COLLOQUIUM MATHEMATICUM

Online First version

JACOBIAN CONDITIONS FOR POLYNOMIALS OVER A UFD

BY

PIOTR JĘDRZEJEWICZ, ŁUKASZ MATYSIAK and JANUSZ ZIELIŃSKI

Abstract. We investigate the ideal generated by Jacobian determinants. We generalize Freudenburg's Lemma to the case of r polynomials in n variables over an arbitrary unique factorization domain. This lemma is useful in reformulations of the Jacobian Conjecture.

1. Introduction. The following lemma of Freudenburg from 1996 was useful in determining the rings of constants of locally nilpotent derivations of the algebra $\mathbb{C}[x_1, \ldots, x_n]$ (see [4]).

THEOREM 1.1 (Freudenburg's Lemma). *Given a polynomial $f \in \mathbb{C}[x, y]$, let $g \in \mathbb{C}[x, y]$ be an irreducible common factor of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Then there exists $c \in \mathbb{C}$ such that g divides $f + c$.*

It was generalized by van den Essen, Nowicki and Tyc in [3] as follows:

THEOREM 1.2 ([3, Theorem 3.1]). *Let k be an algebraically closed field of characteristic zero. Let Q be a prime ideal of the ring $k[x_1, \ldots, x_n]$ and $f \in k[x_1, \ldots, x_n]$. If for each i the partial derivative $\frac{\partial f}{\partial x_i}$ belongs to Q, then there exists $c \in k$ such that $f - c \in Q$.*

It was further generalized, also to positive characteristic, in [5].

THEOREM 1.3 ([5, Theorem 3.1]). *Let K be a unique factorization domain, let Q be a prime ideal of $K[x_1, \ldots, x_n]$. Consider a polynomial $f \in K[x_1, \ldots, x_n]$ such that $\frac{\partial f}{\partial x_i} \in Q$ for $i = 1, \ldots, n$.*

(a) *If $\operatorname{char} K = 0$, then there exists an irreducible polynomial $w \in K[T]$ such that $w(f) \in Q$.*
(b) *If $\operatorname{char} K = p > 0$, then there exist $b, c \in K[x_1^p, \ldots, x_n^p]$ such that $\gcd(b, c) = 1$, $b \notin Q$ and $bf + c \in Q$.*

2020 *Mathematics Subject Classification*: Primary 13F15; Secondary 13F20.

Key words and phrases: polynomial algebra, Jacobian determinant, separable independence, Jacobian Conjecture.
Received 12 December 2025; revised 20 March 2026.
Published online 18 May 2026.

DOI: 10.4064/cm9763-3-2026

A generalization of Freudenburg's Lemma to an arbitrary number of polynomials over a field of characteristic zero was obtained in [8]. Denote by $\mathrm{jac}^{f_1,\ldots,f_r}_{x_{j_1},\ldots,x_{j_r}}$ the Jacobian determinant of the polynomials $f_1,\ldots,f_r$ with respect to $x_{j_1},\ldots,x_{j_r}$.

THEOREM 1.4 ([8, Theorem 2.3]). *Let k be a field of characteristic zero, let $f_1,\ldots,f_r \in k[x_1,\ldots,x_n]$ be arbitrary polynomials, where $r \in \{1,\ldots,n\}$, and let $g \in k[x_1,\ldots,x_n]$ be an irreducible polynomial. The following conditions are equivalent:*

(i) $g \mid \mathrm{jac}^{f_1,\ldots,f_r}_{x_{j_1},\ldots,x_{j_r}}$ *for every* $j_1,\ldots,j_r \in \{1,\ldots,n\}$,
(ii) $g^2 \mid w(f_1,\ldots,f_r)$ *for some irreducible polynomial* $w \in k[x_1,\ldots,x_r]$,
(iii) $g^2 \mid w(f_1,\ldots,f_r)$ *for some square-free polynomial* $w \in k[x_1,\ldots,x_r]$.

See also Jędrzejewicz [6] and de Bondt and Yan [1].

Note moreover that a positive characteristic analog of Freudenburg's Lemma for r polynomials in n variables was obtained in [7]. It was connected with a characterization of p-bases of rings of constants with respect to polynomial derivations.

