

Approximation properties of intermediate β -expansions

by

KARMA DAJANI and YAN HUANG

Abstract. Given $\beta > 1$ and $\alpha \in [0, 1)$, let $T_{\beta, \alpha}(x) := \beta x + \alpha \pmod{1}$. Then under the map $T_{\beta, \alpha}$, each $x \in [0, 1]$ has an *intermediate β -expansion* of the form $x = \sum_{i=1}^{\infty} \frac{c_i - \alpha}{\beta^i}$ with $c_i \in \{0, 1, \dots, \lfloor \beta + \alpha \rfloor\}$ for each i . We study the approximation properties of $T_{\beta, \alpha}$ by considering the expected value $M_{\beta}(\alpha)$ of the *normalized errors* $(\theta_{\beta, \alpha}^n(x))_{n \geq 1}$, where

$$\theta_{\beta, \alpha}^n(x) := \beta^n \left(x - \sum_{i=1}^n \frac{c_i - \alpha}{\beta^i} \right), \quad n \in \mathbb{N}.$$

We prove that $M_{\beta}(\cdot)$ is continuous on $[0, 1)$. As a result, $\mathcal{M}_{\beta} := \{M_{\beta}(\alpha) : \alpha \in [0, 1)\}$ is a closed interval. In particular, if β is a multinacci number, the map $T_{\beta, \alpha}$ has matching for Lebesgue almost every $\alpha \in [0, 1)$, and then $M_{\beta}(\cdot)$ is locally linear Lebesgue almost everywhere on $[0, 1)$.

1. Introduction. The study of non-integer base expansions was initiated by Rényi [25] and Parry [22, 23]. Since then, expansions in non-integer bases have received much attention revealing connections with many areas of mathematics, such as ergodic theory, fractal geometry, symbolic dynamics, and number theory. They have gained momentum after the surprising discovery by Erdős et al. [11, 12] that for any $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $\beta \in (1, 2]$ and $x \in [0, 1/(\beta - 1)]$ such that x has precisely k different β -expansions. This phenomenon is completely different from the integer base expansions, where each x has a unique expansion, except for countably many x having two expansions. Typically each point in $[0, 1]$ has uncountably many representations [26]. The largest one in the lexicographic order is called the *greedy expansion* and the smallest one is called the *lazy expansion*. An interesting feature of these extreme cases is that they can be generated dynamically by iterating the so-called *greedy β -transformation*

2020 *Mathematics Subject Classification*: Primary 11K55; Secondary 28D05, 11A67, 37A10.

Key words and phrases: intermediate β -expansions, approximation properties, matching, normalized errors, invariant ergodic measures.

Received 30 September 2025; revised 28 January 2026.

Published online 20 May 2026.

and *lazy* β -transformation, respectively. It is natural to ask which one has better approximation properties. It has been proven in [10] that “on average” the greedy convergents $(\sum_{i=1}^n a_i/\beta^i)_{n \geq 1}$ approximate x better than the lazy convergents $(\sum_{i=1}^n b_i/\beta^i)_{n \geq 1}$ for almost every $x \in [0, 1]$. So, what about those expansions between greedy and lazy which are generated by iterating an appropriate transformation? There are some results that focus on the Diophantine approximation properties of β -expansions; see [19, 20, 28]. Also Baker [1, 2] studied the approximation properties of β -expansions.

In this paper, we address the above question through an analysis of approximation properties in *intermediate* β -expansions. These expansions were first introduced by Rényi in 1957 (see [25]) and can be dynamically generated by iterating the *intermediate* β -transformations. We begin by presenting the formal definition and fundamental properties of these transformations.

Given $\beta > 1$, $\alpha \in [0, 1)$, the *intermediate* β -transformation $T_{\beta, \alpha} : [0, 1) \rightarrow [0, 1)$ is defined by

$$(1.1) \quad T_{\beta, \alpha}(x) := \beta x + \alpha \pmod{1} = \beta x + \alpha - \lfloor \beta x + \alpha \rfloor.$$

