisting of the sequences of natural numbers indexed by the prime numbers with finite support. By this isomorphism, the relation $a^k = \prod_{i=1}^k b_i$ in A is equivalent to the centroid relation $\boldsymbol{e} = \frac{1}{k} \sum_{i=1}^k \boldsymbol{x}_i$ between elements of $E = \{(\nu_p(a))_{p \in \mathscr{P}} \mid a \in A\}$.

We immerse the set E in the $\mathbb{R}$-vector space $V = \mathbb{R}^{(\mathscr{P})}$. A linear form $\phi : V \to \mathbb{R}$ is defined by the family $(c_p)_{p \in \mathscr{P}}$ of its coefficients, that is, the image of the basis vectors of V. Explicitly, $\phi(x) = \sum_{p \in \mathscr{P}} c_p x_p$ for $x = (x_p)_{p \in \mathscr{P}} \in V$. A particular meaningful linear form is the linear form Λ in the coefficients $(\log p)_{p \in \mathscr{P}}$. This form Λ precomposed with the aforementioned valuation map corresponds to the logarithm over $\mathbb{N}^*$, since $\log a = \sum_{p \in \mathscr{P}} (\log p) \nu_p(a)$ for any $a \in \mathbb{N}^*$. Finally, for a given prime p_0, let π_0 be the projection $(x_p)_{p \in \mathscr{P}} \mapsto (x_p)_{p \geqslant p_0}$ whose kernel is generated by the basis vectors of $\mathbb{R}^{(\mathscr{P})}$ corresponding to primes less than p_0. We are now able to state the previous lemma in this new setting.

Lemma 4.11′. *Let* $E \subset \mathbb{N}^{(\mathscr{P})}$ *be an infinite set,* $p_0 \in \mathscr{P}$ *and* π_0 *the associated projection. Assume that* $\pi_0(E)$ *is also infinite. Then there exists an infinite subset* E^* *of* E *such that any centroid relation* $\boldsymbol{e} = \frac{1}{k}\sum_{i=1}^{k} \boldsymbol{x}_i$ *with* $\boldsymbol{e} \in E^*$ *and* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in E$ *implies that either* $\boldsymbol{x}_1 = \dots = \boldsymbol{x}_k = \boldsymbol{e}$*, or there is an* $\boldsymbol{x}_i$ *with* $\Lambda(\boldsymbol{x}_i) \geqslant \log p_0 + \Lambda(\boldsymbol{e})$*.*

4.2.1. Convex geometry. If we consider the convex hull of E in V, any extreme point $\boldsymbol{e}$ of it is guaranteed to be in E^*, since the centroid relation $\boldsymbol{e} = \frac{1}{k}\sum_{i=1}^{k} \boldsymbol{x}_i$ has no non-trivial solution $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in E$. But the set of extreme points is not always infinite.

To overcome this limitation, we relax our assumption on extreme points. Roughly speaking, an extreme point strictly minimizes a (well-chosen) linear form over E, that is $\operatorname{conv}(E)$ meets the corresponding half-space only at this particular point. We consider *weak extreme points* (*), at which a given *wedge* (intersection of two half-spaces, that is defined by two linear forms ϕ_+ and ϕ_-) does not meet E. Choosing ϕ_+ and ϕ_- appropriately, we can ensure the infinitude of these weak extreme points.

However, at the weak extreme points, the centroid equation may have non-trivial solutions. Later, we will choose the forms ϕ_+ and ϕ_- such that non-trivial solutions of the centroid equation satisfy the property stated in Lemma 4.11′.

Lemma 4.12. *Let* V *be an* $\mathbb{R}$*-vector space and* E *an infinite subset of* V*. Assume that there are two linear forms* ϕ_+ *and* ϕ_- *on* V *such that* $\phi_+(E)$ *is a discrete subset of* $\mathbb{R}_+$*, and* $\phi_-(E)$ *is not bounded below. Let*

$$W = \{\boldsymbol{x} \in V \mid \phi_-(\boldsymbol{x}) \leqslant 0 \text{ and } \phi_+(\boldsymbol{x}) < 0\}.$$

Then there exists an infinite sequence $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ *of distinct elements of* E *such that the set* $(\boldsymbol{e}_n + W) \cap E$ *is empty for every* $n \in \mathbb{N}$*.*

Indeed, we can relax the conditions on the sets $\phi_+(E)$ and $\phi_-(E)$ to: $\phi_+(E)$ *is well-ordered* (*so any non-empty subset has a minimum*) and $\phi_-(E)$ *does not have a minimum.* These conditions only use the intrinsic order of $\phi_+(E)$ and $\phi_-(E)$, and not the topology of $\mathbb{R}$, but the construction of the proof still holds.

Proof of Lemma 4.12. We will construct, for any $\boldsymbol{e} \in E$, an element $\boldsymbol{e}' \in E$ with $\phi_-(\boldsymbol{e}') < \phi_-(\boldsymbol{e})$ and $(\boldsymbol{e}' + W) \cap E = \emptyset$. By iteration of this process, the sequence $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ can be produced with the property that $\phi_-(\boldsymbol{e}_{n+1}) < \phi_-(\boldsymbol{e}_n)$ for all $n \in \mathbb{N}$. In particular, the terms of this sequence are all distinct.

Choose $\boldsymbol{e} \in E$ arbitrarily. Since $\phi_-(E)$ is not bounded below, the set E_0 of elements $\boldsymbol{x} \in E$ with $\phi_-(\boldsymbol{x}) < \phi_-(\boldsymbol{e})$ is infinite, therefore non-empty. The set $\phi_+(E_0) \subseteq \phi_+(E)$ is a non-empty discrete subset of $\mathbb{R}_+$, thus it has a minimum, and there is at least one element $\boldsymbol{e}' \in E_0$ attaining this minimum. By construction, no element $\boldsymbol{x} \in E_0$ satisfies $\phi_+(\boldsymbol{x}) < \phi_+(\boldsymbol{e}')$, thus no element $\boldsymbol{x} \in E$ satisfies both $\phi_-(\boldsymbol{x}) < \phi_-(\boldsymbol{e})$ and $\phi_+(\boldsymbol{x}) < \phi_+(\boldsymbol{e}')$. This means that $(\boldsymbol{e}' + W) \cap E$ is empty. ∎

Actually, this construction (see Figure 5) gives a slightly wider region without elements of E: For any $n \in \mathbb{N}$, no element $\boldsymbol{x}$ of E satisfies both $\phi_-(\boldsymbol{x}) < \phi_-(\boldsymbol{e}_n)$ and $\phi_+(\boldsymbol{x}) < \phi_+(\boldsymbol{e}_{n+1})$.

(*) Which correspond to the *vertex numbers* in [Dah52, Definition 3.2].