Such generalizations of Freudenburg's Lemma are tools in equivalent formulations of the Jacobian Conjecture (see [8, 9]), which is one of the most fascinating problems in algebra (see for instance [2, 11]).

In this paper we present further such generalizations. The main results are Theorem 4.1 and Proposition 4.2, and they are summarized in a more explicit way in Theorem 4.3.

2. Separable dependence. Let A be a domain of arbitrary characteristic $p \geqslant 0$. Let R be a subring of A. If $p > 0$, assume that $A^p \subset R$, where $A^p = \{a^p : a \in A\}$. Put

$$R[T_1,\ldots,T_r]_{(p)} = \begin{cases} \{\sum_{0\leqslant i_1,\ldots,i_r<p} b_{(i)} T_1^{i_1}\ldots T_r^{i_r} : b_{(i)} \in R\} & \text{if } p>0,\\ R[T_1,\ldots,T_r] & \text{if } p=0,\end{cases}$$

where $(i) = (i_1,\ldots,i_r)$. In this case ($A^p \subset R$ if $p > 0$) the definitions of separable dependence and separable algebraicity (compare [10, Ch. 0, Sec. 2, p. 2]) have the following form.

DEFINITION 2.1.

(a) Elements $a_1,\ldots,a_r \in A$ are called *separably dependent* over R if they satisfy $W(a_1,\ldots,a_r) = 0$ for some nonzero polynomial $W(T_1,\ldots,T_r) \in R[T_1,\ldots,T_r]_{(p)}$. Otherwise, they are called *separably independent.*
(b) An element $a \in A$ is called *separably algebraic* over R if it satisfies $W(a) = 0$ for some nonzero polynomial $W(T) \in R[T]_{(p)}$.

Of course, a single element $a \in A$ is separably dependent over R if and only if it is separably algebraic over R. Note also the following easy observation.

LEMMA 2.2. *Given arbitrary elements $a_1, \ldots, a_r \in A$, the following conditions are equivalent:*

(i) *$a_1, \ldots, a_r$ are separably dependent over R,*
(ii) *there exists $i \in \{1, \ldots, r\}$ such that a_i is separably algebraic over $R[a_1, \ldots, \widehat{a_i}, \ldots, a_r]$, where $\widehat{\ }$ means that the respective element is omitted.*

Now, let Q be a prime ideal of A. Put $\overline{A} = A/Q$. For an element $a \in A$ denote $\overline{a} = a + Q$. Denote by $\overline{R}$ the canonical homomorphic image of R in $\overline{A}$. Moreover, let $Q[T_1, \ldots, T_r]$ be an R-module of polynomials in variables $T_1, \ldots, T_r$ with coefficients in Q.

LEMMA 2.3. *Given arbitrary elements $a_1, \ldots, a_r \in A$, the following conditions are equivalent:*

(i) *$\overline{a_1}, \ldots, \overline{a_r}$ are separably dependent over $\overline{R}$,*
(ii) *there exists a polynomial $W(T_1, \ldots, T_r) \in R[T_1, \ldots, T_r]_{(p)} \setminus Q[T_1, \ldots, T_r]$ such that $W(a_1, \ldots, a_r) \in Q$.*

3. The ideal generated by jacobian determinants. In the remaining part of the paper, we let K be a unique factorization domain and $A = K[x_1, \ldots, x_n]$ be the K-algebra of polynomials in n variables. We put $B = K[x_1^p, \ldots, x_n^p]$ if $\operatorname{char} K = p > 0$ and $B = K$ if $\operatorname{char} K = 0$.

Recall that for arbitrary $f_1, \ldots, f_r \in A$ and $j_1, \ldots, j_r \in \{1, \ldots, n\}$ we denote by $\operatorname{jac}^{f_1,\ldots,f_r}_{j_1,\ldots,j_r}$ the Jacobian determinant of $f_1, \ldots, f_r$ with respect to $x_{j_1}, \ldots, x_{j_r}$. Then we denote by $\operatorname{Jac}(f_1, \ldots, f_r)$ the ideal of A generated by all determinants of the form $\operatorname{jac}^{f_1,\ldots,f_r}_{j_1,\ldots,j_r}$, where $1 \leqslant j_1 < \ldots < j_r \leqslant n$.