For a real number r we denote by $\langle r \rangle$ and $\lfloor r \rfloor$ its fractional and integer parts, respectively. For convenience, we define $T_{\beta, \alpha}(1) := \beta + \alpha \pmod{1} = \beta + \alpha - \lfloor \beta + \alpha \rfloor$, which will be used later in the invariant measure of $T_{\beta, \alpha}$. Then by (1.1) it follows that for $\alpha \in [0, 1 - \langle \beta \rangle)$ we have

$$(1.2) \quad T_{\beta, \alpha}(x) = \begin{cases} \beta x + \alpha & \text{if } x \in [0, \frac{1-\alpha}{\beta}), \\ \beta x + \alpha - i & \text{if } x \in [\frac{i-\alpha}{\beta}, \frac{i+1-\alpha}{\beta}), i \in \{1, \dots, \lfloor \beta \rfloor - 1\}, \\ \beta x + \alpha - \lfloor \beta \rfloor, & \text{if } x \in [\frac{\lfloor \beta \rfloor - \alpha}{\beta}, 1], \end{cases}$$

and for $\alpha \in [1 - \langle \beta \rangle, 1)$ we have

$$(1.3) \quad T_{\beta, \alpha}(x) = \begin{cases} \beta x + \alpha & \text{if } x \in [0, \frac{1-\alpha}{\beta}), \\ \beta x + \alpha - i & \text{if } x \in [\frac{i-\alpha}{\beta}, \frac{i+1-\alpha}{\beta}), i \in \{1, \dots, \lfloor \beta \rfloor\}, \\ \beta x + \alpha - \lfloor \beta \rfloor - 1 & \text{if } x \in [\frac{\lfloor \beta \rfloor + 1 - \alpha}{\beta}, 1]. \end{cases}$$

Clearly, the map $T_{\beta, \alpha}$ has $\lfloor \beta \rfloor + 1$ branches if $\alpha \in [0, 1 - \langle \beta \rangle)$, and has $\lfloor \beta \rfloor + 2$ branches if $\alpha \in [1 - \langle \beta \rangle, 1)$. See Figure 1 for an illustration.

It is well known that, in general, $T_{\beta, \alpha}$ does not preserve Lebesgue measure and is not a subshift of finite type with mixing properties. This causes difficulties in studying metrical questions related to β -expansions. We mention a few results regarding these transformations. Hofbauer [16] proved that for any $\alpha \in [0, 1)$ the topological entropy of $T_{\beta, \alpha}$ is $\log \beta$, and $T_{\beta, \alpha}$ has a unique invariant probability measure of maximal entropy. This maximal measure is absolutely continuous with respect to the Lebesgue measure, and its support is a finite union of intervals (Hofbauer [15, 17]). Its density function was given by Parry [22], and was later proved to be non-negative by Halfin [14].

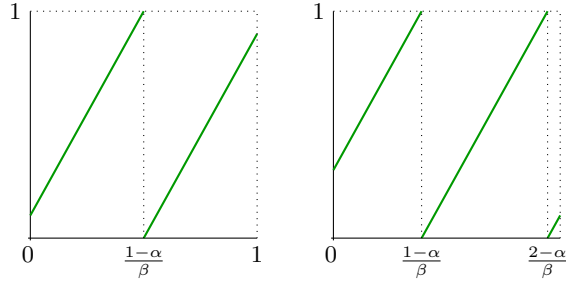


Fig. 1. Fix $\beta = 1.8$. The left figure shows the image of the transformation $T_{\beta, \alpha}$ for $\alpha = 0.1 \in [0, 1 - \langle\beta\rangle)$, and the right figure for $\alpha = 0.3 \in [1 - \langle\beta\rangle, 1)$.

Concerning the ergodic properties of $T_{\beta, \alpha}$, Wilkinson [29, 30] showed that $T_{\beta, \alpha}$ is weak-Bernoulli with respect to the maximal measure when $\beta > 2$, and this was extended by Parry [24] to $\beta > \sqrt{2}$. However, this is not true for $\beta \in (1, \sqrt{2}]$. Palmer (cf. [13]) characterized all pairs $(\beta, \alpha) \in (1, \sqrt{2}) \times [0, 1)$ for which $T_{\beta, \alpha}$ is weak-Bernoulli, and proved that when the pair (β, α) is not in the “bubble”, the map $T_{\beta, \alpha}$ is weak-Bernoulli. Importantly, the map $T_{\beta, \alpha}$ is ergodic for any $\beta > \sqrt{2}$ and $\alpha \in [0, 1)$ ([6]).

The following theorem describes the density function of the unique $T_{\beta, \alpha}$ -invariant probability measure of maximal entropy (see [23]). Throughout this article, we denote the Lebesgue measure on \mathbb{R} by λ .