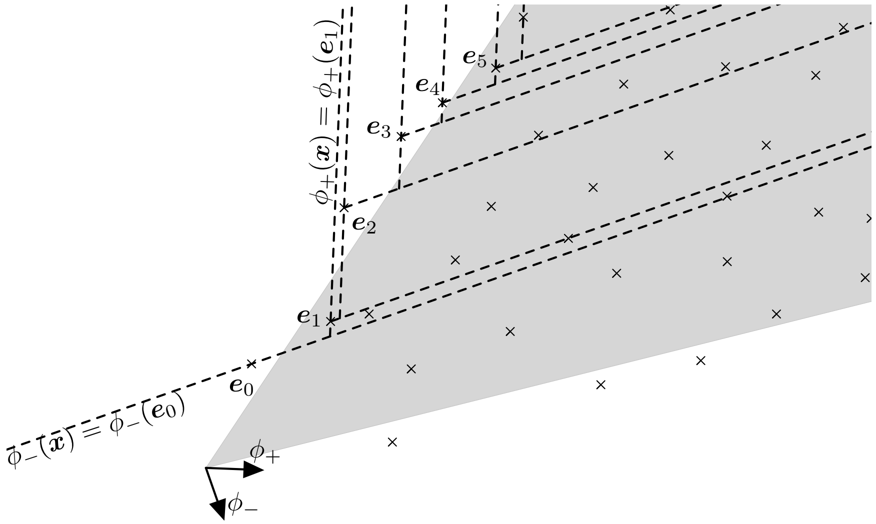

Fig. 5. Linear projection of E on a plane (complementary to $\ker\phi_+ \cap \ker\phi_-$) and construction of the sequence $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$

4.2.2. Diophantine approximation. While the geometric intuition suggests picking ϕ_+ near ϕ_- to widen the wedge, it turns out that the quality of the inequality depends more on the choice of ϕ_+.

The linear forms ϕ_+ and ϕ_- are chosen as linear combinations of Λ (since we want an inequality involving Λ) and a well-chosen form with integer coefficients, allowing the basic trick of diophantine approximation: if $x > 0$ and $x \in \mathbb{Z}$, then $x \geqslant 1$.

Proof of Lemma 4.11′. Let N_0 be the linear form for which the family of coefficients $(c_p)_{p\in\mathscr{P}}$ is defined by

$$c_p = \begin{cases} 0 & \text{if } p < p_0, \\ 1 & \text{if } p = p_0, \\ \lfloor \log p / \log(p_0+1) \rfloor & \text{if } p > p_0. \end{cases}$$

Consider $\phi = \Lambda - (\log p_0) N_0$. The coefficients $(c'_p)_{p\in\mathscr{P}}$ of this linear form satisfy $c'_p = \log p$ if $p < p_0$, $c'_{p_0} = 0$ and

$$c'_p = \log p - \log p_0 \left\lfloor \frac{\log p}{\log(p_0+1)} \right\rfloor \geqslant \left(1 - \frac{\log p_0}{\log(p_0+1)}\right) \log p$$

if $p > p_0$. These coefficients are non-negative, so that $\phi(E) \subseteq \phi(\mathbb{N}^{(\mathscr{P})}) \subseteq \mathbb{R}_+$, and they tend to $+\infty$ when $p \to +\infty$. In particular, for any $M > 0$, there exists P such that $c'_p \geqslant M$ for any $p > P$. We deduce that for any $M > 0$,

$$\phi(\mathbb{N}^{(\mathscr{P})}) \cap [0, M] = \Big(\sum_{p\in\mathscr{P}} c'_p \mathbb{N}\Big) \cap [0, M] \subseteq \Big(\sum_{p\leqslant P} c'_p \mathbb{N}\Big) \cap [0, M] \subseteq \sum_{p\leqslant P} (c'_p \mathbb{N} \cap [0, M])$$

which is a finite sum of finite sets. This implies that $\phi(\mathbb{N}^{(\mathscr{P})})$ and thus $\phi(E)$ is a discrete subset of $\mathbb{R}_+$.

The form $\Lambda - (\log p_0)N_0$ satisfies the condition required for ϕ_+ in Lemma 4.12.

Since $\lfloor t \rfloor \geqslant t/2$ for any $t \geqslant 1$, we have

$$N_0(\boldsymbol{x}) = x_{p_0} + \sum_{p>p_0} c_p x_p > \frac{1}{2\log(p_0+1)} \sum_{p \geqslant p_0} \log p\, x_p = \frac{1}{2\log(p_0+1)} \Lambda(\pi_0(\boldsymbol{x}))$$

for any $\boldsymbol{x} \in \mathbb{N}^{(\mathscr{P})}$. Since $\pi_0(E)$ is not finite, $\Lambda(\pi_0(\boldsymbol{x}))$ is not bounded above, and $-N_0(E)$ is not bounded below.

Therefore, $-N_0$ satisfies the condition for ϕ_- in Lemma 4.12.

Set $\phi_+ = \Lambda - (\log p_0)N_0$ and $\phi_- = -N_0$. Using these two linear forms in Lemma 4.12, we get a sequence of distinct points $\boldsymbol{e}_n \in E$ such that for every $\boldsymbol{x}$ of E and every $n \in \mathbb{N}$, if $\phi_-(\boldsymbol{x}) \leqslant \phi_-(\boldsymbol{e}_n)$ then $\phi_+(\boldsymbol{x}) \geqslant \phi_+(\boldsymbol{e}_n)$.

Let $(\boldsymbol{x}_i)_{i \in I}$ be a finite family of elements of E whose centroid is $\boldsymbol{e}_n$. Then either $\phi_-(\boldsymbol{x}_i) = \phi_-(\boldsymbol{e}_n)$ for every $i \in I$, or there exists some $i \in I$ with $\phi_-(\boldsymbol{x}_i) < \phi_-(\boldsymbol{e}_n)$.

In the formed case, we also have $\phi_+(\boldsymbol{x}_i) \geqslant \phi_+(\boldsymbol{e}_n)$ for any $i \in I$, therefore $\phi_+(\boldsymbol{x}_i) = \phi_+(\boldsymbol{e}_n)$ for $i \in I$. Since Λ can be written as $\phi_+ - (\log p_0)\phi_-$, we get $\Lambda(\boldsymbol{x}_i) = \Lambda(\boldsymbol{e}_n)$ and since the restriction of Λ on $\mathbb{N}^{(\mathscr{P})}$ is injective by the fundamental theorem of arithmetic, we get $\boldsymbol{x}_i = \boldsymbol{e}_n$ for every $i \in I$.

In the latter case, since $\phi_-(\boldsymbol{x}_i) < \phi_-(\boldsymbol{e}_n)$, we also have $\phi_+(\boldsymbol{x}_i) \geqslant \phi_+(\boldsymbol{e}_n)$. Since $N_0(\boldsymbol{x}_i - \boldsymbol{e}_n) = \phi_-(\boldsymbol{e}_n - \boldsymbol{x}_i) > 0$ and $N_0(\boldsymbol{x}_i - \boldsymbol{e}_n) \in \mathbb{Z}$, we deduce that $N_0(\boldsymbol{x}_i - \boldsymbol{e}_n) \geqslant 1$. Then

$$\Lambda(\boldsymbol{x}_i) - \Lambda(\boldsymbol{e}_n) = \phi_+(\boldsymbol{x}_i) - \phi_+(\boldsymbol{e}_n) + (\log p_0)N_0(\boldsymbol{x}_i - \boldsymbol{e}_n) \geqslant 0 + (\log p_0)\cdot 1 = \log p_0$$

as stated. ■

4.3. The remaining case. In the previous cases, the set $\mathcal{P}^*_{\boldsymbol{\nu}}$ satisfied the conditions of Corollary 4.3, because the set $\Theta_{\boldsymbol{\nu}}$ of zeros and poles inside the unit circle was non-empty or because the set of non-zero exponents (indexed by $A_{\boldsymbol{\nu}}$) in $h_{\boldsymbol{\nu}} = \prod_{n\geqslant 1}(1-T^n)^{c(n\boldsymbol{\nu})}$ was not finite.