Recall that a *K-derivation* of a K-algebra A is a K-linear map $d : A \to A$ such that $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. Recall further the following easy fact.

LEMMA 3.1 ([5, Lemma 3.2]). *Let K be a domain, let I be an ideal of the polynomial algebra $K[x_1, \ldots, x_n]$ and let δ be an arbitrary K-derivation of the factor algebra $A = K[x_1, \ldots, x_n]/I$. Then there exists a K-derivation d of $K[x_1, \ldots, x_n]$ such that $\delta(\overline{f}) = \overline{d(f)}$ for every $f \in k[x_1, \ldots, x_n]$, where $\overline{f}$ denotes the coset of f in A.*

The following proposition is a common generalization of [7, Proposition 3.4] and [8, Lemma 2.1].

PROPOSITION 3.2. *Let $f_1, \dots, f_r \in A$ and let Q be a prime ideal of A.*

(a) *The inclusion* $\operatorname{Jac}(f_1, \dots, f_r) \subset Q$ *holds if and only if the following condition is satisfied for some $i \in \{1, \dots, r\}$:*

(∗) *there exist $s_1, \dots, s_r \in A$, where $s_i \notin Q$, such that $s_1 d(f_1) + \dots + s_r d(f_r) \in Q$ for every K-derivation d of A.*

(b) *If (∗) is satisfied for a given $i \in \{1, \dots, r\}$, then $\overline{f_i}$ is separably algebraic over $\overline{R_i}$, where $R_i = B[f_1, \dots, \widehat{f_i}, \dots, f_r]$.*

Proof. (a) The inclusion $\operatorname{Jac}(f_1, \dots, f_r) \subset Q$ holds if and only if the rank of the matrix

$$\begin{bmatrix} \overline{\frac{\partial f_1}{\partial x_1}} & \overline{\frac{\partial f_1}{\partial x_2}} & \cdots & \overline{\frac{\partial f_1}{\partial x_n}} \\ \overline{\frac{\partial f_2}{\partial x_1}} & \overline{\frac{\partial f_2}{\partial x_2}} & \cdots & \overline{\frac{\partial f_2}{\partial x_n}} \\ \vdots & \vdots & & \vdots \\ \overline{\frac{\partial f_r}{\partial x_1}} & \overline{\frac{\partial f_r}{\partial x_2}} & \cdots & \overline{\frac{\partial f_r}{\partial x_n}} \end{bmatrix}$$

over the field $(\overline{A})_0$ is less than r. This is equivalent to the linear dependence of the rows of this matrix:

$$\overline{s_1}\left[\overline{\frac{\partial f_1}{\partial x_1}}, \dots, \overline{\frac{\partial f_1}{\partial x_n}}\right] + \dots + \overline{s_r}\left[\overline{\frac{\partial f_r}{\partial x_1}}, \dots, \overline{\frac{\partial f_r}{\partial x_n}}\right] = \left[\overline{0}, \dots, \overline{0}\right]$$

for some $s_1, \dots, s_r \in A$, where $\overline{s_i} \neq \overline{0}$ for some i. The above equality holds if and only if all the polynomials

$$h_1 = s_1 \frac{\partial f_1}{\partial x_1} + \dots + s_r \frac{\partial f_r}{\partial x_1}, \quad \dots, \quad h_r = s_1 \frac{\partial f_1}{\partial x_n} + \dots + s_r \frac{\partial f_r}{\partial x_n}$$

belong to Q.

Observe that for an arbitrary K-derivation d of A we have

$$s_1 d(f_1) + \dots + s_r d(f_r) = h_1 d(x_1) + \dots + h_r d(x_r).$$

Hence, if $h_1, \dots, h_r \in Q$, then $s_1 d(f_1) + \dots + s_r d(f_r) \in Q$. On the other hand, if $s_1 d(f_1) + \dots + s_r d(f_r) \in Q$ for every K-derivation d, then for the partial derivatives we obtain $h_1, \dots, h_r \in Q$.