THEOREM 1.1 (Parry, 1964). *Given $\beta > 1$ and $\alpha \in [0, 1)$, let $T_{\beta, \alpha}(x) = \beta x + \alpha \pmod{1}$. Then there exists a unique $T_{\beta, \alpha}$ -invariant probability measure $\nu_{\beta, \alpha}$ on $[0, 1]$ such that*

$$\nu_{\beta, \alpha}(E) := \int_E g_{\beta, \alpha}(x) \, d\lambda(x)$$

for every Borel set $E \subset [0, 1]$, where the density function $g_{\beta, \alpha}$ is given by

$$(1.4) \quad g_{\beta, \alpha}(x) := \frac{\tilde{g}_{\beta, \alpha}(x)}{\int_{[0, 1]} \tilde{g}_{\beta, \alpha} \, d\lambda} \quad \text{with} \quad \tilde{g}_{\beta, \alpha}(x) := \sum_{x < T_{\beta, \alpha}^n(1)} \frac{1}{\beta^n} - \sum_{x < T_{\beta, \alpha}^n(0)} \frac{1}{\beta^n}.$$

Furthermore:

- (i) $\tilde{g}_{\beta, \alpha}(x) \geq 0$ for λ -almost every $x \in [0, 1]$.
- (ii) If $\beta > \sqrt{2}$ and $\alpha \in [0, 1)$, then $T_{\beta, \alpha}$ is ergodic with respect to $\nu_{\beta, \alpha}$.

Given $\beta > 1$ and $\alpha \in [0, 1)$, recall that every $x \in [0, 1]$ admits an intermediate β -expansion generated by $T_{\beta, \alpha}$. We now formally describe the dynamical mechanism through which this transformation produces expansions. For $x \in [0, 1]$ and $n \geq 0$ let

$$(1.5) \quad c_{n+1} := \lfloor \beta T_{\beta, \alpha}^n(x) + \alpha \rfloor.$$

Then (1.1) yields $T_{\beta,\alpha}(x) = \beta x + \alpha - c_1$, which gives $x = \frac{T_{\beta,\alpha}(x)}{\beta} + \frac{c_1 - \alpha}{\beta}$. Note that $T_{\beta,\alpha}(x) \in [0, 1]$. Replacing x by $T_{\beta,\alpha}(x)$ in the above argument gives $x = \frac{T_{\beta,\alpha}^2(x)}{\beta^2} + \frac{c_2 - \alpha}{\beta^2} + \frac{c_1 - \alpha}{\beta}$. Iterating this argument, we obtain the intermediate β -expansion of x by

$$(1.6) \quad x = \sum_{i \geq 1} \frac{c_i - \alpha}{\beta^i}, \quad c_i \in \{0, 1, \dots, \lfloor \beta + \alpha \rfloor\}.$$

The infinite sequence (c_i) is also called the *intermediate β -expansion of x with parameter α* .

In [9] Dajani et al. introduced the *normalized errors* of x with respect to $T_{\beta,\alpha}$, which are defined by

$$\theta_{\beta,\alpha}^n(x) := T_{\beta,\alpha}^n(x) = \beta^n \left(x - \sum_{i=1}^n \frac{c_i - \alpha}{\beta^i} \right), \quad n \in \mathbb{N}.$$

Note that the sequence $(\theta_{\beta,\alpha}^n(x))_{n \geq 1}$ reveals the approximation efficiency of the convergents $(\sum_{i=1}^n \frac{c_i - \alpha}{\beta^i})_{n \geq 1}$ to the point x . If $\beta > \sqrt{2}$ and $\alpha \in [0, 1)$, then by Theorem 1.1 and a straightforward application of Birkhoff's ergodic theorem it follows that for $\nu_{\beta,\alpha}$ -almost every $x \in [0, 1]$ we have

$$(1.7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \theta_{\beta,\alpha}^i(x) = \int_{[0,1]} x g_{\beta,\alpha}(x) d\lambda(x) =: M_\beta(\alpha).$$

The smaller $M_\beta(\alpha)$, the better the approximation efficiency of $T_{\beta,\alpha}$. So, the quantity $M_\beta(\alpha)$ characterizes the approximation efficiency of $T_{\beta,\alpha}$. Observe by (1.1) that for $\alpha = 0$ the map $T_{\beta,0}$ yields the greedy β -expansion of x , while for $\alpha = 1 - \langle \beta \rangle$ the map $T_{\beta,1-\langle \beta \rangle}$ gives the lazy β -expansion of x . Dajani et al. [9] showed that $M_\beta(0) < M_\beta(1 - \langle \beta \rangle)$ for any $\beta > 1$. This means in general that the greedy β -transformation has better approximation properties than the lazy β -transformation. In general, what can we say about the approximation efficiency of $T_{\beta,\alpha}$ for a general $\alpha \in [0, 1)$? This will be our main object of study in this paper.

