

Ball covering property on operators and Calkin algebras

by

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Abstract. A Banach space X is said to have the ball covering property (BCP) if the unit sphere of X can be covered by countably many open balls $B(x_i, r_i)$ with $r_i \leq \|x_i\|$ for each $i \in \mathbb{N}$. If there are $R, \delta > 0$ such that $r_i \leq R$ and $\|x_i\| - r_i > \delta$ for all $i \in \mathbb{N}$, then we say that X has the uniform ball covering property (UBCP). In this paper, we show that if X has a 1-unconditional basis or X is a 1-complemented subspace of a Banach space with a shrinking 1-unconditional basis, then the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ fails the BCP. It is also shown that if X has a shrinking unconditional basis with unconditional constant less than 2, then $\mathcal{B}(X)$ has the UBCP.

1. Introduction. A normed space X is said to have the *ball covering property* (BCP) [C06] if its unit sphere can be covered by countably many closed, or equivalently, open balls (B_n) , none of which contains the origin of X . The centers of the balls are called the *BCP points*. If in addition the radii of the covering balls are bounded, then we say X has the *strong ball covering property* (SBCP) [LZ21]. If X has the SBCP and there exists an open ball B_0 containing the origin such that $B_n \cap B_0 = \emptyset$ for all $n \in \mathbb{N}$, then X is said to have the *uniform ball covering property* (UBCP) [LZ21]. It follows directly that the UBCP implies the SBCP and that the SBCP implies the BCP. But in general, these properties are different [LZ21].

Separable normed spaces automatically have the BCP. It was shown in [CCL08] that BCP is not preserved under linear isomorphisms by providing renormings of ℓ_∞ . Actually it is easy to see that ℓ_∞ has the BCP while L_∞ can be shown to fail the BCP [LZ20]. Although the BCP is a geometric property, it is closely related to a topological property of the dual spaces. To be more precise, V. P. Fonf and C. Zanco [FZ09] proved that X^* is w^* -separable if and only if X can be $(1 + \varepsilon)$ -equivalently renormed to have the SBCP for all $\varepsilon > 0$. The BCP is also connected with other commonly

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used properties, such as the Radon–Nikodym property, uniform convexity and dentability [CW⁺11, SC15, SC18].

There are quite a few recent advances about the BCP. It was proved by Z. Luo and B. Zheng [LZ20] that $(\sum \oplus X_k)_E$ has the BCP if and only if each summand X_k has the BCP, where E is either a Lorentz sequence space, a separable Orlicz sequence space, or ℓ_∞ . It was also shown in [LZ20] that if (Ω, Σ, μ) is a separable measure space, then the space of Bochner integrable functions $L^p(\mu, X)$ has the BCP if and only if X has the BCP. In general, the BCP does not pass to subspaces or quotients [CCL08]. If X is the ℓ_p direct sum of Y and Z , then X has the BCP if and only if Y and Z have the BCP. However, in [LL⁺22], M. Liu, R. Liu, J. Lu and B. Zheng proved that $\mathcal{B}(H)$ has the BCP and L_∞ is a 1-complemented subspace of $\mathcal{B}(H)$ which fails the BCP.

It is known that the BCP does not pass to quotient spaces. The first such example is ℓ_∞/c_0 , which was given by L. Cheng, Q. Cheng and X. Liu [CCL08]. Recently, Q. Bao, R. Liu and J. Shen [BLS25] showed that if X has a shrinking 1-unconditional basis, then the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ fails the BCP. However, the classical case $\mathcal{B}(\ell_1)/\mathcal{K}(\ell_1)$ is not included. A more general question is whether the shrinkingness can be removed from the assumption.