In the remaining case, we assume that q^+ and $q^+_{\boldsymbol{\nu}}$ are coprime, that q^- and $q^-_{\boldsymbol{\nu}}$ are coprime, and that $h_{\boldsymbol{\nu}}$ is a quotient of products of cyclotomic factors. This implies that $\Theta_{\boldsymbol{\nu}}$ is empty by Lemma 4.5, and that $A_{\boldsymbol{\nu}}$ is finite by Lemma 3.1′. In that case, the set $\mathcal{P}_{\boldsymbol{\nu}} = A_{\boldsymbol{\nu}} \times G'$ parameterizes a discrete set of hyperplanes. Hence, there cannot be an accumulation of parallel hyperplanes in (f) along the face of Γ with normal vector $\boldsymbol{\nu}$. Of course, one can try to find an accumulation of analytic surfaces (in the divisor (f)) of larger complexity, and to some extent this is the strategy in [Del10, Del13].

Instead, we consider a family of hyperplanes $H_{\boldsymbol{\alpha}}(\rho)$ with $\rho \in G'$ and where the normal vector $\boldsymbol{\alpha}$ is in an affine line directed by $\boldsymbol{\nu}$, that is, $\boldsymbol{\alpha} \in C \cap (\boldsymbol{\mu} + \boldsymbol{\nu}\mathbb{R}_+)$ with $C = \operatorname{xsupp} h$ and $\boldsymbol{\mu}$ non-collinear with $\boldsymbol{\nu}$. Any pair of distinct normal vectors in this family are non-parallel. This will correspond to an accumulation of hyperplanes on the points of the corresponding face whose real part is sufficiently close to the origin (see Figure 1).

Lemma 4.13. *If q^+ and $q^+_{\boldsymbol{\nu}}$ are coprime, q^- and $q^-_{\boldsymbol{\nu}}$ are coprime, and $h_{\boldsymbol{\nu}}$ is a quotient of products of cyclotomic factors, then there exist $\boldsymbol{\mu} \in \mathbb{N}^k$ and $\mathcal{S} = A_0 \times G_0$ with $A_0 \subseteq \{a \in \mathbb{N} \mid \boldsymbol{\mu} + a\boldsymbol{\nu} \in C\}$ and $G_0 \subseteq G'$ such that $\mathcal{S} \subseteq \mathcal{P}^*_{\boldsymbol{\mu},\boldsymbol{\nu}}$, $\mathcal{S}$ satisfies the vertical accumulation property, and for any $(a,g) \in \mathcal{S}$, $\Re e(g) \in \left[\frac{1}{2}, 1\right]$.*

Notice that the conditions on $q_{\boldsymbol{\nu}}^+$ and $q_{\boldsymbol{\nu}}^-$ do not mean that $q_{\boldsymbol{\nu}}^+$ and $q_{\boldsymbol{\nu}}^-$ are products of cyclotomic factors: they could have a non-trivial common factor which is not a product of cyclotomic polynomials, and which is coprime to q^+ and q^-.

Proof of Lemma 4.13. By Lemma 3.1′, $C \cap \boldsymbol{\nu}\mathbb{N}$ is finite, and assertion (i) of Proposition 3.22 is not satisfied. Therefore, assertion (ii) holds, and there exists $\boldsymbol{\mu} \in \mathbb{N}^k$ such that

- the set $A_1 = \{a \in \mathbb{N} \mid \boldsymbol{\mu} + a\boldsymbol{\nu} \in C\}$ is infinite;
- the set A_1 is syndetic, that is, there exist $a_0 \in \mathbb{N}$ and $d \geqslant 1$ such that for any $a \geqslant a_0$ the intersection $A_1 \cap \,]a, a+d]$ is non-empty;
- $\boldsymbol{\mu}$ can be regarded as "minimal" in the sense that there are only finitely many elements of C of the form $t\boldsymbol{\mu} + x\boldsymbol{\nu}$, with $t \in [0,1[$ and $x \in \mathbb{R}_+$.

Notice that $\boldsymbol{\mu}$ cannot be collinear with $\boldsymbol{\nu}$.

To any parameter pair $(a, g) \in A_1 \times G'$ we associate the hyperplane $H_{\boldsymbol{\alpha}_0}(g)$ with $\boldsymbol{\alpha}_0 = \boldsymbol{\mu} + a\boldsymbol{\nu}$. By construction, this hyperplane appears in a term of our expression of the divisor of f,

$$(f) = \sum_p (h_p) - \sum_{\boldsymbol{\alpha} \in C} \sum_{\gamma \in G'} m_\gamma c(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}(\gamma).$$

However, it may occur in multiple terms. We distinguish three types of terms where the hyperplane may also appear (†):

(i) Terms of the first sum $\sum_p (h_p)$.
(ii) Terms $H_{t\boldsymbol{\alpha}_0}(tg)$ in the second sum with $t < 1$, $t\boldsymbol{\alpha}_0 \in C$ and $tg \in G'$.
(iii) Terms $H_{t\boldsymbol{\alpha}_0}(tg)$ in the second sum with $t > 1$, $t\boldsymbol{\alpha}_0 \in C$ and $tg \in G'$.

We first construct a cofinite subset A_0 of A_1 such that no hyperplane corresponding to a parameter pair $(a, g) \in A_0 \times G'$ appears in (f) as a term of type (i) or (ii).

Let $(a, g) \in A_1 \times G'$, and assume that the hyperplane $H_{\boldsymbol{\alpha}_0}(g)$ appears in the first sum $\sum_p (h_p)$. We denote by d the greatest common divisor of the coordinates of $\boldsymbol{\alpha}_0$, so that $\boldsymbol{\nu}' = \frac{1}{d}\boldsymbol{\alpha}_0$ is primitive. Thus, the hyperplane $H_{\boldsymbol{\nu}'}(g/d)$ appears in the divisor of h_p for some p. By Lemma 4.1, the polynomial q^+q^- is divisible by the irreducible polynomial $\boldsymbol{X}^{\boldsymbol{\nu}'} - p^{-g/d}$. The set C' of primitive vectors $\boldsymbol{\nu}'$ such that q^+q^- is divisible by $\boldsymbol{X}^{\boldsymbol{\nu}'} - \theta$ for some $\theta \in \mathbb{C}$ is finite. And since $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are non-collinear, for each such $\boldsymbol{\nu}' \in C'$, the relation $\boldsymbol{\alpha}_0 = \boldsymbol{\mu} + a\boldsymbol{\nu} = d\boldsymbol{\nu}'$ is satisfied for at most one couple $(a, d) \in A_1 \times \mathbb{N}$. Therefore, only finitely many values of $a \in A_1$ are concerned with terms of type (i).