Our first result states that M_β is continuous.

THEOREM 1.2. *Let $\beta > \sqrt{2}$. Then M_β is continuous on $[0, 1)$. Furthermore, $M_\beta(\alpha) + M_\beta(1 - \langle \beta + \alpha \rangle) = 1$ for any $\alpha \in [0, 1)$.*

REMARK 1.3. An adaption of the proof of Theorem 1.2 shows that the map $\beta \mapsto M_\beta(\alpha)$ is also continuous on $(\sqrt{2}, \infty)$ for any $\alpha \in [0, 1)$.

In the following we fix $\beta > \sqrt{2}$, and set

$$\mathcal{M}_\beta := \{M_\beta(\alpha) : \alpha \in [0, 1)\}.$$

Note by Theorem 1.2 that

$$M_\beta(\alpha) + M_\beta(1 - \langle \beta + \alpha \rangle) = 1 \quad \text{for any } \alpha \in [0, 1).$$

Then

$$\begin{cases} M_\beta(\alpha) + M_\beta(1 - \langle \beta \rangle - \alpha) = 1 & \text{if } \alpha \in [0, 1 - \langle \beta \rangle), \\ M_\beta(\alpha) + M_\beta(2 - \langle \beta \rangle - \alpha) = 1 & \text{if } \alpha \in [1 - \langle \beta \rangle, 1). \end{cases}$$

This, together with the continuity property of M_β in Theorem 1.2, implies that \mathcal{M}_β is a closed interval in $[0, 1)$ symmetric about the line $y = 1/2$ (see Figure 2 for an example).

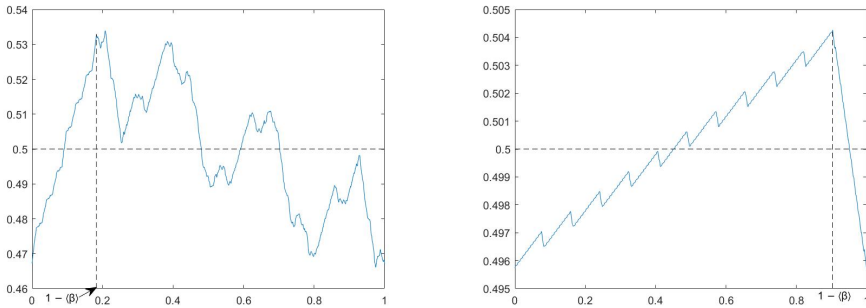


Fig. 2. The graphs of $M_\beta(\alpha)$ for $\alpha \in [0, 1)$ with $\beta \approx 2.8177$ (left) and $\beta \approx 11.0976$ (right).

Note that M_β depends on the density function $g_{\beta,\alpha}$, which is in general an infinite series (see Theorem 1.1). This leads to difficulties in studying extreme values of \mathcal{M}_β . However, $g_{\beta,\alpha}$ becomes a finite sum if the orbits of 0 and 1 meet after a finite number of iterations; this phenomenon is called *matching*. Especially in the case of α -continued fraction maps, the concept of matching has proven to be very useful (see [7, 8, 18, 21]). For piecewise linear transformations, prevalent matching appears to be rare: for instance it can occur for $T_{\beta,\alpha}$ only if the slope β is an algebraic integer. In [4] it is shown that matching occurs for all quadratic Pisot slopes and a set of translation parameters whose complement has Hausdorff dimension smaller than 1. Later, Bruin and Keszthelyi [5] extended this result. Sun et al. [27] gave some dense fibre results of matching for fixed $\beta \in (1, 2)$ and $\alpha \in [0, 1 - \langle \beta \rangle)$. We point out that all of these results for $T_{\beta,\alpha}$ focus on either $\beta \in (1, 2)$ or $\alpha \in [0, 1 - \langle \beta \rangle)$.

In the following we assume β belongs to a special class of algebraic integers. Given $q, m \in \mathbb{N}$, we define the (q, m) -*multinacci number* $\beta_{q,m} \in (q, q+1)$ to be the real root of

$$x^m = qx^{m-1} + qx^{m-2} + \dots + q.$$

Our second result shows that $T_{\beta_{q,m},\alpha}$ has matching for typical $\alpha \in [0, 1)$.