QUESTION 1. *Let X be a Banach space with a 1-unconditional basis. Does the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ fail the BCP?*

In Section 2, we will give an affirmative answer to Question 1. Moreover, by using the machinery of Markushevich bases, we also prove that if X is a 1-complemented subspace of a Banach space with a shrinking 1-unconditional basis, then the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ fails the BCP. It should be pointed out that the unconditionality requirement in Question 1 is essential and cannot be replaced by a shrinking Schauder basis. For example, if we take X to be the Argyros–Haydon space [AH11], then every bounded linear operator on X is a scalar multiple of the identity plus a compact operator. An immediate observation is that $\mathcal{B}(X)$, $\mathcal{K}(X)$ and $\mathcal{B}(X)/\mathcal{K}(X)$ all have the BCP. P. Motakis [M24] proved that for every compact metric space K there exists a Banach space X_K whose Calkin algebra is isometric to $C(K)$. It is obvious that such $C(K)$ spaces are separable and hence have the BCP.

It was proved in [LL⁺22] that $\mathcal{B}(c_0)$ and $\mathcal{B}(\ell_p)$ ($1 \leq p < \infty$) have the UBCP. All these spaces have 1-unconditional bases. A natural question is the following:

QUESTION 2. *Let X be a Banach space with an unconditional basis. Does the space $\mathcal{B}(X)$ have the UBCP?*

In [BLS25], it was shown that if X has a monotone shrinking unconditional basis with unconditional constant less than 2, then $\mathcal{B}(X)$ has the BCP. In Section 3, we prove that the monotonicity is redundant and $\mathcal{B}(X)$ has the UBCP. Actually, we show that if X is a Banach space with X^* separable and Y has an unconditional basis with unconditional constant less than 2, then the space $\mathcal{B}(X, Y)$ has the UBCP.

2. BCP of Calkin algebras. Let X be a Banach space with a normalized Schauder basis $(e_n)_n$. Then every $x \in X$ can be written uniquely as $x = \sum_n e_n^*(x)e_n$, where $(e_n^*)_n$ denotes the sequence of biorthogonal functionals for $(e_n)_n$. As usual, we define

$$\text{supp } x = \{n \in \mathbb{N} : e_n^*(x) \neq 0\}.$$

We consider the space $c_{00}(X)$ of all elements of X with finite support. For a subset A of X , $[A]$ denotes the linear span of A . For $x \in c_{00}(X)$, we use $m(x, (e_i))$ and $M(x, (e_i))$ to denote respectively the minimum and maximum of $\text{supp } x$ with respect to the basis (e_i) . The set $\{i \in \mathbb{N} : m(x, (e_i)) \leq i \leq M(x, (e_i))\}$ is denoted by $I(x, (e_i))$. When there is no confusion about the basis, we simply write $m(x)$, $M(x)$ and $I(x)$ respectively. We say that a sequence $(x_j)_j$ in $c_{00}(X)$ is a *block sequence* of $(e_n)_n$ if there exist $0 = q_1 < q_2 < \dots$ such that, for all $j \geq 1$, x_j is in $[e_i : q_{j-1} < i \leq q_j]$.

Let T be a bounded linear operator from a Banach space X into a Banach space Y . The *essential norm* of T is defined by

$$\|T\|_e = \inf \{\|T - K\| : K \in \mathcal{K}(X, Y)\}.$$

DEFINITION 2.1. A Banach space X is said to have *property (E)* if for every bounded linear operator $T \in \mathcal{B}(X)$, there exists a normalized weakly null sequence $(x_n) \subset X$ such that

$$\|T\|_e = \lim \|Tx_k\|.$$

LEMMA 2.2. *Let X be a 1-complemented subspace of a Banach space Y with a bimonotone shrinking Schauder basis $(e_n)_n$. Then X has property (E).*

