

Hardy spaces meet harmonic weights revisited

by

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Abstract. We investigate Hardy spaces $H_L^1(X)$ corresponding to self-adjoint operators L . Our main aim is to obtain a description of $H_L^1(X)$ in terms of atomic decompositions similar to such characterization of the classical Hardy spaces $H^1(\mathbb{R}^d)$. Under suitable assumptions, such a description was obtained by Yan and the authors in [Trans. Amer. Math. Soc. 375 (2022), 6417–6451], where the atoms associated with an L -harmonic function are considered. Here we continue this study and modify the previous definition of atoms.

The modified approach allows us to investigate settings when the generating operator is related to a system of linearly independent harmonic functions. In this context, the cancellation condition for atoms is adjusted to fit this system. In an explicit example, we consider a symmetric manifold with ends $\mathbb{R}^d \# \mathbb{R}^d$. For this manifold the space of bounded harmonic functions is two-dimensional. Any element from the Hardy space $H_L^1(X)$ has to be orthogonal to all of the harmonic functions in the system.

1. Introduction

1.1. Background and assumptions. Hardy and BMO space theory is a significant mainstream area of harmonic analysis. Its origin can be traced back to a notion of complex analysis initiated by Hardy. For an elegant and comprehensive description of the classical theory of real Hardy and BMO spaces, we refer readers to [S93]. A recent new direction in this area is to extend the investigation to Hardy and BMO spaces corresponding to semigroups generated by some differential operators. It is a natural idea. Indeed, one of the equivalent forms of the classical definition can be stated in terms of the maximal function corresponding to the heat semigroup. That is, we

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define the maximal function M by the formula

$$Mf(x) = \sup_{t>0} |\exp(t\Delta)f(x)|, \quad x \in \mathbb{R}^n,$$

where Δ is the standard Laplace operator and $\exp(t\Delta)$ is the corresponding heat semigroup. Then we define the $H^1(\mathbb{R}^n)$ -norm by setting

$$\|f\|_{H^1(\mathbb{R}^n)} = \|Mf(x)\|_{L^1(\mathbb{R}^n)}.$$

This definition can be generalized for any semigroup generated by a non-negative self-adjoint operator L defined on $L^2(X, \mu)$, where (X, μ) is a measure space. There are many papers devoted to the study of Hardy and BMO spaces corresponding to the semigroup of operators; see for example [DZ13, DZ12, SY18, DY05, CKS93, AR03, HMM11] and references therein. A general approach in this direction is discussed in the seminal paper [HL⁺11].

Our main goal is to describe the Hardy spaces $H_L^1(X)$ associated with an operator L in terms of atomic decompositions. For the classical Hardy space $H^1(\mathbb{R}^n)$ such theory was obtained in [C74, L78]. In [PSY22] the authors described the Hardy space $H_L^1(X)$ by using atoms that are related to a non-negative L -harmonic function h . Here we continue this study and modify the notion of atoms from there. The new definition is more natural and its geometrical aspects are more evident. The approach which we develop in this paper allows us to study the cases when there exist two or more linearly independent harmonic functions related to the operator L . In this situation, atoms satisfy the cancellation conditions with respect to all these functions. Additionally, we study dual BMO-type spaces.

We shall work under the following assumptions. Let (X, d, μ) be a metric measure space of homogeneous type in the sense of Coifman and Weiss. In particular, μ satisfies the doubling condition and there exist $C, D > 0$ such that

$$(1.1) \quad \mu(B(x, R)) \leq C \left(\frac{R}{r} \right)^D \mu(B(x, r)), \quad x \in X, 0 < r \leq R,$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$.

In what follows we study a positive self-adjoint operator L acting on $L^2(X, \mu) = L^2(\mu)$ and the semigroup $T_t = \exp(-tL)$ generated by L . We assume that there exists a non-negative integral kernel $T_t(x, y)$ related to T_t that satisfies the upper Gaussian bounds, i.e. there exist $C, c > 0$ such that

$$(UG) \quad 0 \leq T_t(x, y) \leq C \mu(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{d(x, y)^2}{ct}\right),$$

$t > 0, x, y \in X.$

The maximal function corresponding to the semigroup $T_t = \exp(-tL)$ is given by the formula $M_L f(x) = \sup_{t>0} |T_t f(x)|$. For $f \in L^2(\mu)$ we set

$$\|f\|_{H_L^1(X)} = \|M_L f\|_{L^1(\mu)}.$$

Define the Hardy space $H_L^1(X)$ as the completion of the set $\{f \in L^2(\mu) : \|f\|_{H_L^1(X)} < \infty\}$ with respect to this norm; cf. [HL⁺11, PSY22].

In this paper we shall consider the following atomic Hardy spaces.

DEFINITION 1.1. Let $r, r' > 1$ be conjugate exponents, i.e. $1/r + 1/r' = 1$, let $m \in \mathbb{N}$, and $h_1, \dots, h_m \in L_{\text{loc}}^{r'}(\mu)$. We say that a function $a : X \rightarrow \mathbb{C}$ is a $(\mu; h_1, \dots, h_m; r)$ -atom if there exists a ball B in X such that

$$(1.2) \quad \text{supp } a \subseteq B, \quad \|a\|_{L^r(\mu)} \leq \mu(B)^{-1+1/r}, \quad \int a(x) h_j(x) d\mu(x) = 0$$

for $j = 1, \dots, m$. The space $H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$ is defined as follows: a function f is in $H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$ if $f(x) = \sum_k \lambda_k a_k(x)$, where $\sum_k |\lambda_k| < \infty$, and a_k are $(\mu; h_1, \dots, h_m; r)$ -atoms. We set

$$\|f\|_{H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)} = \inf \sum_k |\lambda_k|,$$

where the infimum is taken over all representations as above.

Note that for an atom a as in (1.2) we have $\|a\|_{L^1(\mu)} \leq 1$ and thus the series $\sum_k \lambda_k a_k(x)$ converges in $L^1(\mu)$. Moreover, by a standard argument, $H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$ is a Banach subspace of $L^1(\mu)$. We shall also study related BMO-type spaces that are defined as follows.

DEFINITION 1.2. Let $m \in \mathbb{N}$, $r' > 1$, and $h_1, \dots, h_m \in L_{\text{loc}}^{r'}(\mu)$. Define $V = \text{span}(h_1, \dots, h_m)$. For $g \in L_{\text{loc}}^{r'}(\mu)$ we define

$$\|g\|_{\text{BMO}(\mu; h_1, \dots, h_m; r')} = \sup_B \left(\inf_{u \in V} \left(\mu(B)^{-1} \int_B |g(x) - u(x)|^{r'} d\mu(x) \right)^{1/r'} \right),$$

where the supremum is taken over all balls in X . By definition, the elements of the space $\text{BMO}(\mu; h_1, \dots, h_m; r')$ are classes $\{g + u : u \in V\}$ such that $\|g\|_{\text{BMO}(\mu; h_1, \dots, h_m; r')} < \infty$.