THEOREM 1.4. *For all $m, q \in \mathbb{N}$, the transformation $T_{\beta_{q,m},\alpha}$ has matching for Lebesgue almost every $\alpha \in [0, 1)$.*

Given $\beta > \sqrt{2}$, a subinterval $I \subset (0, 1]$ is called a *matching interval* if there exists $k \in \mathbb{N}$ such that for every $\alpha \in I$,

$$T_{\beta,\alpha}^i(0) \neq T_{\beta,\alpha}^i(1) \quad \forall 1 \leq i \leq k \quad \text{and} \quad T_{\beta,\alpha}^{k+1}(0) = T_{\beta,\alpha}^{k+1}(1),$$

where I is maximal with this property. By (1.4), the density function $g_{\beta,\alpha}$ is then a sum of at most $k + 1$ terms for all $\alpha \in I$.

We establish that M_β is linear on each matching interval. Theorem 1.4 implies that matching intervals cover $[0, 1)$ up to a Lebesgue null set when $\beta = \beta_{q,m}$. Consequently, $M_{\beta_{q,m}}(\cdot)$ is linear for almost every $\alpha \in [0, 1)$. Furthermore, we characterize the monotonicity of $M_{\beta_{q,m}}$ on each matching interval (see details in Section 4). In particular, $[0, 1 - \langle \beta_{q,m} \rangle)$ is a matching interval where $M_{\beta_{q,m}}$ is strictly increasing (see Section 3). For fixed $q, m \in \mathbb{N}$, write

$$M_{\beta_{q,m}}(\alpha) := k_{q,m}\alpha + b_{q,m} \quad \text{for } \alpha \in [0, 1 - \langle \beta_{q,m} \rangle).$$

Our final result states that for $q \in \mathbb{N}$ and any $m \geq 2$,

$$k_{q,m} < k_{q,m+1} \quad \text{and} \quad b_{q,m} < b_{q,m+1}.$$

This implies that $T_{\beta_{q,2},\alpha}$ achieves superior approximation efficiency for $\alpha \in [0, 1 - \langle \beta_{q,m} \rangle)$ among $m \geq 2$ (see Figure 3).

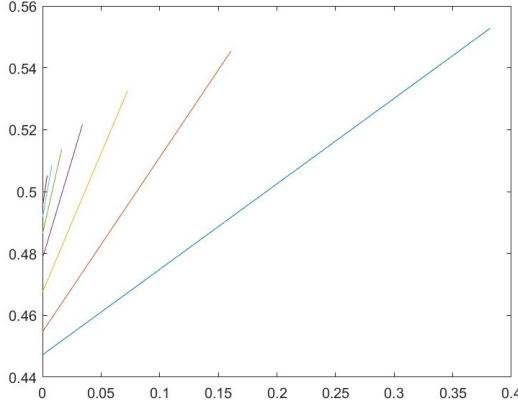


Fig. 3. The graph of $M_{\beta_{1,m}}(\alpha) = k_{1,m}\alpha + b_{1,m}$ over $\alpha \in [0, 1 - \langle \beta_{1,m} \rangle)$ for $m = 2, \dots, 8$ (from bottom to top).

THEOREM 1.5. *Given $q \geq 1$, both sequences $\{k_{q,m}\}_{m \geq 1}$ and $\{b_{q,m}\}_{m \geq 1}$ are strictly increasing.*

The rest of the paper is organized as follows. In the next section we study the continuity properties of M_β , and then prove Theorem 1.2. The matching result in Theorem 1.4 is split into two parts, $\alpha \in [0, 1 - \langle \beta_{q,m} \rangle)$ and $\alpha \in [1 - \langle \beta_{q,m} \rangle, 1)$. In Section 3 we show that $M_{\beta_{q,m}}$ is linear and increasing on $[0, 1 - \langle \beta_{q,m} \rangle)$ and prove Theorem 1.5. The matching results for $\alpha \in [1 - \langle \beta_{q,m} \rangle, 1)$ are proven in Section 4. We give a further question in Section 5.

2. The continuity of M_β . This section investigates the continuity of the function M_β , culminating in the proof of Theorem 1.2. Recall from (1.7) that

$$M_\beta(\alpha) = \int_{[0,1]} x g_{\beta,\alpha}(x) d\lambda(x),$$

where $g_{\beta,\alpha}$ is the density of the invariant probability measure $\nu_{\beta,\alpha}$ for $T_{\beta,\alpha}$ (see Theorem 1.1). So, to prove the continuity of M_β it suffices to prove the continuity of the map $\alpha \mapsto g_{\beta,\alpha}$ in $L^1(\lambda)$. We first establish the right continuity of this mapping.