Now assume that the hyperplane $H_{\boldsymbol{\alpha}_0}(g)$ appears in the second sum with parameters $t\boldsymbol{\alpha}_0 \in C$ and $tg \in G'$ with $t < 1$. By the minimality of the chosen $\boldsymbol{\mu}$, there are only finitely many elements of C of the form $t\boldsymbol{\mu} + x\boldsymbol{\nu}$, with $t \in [0,1[$ and $x \in \mathbb{R}_+$. For each such t and x, there is at most one element $a \in A_1$ such that $\boldsymbol{\mu} + a\boldsymbol{\nu}$ is collinear with $t\boldsymbol{\mu} + x\boldsymbol{\nu}$ (a must satisfy $ta = x$). Therefore, only finitely many values of $a \in A_1$ are concerned with terms of type (ii).

Let A_0 be the set of elements of A_1 which are not in one of the two finite sets described above. Since A_0 is cofinite in A_1, A_0 is infinite and syndetic.

(†) Obviously, types (ii) and (iii) occur only if the Riemann Hypothesis is not satisfied, which we cannot yet rule out.

We now construct an infinite subset G_0 of G' such that no hyperplane corresponding to a parameter pair $(a,g) \in A_1 \times G_0$ also appears in (f) as a term of type (iii). We also require that for any $g \in G_0$, $\Re e(g) \geqslant \frac{1}{2}$. For this purpose, we define G'' as $\{\gamma \in G' \mid \Re e(\gamma) \geqslant 1/2\}$, and we construct G_0 as a subset of G''. Since there are infinitely many zeros of ζ with real part $\geqslant 1/2$, G'' is infinite. It is also closed under complex conjugation.

The key argument is the existence of $\eta > 1$ such that for any $a \in \mathbb{Z}$ and any $t > 1$ such that $t(\boldsymbol{\mu} + a\boldsymbol{\nu}) \in \mathbb{Z}^k$, we have $t \geqslant \eta$.

Consider the 2-dimensional lattice $\mathbb{Z}^k \cap \mathrm{vect}_{\mathbb{R}}\{\boldsymbol{\nu}, \boldsymbol{\mu}\}$. Since $\boldsymbol{\nu}$ is primitive, in this lattice we can find $\boldsymbol{\nu}'$ such that $(\boldsymbol{\nu}, \boldsymbol{\nu}')$ is a basis of it. The vector $\boldsymbol{\mu}$ can be written as $d\boldsymbol{\nu} + d'\boldsymbol{\nu}'$ with $d, d' \in \mathbb{Z}$. Notice that $|d'|$ does not depend on the choice of ν', but only on the choice of $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$. For any $a \in \mathbb{Z}$, if the vector $t(\boldsymbol{\mu} + a\boldsymbol{\nu})$ is in $\mathbb{Z}^k$, then it is in that lattice, and its coordinates with respect to that basis are $(ta + td, td')$. Therefore, td' is an integer, and since $t > 1$, we have $t \geqslant 1 + 1/|d'|$.

Setting $\eta = 1 + 1/|d'| > 1$, we partition the interval $]0,1]$ into $\bigcup_{n \in \mathbb{N}} I_n$ with $I_n =]\eta^{-(n+1)}, \eta^{-n}]$ and define G''_n as $\{g \in G'' \mid \Re e(g) \in I_n\}$. If $\eta^n > 2$, then G''_n is empty. The infinite set G'' can be written as the finite union of G''_n with n such that $\eta^n \leqslant 2$. There is at least one set G''_n which is infinite. We set n_1 to be the minimal n such that G''_n is infinite, and define

$$G_1 = G''_{n_1} \qquad \text{and} \qquad G_2 = \bigcup_{n < n_1} G''_n = \{g \in G' \mid \Re e(g) > \eta^{-n_1}\}.$$

By construction, the set G_1 is infinite, and by the minimality of n_1, the set G_2 is finite. Both sets are closed under complex conjugation. Finally, we set

$$G_0 = G_1 \smallsetminus \{rg \mid r \in [0,1],\ g \in G_2\}.$$

As a subset of G', G_1 is discrete, and the removed set is compact, therefore G_0 is cofinite in G_1. We deduce that G_0 is infinite and closed under complex conjugation.

Let $(a,g) \in A_1 \times G_0$, and assume that the associated hyperplane $H_{\boldsymbol{\alpha}_0}(g)$ also appears in (f) as $H_{t\boldsymbol{\alpha}_0}(tg)$ with $t > 1$, $t\boldsymbol{\alpha}_0 \in C$ and $tg \in G'$. We set $g' = tg$. Since $t\alpha_0 \subseteq \mathbb{Z}^k$, we have $t \geqslant \eta$. Since $g \in G_0 \subseteq G''_{n_1}$, we have $\Re e(g) > \eta^{-(n_1+1)}$. We deduce that $\Re e(g') > \eta^{-n_1}$, therefore $g' \in G_2$. Finally, g can be written as rg' with $r = 1/t \in [0,1]$ and $g' \in G_2$, which is impossible by the construction of G_0.

Therefore, no hyperplane $H_{\boldsymbol{\alpha}_0}(g)$ corresponding to $(a,g) \in A_1 \times G_0$ also appears in (f) as a term of type (iii).

Putting all these properties together, we have proven that $A_0 \times G_0$ is a set of non-recurring pairs in $\mathcal{P}^*_{\boldsymbol{\mu},\boldsymbol{\nu}}$, and that for any $(a,g) \in A_0 \times G_0$, $\Re e(g) \in [1/2, 1]$. It remains to prove that this set has the vertical accumulation property, that is, for any $x \in \mathbb{R}$, $\mathrm{i}x$ is a cluster point of $\{g/a \mid a \in A_0,\ g \in G_0\}$.

Since G_0 is closed under complex conjugation, we may consider only $x > 0$ (the case $x = 0$ is trivial since A_0 is infinite).

Actually, we show the vertical accumulation property for a subset $A_0 \times G'_0$ where

$$G'_0 = \{g \in G_0 \mid \forall g' \in G_0, \forall r \in \mathbb{R}_+, rg' = g \Rightarrow r \leqslant 1\},$$

so that each real line $\gamma\mathbb{R} \subseteq \mathbb{C}$ which intersects G_0 intersects G'_0 in exactly one point, the furthest from the origin. Note that $\gamma\mathbb{R} \cap G_0$ is finite for any $\gamma \in \mathbb{C}^*$, which implies that G_0' is an infinite set. Just like G_0, the set G_0' is also discrete and closed under complex conjugation. Since the real parts of its elements are $[0,1]$, their imaginary parts are unbounded above. This set has been constructed so that the function $\rho : (a,g) \mapsto g/a$ defined over $A_0 \times G_0'$ is one-to-one.