1.2. Operators related to one harmonic function. In this section, as in [PSY22], we assume that there exists a non-negative harmonic function h such that the heat semigroup T_t , after applying the Doob transform (or h -transform) corresponding to h , satisfies the upper and lower Gaussian estimates. More precisely, we assume that there exists a function $h \in L_{\text{loc}}^\infty(X)$ that is positive a.e., L -harmonic in the sense that for all $t > 0$,

$$(1.3) \quad T_t h(x) = h(x) \quad \text{for a.e. } x \in X,$$

and that there exist $c_1, c_2, C > 0$ such that for a.e. $x, y \in X$ and $t > 0$ we have

$$(ULG_h) \quad \begin{aligned} \frac{T_t(x, y)}{h(x)h(y)} &\geq \frac{C^{-1}}{\mu_{h^2}(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{c_1 t}\right), \\ \frac{T_t(x, y)}{h(x)h(y)} &\leq \frac{C}{\mu_{h^2}(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{c_2 t}\right). \end{aligned}$$

Recall that the notion of atoms introduced in [PSY22] was different from the atoms introduced in Definition 1.1. The atoms from [PSY22] are denoted as $[\mu, h]$ -atoms in contrast to $(\mu; h; 2)$ -atoms that we consider here. The $[\mu, h]$ -atom is defined by the conditions

$$(1.4) \quad \text{supp } a \subseteq B, \quad \|a\|_{L^2(h^{-1}\mu)} \leq \mu_h(B)^{-1/2}, \quad \int a(x)h(x) d\mu(x) = 0,$$

where μ_h is the measure on X with density $h(x)d\mu(x)$. Notice that the difference between $[\mu, h]$ -atoms and $(\mu; h; 2)$ -atoms lies in the L^2 -size condition. The space $H_{\text{at}}^1[\mu, h]$ is defined analogously as in Definition 1.1. It appears that, under suitable assumptions, the spaces $H_L^1(X)$, $H_{\text{at}}^1(\mu; h; 2)$, and $H_{\text{at}}^1[\mu, h]$ are equivalent.

At this point let us also mention that the assumption (1.3) is, in some sense, superfluous. If a function h satisfies (ULG_h) then one can find $\tilde{h}$ that is comparable to h and satisfies (1.3) (see [PSY22, Proposition 2.3]).

In order to state our results, recall that a non-negative function $w \in L_{\text{loc}}^1(\mu)$ on X is in the *Reverse Hölder class* $\text{RH}_q(\mu)$, $1 < q < \infty$, if there is a constant $C > 0$ such that

$$(1.5) \quad \left(\frac{1}{\mu(B)} \int_B w(x)^q d\mu(x) \right)^{1/q} \leq \frac{C}{\mu(B)} \int_B w(x) d\mu(x)$$

for every ball B in X . Moreover, $\text{RH}_\infty(\mu)$ is defined by the condition

$$(1.6) \quad \sup_{x \in B} w(x) \leq \frac{C}{\mu(B)} \int_B w(x) d\mu(x).$$

Recall that $\text{RH}_{q_2}(\mu) \subseteq \text{RH}_{q_1}(\mu)$ for $1 < q_1 \leq q_2 \leq \infty$.

We are ready to state a theorem that establishes the equality $H_L^1(X) = H_{\text{at}}^1(\mu; h; 2)$.

THEOREM A. *Suppose that a metric measure space (X, d, μ) satisfies the doubling condition (1.1). Let L be a self-adjoint densely defined operator on $L^2(X)$ such that the upper Gaussian bounds (UG) hold for the corresponding heat kernel. Assume that $h : X \rightarrow [0, \infty)$ is an L -harmonic function in the sense of (1.3), the estimates (ULG_h) are satisfied, and $h \in \text{RH}_\infty(\mu)$. Then the spaces $H_L^1(X)$ and $H_{\text{at}}^1(\mu; h; 2)$ coincide and have comparable norms.*

The proof of Theorem A will be given in Section 2 below. We additionally prove a similar result with a slightly weaker assumption, namely $h \in \text{RH}_q(\mu)$ with $q < \infty$ large enough; see Theorem 2.1. Let us point out the differences between Theorem A and [PSY22, Theorem A], which we recall in Theorem 2.2 below. Firstly and most significantly, the $(\mu; h; 2)$ -atoms in Theorem A satisfy $\|a\|_{L^2(\mu)} \leq \mu(B)^{-1/2}$, whereas the $[\mu, h]$ -atoms from [PSY22] have a more complicated size condition $\|a\|_{L^2(h^{-1}\mu)} \leq \mu_h(B)^{-1/2}$. The latter in-

volved the function h . We shall see that this change opens a possibility of studying manifolds in the case when there exist two or more linearly independent bounded harmonic functions. It also has clearer geometrical meaning when h is used only in the cancellation condition. Secondly, the assumption $h^{-1} \in A_p(\mu_{h^2})$ from Theorem 2.2 is replaced by $h \in \text{RH}_\infty(\mu)$. Thirdly, a simple fact stated in Proposition 2.4 allows us to get rid of the assumption that μ_{h^2} is a measure that satisfies the doubling property. Let us mention that for all the applications known to us the condition $h \in \text{RH}_\infty(\mu)$ holds.

As a consequence of our new atomic characterization of $H_L^1(X)$ we also obtain a more natural description of the dual space.

THEOREM B. *Assume that $h \in L_{\text{loc}}^2(\mu)$. Then the space $\text{BMO}(\mu; h; 2)$ is dual to $H_{\text{at}}^1(\mu; h; 2)$.*

Let us notice that the above statement does not refer to the operator L and the corresponding semigroup T_t . However, its main motivation comes from Theorem A. We shall prove Theorem B in a more general setting (with a set of functions $h_1, \dots, h_m$ instead of just one function h) in Section 3.

1.3. Two harmonic functions. Another goal of this paper is to study operators that are related to two (or more) linearly independent harmonic functions. We shall discuss this situation in the particular case of manifolds with ends, but one can apply our result in other similar circumstances. Analysis on manifolds with ends has recently attracted a significant broad interest. The theory of heat kernel bounds on such manifolds was studied, among others, by Davies, Grigor'yan, Ishiwata, and Saloff-Coste in [D97, GS09, GIS23, GIS21]. The Riesz transform in this context was studied in e.g. [BS23, H23, CCH06, HS19, HNS24].

We shall consider a symmetric Riemannian manifold with two ends of the form $M = \mathbb{R}^n \# \mathbb{R}^n$, $n \geq 3$. Such a manifold is obtained as a connected sum of two copies of $\mathbb{R}^n$ combined by a compact smooth connection. We shall also assume that there exists a symmetry $\sigma : M \rightarrow M$ that interchanges one end with the other and that preserves both the metric and the measure. For precise descriptions, definitions, and assumptions, see Section 5.2.

Let L_M be a Laplace–Beltrami operator on $M = \mathbb{R}^n \# \mathbb{R}^n$. Denote the associated heat semigroup by $T_t = \exp(-tL_M)$. It is known that the corresponding heat kernel $T_t(x, y)$ satisfies (UG) (see e.g. [GS09]). In addition, for L_M there exist two linearly independent non-negative bounded harmonic functions h_+ and h_- . One possible choice of h_+ can be given in probability language, where $h_+(x)$ is the probability that the process starting at $x \in M$ will escape as $t \rightarrow \infty$ via “the right end”. For an analytic description of these functions, see e.g. [HS19].

The following theorem establishes the equivalence of $H_{L_M}^1(M)$ and the atomic Hardy space $H_{\text{at}}^1(\mu; h_+, h_-; 2)$ from Definition 1.1.

THEOREM C. *Let L_M be the Laplace–Beltrami operator on a symmetric manifold $M = \mathbb{R}^n \# \mathbb{R}^n$ as in Section 5.2. Denote by h_+ and h_- the bounded L_M -harmonic functions. Then the Hardy spaces $H_{L_M}^1(M)$, $H_{\text{at}}^1(\mu; h_+, h_-; 2)$ coincide and there exists $C > 0$ such that*

$$C^{-1} \|f\|_{H_{L_M}^1(M)} \leq \|f\|_{H_{\text{at}}^1(\mu; h_+, h_-; 2)} \leq C \|f\|_{H_{L_M}^1(M)}.$$

The proof of Theorem C is discussed in Section 5.2 below. Using Theorem 3.1 one can describe the dual space as follows.

COROLLARY 1.3. *For the atomic Hardy space $H_{\text{at}}^1(\mu; h_+, h_-; 2)$ from Theorem C the dual space is $\text{BMO}(\mu; h_+, h_-; 2)$.*

2. Atomic characterizations of $H_L^1(X)$. In this section we consider the metric measure space (X, d, μ) and operators L, T_t as in Sections 1.1 and 1.2. In particular, we assume that there exists h such that (1.3) and (ULG_h) are satisfied. Our main goal here is to prove Theorem A. Actually, we shall prove a slightly more general Theorem 2.1 that covers Theorem A as the special case $q = \infty$ and $r = 2$.