Proof. For each $n \in \mathbb{N}$, let P_n be the natural projection from Y onto $[e_i : 1 \leq i \leq n]$. Since $(e_n)_n$ is bimonotone, $\|P_n\| = \|I - P_n\| = 1$. Let P be a norm 1 projection from Y onto X and let T be any bounded linear operator in $\mathcal{B}(X)$. If T is compact, then the conclusion is obvious. So we may assume that T is not compact and $\|T\|_e > 0$. Fix a sequence (ε_n) of positive real numbers which converges to 0. Since $\|T\|_e = \|T - T P P_n|_X\|_e = \|T P - T P P_n|_X\|_e \leq \|T P (I - P_n)\|$, we can find a normalized sequence $(y_n)_n$ in X such that for each $n \in \mathbb{N}$,

$$\|T\|_e < \|T P (I - P_n) y_n\| + \varepsilon_n.$$

Let $x_n = P(I - P_n)y_n/\|P(I - P_n)y_n\|$. Then (x_n) is a normalized weakly null sequence in X and

$$\|T\|_e \leq \limsup \|Tx_n\|.$$

On the other hand, for any compact operator L on X , we have

$$\limsup \|Tx_n\| = \limsup \|(T - L)x_n\| \leq \|T - L\|.$$

By taking the infimum on the right-hand side, we get

$$(2.1) \quad \limsup \|Tx_n\| \leq \|T\|_e.$$

Combining the results above, we have

$$\|T\|_e = \limsup \|Tx_n\|.$$

The conclusion follows by passing to a subsequence of (x_n) . ■

Let (e_n) be a Schauder basis for a Banach space X . We say that (e_n) is *unconditional* if there exists a constant $C \geq 1$ such that for any sequences (a_n) and (b_n) with $|a_n| \leq |b_n|$ for all $n \in \mathbb{N}$,

$$\left\| \sum a_n e_n \right\| \leq C \left\| \sum b_n e_n \right\|.$$

The best constant C is called the *unconditional constant* for (e_n) .

Recall that an *M-basis* for a separable Banach space X is a biorthogonal system $(x_i, x_i^*) \subset X \times X^*$ such that $[x_i] = X$ and $[\overline{x_i^*}]^* = X^*$; here $[x_i]$ denotes the closed linear span of the x_i 's and $[\overline{x_i^*}]^*$ is the weak* closure of the linear span of the x_i^* 's. Markushevich [M24] (or see [LT77, Proposition 1.f.3]) proved that every separable Banach space admits an M-basis. If X^* is separable, we can choose an M-basis (x_i, x_i^*) such that $[x_i^*] = X^*$. In that case, we call (x_i, x_i^*) a *shrinking M-basis* for X . If (x_i, x_i^*) is a shrinking M-basis for X , then every normalized block sequence (y_i) of (x_i) converges weakly to 0. Given an M-basis (x_i, x_i^*) and $1 = n_1 < n_2 < \dots$, the sequence (X_i, X_i^*) defined by $X_i = [x_j]_{j=n_i}^{n_{i+1}-1}$, $X_i^* = [x_j^*]_{j=n_i}^{n_{i+1}-1}$ is called a *blocking* of (x_i, x_i^*) .

THEOREM 2.3. *Let X be an infinite-dimensional complemented subspace of a Banach space Y with a shrinking unconditional basis $\{e_n\}$. If X has property (E), then the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ does not have the ball covering property.*

Proof. Assume that the sequence $(K_n)_n$ is a ball covering of $\mathcal{B}(X)/\mathcal{K}(X)$. We will prove that there exists an operator $T \in \mathcal{B}(X)$ such that

- $\|T\|_e = 1$,
- $\|K_n - T\|_e \geq \|K_n\|_e$ for all n .

Let (f_i, f_i^*) be a shrinking M-basis for X such that $\|f_i\| = 1$ for all $i \in \mathbb{N}$. Since X has property (E), by a standard perturbation argument and induction, we can select for each n a normalized block sequence $(y_k^n)_k$ of (f_i) such that

- (1) $\|K_n\|_e = \lim_k \|K_n y_k^n\|$,
- (2) $I(y_k^n, (f_i)) \cap I(y_j^m, (f_i)) = \emptyset$ whenever $n \neq m$.