Fix $x \in \mathbb{R}_+^*$ and $\varepsilon > 0$, and for any $a > 0$ set $R_a = [0,1] + \mathrm{i}[(1-\varepsilon)ax, ax] \subseteq \mathbb{C}$. We have the following composition of one-to-one functions:

$$\bigcup_{a \in A_0} (G_0' \cap R_a) \xrightarrow{\pi_2^{-1}} \bigcup_{a \in A_0} \{(a,g) \mid g \in G_0' \cap R_a\} \xrightarrow{\rho} \{g/a \mid (a,g) \in A_0 \times G_0'\} \cap R_1,$$

where the first injection is a section of the projection π_2 onto the second coordinate.

Since A_0 is syndetic, there exist $a_0 \in \mathbb{N}$ and $d \geqslant 1$ such that for any $a \in A_0$ with $a \geqslant a_0$, there exists $a' \in A_0 \cap]a, a+d]$. If moreover $a \geqslant d/\varepsilon$, then

$$(1-\varepsilon)a' \leqslant (1-\varepsilon)(a+d) \leqslant (1-\varepsilon)(a+a\varepsilon) < a.$$

Therefore, the union $\bigcup_{a \in A_0} R_a$ contains the half-strip $S = [0,1] + \mathrm{i}[T_0 x, +\infty[$ with $T_0 = \max(a_0, d/\varepsilon)$. The set $G_0' \cap S$ is infinite, and its image by the one-to-one function

$$G_0' \cap S \subseteq G_0' \cap \Big(\bigcup_{a \in A_0} R_a \Big) \xrightarrow{\rho \circ \pi_2^{-1}} \{g/a \mid (a,g) \in A_0 \times G_0'\} \cap R_1,$$

is also infinite. Thus, the set $\{g/a \mid a \in A_0,\ g \in G_0\}$ admits a cluster point in R_1, which can only be on the imaginary line with imaginary part in $\big[(1-\varepsilon)x, x\big]$. Since the result is true for any $x > 0$ and any $\varepsilon > 0$, we conclude that each point of the vertical half-line $\mathrm{i}\mathbb{R}_+$ is such a cluster point.

Therefore, the set $\mathcal{S} = A_0 \times G_0$ satisfies the three aimed properties. ■

This Lemma alone is insufficient to conclude that every point of the face of Γ of normal vector $\boldsymbol{\nu}$ is a singular point of f. In the proof of Corollary 4.3, we use a somewhat surprising property of the singular set of a meromorphic function defined on a tubular domain, which remains to be proven. The next subsection is dedicated to this proof. It can be seen as an extension of this subsection, since it completes the argument of Corollary 4.3 in this case. However, since this result may be of independent interest, and since it is better stated in a general setting, we have set its treatment apart.

4.4. Singularities along faces of tubular domains. A celebrated theorem of Bochner [Boc38] asserts that any holomorphic function on a tubular domain can be holomorphically continued to the convex hull of this domain. Consider a tubular convex domain $\Omega = K + \mathrm{i}\mathbb{R}^k$, where K is a convex open subset of $\mathbb{R}^k$. As a convex domain, Ω is also a *domain of holomorphy*, which means that for any $\boldsymbol{b}$ on the boundary $\partial\Omega$, we can find a function which is holomorphic on Ω but not at $\boldsymbol{b}$. And indeed, for a convex tubular domain, finding such functions is straightforward.

Let $\boldsymbol{b}$ be on the boundary $\partial\Omega$. We can write $\boldsymbol{b}$ as $\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$ with $\boldsymbol{x} \in \partial K$ and $\boldsymbol{y} \in \mathbb{R}^k$. Since K is convex, by the hyperplane separation theorem, there exists $\boldsymbol{c} \in \mathbb{R}^k$ such that $\langle \boldsymbol{c}, \boldsymbol{\sigma} - \boldsymbol{x} \rangle < 0$ for every $\boldsymbol{\sigma} \in K$. Defining V as $\{\boldsymbol{c}\}^\perp \subseteq \mathbb{R}^k$, we find that V is a linear subspace of codimension 1 in $\mathbb{R}^k$ such that $K \cap (\boldsymbol{x} + V)$ is empty and the function $\boldsymbol{s} \mapsto 1/\langle \boldsymbol{c}, \boldsymbol{s} - \boldsymbol{b} \rangle$ is obviously holomorphic on Ω, but not at $\boldsymbol{b}$.

The choice of functions $\xi_{\boldsymbol{b}} : \boldsymbol{s} \mapsto \exp(1/\langle \boldsymbol{c}, \boldsymbol{s} - \boldsymbol{b}\rangle)$ for any $\boldsymbol{b} \in \partial\Omega$ proves that Ω is a domain of meromorphy. More generally, by a similar argument, a domain of holomorphy $\Omega \subseteq \mathbb{C}^k$ is also a domain of meromorphy. The converse is also true, but more elaborate (‡).

Observe that this elementary example can produce a large set of singular points in $\partial\Omega$, with a particular structure. Let F be the face of $\overline{K}$ defined by $\overline{K} \cap (\boldsymbol{x} + V)$, which contains at least $\boldsymbol{x}$. The set of singular points of the function $\xi_{\boldsymbol{b}}$ over $\partial\Omega$ is exactly $F + \mathrm{i}(\boldsymbol{y} + V)$.

A different choice of $\boldsymbol{c}$ would have led to a different choice of $\xi_{\boldsymbol{b}}$, and a different set of singular points. But if $\boldsymbol{x}$ is in the relative interior of the face F, the relation $F = \overline{K} \cap (\boldsymbol{x} + V)$ implies that V should contain the linear subspace W associated to the affine hull of F, its direction (once again by [Roc97, Theorem 6.1]). In that case, for any choice of $\boldsymbol{c}$, thus of $\xi_{\boldsymbol{b}}$, the set of singular points in $\partial\Omega$ always contains the set $F + \mathrm{i}(\boldsymbol{y} + W)$, which is independent of $\boldsymbol{c}$ and is of dimension $2\dim_{\mathbb{R}} F$.

Even if this property seems very particular to the functions $\xi_{\boldsymbol{b}}$, we show that for any holomorphic function ξ defined on Ω, the set of singular points has a similar property over the face $F + \mathrm{i}\mathbb{R}^k$.

THEOREM 4.14. *Let ξ be a function holomorphic on a tubular domain $K + \mathrm{i}\mathbb{R}^k$, with $K \subseteq \mathbb{R}^k$ convex, and let F be a proper face of $\overline{K}$ and denote W the direction of the affine hull of F in $\mathbb{R}^k$.*

If there exist $\boldsymbol{x} \in F$ and $\boldsymbol{y} \in \mathbb{R}^k$ such that ξ can be extended holomorphically around $\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$, then, for any $\boldsymbol{x}'$ in the relative interior of F, and for any $\boldsymbol{y}' \in \boldsymbol{y} + W$, the function ξ can be extended holomorphically around $\boldsymbol{x}' + \mathrm{i}\boldsymbol{y}'$.