THEOREM 2.1. *Suppose that a metric measure space (X, d, μ) satisfies the doubling condition (1.1). Let L be a self-adjoint operator on $L^2(X)$ such that the upper Gaussian bounds (UG) hold for the corresponding heat kernel semigroup. Assume that $h : X \rightarrow [0, \infty)$ is an L -harmonic function in the sense of (1.3), the estimates (ULG_h) are satisfied, and $h \in \text{RH}_q(\mu)$ for $q \in (3, \infty]$ sufficiently large. Then the Hardy space associated with L satisfies*

$$H_L^1(X) = H_{\text{at}}^1(\mu; h; r)$$

with $r = 2q/(q+1)$ if $q < \infty$ and $r = 2$ if $q = \infty$. Moreover, the corresponding norms are equivalent.

In this note the parameters $q \in (1, \infty]$ from $\text{RH}_q(\mu)$ (see (1.5)) and $r \leq 2$ from $(\mu; h; r)$ -atoms are always related by

$$(2.1) \quad r = \frac{2q}{q+1}.$$

Notice that if r and $q > 3$ satisfy (2.1), then $r' < q$. Thus $h \in \text{RH}_q(\mu)$ implies that $h \in \text{RH}_{r'}(\mu)$, and $h \in L^{r'}(\mu)$ as is required in Definition 1.1. There is a natural question whether the spaces $H_{\text{at}}^1(\mu; h; r)$ are equivalent for the full range $r \in (1, \infty]$, but we do not investigate this issue here.

2.1. Auxiliary results. Before proceeding to the proof of Theorems A and 2.1 we shall recall some results and prove auxiliary estimates.

By definition, a locally integrable non-negative function w belongs to the Muckenhoupt class $A_p(\nu)$ if

$$(2.2) \quad \left(\frac{1}{\nu(B)} \int_B w(x) d\nu(x) \right) \left(\frac{1}{\nu(B)} \int_B w(x)^{-1/(p-1)}(x) d\nu(x) \right)^{p-1} \leq C$$

for all balls B .

Recall that the atomic Hardy spaces $H_{\text{at}}^1[\mu, h]$ with $[\mu, h]$ -atoms (see (1.4)) are considered in [PSY22], where the following theorem is proved.

THEOREM 2.2 ([PSY22, Theorem A]). *Let (X, μ, d) be a space satisfying (1.1), and L be a non-negative self-adjoint operator related to the semigroup T_t with a kernel $T_t(x, y)$ satisfying (UG). Assume that there exists a function $h : X \rightarrow (0, \infty)$ such that h is L -harmonic in the sense of (1.3), the measure μ_{h^2} is doubling, and (ULG_h) is satisfied. Then there exists $p_0 \in (1, 2]$ such that if $h^{-1} \in A_{p_0}(\mu_{h^2})$, then $H_L^1(X)$ and $H_{\text{at}}^1[\mu, h]$ coincide and*

$$\|f\|_{H_L^1(X)} \simeq \|f\|_{H_{\text{at}}^1[\mu, h]}.$$

Recall the notion of the Doob transform (or h -transform) technique (see e.g. [GS11, Section 5.1.2]). This is one of the main tools in the proof of Theorem 2.2. For the semigroup T_t on $L^2(\mu)$ one can define a new kernel

$$\tilde{T}_t(x, y) = \frac{T_t(x, y)}{h(x)h(y)},$$

which defines a semigroup on $L^2(\mu_{h^2})$. The semigroup $\tilde{T}_t$ is conservative, i.e.

$$\int_X \tilde{T}_t(x, y) d\mu_{h^2}(x) = 1.$$

Notice that the assumption (ULG_h) translates into the upper and lower Gaussian bounds for $\tilde{T}_t(x, y)$, i.e.

$$(2.3) \quad \begin{aligned} \tilde{T}_t(x, y) &\geq C^{-1} \mu_{h^2}(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{d(x, y)^2}{c_1 t}\right), \\ \tilde{T}_t(x, y) &\leq C \mu_{h^2}(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{d(x, y)^2}{c_2 t}\right). \end{aligned}$$

It is known that the estimates (ULG_h) automatically give the following Hölder-type estimates; for a short proof see e.g. [DP18, Section 4]. See also [GS11, Theorem 5.11].

PROPOSITION 2.3. *Assume that (X, d, μ) is a metric measure space that satisfies (1.1). Let T_t be a semigroup related to a kernel $T_t(x, y)$ such that (ULG_h) holds for a function h . Then there exist $C, \delta > 0$ such that for $t > 0$*

and $x, y, y_0 \in X$ satisfying $d(y, y_0) \leq \sqrt{t}$ we have

$$(2.4) \quad \left| \frac{T_t(x, y)}{h(x)h(y)} - \frac{T_t(x, y_0)}{h(x)h(y_0)} \right| \leq \frac{C}{\mu_{h^2}(B(x, \sqrt{t}))} \left(\frac{d(y, y_0)}{\sqrt{t}} \right)^\delta \exp\left(-\frac{d(x, y)^2}{c_3 t}\right).$$

Proof. Note that $\int_X \tilde{T}_t(x, y) d\mu_{h^2}(x) = 1$ a.e. and the semigroup $\tilde{T}_t$ satisfies (2.3). Therefore (2.4) follows from [DP18, Theorem 5]. ■

Another known consequence of (ULG_h) is the following proposition.

PROPOSITION 2.4. *Assume that a semigroup T_t related to a kernel $T_t(x, y)$ satisfies (ULG_h) for some function h . Then the measure μ_{h^2} satisfies the doubling condition (1.1).*

Proof. The proposition is well-known and follows from (2.3). We shall provide a short proof for the sake of completeness. Let $x \in X$, $r > 0$. By (2.3) for $y \in B(x, r)$ we have $\mu_{h^2}(B(x, \sqrt{2}r))^{-1} \gtrsim \tilde{T}_{2r^2}(x, y)$. By averaging this inequality over $B(x, r)$ and using the semigroup property we obtain

$$\begin{aligned} \mu_{h^2}(B(x, \sqrt{2}r))^{-1} &\gtrsim \mu_{h^2}(B(x, r))^{-1} \int_{B(x, r)} \tilde{T}_{2r^2}(x, y) d\mu_{h^2}(y) \\ &\gtrsim \mu_{h^2}(B(x, r))^{-1} \int_{B(x, r)} \int_{B(y, r)} \tilde{T}_{r^2}(x, z) \tilde{T}_{r^2}(z, y) d\mu_{h^2}(z) d\mu_{h^2}(y) \\ &\gtrsim \mu_{h^2}(B(x, r))^{-2} \int_{B(x, r)} \int_{B(y, r)} \mu_{h^2}(B(y, r))^{-1} d\mu_{h^2}(z) d\mu_{h^2}(y) \\ &\simeq \mu_{h^2}(B(x, r))^{-1}. \quad \blacksquare \end{aligned}$$

The next lemma describes the relation between $[\mu, h]$ -atoms and $(\mu; h; r)$ -atoms. This is required in the first part of the proof of Theorems A and 2.1.

LEMMA 2.5.