(b) Assume that (∗) holds for a given i. Consider an arbitrary $\overline{R_i}$-derivation δ of $\overline{A}$. By Lemma 3.1, there exists a K-derivation d of A such that $\delta(\overline{f}) = \overline{d(f)}$ for every $f \in A$. We have $\overline{d(f_j)} = \delta(\overline{f_j}) = \overline{0}$, that is, $d(f_j) \in Q$, for each $j \neq i$. Hence, (∗) yields $s_i d(f_i) \in Q$, so $d(f_i) \in Q$, because $s_i \notin Q$. This means that $\delta(\overline{f_i}) = \overline{d(f_i)} = \overline{0}$. Hence, $\overline{f_i}$ belongs to the smallest ring of constants of an $\overline{R_i}$-derivation of $\overline{A}$, that is, $\overline{f_i}$ is separably algebraic over $\overline{R_i}$. ∎

PROPOSITION 3.3. *Suppose that* $\operatorname{char} K = p \geqslant 0$. *For every $f \in A \setminus \{0\}$ there exist a unique (up to a multiplicative constant) polynomial $a \in B$ and pairwise different irreducible polynomials $g_1, \dots, g_m \in A \setminus B$ such that $f = a g_1^{l_1} \cdot \ldots \cdot g_m^{l_m}$, where $1 \leqslant l_1, \dots, l_m$, and $l_1, \dots, l_m < p$ if $p > 0$.*

Proposition 3.4. *Assume that* $n \geqslant 2$ *and* $0 \leqslant r \leqslant n-2$. *Consider polynomials* $u_1 \in K[x_1, \ldots, x_r, x_{r+1}] \setminus B[x_1, \ldots, x_r]$, $u_2 \in K[x_1, \ldots, x_r, x_{r+2}] \setminus B[x_1, \ldots, x_r]$. *Assume that the degrees of* u_1 *with respect to* x_{r+1} *and of* u_2 *with respect to* x_{r+2} *are relatively prime and, if* $p > 0$, *not divisible by* p. *Then there exist nonzero polynomials* $w_1 \in B[x_1, \ldots, x_r]$ *and* $w_2 \in K[x_1, \ldots, x_r, x_{r+1}, x_{r+2}] \setminus B[x_1, \ldots, x_r]$ *such that* $u_1 + u_2 = w_1 w_2$ *and* w_2 *is irreducible in* $K[x_1, \ldots, x_r, x_{r+1}, x_{r+2}]$.

Proof. By Proposition 3.3 we obtain $u_1 + u_2 = a v_1^{l_1} \ldots v_s^{l_s} v_{s+1}^{l_{s+1}} \ldots v_t^{l_t}$, where $l_i < p$, $a \in B$, $v_1, \ldots, v_t$ are pairwise different irreducible polynomials, $v_1, \ldots, v_s \in k[x_1, \ldots, x_r] \setminus B$ and $v_{s+1}, \ldots, v_t \in k[x_1, \ldots, x_r, x_{r+1}, x_{r+2}] \setminus B[x_1, \ldots, x_r]$, $0 \leqslant s \leqslant t$. Observe that $s < t$, because $u_1 + u_2 \notin B[x_1, \ldots, x_r]$.

Now, consider the field $L = K_0(x_1, \ldots, x_r)$. Since the degrees of the polynomials: u_1 in $L[x_{r+1}]$ and u_2 in $L[x_{r+2}]$ are positive and relatively prime, the polynomial $u_1 + u_2$ is irreducible in $L[x_{r+1}, x_{r+2}]$, by [12, Corollary 3 to Theorem 21, p. 94]. Hence $t = s + 1$ and $l_t = 1$. Finally, put $w_1 = a v_1^{l_1} \ldots v_s^{l_s}$ and $w_2 = v_t$. ∎