By (1) and (2), we can relabel the set $\{y_k^n\}_{k,n}$ as a block sequence (y_i) of (f_i) so that

$$\|K_n\|_e = \limsup_m \|K_n y_m\| \quad \text{for all } n.$$

Let (ε_i) be a sequence of positive real numbers decreasing to 0. Since both (e_i) and (f_i) are shrinking, again by induction we are able to find a lacunary subsequence (x_i) of (y_i) such that

- (i) $\|K_n\|_e = \limsup_m \|K_n x_m\|$ for all n ,
- (ii) there are increasing sequences (m_j) and (M_j) of natural numbers such that $m_j < M_j < m_{j+1}$ for all j and

$$\left\| \left(\sum_{i \notin [m_n, M_n]} e_i^* \otimes e_i \right) \Big|_{[f_i]_{i \in I(x_n, (f_i))}} \right\| < \varepsilon_n \quad \text{for all } n,$$

- (iii) for each $j \in \mathbb{N}$, there is a $t_j \in \mathbb{N}$ such that $M(x_j, (f_i)) < t_j < m(x_{j+1}, (f_i))$ and

$$\left\| \left(\sum_{i \notin (M_j, m_{j+1})} e_i^* \otimes e_i \right) (f_{t_j}) \right\| < \varepsilon_j.$$

Now we define an operator S on X as follows:

$$S = P \circ \left(\sum_{i \in \bigcup_j (M_j, m_{j+1})} e_i^* \otimes e_i \right),$$

where P is a bounded linear projection from Y onto X .

Since (e_i) is unconditional, S is bounded. Let C be the unconditional constant for (e_i) . By (ii),

$$\|S(x_j)\| < C\|P\|\varepsilon_j \quad \text{for all } j.$$

So we have $\lim_j \|S(x_j)\| = 0$. Next we show that S is not compact. Actually by (iii), we have

$$\begin{aligned} \|S(f_{t_j})\| &= \left\| P \circ \left(I - \left(\sum_{i \notin (M_j, m_{j+1})} e_i^* \otimes e_i \right) \right) (f_{t_j}) \right\| \\ &= \left\| f_{t_j} - P \circ \left(\sum_{i \notin (M_j, m_{j+1})} e_i^* \otimes e_i \right) (f_{t_j}) \right\| \geq 1 - \|P\|\varepsilon_j. \end{aligned}$$

Therefore $\limsup_j \|S(f_{t_j})\| \geq 1$. Since $(f_{t_j})_j$ is weakly null, S is not compact and $\|S\|_e > 0$.

Let $T = S/\|S\|_e$. Hence, for each n ,

$$\begin{aligned} \|K_n - T\|_e &= \left\| K_n - \frac{S}{\|S\|_e} \right\|_e \\ &\geq \limsup_j \left\| K_n x_j - \frac{S}{\|S\|_e} x_j \right\| = \limsup_j \|K_n x_j\| = \|K_n\|_e. \blacksquare \end{aligned}$$

By Lemma 2.2 and Theorem 2.3, we have an immediate corollary:

COROLLARY 2.4. *Let X be a 1-complemented subspace of a Banach space Y with a shrinking 1-unconditional basis. Then the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ fails the BCP.*

When we do not have shrinkingness for the unconditional basis, situation is different. Next we will show the general case. That is, if X has an 1-unconditional basis, then the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ does not have the BCP. The proof of the first lemma below is standard and will be omitted.