- (a) *Assume that $h \in \text{RH}_\infty(\mu)$. Then there exists C such that for a ball B and a function a supported in B we have*

$$\|a\|_{L^2(h^{-1}\mu)} \leq \mu_h(B)^{-1/2} \implies \|a\|_{L^2(\mu)} \leq C\mu(B)^{-1/2}.$$

- (b) *Assume that $h \in \text{RH}_q(\mu)$ for $q > 1$ and $r < 2$ is as in (2.1). Then there exists $C > 0$ such that for any ball B and any function a supported in B we have*

$$\|a\|_{L^2(h^{-1}\mu)} \leq \mu_h(B)^{-1/2} \implies \|a\|_{L^r(\mu)} \leq C\mu(B)^{-1+1/r}.$$

Proof. Part (a) follows immediately from the definition of $\text{RH}_\infty(\mu)$. Indeed,

$$\|a\|_{L^2(\mu)}^2 \leq \sup_{x \in B} h(x) \cdot \|a\|_{L^2(h^{-1}\mu)}^2 \leq \sup_{x \in B} h(x) \cdot \mu_h(B)^{-1} \lesssim \mu(B)^{-1}.$$

To prove (b) we use Hölder's inequality with the exponents $2/r$ and $2/(2-r)$ and the definition of $\text{RH}_q(\mu)$ with $q = r/(2-r) > 1$ (see (1.5)). Then

$$\begin{aligned} \|a\|_{L^r(\mu)}^r &= \int_B |a|^r h^{-r/2} h^{r/2} d\mu \leq \left(\int_B |a|^2 h^{-1} d\mu \right)^{r/2} \cdot \left(\int_B h^{\frac{r}{2-r}} d\mu \right)^{\frac{2-r}{2}} \\ &\lesssim \mu_h(B)^{-r/2} \cdot \mu(B)^{1-r} \cdot \mu_h(B)^{r/2} = \mu(B)^{1-r}. \blacksquare \end{aligned}$$

In the whole paper p and q denote the parameters from the Muckenhoupt class A_p and the Reverse Hölder class RH_q , respectively. They are always related by

$$(2.5) \quad p = \frac{q-1}{q-2}.$$

LEMMA 2.6. *Assume that $h \in \text{RH}_q(\mu)$ for $q > 2$. Then $h^{-1} \in A_p(\mu_{h^2})$ for p as in (2.5). In particular, if $h \in \text{RH}_\infty(\mu)$, then $h^{-1} \in A_p(\mu_{h^2})$ for all $p > 1$.*

Proof. Let $p > 1$ and $q > 2$ satisfy (2.5), i.e. $q = 2 + (p-1)^{-1}$. Then the condition $h^{-1} \in A_p(\mu_{h^2})$ is equivalent to

$$(2.6) \quad \mu(B)^{-1} \int_B h d\mu \cdot \left(\mu(B)^{-1} \int_B h^q d\mu \right)^{p-1} \lesssim \left(\mu(B)^{-1} \int_B h^2 d\mu \right)^p$$

(see (2.2)). By the Reverse Hölder assumption the left-hand side of (2.6) can be estimated by $(\mu(B)^{-1} \int_B h d\mu)^{2p}$. Then (2.6) follows from the Cauchy–Schwarz inequality. The second statement follows immediately, since if $q \rightarrow \infty$ then $p = 1 + (q-2)^{-1}$ tends to 1. $\blacksquare$

2.2. Proof of Theorems A and 2.1

CASE 1: $q < \infty$. Let $q_0 = \max(D/\delta, 2 + (p_0 - 1)^{-1})$, where D is from (1.1), δ is the Hölder exponent from (2.4), and p_0 is as in Theorem 2.2. Notice that since $p_0 \leq 2$, we have $q_0 \geq 3$. Assume that $h \in \text{RH}_q(\mu)$ with $q > q_0$ and, as before, $r = 2 - 2/(q+1)$, see (2.1).

STEP 1. Since $q > q_0$, we see that $h \in \text{RH}_{q_0}(\mu)$ and, by Lemma 2.6, we obtain $h^{-1} \in A_{p_0}(\mu_{h^2})$. Recall that, by Proposition 2.4, μ_{h^2} is doubling. By Theorem 2.2, we have $H_L^1(X) = H_{\text{at}}^1[\mu, h]$. Moreover, Lemma 2.5(b) states that every $[\mu, h]$ -atom is also a $(\mu; h; r)$ -atom, so $H_L^1(X) \subseteq H_{\text{at}}^1(\mu; h; r)$.

STEP 2a. In this step we prove that there exists $C > 0$ such that

$$(2.7) \quad \|M_L a\|_{L^1(\mu)} \leq C$$

for every $(\mu; h; r)$ -atom a with C independent of a . Assume that a satisfies (1.2) for a ball $B = B(y_0, R)$. Denote $\lambda B = B(y_0, \lambda R)$, $\lambda > 0$. By (1.1) and the boundedness of M_L on $L^r(\mu)$ we have

$$\|M_L a\|_{L^1(2B, \mu)} \leq \mu(2B)^{1-1/r} \|M_L a\|_{L^r(\mu)} \lesssim \mu(B)^{1-1/r} \|a\|_{L^r(\mu)} \leq C.$$

Our next goal is to estimate $M_L a(x)$ for $x \notin 2B$. Notice that if $y \in B$, then $d(x, y) \simeq d(x, y_0)$. First consider t such that $\sqrt{t} > R$. In this case, the cancellation $\int a h d\mu = 0$ and (2.4) yield

$$\begin{aligned} (2.8) \quad |T_t a(x)| &= h(x) \left| \int_B \left(\frac{T_t(x, y)}{h(x)h(y)} - \frac{T_t(x, y_0)}{h(x)h(y_0)} \right) a(y) h(y) d\mu(y) \right| \\ &\leq h(x) \int_B \left| \frac{T_t(x, y)}{h(x)h(y)} - \frac{T_t(x, y_0)}{h(x)h(y_0)} \right| |a(y)| h(y) d\mu(y) \\ &\lesssim \int_B \left(\frac{d(y, y_0)}{\sqrt{t}} \right)^\delta \frac{h(x)}{\mu_{h^2}(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{ct}\right) |a(y)| h(y) d\mu(y) \\ &\lesssim \left(\frac{R}{\sqrt{t}} \right)^\delta \frac{h(x)}{\mu_{h^2}(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y_0)^2}{ct}\right) \int_B |a| h d\mu. \end{aligned}$$

Next suppose $\sqrt{t} \leq R$. By (ULG_h) , we get

$$\begin{aligned} (2.9) \quad |T_t a(x)| &\leq h(x) \int_B \frac{T_t(x, y)}{h(x)h(y)} |a(y)| h(y) d\mu(y) \\ &\lesssim h(x) \left(\frac{R}{\sqrt{t}} \right)^\delta \mu_{h^2}(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{d(x, y_0)^2}{ct}\right) \int_B |a| h d\mu. \end{aligned}$$

Notice that Proposition 2.4 implies

$$(2.10) \quad \frac{\mu_{h^2}(B(x, d(x, y_0)))}{\mu_{h^2}(B(x, \sqrt{t}))} \lesssim \left(1 + \frac{d(x, y_0)}{\sqrt{t}} \right)^D.$$

By combining (2.8)–(2.10), and using Hölder's inequality we arrive at

$$\begin{aligned} (2.11) \quad M_L a(x) &\lesssim \sup_{t>0} \left(\left(\frac{R}{\sqrt{t}} \right)^\delta \frac{h(x)}{\mu_{h^2}(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y_0)^2}{ct}\right) \right) \int_B |a| h d\mu \\ &\lesssim \frac{h(x)}{\mu_{h^2}(B(x, d(x, y_0)))} \sup_{t>0} \left(\left(\frac{R}{\sqrt{t}} \right)^\delta \exp\left(-\frac{d(x, y_0)^2}{c't}\right) \right) \int_B |a| h d\mu, \\ &\lesssim \left(\frac{R}{d(x, y_0)} \right)^\delta \frac{h(x)}{\mu_{h^2}(B(x, d(x, y_0)))} \|a\|_{L^r(B, \mu)} \|h\|_{L^{r'}(B, \mu)}, \end{aligned}$$

where $1/r + 1/r' = 1$.