Consequently, if there exist $\boldsymbol{x}$ in the relative interior of F and $\boldsymbol{y} \in \mathbb{R}^k$ such that $\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$ is a singular point of ξ, then for any $\boldsymbol{x}' \in F$, and for any $\boldsymbol{y}' \in \boldsymbol{y} + W$, $\boldsymbol{x}' + \mathrm{i}\boldsymbol{y}'$ is a singular point of ξ.

The theorem is obviously true (but useless) if F is a singleton.

Notice that the second assertion of the theorem is just the contrapositive of the first one. We will only give a proof for the first statement.

Remark. Observe also that this theorem can be stated *mutatis mutandis* for meromorphic functions, in the sense that every occurrence of the term "holomorph-" in the theorem must be replaced by "meromorph-". This version is the one required in the proof of Corollary 4.3.

As we have already seen, any domain of meromorphy on $\mathbb{C}^k$ is a domain of holomorphy, and conversely. Since the first statement of the theorem can be phrased in terms of a domain of holomorphy: *Any domain of holomorphy containing $\Omega \cup \{\boldsymbol{x} + \mathrm{i}\boldsymbol{y}\}$ also contains* $\mathrm{ri}(F) + \mathrm{i}(\boldsymbol{y} + W)$, it can be written in terms of domains of meromorphy as well.

To sum up both assertions of the theorem, we state a direct consequence, which may be more useful in a large variety of contexts.

(‡) Levi has shown that every domain of meromorphy is pseudoconvex (see for example [Nis01, §4.1], also for the equivalence of the different notions of pseudoconvexity) and Oka has succeeded in proving that every pseudoconvex domain is a domain of holomorphy (see [Sha92, 39.Theorem 4]).

Corollary 4.15. *Let ξ be a function holomorphic on a tubular domain $K + \mathrm{i}\mathbb{R}^k$, with $K \subseteq \mathbb{R}^k$ convex, let F be a proper face of $\overline{K}$ and denote by W the direction of the affine hull of F in $\mathbb{R}^k$. Then there is a closed subset $Y \subseteq \mathbb{R}^k$, invariant under translation by elements of W, such that the set of singular points of ξ inside* $\mathrm{ri}(F) + \mathrm{i}\mathbb{R}^k$ *is exactly* $\mathrm{ri}(F) + \mathrm{i}Y$.

To prove Theorem 4.14, we use a result of Siciak [Sic69, Theorem 5], which can be seen as a local version of Bochner's theorem. We have to introduce some geometric objects first.

Let c be a positive real number. For any $R \geqslant c$, set $D(R)$ as the compact region of $\mathbb{C}$ enclosed by the ellipse of foci $-\mathrm{i}c$ and $\mathrm{i}c$ and for which the sum the small half-axis and large half-axis is R. In that case, the small half-axis is $r = \frac{1}{2}(R - c^2/R)$, and the projection of $D(R)$ over $\mathbb{R}$ is the intersection of $D(R)$ with $\mathbb{R}$, i.e. $[-r, r]$. The large half-axis is $\frac{1}{2}(R + c^2/R) = \sqrt{r^2 + c^2}$ and $D(R)$ is inside the closed ball $\overline{B}_{\mathbb{C}}(0, \sqrt{r^2 + c^2}\,)$.

This ellipse is described by the equation $\left|z + (z^2 + c^2)^{1/2}\right| = R$ where the chosen branch of $z \mapsto (z^2 + c^2)^{1/2}$ on $\mathbb{C} \smallsetminus [-\mathrm{i}c, \mathrm{i}c]$ is determined by the requirement that $(z^2 + c^2)^{1/2}$ is a positive real number when z is a positive real number (to avoid confusion, we will use the square root sign only for non-negative arguments). For $z \in [-\mathrm{i}c, \mathrm{i}c]$ we have $|z + (z^2 + c^2)^{1/2}| = c$. Indeed, $D(c) = [-\mathrm{i}c, \mathrm{i}c]$.

Let $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ be an orthonormal basis of $\mathbb{R}^k$ (as an $\mathbb{R}$-vector space), therefore of $\mathbb{C}^k$ (as a $\mathbb{C}$-vector space), and $R_1, \dots, R_k$ be k real numbers larger than c. For any $m \in [\![1, k]\!]$, we let E_m be the set of $\boldsymbol{z}$ of coordinates $(z_1, \dots, z_k)$ with respect to the basis $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$, with z_i in the interval $D(c) = [-\mathrm{i}c, \mathrm{i}c]$ for all $i \neq m$ and with $z_m \in D(R_m)$. In particular, E_m is inside the closed ball $\overline{B}_{\mathbb{C}^k}(\mathbf{0}, \sqrt{kc^2 + r_m^2}\,)$, where r_m is the small half-axis of $D(R_m)$. And since $D(R_m)$ is contained in $[-r_m, r_m] + \mathrm{i}\mathbb{R}$, we also have $E_m \subseteq [-r_m, r_m]\boldsymbol{u}_m + \mathrm{i}\mathbb{R}^k$.

The theorem of Siciak states that if a function ξ is defined on $\bigcup_{m=1}^k E_m$ and holomorphic with respect to the mth coordinate on E_m for any $m \in [\![1, k]\!]$, then ξ can be extended to a holomorphic function defined over

$$\Xi = \left\{ z_1\boldsymbol{u}_1 + \dots + z_k\boldsymbol{u}_k \in \mathbb{C}^k \;\middle|\; \sum_{m=1}^{k} \frac{\log(|z_m + (z_m^2 + c^2)^{1/2}|/c)}{\log(R_m/c)} < 1 \right\}.$$

We will only use a much weaker version, which focuses on the real parts and isolates the first coordinate.

Lemma 4.16. *Let ξ be a function holomorphic on $\big([-r, r]\boldsymbol{u}_1 + \mathrm{i}\mathbb{R}^k\big) \cup B_{\mathbb{C}^k}(0, \rho)$. Then ξ can be holomorphically extended to the polyhedron*

$$\Sigma = \left\{ x_1\boldsymbol{u}_1 + \dots + x_k\boldsymbol{u}_k \in \mathbb{R}^k \;\middle|\; \frac{r - |x_1|}{\sqrt{r^2 + c^2}\log(R_1/c)} > \sum_{m=2}^{k} \frac{|x_m|}{c\log 2} \right\} \subseteq \mathbb{C}^k$$

where $c = \frac{1}{\sqrt{k+1}}\rho$ and $R_1 = r + \sqrt{r^2 + c^2}$. It can also be holomorphically extended to the polyhedron $\Sigma' = \{(\mathrm{i}z_1, z_2, \dots, z_k) \mid (z_1, \dots, z_k) \in \Sigma\}$.

Proof. We fix $R_m = 2c$ for $m \geqslant 2$ (we have already set $R_1 = r + \sqrt{r^2 + c^2}$), and we consider the sets E_m as defined above. In that case, we have $r_1 = r$ and E_1 is inside $[-r,r]\boldsymbol{u}_1+\mathrm{i}\mathbb{R}^k$, and for any $m\geqslant 2$, we have $r_m<c$, so $E_m\subseteq B_{\mathbb{C}^k}(\mathbf{0},\sqrt{k+1}\,c)$ $=B_{\mathbb{C}^k}(\mathbf{0},\rho)$. Therefore, the function ξ satisfies the conditions of the theorem of Siciak, and it can be extended to Ξ.