LEMMA 2.5. *Let X be a Banach space with a Schauder basis (e_n) , and (P_n) be the natural basis projection from X onto $[e_i]_{i=1}^n$. Let K be any compact operator in $\mathcal{B}(X)$. Then*

$$\lim_n \|(I - P_n)K\| = 0.$$

LEMMA 2.6. *Let X be a Banach space with a bimonotone basis (e_n) , and (P_n) be the natural basis projection from X onto $[e_i]_{i=1}^n$. Then for every operator $T \in \mathcal{B}(X)$, there exist a normalized block sequence (x_n) of (e_n) and an increasing sequence (k_n) of natural numbers such that*

$$\|T\|_e = \lim_n \|(I - P_{k_n})Tx_n\|.$$

Proof. By the definition of the essential norm, for any $n \in \mathbb{N}$, there exists a compact operator S_n such that

$$\|T\|_e > \|T - S_n\| - 1/n.$$

By Lemma 2.5, there exists a $t_n \in \mathbb{N}$ such that

$$\|(I - P_{t_n})S_n\| < 1/n.$$

So we have

$$\|T\|_e > \|T - S_n\| - 1/n \geq \|(I - P_{t_n})(T - S_n)\| - 1/n \geq \|(I - P_{t_n})T\| - 2/n.$$

On the other hand,

$$\|T\|_e = \|(I - P_{t_n})T\|_e \leq \|(I - P_{t_n})T\|.$$

Therefore, we get

$$\|(I - P_{t_n})T\| - 2/n \leq \|T\|_e \leq \|(I - P_{t_n})T\|.$$

Then we can choose a finitely supported norm 1 vector y_n in X such that

$$\|(I - P_{t_n})Ty_n\| > \|(I - P_{t_n})T\| - 1/n$$

and

$$\|(I - P_{t_n})Ty_n\| - 2/n \leq \|T\|_e \leq \|(I - P_{t_n})Ty_n\| + 1/n.$$

By taking the limit over n , we have

$$\|T\|_e = \lim_n \|(I - P_{t_n})Ty_n\| = \lim_n \|(I - P_{t_n})T\|.$$

Now we will show that a subsequence of (y_n) is a small perturbation of a block basis.

Let $r_1 = 1$. Since y_1 is finitely supported, $M(y_1)$ is finite. By Lemma 2.5, there is an $r_2 > r_1$ such that

$$\|(I - P_{t_{r_2}})TP_{M(y_1)}\| < 1/2.$$

In general, since y_{r_n} is finitely supported, we can find $r_{n+1} > r_n$ such that

$$\|(I - P_{t_{r_{n+1}}})TP_{M(y_{r_n})}\| < 1/n.$$

So we have

$$\begin{aligned} \|(I - P_{t_{r_{n+1}}})Ty_{r_{n+1}}\| &= \|(I - P_{t_{r_{n+1}}})TP_{M(y_{r_n})}y_{r_{n+1}} + (I - P_{t_{r_{n+1}}})T(I - P_{M(y_{r_n})})y_{r_{n+1}}\| \\ &< \|(I - P_{t_{r_{n+1}}})T(I - P_{M(y_{r_n})})y_{r_{n+1}}\| + 1/n. \end{aligned}$$

Let $x_1 = y_1, x_{n+1} = (I - P_{M(y_{r_n})})y_{r_{n+1}}/\|(I - P_{M(y_{r_n})})y_{r_{n+1}}\|$ and $k_n = t_{r_n}$. Then (x_n) is a normalized block basis and

$$\|(I - P_{t_{r_{n+1}}})Ty_{r_{n+1}}\| - 1/n < \|(I - P_{k_{n+1}})Tx_{n+1}\| \leq \|(I - P_{t_{r_{n+1}}})T\|.$$

Hence

$$\|T\|_e = \lim_n \|(I - P_{k_n})Tx_n\| = \lim_n \|(I - P_{k_n})T\|. \quad \blacksquare$$

THEOREM 2.7. *Let X be a Banach space with a 1-unconditional basis. Then the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ does not have the BCP.*

Proof. We assume that there is a ball covering $(K_n)_n$ of $\mathcal{B}(X)/\mathcal{K}(X)$. To reach a contradiction, we will construct an operator $T \in \mathcal{B}(X)$ such that

- $\|T\|_e = 1$,
- $\|K_n - T\|_e \geq \|K_n\|_e$ for all n .