Proposition 3.5. *Let* $w \in K[x_1, \ldots, x_m]$ *be an irreducible polynomial such that* $\frac{\partial w}{\partial x_i} \neq 0$ *for some* $i \in \{1, \ldots, m\}$. *Then there exist polynomials* $v_1, v_2 \in K[x_1, \ldots, x_m]$ *and* $v \in K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m] \setminus \{0\}$ *such that*

$$v_1 w + v_2 \frac{\partial w}{\partial x_i} = v.$$

Proof. Consider the field $L = K_0(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$. The polynomial w is irreducible in $L[x_i]$, and the polynomial $\frac{\partial w}{\partial x_i}$ is nonzero, so they are relatively prime in $L[x_i]$. Hence there exist polynomials $u_1, u_2 \in L[x_i]$ such that

$$u_1 w + u_2 \frac{\partial w}{\partial x_i} = 1.$$

Let v ($v \in K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m]$) be the least common denominator of the coefficients of u_1, u_2. Multiplying the above equality by v and denoting $v_1 = u_1 v$, $v_2 = u_2 v$ we get the proposition. ∎

4. Generalizations of Freudenburg's Lemma. Recall that K is a UFD, $A = K[x_1, \ldots, x_n]$ is the polynomial K-algebra in n variables, $B = K[x_1^p, \ldots, x_n^p]$ if $\operatorname{char} K = p > 0$ and $B = K$ if $\operatorname{char} K = 0$.

Theorem 4.1. *Assume that polynomials* $f_1, \ldots, f_r \in A$ *are separably independent over* B. *Let* Q *be a prime ideal of* A. *If* $\operatorname{Jac}(f_1, \ldots, f_r) \subset Q$, *then* $B[f_1, \ldots, f_r] \cap Q \not\subset (B \cap Q)[f_1, \ldots, f_r]$.

Proof. Assume that $\operatorname{Jac}(f_1, \ldots, f_r) \subset Q$. By Proposition 3.2 there exists $i \in \{1, \ldots, m\}$ such that $\overline{f_i}$ is separably algebraic over $\overline{R_i}$, where $R_i = B[f_1, \ldots, \widehat{f_i}, \ldots, f_r]$. By Lemma 2.2 we know that $\overline{f_1}, \ldots, \overline{f_r}$ are separably dependent over $\overline{B}$. By Lemma 2.3 there exists a polynomial $W(T_1,\dots,T_r) \in B[T_1,\dots,T_r]_{(p)} \setminus Q[T_1,\dots,T_r]$ such that $W(f_1,\dots,f_r) \in Q$.

Put $w = W(f_1,\dots,f_r)$. Then $w \in B[f_1,\dots,f_r]$, so $w \in B[f_1,\dots,f_r]\cap Q$. We have $W(T_1,\dots,T_r) = \sum_{0\leqslant i_1,\dots,i_r<p} b_{(i)} T_1^{i_1}\dots T_r^{i_r}$, where $b_{(i)} \in B$ and $b_{(j)} \notin Q$ for some j. It is enough to show that $w \notin (B\cap Q)[f_1,\dots,f_r]$.

Suppose $w \in (B\cap Q)[f_1,\dots,f_r]$. Then $w = \sum_{l_1,\dots,l_r\geqslant 0} c_{(l)} f_1^{l_1}\dots f_r^{l_r}$, where $c_{(l)} \in B\cap Q$, so

$$w = \sum_{0\leqslant i_1,\dots,i_r<p} \Big(\sum_{s_1,\dots,s_r\geqslant 0} c_{(i)+p(s)} (f_1^p)^{s_1}\dots(f_r^p)^{s_r} \Big) f_1^{i_1}\dots f_r^{i_r}.$$

Since $f_1,\dots,f_r$ are separably algebraic independent over B, by the uniqueness of presentation we obtain $b_{(i)} = \sum_{s_1,\dots,s_r\geqslant 0} c_{(i)+p(s)} (f_1^p)^{s_1}\dots(f_r^p)^{s_r}$. For each (i) and (s) we have $c_{(i)+p(s)} \in Q$, so $b_{(i)} \in Q$, which is a contradiction. ■

We can strengthen the assertion of Theorem 4.1 as follows. Here, we denote by $\operatorname{Irr} R$ the set of all irreducible elements of a domain R.