For $n = 1, 2, \dots$ denote $S_n = 2^{n+1}B \setminus 2^n B$. Then

$$\|M_L a\|_{L^1((2B)^c, \mu)} \leq \sum_{n=1}^{\infty} \underbrace{\|M_L a\|_{L^1(S_n, \mu)}}_{P_n}$$

We claim that $\sum_{n \geq 1} P_n \leq C$. Notice that

$$\mu_{h^2}(B(x, d(x, y_0))) \simeq \mu_{h^2}(B(y_0, d(x, y_0)));$$

see Proposition 2.4. For $x \in S_n$ we have $2^n R \leq d(x, y_0) \leq 2^{n+1} R$, so by (2.11) and (1.2) we obtain

$$\begin{aligned} P_n &\lesssim 2^{-n\delta} \frac{\mu_h(2^{n+1}B)}{\mu_{h^2}(2^n B)} \mu(B)^{-1+1/r} \|h\|_{L^{r'}(B, \mu)} \\ &\simeq 2^{-n\delta} \frac{(\mu(2^{n+1}B)^{-1} \int_{2^{n+1}B} h \, d\mu) \cdot (\mu(B)^{-1} \int_B h^{r'} \, d\mu)^{1/r'}}{\mu(2^{n+1}B)^{-1} \int_{2^{n+1}B} h^2 \, d\mu} \\ &\simeq 2^{-n\delta} \frac{(\mu(B)^{-1} \int_B h^{r'} \, d\mu)^{1/r'}}{(\mu(2^{n+1}B)^{-1} \int_{2^{n+1}B} h^2 \, d\mu)^{1/2}}. \end{aligned}$$

In the estimate above we used Proposition 2.4 and Hölder's inequality. Recall that $q > q_0 \geq 3$, $r' = 2q/(q-1) \leq q$, and $h \in \text{RH}_q(\mu)$. By Hölder's inequality and Reverse Hölder's assumption we arrive at

$$\begin{aligned} P_n &\lesssim 2^{-n\delta} \frac{(\mu(B)^{-1} \int_B h^q \, d\mu)^{1/q}}{\mu(2^{n+1}B)^{-1} \int_{2^{n+1}B} h \, d\mu} \\ &\lesssim 2^{-n\delta} \left(\frac{\mu(2^{n+1}B)}{\mu(B)} \cdot \frac{\int_B h^q \, d\mu}{\int_{2^{n+1}B} h^q \, d\mu} \right)^{1/q} \\ &\lesssim 2^{-n\delta} 2^{nD/q}, \end{aligned}$$

where D is from (1.1). The claim $\sum_{n \geq 1} P_n \leq C$ is proved, since $\delta - D/q > 0$. The proof of (2.7) is finished.

STEP 2b. The inclusion $H_{\text{at}}^1(\mu; h; r) \subseteq H_L^1(X)$ follows from (2.7) by a standard argument. We present the details for the sake of completeness. Take $f(x) = \sum_k \lambda_k a_k(x)$, where $\sum_k |\lambda_k| < \infty$ and a_k are $(\mu; h; r)$ -atoms. Recall that $H_L^1(X)$ embeds continuously into $L^1(\mu)$ (see [AMM15] and [PSY22, Section 8]). Let $f_n(x) = \sum_{k=1}^n \lambda_k a_k(x)$. By (2.7) for $n < m$ we have

$$\|M_L(f_m - f_n)\|_{L^1(\mu)} \lesssim \sum_{k=n+1}^m |\lambda_k|.$$

Thus f_n is a Cauchy sequence in $H_L^1(X)$ and there is a limit $g \in H_L^1(X)$ in the $H_L^1(X)$ -norm. Also, $f_n \rightarrow f$ in $L^1(\mu)$, and since $H_L^1(X)$ embeds continuously

into $L^1(\mu)$, we have $f = g$. Moreover,

$$\|f\|_{H_L^1(X)} \leq \sup_n \|f_n\|_{H_L^1(X)} \lesssim \sum_{k=1}^{\infty} |\lambda_k| \lesssim \|f\|_{H_{\text{at}}^1(\mu; h; r)},$$

which ends the proof of Theorem 2.1 in the case $q < \infty$.

CASE 2: $q = \infty$. This case (equivalent to Theorem A) can be verified by the same argument with appropriate modifications and we only comment on the differences. The assumption $h \in \text{RH}_{\infty}(\mu)$ implies $h \in \text{RH}_q(\mu)$ for all $q > 1$. Then $h^{-1} \in A_p(\mu_{h^2})$ for all $p > 1$, see Lemma 2.6. The main change in the proof is that we use Lemma 2.5(a) (instead of (b)), and therefore we may keep $r = 2$. We leave the details to the reader. ■

3. BMO spaces and harmonic functions. Let $r, r' > 1$ be conjugate exponents, i.e. $1/r + 1/r' = 1$. In this section we prove that the dual of the atomic Hardy space $H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$ (see Definition 1.1) is the space $\text{BMO}(\mu; h_1, \dots, h_m; r')$ (see Definition 1.2).

THEOREM 3.1. *Let $m \in \mathbb{N}$ and $h_1, \dots, h_m \in L_{\text{loc}}^{r'}(\mu)$ be functions that are linearly independent on X . Then the space $\text{BMO}(\mu; h_1, \dots, h_m; r')$ is dual to $H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$.*

The proof of Theorem 3.1 consists of verifying two statements:

- (1) For each $g \in \text{BMO}(\mu; h_1, \dots, h_m; r')$ one can define (in a linear way) a bounded linear functional on $H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$, which we shall denote by $\langle f, g \rangle$, such that

$$|\langle f, g \rangle| \leq \|f\|_{H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)} \|g\|_{\text{BMO}(\mu; h_1, \dots, h_m; r')}.$$

- (2) Conversely, for every $l \in (H_{\text{at}}^1(\mu; h_1, \dots, h_m; r))^*$ there exists $g \in \text{BMO}(\mu; h_1, \dots, h_m; r')$ such that l is defined as above, i.e. $l(f) = \langle f, g \rangle$ with

$$\|g\|_{\text{BMO}(\mu; h_1, \dots, h_m; r')} \lesssim \|l\|_{H_{\text{at}}^1(\mu; h_1, \dots, h_m; r) \rightarrow \mathbb{C}}.$$

Theorem B follows by a proper modification of standard techniques (see e.g. [S93, pp. 142–145], [PSY22, Section 5]).

Proof of Theorem 3.1. Denote $V = \text{span}(h_1, \dots, h_m)$ as in Definition 1.2.

To prove (1) take $g \in \text{BMO}(\mu; h_1, \dots, h_m; r')$. For a $(\mu; h_1, \dots, h_m; r)$ -atom a and $u \in V$ we have

$$\begin{aligned}
\left| \int_X a(x)g(x) d\mu(x) \right| &= \left| \int_X a(x)(g(x) - u(x)) d\mu(x) \right| \\
&\leq \|a\|_{L^r(\mu)} \left(\int_B |g(x) - u(x)|^{r'} d\mu(x) \right)^{1/r'} \\
&\leq \left(\mu(B)^{-1} \int_B |g(x) - u(x)|^{r'} d\mu(x) \right)^{1/r'}.
\end{aligned}$$

The estimate above holds for all balls B in X and all $u \in V$, therefore we see that

$$(3.1) \quad \left| \int_X a(x)g(x) d\mu(x) \right| \leq \|g\|_{\text{BMO}(\mu; h_1, \dots, h_m; r')}.$$

Let $f(x) = \sum_k \lambda_k a_k(x)$, where $\sum_k |\lambda_k| < \infty$. From (3.1) we deduce that

$$\int_X \sum_{k=1}^n \lambda_k a_k(x) g(x) d\mu(x)$$

is a Cauchy sequence as $n \rightarrow \infty$ and we define the pairing $\langle f, g \rangle$ as its limit for an arbitrary $f \in H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$. It is easy to check that this pairing is well-defined, linear, and

$$|\langle f, g \rangle| \leq \|f\|_{H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)} \|g\|_{\text{BMO}(\mu; h_1, \dots, h_m; r')}.$$

Notice that $\langle f, g \rangle = \int f(x)g(x) d\mu(x)$ when f is a finite linear combination of $(\mu; h_1, \dots, h_m; r)$ -atoms (this subspace is dense in $H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$).