For each K_n , by Lemma 2.6, there exist a normalized block sequence (y_i^n) of (e_n) and an increasing sequence (k_i^n) of natural numbers such that

$$\|K_n\|_e = \lim_i \|(I - P_{k_i^n})K_n y_i^n\|.$$

Inductively, we can select a block sequence (x_i) such that

- (x_i) contains a subsequence of (y_i^n) for each n and hence

$$\|K_n\|_e \leq \limsup_m \|(I - P_{k_m})K_n x_m\| \quad \text{for all } n,$$

- for each i ,

$$M(x_i) + 1 < m(x_{i+1}).$$

Let $S \in \mathcal{B}(X)$ be defined by

$$S = \sum_i e_{M(x_i)+1}^* \otimes e_{M(x_i)+1}.$$

Since (e_i) is unconditional, S is bounded. Since $S(e_{M(x_i)+1}) = e_{M(x_i)+1}$ for each i , S is not compact. Let $T = S/\|S\|_e$. Then $\|T\|_e = 1$ and $T(x_i) = 0$ for each i . We will show $\|K_n - T\|_e \geq \|K_n\|_e$ for each n . For any $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists a compact operator K_ε such that

$$\|K_n - T\|_e > \|K_n - T - K_\varepsilon\| - \varepsilon.$$

Then by Lemma 2.5, there exists an $N_\varepsilon \in \mathbb{N}$ such that if $m > N_\varepsilon$, then

$$\|(I - P_m)K_\varepsilon\| < \varepsilon.$$

So for any $m > N_\varepsilon$ we have

$$\begin{aligned} \|K_n - T\|_e &> \|K_n - T - K_\varepsilon\| - \varepsilon \geq \|(I - P_{k_m})(K_n - T - K_\varepsilon)\| - \varepsilon \\ &\geq \|(I - P_{k_m})(K_n - T)\| - 2\varepsilon \geq \|(I - P_{k_m})(K_n - T)x_m\| - 2\varepsilon \\ &\geq \|(I - P_{k_m})K_n x_m\| - 2\varepsilon. \end{aligned}$$

Taking the limit over m , we get

$$\|K_n - T\|_e \geq \limsup_m \|(I - P_{k_m})K_n x_m\| - 2\varepsilon = \|K_n\|_e - 2\varepsilon.$$

Since ε is arbitrary, we have

$$\|K_n - T\|_e \geq \|K_n\|_e. \quad \blacksquare$$

REMARK 2.8. Note that the theorem above also holds if X has only an unconditional finite-dimensional decomposition (FDD). The unconditionality assumption in the theorem cannot be removed without adding other assumptions. Indeed, the space $\mathfrak{X}_{\text{AH}}$, introduced by Argyros and Haydon [AH11], has a shrinking Schauder basis but its Calkin algebra is one-dimensional, which of course has the BCP.

PROBLEM 2.9. *Does there exist a non-separable Banach space X whose Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ has the BCP?*

3. BCP for $\mathcal{B}(X)$. As mentioned in the introduction, it is known that $\mathcal{B}(c_0)$ and $\mathcal{B}(\ell_p)$ ($1 \leq p < \infty$) have the BCP. The spaces c_0 and ℓ_p ($1 \leq p < \infty$) have 1-unconditional bases. In this section, we will prove that $\mathcal{B}(X)$ has the BCP if X has a shrinking unconditional basis with unconditional constant less than 2. Actually, we have the following more general result.