PROPOSITION 4.2. *Let $f_1,\dots,f_r \in A$ and Q be a prime ideal of A. If $B[f_1,\dots,f_r]\cap Q \not\subset (B\cap Q)[f_1,\dots,f_r]$, then we have* $(\operatorname{Irr} B[f_1,\dots,f_r])\cap Q \not\subset (B\cap Q)[f_1,\dots,f_r]$.

Proof. Consider $w \in B[f_1,\dots,f_r]\cap Q$ such that $w \notin (B\cap Q)[f_1,\dots,f_r]$. Suppose $w = w_1\dots w_m$, where $w_1,\dots,w_m \in (\operatorname{Irr} B[f_1,\dots,f_r])$. Since $w_1\dots w_m \in Q$, there exists $i \in \{1,\dots,m\}$ with $w_i \in Q$. If $w_i \in (B\cap Q)[f_1,\dots,f_r]$, then also $w \in (B\cap Q)[f_1,\dots,f_r]$. It follows that $w_i \notin (B\cap Q)[f_1,\dots,f_r]$. ■

As a consequence of Theorem 4.1 and Proposition 4.2 we obtain the following theorem.

THEOREM 4.3. *Let K be a unique factorization domain, Q be a prime ideal of $K[x_1,\dots,x_n]$, and $B = K[x_1^p,\dots,x_n^p]$ if* $\operatorname{char} K = p > 0$ *and $B = K$ if* $\operatorname{char} K = 0$. *Let $f_1,\dots,f_r \in K[x_1,\dots,x_n]$ be separably independent over B with* $\operatorname{Jac}(f_1,\dots,f_r) \subset Q$. *Then there is an irreducible polynomial w, whose coefficients are in B, but not all in Q, such that $w(f_1,\dots,f_r) \in Q$.*

We believe that Theorems 4.1, 4.3 and Proposition 4.2 can be useful in proofs of further equivalent formulations of the Jacobian conditions. We have several conjectures in terms of irreducibility and square-freeness that are too complicated to present here.

REFERENCES

[1] M. de Bondt and D. Yan, *Irreducibility properties of Keller maps*, Algebra Colloq. 23 (2016), 663–680.

[2] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, Birkhäuser, Basel, 2000.
[3] A. van den Essen, A. Nowicki and A. Tyc, *Generalizations of a lemma of Freudenburg*, J. Pure Appl. Algebra 177 (2003), 43–47.
[4] G. Freudenburg, *A note on the kernel of a locally nilpotent derivation*, Proc. Amer. Math. Soc. 124 (1996), 27–29.
[5] P. Jędrzejewicz, *A characterization of one-element p-bases of rings of constants*, Bull. Polish Acad. Sci. Math. 59 (2011), 19–26.
[6] P. Jędrzejewicz, *A characterization of Keller maps*, J. Pure Appl. Algebra 217 (2013), 165–171.
[7] P. Jędrzejewicz, *A characterization of p-bases of rings of constants*, Cent. Eur. J. Math. 11 (2013), 900–909.
[8] P. Jędrzejewicz and J. Zieliński, *Analogs of Jacobian conditions for subrings*, J. Pure Appl. Algebra 221 (2017), 2111–2118.
[9] P. Jędrzejewicz and J. Zieliński, *An approach to the Jacobian Conjecture in terms of irreducibility and square-freeness*, Eur. J. Math. 3 (2017), 199–207.
[10] E. R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.
[11] A. Nowicki, *Polynomial Derivations and Their Rings of Constants*, Nicolaus Copernicus Univ., Toruń, 1994.
[12] A. Schinzel, *Polynomials with Special Regard to Reducibility*, Cambridge Univ. Press, Cambridge, 2000.

Piotr Jędrzejewicz, Janusz Zieliński
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
87-100 Toruń, Poland
E-mail: pjedrzej@mat.umk.pl
ubukrool@mat.umk.pl

Łukasz Matysiak
Institute of Mathematics and Cryptology
Faculty of Cybernetics
Military University of Technology
00-908 Warszawa, Poland
E-mail: lukasz.matysiak@wat.edu.pl