We turn to proving (2). Let l be a linear functional on $H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$. We may assume that $\|l\|_{H_{\text{at}}^1(\mu; h_1, \dots, h_m; r) \rightarrow \mathbb{C}} \leq 1$. Let B be a ball and consider the space

$$\mathcal{H}_B = \left\{ f \in L^r(\mu|_B) : \forall u \in V \quad \int_B f(x)u(x) d\mu(x) = 0 \right\}.$$

Obviously, if $f \in \mathcal{H}_B$, then $f \in H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)$ with

$$\|f\|_{H_{\text{at}}^1(\mu; h_1, \dots, h_m; r)} \leq \mu(B)^{1-1/r} \|f\|_{L^r(\mu)}.$$

Therefore, by the Hahn–Banach extension and the Riesz representation theorems, there exists $\tilde{g}_B$ such that

$$l(f) = \int_B f(x) \tilde{g}_B(x) d\mu(x), \quad f \in \mathcal{H}_B,$$

$$\|\tilde{g}_B\|_{L^{r'}(\mu)} = \|l\|_{\mathcal{H}_B \rightarrow \mathbb{C}} \leq \mu(B)^{1-1/r}.$$

Since the functions $h_1, \dots, h_m$ are linearly independent on X , there exists a ball such that $h_1, \dots, h_m$ are linearly independent on this ball. Fix such a ball B_0 . Then $M = (\int_{B_0} h_i h_j d\mu)_{i,j=1}^m$ is a non-degenerate $m \times m$ matrix.

We choose constants $c_B = (c_{B,1}, \dots, c_{B,m})$ such that

$$Mc_B = -\left(\int_{B_0} \tilde{g}_B h_1 d\mu, \dots, \int_{B_0} \tilde{g}_B h_m d\mu\right).$$

Set $g_B = \tilde{g}_B + c_{B,1}h_1 + \dots + c_{B,m}h_m$. Note that $l(f) = \int_B f(x)g_B(x) d\mu(x)$ for $f \in \mathcal{H}_B$. According to our choice of c_B we have

$$(3.2) \quad \int_{B_0} g_B h_j d\mu = 0, \quad j = 1, \dots, m.$$

Take an increasing family of balls $B_0 \subseteq B_1 \subseteq \dots$ such that $\bigcup_{n \in \mathbb{N}} B_n = X$. We have $g_{B_n} - g_{B_{n+1}} = d_{n,1}h_1 + \dots + d_{n,m}h_m$ on $B_n \supseteq B_0$. By integrating on B_0 with respect to $h_1, \dots, h_m$ we obtain $M(d_{n,1}, \dots, d_{n,m}) = 0$ and thus $d_{n,j} = 0$ for $n \in \mathbb{N}$ and $j = 1, \dots, m$. Since g_{B_n} and $g_{B_{n+1}}$ coincide on B_n , we can define the limit

$$g(x) = \lim_{n \rightarrow \infty} g_{B_n}(x).$$

Moreover, for any ball B ,

$$(3.3) \quad \left(\frac{1}{\mu(B)} \int_B |g - c_{B,1}h_1 - \dots - c_{B,m}h_m|^{r'} d\mu\right)^{\frac{1}{r'}} = \left(\frac{1}{\mu(B)} \|\tilde{g}_B\|_{L^{r'}(\mu)}^{r'}\right)^{\frac{1}{r'}} \leq C.$$

We have proved $g \in \text{BMO}(\mu; h_1, \dots, h_m; r')$ and $\|g\|_{\text{BMO}(\mu; h_1, \dots, h_m; r')} \leq C$. Also, $l(f) = \int_X f(x)g(x) d\mu(x)$ whenever f is a finite combination of atoms.

This ends the proof of Theorem 3.1. ■

4. Spaces with symmetry and operators related to two harmonic functions. Let (X, d, μ) be a space of homogeneous type, see (1.1). Let us start with a general proposition, that relies on (UG). More assumptions (e.g. symmetry of X) will be added later.

PROPOSITION 4.1. *Suppose that L is a non-negative self-adjoint operator acting on $L^2(\mu)$ such that the integral kernel $T_t(x, y)$ corresponding to the semigroup $T_t = e^{-tL}$ satisfies the upper Gaussian estimates (UG). Assume that the functions $h_1, \dots, h_m \in L_{\text{loc}}^\infty(\mu)$ are L -harmonic in the sense of (1.3). Then $H_L^1(X) \subseteq H_{\text{at}}^1(\mu; h_1, \dots, h_m; 2)$ and there exists $C > 0$ such that*

$$\|f\|_{H_{\text{at}}^1(\mu; h_1, \dots, h_m; 2)} \leq C \|f\|_{H_L^1(X)}.$$

Proof. The proof follows from [HL⁺11] and the argument is similar to [HL⁺11, Lemma 9.1] and [PSY22, Theorem 3.5]. For completeness we present a sketch of the argument.

Recall that the space $H_L^1(X)$ with T_t satisfying (UG) has many equivalent characterizations, e.g. by maximal functions, Luzin area functions, square functions, etc. (see [HL⁺11, SY18]). By [HL⁺11, Theorem 7.1] for f from

a certain dense subset $\mathbb{H}_L^1(X)$ of $H_L^1(X)$ we have $f(x) = \sum_k \lambda_k a_k(x)$ (with convergence in both $L^1(\mu)$ and $L^2(\mu)$), where a_k are such that there exist balls $B_k = B(y_k, r_k)$ and functions $b_k \in \text{Dom}(L)$ with

$$a_k = Lb_k, \quad \text{supp}(L^i b_k) \subseteq B_k, \quad \|(r_k^2 L)^i b_k\|_{L^2(\mu)} \leq r_k^2 \mu(B_k)^{-1/2}$$

for $i = 0, 1$ and $k = 1, 2, \dots$. We can assume that $\|f\|_{H_L^1(X)} \simeq \sum_k |\lambda_k|$. Since a_k are properly localized and satisfy the size condition that we need, it is enough to show that $\int a_k h_j d\mu = 0$ for $j = 1, \dots, m$ and $k \in \mathbb{N}$. For fixed j from (1.3) we have

$$(I + L)^{-1} h_j = \int_0^\infty e^{-t} T_t h_j dt = h_j.$$

Then

$$\begin{aligned} \int_X a_k(x) h_j(x) d\mu(x) &= \int_X a_k(x) (I + L)^{-1} h_j(x) d\mu(x) \\ &= \int_X (I + L)^{-1} L b_k(x) h_j(x) d\mu(x) \\ &= \int_X (I + L)^{-1} (I + L) b_k(x) h_j(x) d\mu(x) \\ &\quad - \int_X (I + L)^{-1} b_k(x) h_j(x) d\mu(x) \\ &= \int_X b_k(x) h_j(x) d\mu(x) - \int_X b_k(x) (I + L)^{-1} h_j(x) d\mu(x) \\ &= 0. \end{aligned}$$

Notice that the calculation above is justified since $h_j \in L_{\text{loc}}^\infty(\mu)$. What we have just proved is that each a_k is a $(\mu; h_1, \dots, h_m; 2)$ -atom. Thus, for $f \in \mathbb{H}_L^1(X)$ we have $f \in H_{\text{at}}^1(\mu; h_1, \dots, h_m; 2)$ and

$$\|f\|_{H_{\text{at}}^1(\mu; h_1, \dots, h_m; 2)} \leq C \|f\|_{H_L^1(X)}.$$

The proof is finished, since $\mathbb{H}_L^1(X)$ is a dense subset of $H_L^1(X)$. ■

COROLLARY 4.2. *Let L be an operator as in Proposition 4.1. If h is a bounded L -harmonic function then*

$$\int f(x) h(x) d\mu(x) = 0$$

for $f \in H_L^1(X)$.