THEOREM 3.1. *Let X and Y be Banach spaces such that X^* is separable and Y has an unconditional basis with unconditional constant less than 2. Then any subspace E of $\mathcal{B}(X, Y)$ containing $\mathcal{F}(X, Y)$ has the UBCP.*

Proof. It is sufficient to show that the UBCP points of $\mathcal{B}(X, Y)$ can be taken in $\mathcal{F}(X, Y)$. Let (e_n) be a normalized γ -unconditional basis of Y with $1 \leq \gamma < 2$ and let (e_n^*) be the corresponding biorthogonal functionals in Y^* . Since X^* is separable, there exists a countable dense subset (x_n^*) in $B(0, \gamma)$ of X^* . We will show that the countable set $\mathcal{A}$ given by

$$\mathcal{A} = \left\{ \lambda \sum_{i \in F} x_{m_i}^* \otimes e_i : \lambda \in \mathbb{Q} \cap [0, 2\gamma], F \subset \mathbb{N} \text{ finite}, m_i \in \mathbb{N} \right\}$$

is a set of UBCP points for $\mathcal{B}(X, Y)$.

Let $T = \sum_{n=1}^{\infty} e_n^* T \otimes e_n \in S_{\mathcal{B}(X, Y)}$, where the series converges in the strong operator topology. Let

$$\begin{aligned} \beta_1 &= \left\{ \sup_F \left\| \sum_{i \in F} e_i^* T \otimes e_i \right\| : F \subset \mathbb{N} \text{ finite} \right\}, \\ \beta &= \left\{ \sup_F \left\| \sum_{i \in F} \varepsilon_i e_i^* T \otimes e_i \right\| : F \subset \mathbb{N} \text{ finite}, \varepsilon_i = \pm 1 \right\}. \end{aligned}$$

Since $\|T\| = 1$ and (e_i) is γ -unconditional, by the definition of β_1, β and by the triangle inequality it is easy to see that

$$\beta \leq \gamma < 2 \quad \text{and} \quad 1 \leq \beta_1 \leq \beta \leq 2\beta_1.$$

CASE 1: $\beta < \frac{7}{4}\beta_1$. We can choose a $\lambda \in \mathbb{Q}$ and $\frac{1}{4} > \delta > 0$ such that $2\beta_1 > \lambda > \frac{\beta}{1-2\delta} + \frac{1}{8}$, which is possible since $\frac{1}{8} + \beta < \frac{1}{4} + \frac{7}{4}\beta_1 \leq \frac{\beta_1}{4} + \frac{7}{4}\beta_1 = 2\beta_1$.

Pick a finite subset $F \subset \mathbb{N}$ such that

$$\lambda < 2\theta,$$

where $\theta = \|\sum_{i \in F} e_i^* T \otimes e_i\| \leq \beta_1$. Since $\|\frac{e_i^* T}{\theta}\| \leq \gamma$ for all $i \in F$, we can find $m_i \in \mathbb{N}$ such that

$$\sum_{i \in F} \left\| x_{m_i}^* - \frac{e_i^* T}{\theta} \right\| < \delta.$$

Then

$$\begin{aligned} \left\| \lambda \sum_{i \in F} x_{m_i}^* \otimes e_i \right\| &= \left\| \lambda \sum_{i \in F} \left(x_{m_i}^* \otimes e_i - \frac{e_i^* T}{\theta} \otimes e_i + \frac{e_i^* T}{\theta} \otimes e_i \right) \right\| \\ &\geq \lambda \left\| \sum_{i \in F} \frac{e_i^* T}{\theta} \otimes e_i \right\| - \lambda \left\| \sum_{i \in F} \left(x_{m_i}^* \otimes e_i - \frac{e_i^* T}{\theta} \otimes e_i \right) \right\| \\ &\geq \lambda(1 - \delta). \end{aligned}$$