The projection of Ξ onto $\mathbb{R}^k$ is its intersection with $\mathbb{R}^k$:

$$\Xi\cap\mathbb{R}^k=\left\{x_1\boldsymbol{u}_1+\cdots+x_k\boldsymbol{u}_k\in\mathbb{R}^k\;\middle|\;\sum_{m=1}^k\frac{\log(|x_m/c|+\sqrt{(x_m/c)^2+1})}{\log(R_m/c)}<1\right\}.$$

The function $x\mapsto\log(x+\sqrt{x^2+c^2}\,)$ maps 0 to c and r to R_1, and its derivative $x\mapsto 1/\sqrt{x^2+c^2}$ is decreasing on $\mathbb{R}_+$. Therefore, by concavity, we deduce that

$$\log(|x|+\sqrt{x^2+c^2}\,)\leqslant\log c+|x|/c$$

and

$$\log(|x|+\sqrt{x^2+c^2}\,)\leqslant\log R_1-(r-|x|)/\sqrt{r^2+c^2}$$

for any $x\in\mathbb{R}$. Therefore, any point in Σ satisfies

$$\begin{aligned}\sum_{m=1}^k&\frac{\log(|x_m/c|+\sqrt{(x_m/c)^2+1}\,)}{\log(R_m/c)}\\&\leqslant\frac{\log(R_1/c)-(r-|x_1|)/\sqrt{r^2+c^2}}{\log(R_1/c)}+\sum_{m=2}^k\frac{|x_m|/c}{\log(2c/c)}<1\end{aligned}$$

and therefore is in $\Xi\cap\mathbb{R}^k$.

For $z=\mathrm{i}x$ with $x\in\mathbb{R}$, we have

$$\log|z+(z^2+c^2)^{1/2}|=\begin{cases}\log c & \text{if } |x|\leqslant c\\ \log(|x|+\sqrt{x^2-c^2}\,) & \text{if } |x|\geqslant c\end{cases}\leqslant\log(|x|+\sqrt{x^2+c^2}\,),$$

and the same inequality is valid for elements of Σ'. ∎

Lemma 4.17. *Let $L\subset\partial K$ be a line segment with midpoint $\boldsymbol{x}$. If there is $\boldsymbol{y}\in\mathbb{R}^k$ such that ξ can be holomorphically extended around $\boldsymbol{x}+\mathrm{i}\boldsymbol{y}$, then ξ can be holomorphically extended around any point $\boldsymbol{x}'+\mathrm{i}\boldsymbol{y}$ with $\boldsymbol{x}'$ in the relative interior of L, and around any point $\boldsymbol{x}+\mathrm{i}\boldsymbol{y}'$ with $\boldsymbol{x}+\boldsymbol{y}'-\boldsymbol{y}$ in the relative interior of L.*

Proof. Let $\boldsymbol{\sigma}\in K$; $\boldsymbol{\sigma}$ is not aligned with the extremities $\boldsymbol{a}$ and $\boldsymbol{b}$ of L. For any $\lambda\in[0,1]$ let L_λ be the line segment $\big[\lambda\boldsymbol{a}+(1-\lambda)\boldsymbol{\sigma},\lambda\boldsymbol{b}+(1-\lambda)\boldsymbol{\sigma}\big]$. In particular, since K is an open convex set, $L_\lambda\subset K$ for any $\lambda<1$. Set $\boldsymbol{x}_\lambda=\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{\sigma}$; it is the midpoint of L_λ. We denote by $2r$ the length of L, therefore L_λ has length $2\lambda r$.

Let $\rho>0$, such that ξ can be extended holomorphically to $B_{\mathbb{C}^k}(\boldsymbol{x}+\mathrm{i}\boldsymbol{y},\rho)$. In particular, ξ can be extended to $B_{\mathbb{C}^k}(\boldsymbol{x}_\lambda+\mathrm{i}\boldsymbol{y},\rho_\lambda)$ with $\rho_\lambda=\rho-(1-\lambda)\|\boldsymbol{x}-\boldsymbol{\sigma}\|$. We will consider λ larger than $1-\frac{1}{2}\rho/\|\boldsymbol{x}-\boldsymbol{\sigma}\|$, so that $\rho_\lambda>\rho/2$.

Let $\boldsymbol{u}_1=(\boldsymbol{b}-\boldsymbol{a})/\|\boldsymbol{b}-\boldsymbol{a}\|$ be a unit vector in the direction of L. Let $\boldsymbol{u}_2$ be a unit vector in the plane generated by $\boldsymbol{a}-\boldsymbol{\sigma}$ and $\boldsymbol{b}-\boldsymbol{\sigma}$, and orthogonal to $\boldsymbol{u}_1$. We complete this family to an orthonormal basis $\{\boldsymbol{u}_1,\dots,\boldsymbol{u}_k\}$ of $\mathbb{R}^k$ (and of $\mathbb{C}^k$). For $\lambda<1$, the function $\boldsymbol{s}\mapsto\xi(\boldsymbol{s}+\boldsymbol{x}_\lambda+\mathrm{i}\boldsymbol{y})$ is holomorphic on $[-\lambda r,\lambda r]\boldsymbol{u}_1+\mathrm{i}\mathbb{R}^k$, since $L_\lambda\subset K$, and holomorphic on $B_{\mathbb{C}^k}(\mathbf{0},\rho/2)$.

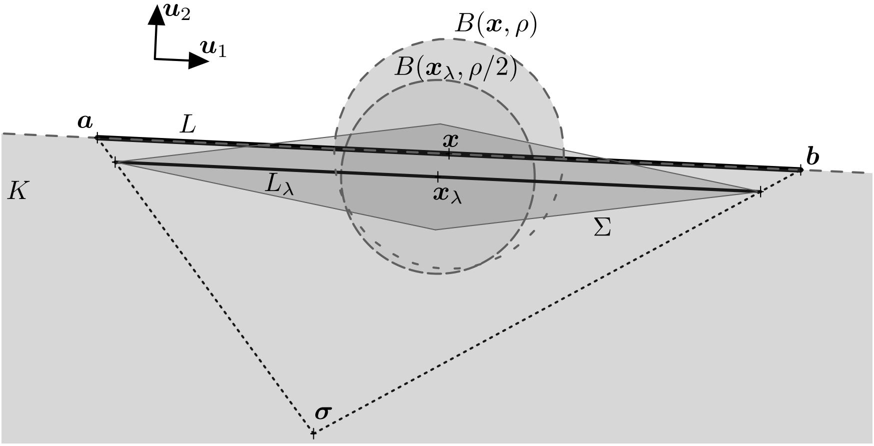

Fig. 6. Getting a large intersection of $\boldsymbol{x}_\lambda + \Sigma$ and of L

By Lemma 4.16, this function is also holomorphically extended to Σ, which contains the points $x_1\boldsymbol{u}_1 + \langle \boldsymbol{x} - \boldsymbol{x}_\lambda, \boldsymbol{u}_2\rangle \boldsymbol{u}_2$ with x_1 such that

$$\frac{\lambda r - |x_1|}{\sqrt{\lambda^2 r^2 + c^2}\log(\lambda r + \sqrt{\lambda^2 r^2 + c^2})} > \frac{\langle \boldsymbol{x} - \boldsymbol{x}_\lambda, \boldsymbol{u}_2\rangle}{c\log 2},$$

with $c = \rho/2\sqrt{k+1}$. It is also extended to Σ', which contains the points $\mathrm{i}x_1\boldsymbol{u}_1 + \langle \boldsymbol{x} - \boldsymbol{x}_\lambda, \boldsymbol{u}_2\rangle \boldsymbol{u}_2$ with x_1 satisfying the same inequality.