Proof. Let $f \in H_L^1(X)$. From Proposition 4.1 we have $f = \sum_k \lambda_k a_k$, where $\sum_k |\lambda_k| < \infty$ and a_k are $(\mu; h; 2)$ -atoms. Then

$$\int f(x) h(x) d\mu(x) = \lim_{n \rightarrow \infty} \int \sum_{k=1}^n \lambda_k a_k(x) h(x) d\mu(x) = 0,$$

since $\sum_{k=1}^n \lambda_k a_k(x)$ converges to f in $L^1(\mu)$ and $h \in L^\infty(\mu)$. ■

4.1. Background and assumptions. In this section we shall consider spaces of homogeneous type with an additional symmetry. These assumptions are modelled by symmetric manifolds $\mathbb{R}^n \# \mathbb{R}^n$ with ends that are described in Sections 1.3 above and 5.2 below. The proofs do not rely on the manifold structure, thus we state some general assumptions that guarantee the atomic characterization of the Hardy space $H_L^1(X)$, where atoms have cancellations with respect to two linearly independent harmonic functions. According to our knowledge, such atomic Hardy spaces have not been intensively studied (however, see [PZ21]).

Let (X, d, μ) be a space of homogeneous type and $X = X_+ \cup X_-$. Assume that $\Gamma := X_+ \cap X_-$, $\mu(\Gamma) = 0$, and $\Gamma = \text{bd } X_+ = \text{bd } X_-$. Denote by μ_+ the measure μ restricted to X_+ . We additionally assume that there exists a bijection $\sigma : X \rightarrow X$ that preserves both the metric and the measure, $\sigma^2 = \text{id}$, $\sigma(x) = x$ for $x \in \Gamma$, and $\sigma(X_+) = X_-$. In what follows we shall denote $\sigma(x) = \tilde{x}$. We shall work under the assumption that

$$(4.1) \quad \forall x, y \in X_+ \quad d(x, y) \leq d(x, \tilde{y}).$$

We start by considering two operators on X_+ that satisfy the assumptions of Theorem A. For further reference let us recall these assumptions. Let L_1 and L_2 be two self-adjoint, non-negative operators that are densely defined on $L^2(\mu_+)$. Denote by $T_t^{[1]}$ and $T_t^{[2]}$ the semigroups related to L_1 and L_2 , and let $T_t^{[1]}(x, y), T_t^{[2]}(x, y)$ for $t > 0, x, y \in X_+$ be the integral kernels related to these semigroups, respectively. Suppose that non-negative functions h_1, h_2 on X_+ are in $\text{RH}_\infty(\mu_+)$. Assume that, for $j = 1, 2, t > 0$, and $x, y \in X_+ \setminus \Gamma$, we have

$$(4.2) \quad T_t^{[j]} h_j(x) = h_j(x),$$

$$(4.3) \quad 0 \leq T_t^{[j]}(x, y) \leq C\mu(B(x, \sqrt{t}))^{-1} e^{-\frac{d(x, y)^2}{c_1 t}},$$

$$(4.4) \quad C^{-1} \mu_{h_j^2}(B(x, \sqrt{t}))^{-1} e^{-\frac{d(x, y)^2}{c_2 t}} \leq \frac{T_t^{[j]}(x, y)}{h_j(x)h_j(y)} \\ \leq C\mu_{h_j^2}(B(x, \sqrt{t}))^{-1} e^{-\frac{d(x, y)^2}{c_3 t}}.$$

Notice that under the above assumptions Theorem A describes $H_{L_j}^1(X_+)$ for $j = 1, 2$.

We say that a function f on X is *even* (or *odd*) if $f(\tilde{x}) = f(x)$ (respectively, $f(\tilde{x}) = -f(x)$). We decompose each $f \in L^2(\mu)$ into even and odd parts, $f = f_e + f_o$, getting $L^2(\mu) \simeq L^2(\mu_+) \oplus L^2(\mu_+)$, a direct sum of two Hilbert subspaces. We define L as $L_1 \otimes L_2$, where L_1 acts on the even functions and L_2 acts on odd functions. Then L is self-adjoint and non-negative on $L^2(\mu)$. Moreover, L is related to the semigroup T_t whose integral kernel

is given by

$$(4.5) \quad T_t(x, y) = \frac{1}{2} \begin{cases} T_t^{[1]}(x, y) + T_t^{[2]}(x, y) & \text{for } x, y \in X_+, \\ T_t^{[1]}(x, \tilde{y}) - T_t^{[2]}(x, \tilde{y}) & \text{for } x, \tilde{y} \in X_+, \\ T_t^{[1]}(\tilde{x}, y) - T_t^{[2]}(\tilde{x}, y) & \text{for } \tilde{x}, y \in X_+, \\ T_t^{[1]}(\tilde{x}, \tilde{y}) + T_t^{[2]}(\tilde{x}, \tilde{y}) & \text{for } \tilde{x}, \tilde{y} \in X_+. \end{cases}$$

We shall also assume that $T_t(x, y)$ satisfies (UG). Notice that the functions

$$\hat{h}_1(x) = \begin{cases} h_1(x) & \text{for } x \in X_+, \\ h_1(\tilde{x}) & \text{for } x \in X_-, \end{cases} \quad \hat{h}_2(x) = \begin{cases} h_2(x) & \text{for } x \in X_+, \\ -h_2(\tilde{x}) & \text{for } x \in X_-, \end{cases}$$

are L -harmonic in the sense of (1.3).

4.2. Hardy space related to $\hat{h}_1$ and $\hat{h}_2$. Our main goal of this section is to prove the following atomic characterization of $H_L^1(X)$.

THEOREM 4.3. *Let a space $X = X_+ \cup X_-$ and a symmetry σ satisfy the assumptions listed in Section 4.1, including (4.1). Suppose that the conditions (4.2)–(4.4) hold for operators L_1, L_2 on $L^2(\mu_+)$ and functions $h_1, h_2 \in \text{RH}_\infty(\mu_+)$ (see (1.6)). Let $L = L_1 \otimes L_2$ be the operator defined above and assume that the related semigroup kernel satisfies the upper Gaussian estimates (UG). Then the Hardy space $H_L^1(X)$ coincides with $H_{\text{at}}^1(\mu; \hat{h}_1, \hat{h}_2; 2)$ and there exists C such that*

$$C^{-1} \|f\|_{H_L^1(X)} \leq \|f\|_{H_{\text{at}}^1(\mu; \hat{h}_1, \hat{h}_2; 2)} \leq C \|f\|_{H_L^1(X)}.$$

Proof. The inclusion $H_L^1(X) \subseteq H_{\text{at}}^1(\mu; \hat{h}_1, \hat{h}_2; 2)$ follows directly from Proposition 4.1, since (UG) holds for L and since both $\hat{h}_1$ and $\hat{h}_2$ are L -harmonic functions.

Now we prove that $H_{\text{at}}^1(\mu; \hat{h}_1, \hat{h}_2; 2) \subseteq H_L^1(X)$. By the same continuity argument as in the proof of Theorem A (Step 2b in Section 2.2) it is enough to prove that there exists $C > 0$ such that

$$(4.6) \quad \|M_L a\|_{L^1(X)} \leq C$$

for every $(\mu; \hat{h}_1, \hat{h}_2; 2)$ -atom a .