Also, since $|\frac{\lambda}{\theta} - 1| = \frac{\lambda}{\theta} - 1 \leq 1$, we get

$$\begin{aligned}
\left\| T - \sum_{i \in F} \lambda x_{m_i}^* \otimes e_i \right\| &= \left\| \sum_{i=1}^{\infty} e_i^* T \otimes e_i - \sum_{i \in F} \lambda x_{m_i}^* \otimes e_i \right\| \\
&\leq \left\| \left(1 - \frac{\lambda}{\theta}\right) \sum_{i \in F} e_i^* T \otimes e_i + \sum_{i \in F^c} e_i^* T \otimes e_i \right\| \\
&\quad + \sum_{i \in F} \left\| \lambda \left(\frac{e_i^* T}{\theta} - x_{m_i}^* \right) \otimes e_i \right\| \\
&\leq \left\| \sum_{i \in F} \left(1 - \frac{\lambda}{\theta}\right) e_i^* T \otimes e_i + \sum_{i \in F^c} e_i^* T \otimes e_i \right\| + \lambda \delta \\
&\leq \left(\frac{\lambda}{\theta} - 1 \right) \left\| - \sum_{i \in F} e_i^* T \otimes e_i + \sum_{i \in F^c} e_i^* T \otimes e_i \right\| \\
&\quad + \left(2 - \frac{\lambda}{\theta} \right) \left\| \sum_{i \in F^c} e_i^* T \otimes e_i \right\| + \lambda \delta \\
&\leq \beta + \lambda \delta \leq \left(\lambda - \frac{1}{8} \right) (1 - 2\delta) + \lambda \delta \\
&\leq \lambda(1 - \delta) - \frac{1}{8}(1 - 2\delta) < \lambda(1 - \delta) - \frac{1}{8} \\
&< \left\| \lambda \sum_{i \in F} x_{m_i}^* \otimes e_i \right\| - \frac{1}{8}.
\end{aligned}$$

CASE 2: $\beta \geq \frac{7}{4}\beta_1$. Let $\delta = 2 - \gamma$. By the definition of β_1 , we choose a finite subset $F \subset \mathbb{N}$ such that

$$\left\| \sum_{i \in F} e_i^* T \otimes e_i \right\| > \beta_1 - \frac{\delta}{8}.$$

Since $\|e_i^* T\| \leq \gamma$ for each $i \in F$, we can find $(x_{m_i}^*)_{i \in F}$ such that

$$\sum_{i \in F} \|e_i^* T - x_{m_i}^*\| < \frac{\delta}{8}.$$

Then

$$\left\| \sum_{i \in F} 2x_{m_i}^* \otimes e_i \right\| > 2 \left(\beta_1 - \frac{\delta}{8} \right) - 2 \frac{\delta}{8} = 2\beta_1 - \frac{\delta}{2}.$$

Now we show that T can be covered by the ball centered at $2 \sum_{i \in F} x_{m_i}^* \otimes e_i$:

$$\begin{aligned}
\left\| T - 2 \sum_{i \in F} x_{m_i}^* \otimes e_i \right\| &= \left\| \sum_{i=1}^{\infty} e_i^* T \otimes e_i - 2 \sum_{i \in F} x_{m_i}^* \otimes e_i \right\| \\
&\leq \left\| \sum_{i=1}^{\infty} e_i^* T \otimes e_i - 2 \sum_{i \in F} e_i^* T \otimes e_i \right\| \\
&\quad + 2 \left\| \sum_{i \in F} x_{m_i}^* \otimes e_i - \sum_{i \in F} e_i^* T \otimes e_i \right\| \\
&\leq \left\| \sum_{i \in F^c} e_i^* T \otimes e_i - \sum_{i \in F} e_i^* T \otimes e_i \right\| + 2 \frac{\delta}{8} \\
&\leq \beta + \frac{\delta}{4} \leq \gamma + \frac{\delta}{4} = 2 - \frac{3}{4} \delta \leq 2\beta_1 - \frac{3}{4} \delta \\
&< \left\| 2 \sum_{i \in F} x_{m_i}^* \otimes e_i \right\| - \frac{\delta}{4}. \blacksquare
\end{aligned}$$

COROLLARY 3.2. *Let X be a Banach space which has a shrinking unconditional basis with unconditional constant less than 2. Then $\mathcal{B}(X)$ has the UBCP.*

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