This means that ξ is holomorphic at any point

$$\boldsymbol{x}_\lambda + \mathrm{i}\boldsymbol{y} + x_1\boldsymbol{u}_1 + \langle \boldsymbol{x} - \boldsymbol{x}_\lambda, \boldsymbol{u}_2\rangle \boldsymbol{u}_2 = \boldsymbol{x} + \mathrm{i}\boldsymbol{y} + (x_1 + \langle \boldsymbol{x}_\lambda - \boldsymbol{x}, \boldsymbol{u}_1\rangle)\boldsymbol{u}_1,$$

and any point

$$\boldsymbol{x} + \mathrm{i}\boldsymbol{y} + (\mathrm{i}x_1 + \langle \boldsymbol{x}_\lambda - \boldsymbol{x}, \boldsymbol{u}_1\rangle)\boldsymbol{u}_1,$$

with x_1 satisfying the previous inequality, that is,

$$|x_1| < \lambda r - \langle \boldsymbol{x}_\lambda - \boldsymbol{x}, \boldsymbol{u}_2\rangle \frac{\sqrt{\lambda^2 r^2 + c^2}\log(\lambda r + \sqrt{\lambda^2 r^2 + c^2}\,)}{c\log 2}.$$

Since this is satisfied for any $\lambda < 1$, and $\|\boldsymbol{x}_\lambda - \boldsymbol{x}\|$ tends to 0 as λ gets close to 1, we deduce that ξ can be holomorphically continued to any point of

$$]\boldsymbol{x} + \mathrm{i}\boldsymbol{y} - r\boldsymbol{u}_1, \boldsymbol{x} + \mathrm{i}\boldsymbol{y} + r\boldsymbol{u}_1[\; = \;]\boldsymbol{a}, \boldsymbol{b}[,$$

that is, the relative interior of L, and to any point of $]\boldsymbol{x} + \mathrm{i}(\boldsymbol{y} - r\boldsymbol{u}_1), \boldsymbol{x} + \mathrm{i}(\boldsymbol{y} + r\boldsymbol{u}_1)[$. ■

Proof of Theorem 4.14. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be two distinct points in the relative interior of the face F. Let L be the intersection of F with the affine line of $\mathbb{R}^k$ generated by $\boldsymbol{x}$ and $\boldsymbol{x}'$, which is a convex set. We assume that for a given $\boldsymbol{y} \in \mathbb{R}^k$, ξ can be holomorphically continued to the point $\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$. From the previous Lemma 4.17, ξ can also be holomorphically continued to any point $\boldsymbol{x}_1 + \mathrm{i}\boldsymbol{y}$, with $\boldsymbol{x}_1$ in any open segment with midpoint $\boldsymbol{x}$ included in L. Such a segment contains the point $\boldsymbol{x} + (\boldsymbol{x}' - \boldsymbol{x})/2^m$ for some integer $m \geqslant 0$. If $m = 0$, then it contains $\boldsymbol{x}'$, and the statement is proven. Otherwise, ξ can also be holomorphically continued at $\boldsymbol{x}+(\boldsymbol{x}'-\boldsymbol{x})/2^m+\mathrm{i}\boldsymbol{y}$, and we use Lemma 4.17 again for a segment with midpoint $\boldsymbol{x}+(\boldsymbol{x}'-\boldsymbol{x})/2^m$ containing $\boldsymbol{x}$. This segment contains $\boldsymbol{x}+(\boldsymbol{x}'-\boldsymbol{x})/2^{m-1}$. And the argument can be reiterated until we can apply it to a segment with midpoint $\boldsymbol{x}+(\boldsymbol{x}'-\boldsymbol{x})/2$ containing $\boldsymbol{x}$ and $\boldsymbol{x}'$. Therefore, ξ can be holomorphically continued to the point $\boldsymbol{x}'+\mathrm{i}\boldsymbol{y}$.

If $\boldsymbol{x}$ is on the boundary of F and ξ can be holomorphically continued to the point $\boldsymbol{x}+\mathrm{i}\boldsymbol{y}$, then ξ can be holomorphically continued to any point of a neighborhood of $\boldsymbol{x}+\mathrm{i}\boldsymbol{y}$ which contains an element $\tilde{\boldsymbol{x}}+\mathrm{i}\boldsymbol{y}$ with $\tilde{\boldsymbol{x}}$ in the relative interior of F, and the previous argument can be used starting with $\tilde{\boldsymbol{x}}$ instead of $\boldsymbol{x}$.

We now choose $\boldsymbol{y}'$ such that $\boldsymbol{y}'-\boldsymbol{y}\in W$. Since W is the affine hull of F and $\boldsymbol{x}'$ is in the relative interior of F, there exists a positive integer $m\geqslant 2$ such that

$$L'=\left[\boldsymbol{x}'-\frac{1}{m-1}(\boldsymbol{y}'-\boldsymbol{y}),\boldsymbol{x}'+\frac{1}{m-1}(\boldsymbol{y}'-\boldsymbol{y})\right]$$

is a subset of F whose midpoint is $\boldsymbol{x}'$. For every j between 0 and m, we set

$$\boldsymbol{y}_j=\boldsymbol{y}+\frac{j}{m}(\boldsymbol{y}'-\boldsymbol{y}),$$

so that for every j with $1\leqslant j\leqslant m$ we find that

$$\boldsymbol{x}'+\boldsymbol{y}_j-\boldsymbol{y}_{j-1}=\boldsymbol{x}'+\frac{1}{m}(\boldsymbol{y}'-\boldsymbol{y})$$

is in the relative interior of L'. The second part of Lemma 4.17 implies that if ξ is holomorphic at $\boldsymbol{x}'+\mathrm{i}\boldsymbol{y}_{j-1}$, then ξ is holomorphic at $\boldsymbol{x}'+\mathrm{i}\boldsymbol{y}_j$. We have just proven that ξ is holomorphic at $\boldsymbol{x}'+\mathrm{i}\boldsymbol{y}_0=\boldsymbol{x}'+\mathrm{i}\boldsymbol{y}$, therefore ξ is holomorphic at $\boldsymbol{x}'+\mathrm{i}\boldsymbol{y}_m=\boldsymbol{x}'+\mathrm{i}\boldsymbol{y}'$. ■

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