Assume then that an atom a is such that $\text{supp } a \subseteq B = B(y, r) \subseteq X$. Denote $\tilde{B} = \sigma(B)$ and

$$a_e(x) = \frac{1}{2}(a(x) + a(\tilde{x})), \quad a_o(x) = \frac{1}{2}(a(x) - a(\tilde{x})).$$

Notice that a_e and a_o are supported in $B \cup \tilde{B}$ and

$$\|M_L a\|_{L^1(\mu)} \leq \|M_L a_e\|_{L^1(\mu)} + \|M_L a_o\|_{L^1(\mu)}.$$

First we shall estimate $\|M_L a_e\|_{L^1(\mu)}$. Since $a_e, \widehat{h}_1$ are even, a_o is odd, and $\int \widehat{a} \widehat{h}_1 d\mu(x) = 0$, we obtain

$$\int_{X_+} a_e h_1 d\mu = \frac{1}{2} \int_X a_e \widehat{h}_1 d\mu = \frac{1}{2} \int_X (a_e + a_o) \widehat{h}_1 d\mu = 0.$$

Denote by $a_{e,+}$ the restriction of a_e to X_+ . Observe that, by (4.1), $a_{e,+}$ is supported on $B_+ := B(z, r) \cap X_+ = (B \cup \widetilde{B}) \cap X_+$, where $z = y$ if $y \in X_+$, and $z = \widetilde{y}$ otherwise. In addition, $\mu(B_+) \simeq \mu(B)$. Note that $T_t a_e(x)$ is an even function and $T_t a_e(x) = T_t^{[1]} a_{e,+}(x)$ for $x \in X_+$, so it is enough to show

$$\|M_{L_1} a_{e,+}\|_{L^1(\mu_+)} \leq C.$$

The last estimate is a consequence of Theorem A, since $a_{e,+}$ is a $(\mu; h_1; 2)$ -atom related to the ball $B_+ \subseteq X_+$ (see Definition 1.1). The estimate for $\|M_L a_o\|_{L^1(\mu)}$ is similar. We skip the details. ■

Using Theorem 3.1, we describe the dual space as follows.

COROLLARY 4.4. *Suppose $h_1, h_2 \in L^2_{\text{loc}}(\mu_+)$. The space $\text{BMO}(\mu; \widehat{h}_1, \widehat{h}_2; 2)$ is dual to $H^1_{\text{at}}(\mu; \widehat{h}_1, \widehat{h}_2; 2)$.*

5. Applications. In this section we present exemplary applications of our theory, i.e. of Theorems A and 4.3.

5.1. Operators with one harmonic function. It is easy to check that all of the examples discussed in [PSY22, Section 6] satisfy the assumptions of Theorem A, including the condition $h \in \text{RH}_\infty(\mu)$. Notice that in many cases the related harmonic function is unbounded either from above or from below. We do not list all the examples here and just quote one application.

Exterior domain outside a bounded convex $C^{1,1}$ set. Assume that $\Omega \subset \mathbb{R}^n$ is such that Ω^c is convex, compact, and its boundary is locally a $C^{1,1}$ function. We consider the Laplace operator Δ_Ω on Ω with the Dirichlet boundary condition and the corresponding Hardy space $H^1_{\Delta_\Omega}(\Omega)$.

THEOREM 5.1. *Assume that for $n \geq 3$, $\Omega \subset \mathbb{R}^n$ is a domain that is the exterior of a $C^{1,1}$ compact convex set. Then there exists a function $h : \Omega \rightarrow (0, \infty)$ such that*

$$h(x) \simeq \min(1, \text{dist}(x, \Omega^c))$$

and the Hardy space $H^1_{\Delta_\Omega}(\Omega)$ coincides with $H^1_{\text{at}}(\mu; h; 2)$, where μ is the Lebesgue measure on Ω .

The estimates for the heat kernel were given in [Z03] (see also [PSY22, Section 6.1.2]). It is not difficult to check that for Ω as in the theorem above we have $h \in \text{RH}_\infty(\mu)$. Hence Theorem 5.1 is a direct consequence of Theorem A. Also, Theorem B describes the dual space as $\text{BMO}(\mu; h; 2)$. For

other results concerning Hardy spaces on domains on $\mathbb{R}^n$ we refer the reader to [AR03, CKS93] and references therein.

5.2. Symmetric manifolds with ends. In this section we consider a Riemmanian manifold with ends of the form $M = \mathbb{R}^n \# \mathbb{R}^n$. The results for this context were stated in Section 1.3 and here we provide some details and precise assumptions. We briefly discuss the proof of Theorem C. By definition, the manifold $\mathbb{R}^n \# \mathbb{R}^n$ has a central compact part K such that $M \setminus K$ is a disjoint sum of two ends E_+ and E_- , where $E_+ = E_- = \mathbb{R}^n \setminus B(0, r)$ and $r > 0$. Let μ be a measure on M that has a smooth positive density and coincides with the Lebesgue measure on E_+ and E_- . On the other hand we suppose that $M = M_+ \cup M_-$, where $E_+ \subseteq M_+$ and $E_- \subseteq M_-$. The halves M_+ and M_- are Riemmanian manifolds with boundaries and the set $\Gamma := M_+ \cap M_- = \text{bd } M_+ = \text{bd } M_- \subseteq K$ is a compact smooth submanifold of order $n - 1$, i.e. $\mu(\Gamma) = 0$. Moreover, we assume that there exists a symmetry $\sigma : M \rightarrow M$ such that $\sigma^2 = \text{id}$, σ preserves both the metric and the measure, $\sigma(x) = x$ for $x \in \Gamma$, and $\sigma(M_+) = M_-$. Notice that in this context the property (4.1) holds.

Denote by L_M the Laplace–Beltrami operator on M . It is known that the related heat kernel satisfies (UG) ([GS09, CCH06]).

For a moment consider the half-space M_+ described above. Let L_N and L_D be the Laplace–Beltrami operators on M_+ with either Neumann or Dirichlet boundary conditions on Γ , respectively. We assume that M_+ is an inner uniform domain in the sense of [GS11]. By [GS11, Theorem 3.10], the heat kernel $T_{t,N}(x, y)$ corresponding to the semigroup $T_{t,N} = \exp(-tL_N)$ satisfies

$$\begin{aligned} T_{t,N}(x, y) &\geq C^{-1} \mu(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{\rho(x, y)^2}{c_1 t}\right), \\ T_{t,N}(x, y) &\leq C \mu(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{\rho(x, y)^2}{c_2 t}\right) \end{aligned}$$

for $x, y \in M_+$ and $t > 0$. Moreover, by [GS11, Section 4 and Corollary 5.10] there exists a function $h_D : M_+ \rightarrow [0, \infty)$ such that

$$\begin{aligned} \frac{T_{t,D}(x, y)}{h_D(x)h_D(y)} &\geq C^{-1} \mu_{h_D^2}(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{\rho(x, y)^2}{c_1 t}\right), \\ \frac{T_{t,D}(x, y)}{h_D(x)h_D(y)} &\leq C \mu_{h_D^2}(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{d(x, y)^2}{c_2 t}\right), \end{aligned}$$

where $t > 0$ and $x, y \in M_+ \setminus \Gamma$. The function h_D is continuous, positive on $M_+ \setminus \Gamma$, and vanishes on Γ . Moreover, h_D is in $\text{RH}_\infty(\mu)$, similarly to the case of Theorem 5.1.

Observe that, thanks to symmetry, the analysis of T_t on even functions on M is equivalent to the analysis of $T_{t,N}$ on M_+ ; compare Section 4.1. Similarly, the analysis of T_t on odd functions on M is equivalent to the analysis of $T_{t,D}$ on M_+ . The kernels $T_t(x, y)$, $T_{t,N}(x, y)$ and $T_{t,D}(x, y)$ are related as in (4.5) with $T_t^{[1]} = T_{t,N}$ and $T_t^{[2]} = T_{t,D}$. On M_+ the related harmonic functions are $h_1 \equiv 1$ and $h_2 = h_D$, respectively. Let $\widehat{h}_1 \equiv 1$ (the even extension of h_1 to M) and denote by $\widehat{h}_2$ the odd extension of h_D to M .

Proof of Theorem C. It is enough to observe that all the assumptions of Theorem 4.3 are satisfied with $T_t^{[1]} = T_{t,N}$ and $T_t^{[2]} = T_{t,D}$. Therefore $\widehat{h}_1 \equiv 1$ and $\widehat{h}_2$ are L -harmonic. Finally, observe that the functions h_+ and h_- defined in Section 1.3 are linear combinations of $\widehat{h}_1$ and $\widehat{h}_2$, i.e.

$$h_+ = \frac{1}{2}(\widehat{h}_1 + \widehat{h}_2), \quad h_- = \frac{1}{2}(\widehat{h}_1 - \widehat{h}_2). \quad \blacksquare$$

REMARK 5.2. We intend to study the Hardy spaces on manifolds with ends without the symmetry assumption in another project.

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