LEMMA 2.1. *Let $\beta > 1$ and $\alpha \in [0, 1)$. For any sequence $(\alpha_k)_{k \geq 1} \subset (\alpha, 1)$ satisfying $\lim_{k \rightarrow \infty} \alpha_k = \alpha$, we have $\lim_{k \rightarrow \infty} \|g_{\beta,\alpha_k} - g_{\beta,\alpha}\|_1 = 0$.*

Proof. Since the proof for $\alpha \in [1 - \langle \beta \rangle, 1)$ is similar, we only focus on $\alpha \in [0, 1 - \langle \beta \rangle)$. Fix $\alpha \in [0, 1 - \langle \beta \rangle)$, and take a sequence $(\alpha_k) \subset (\alpha, 1 - \langle \beta \rangle)$ converging to α as $k \rightarrow \infty$. By Theorem 1.1, we have

$$g_{\beta,\alpha_k}(x) = \frac{\tilde{g}_{\beta,\alpha_k}(x)}{K_{\beta,\alpha_k}} \quad \text{for } x \in [0, 1),$$

where $K_{\beta,\alpha_k} := \int_{[0,1]} \tilde{g}_{\beta,\alpha_k} d\lambda \in (0, \infty)$. Then

$$\begin{aligned} \|g_{\beta,\alpha_k} - g_{\beta,\alpha}\|_1 &= \int_{[0,1]} \left| \frac{\tilde{g}_{\beta,\alpha_k}}{K_{\beta,\alpha_k}} - \frac{\tilde{g}_{\beta,\alpha}}{K_{\beta,\alpha}} \right| d\lambda \\ &\leq \int_{[0,1]} \left| \frac{\tilde{g}_{\beta,\alpha_k}}{K_{\beta,\alpha_k}} - \frac{\tilde{g}_{\beta,\alpha_k}}{K_{\beta,\alpha}} \right| d\lambda + \int_{[0,1]} \left| \frac{\tilde{g}_{\beta,\alpha_k}}{K_{\beta,\alpha}} - \frac{\tilde{g}_{\beta,\alpha}}{K_{\beta,\alpha}} \right| d\lambda \\ &\leq \frac{\|\tilde{g}_{\beta,\alpha_k}\|_\infty}{K_{\beta,\alpha} K_{\beta,\alpha_k}} \|\tilde{g}_{\beta,\alpha_k} - \tilde{g}_{\beta,\alpha}\|_1 + \frac{1}{K_{\beta,\alpha}} \|\tilde{g}_{\beta,\alpha_k} - \tilde{g}_{\beta,\alpha}\|_1, \end{aligned}$$

where $\|\tilde{g}_{\beta,\alpha_k}\|_\infty = \sup_{x \in [0,1]} |\tilde{g}_{\beta,\alpha_k}(x)|$. Thus, it suffices to prove

$$(2.1) \quad \lim_{k \rightarrow \infty} \|\tilde{g}_{\beta,\alpha_k} - \tilde{g}_{\beta,\alpha}\|_1 = 0.$$

Note that $\alpha, \alpha_k \in [0, 1 - \langle \beta \rangle)$ for all $k \in \mathbb{N}$. Let $(a_i), (b_i) \in \{0, \dots, \lfloor \beta \rfloor\}^{\mathbb{N}}$ be the intermediate β -expansions with parameter α of 0 and 1, respectively.

Then by (1.6) we have

$$0 = \sum_{i=1}^{\infty} \frac{a_i - \alpha}{\beta^i}, \quad 1 = \sum_{i=1}^{\infty} \frac{b_i - \alpha}{\beta^i}.$$

Similarly, for $k \in \mathbb{N}$ let $(a_{k,i}), (b_{k,i}) \in \{0, \dots, \lfloor \beta \rfloor\}^{\mathbb{N}}$ be the intermediate β -expansions with parameter α_k of 0 and 1, respectively. Then

$$0 = \sum_{i=1}^{\infty} \frac{a_{k,i} - \alpha_k}{\beta^i}, \quad 1 = \sum_{i=1}^{\infty} \frac{b_{k,i} - \alpha_k}{\beta^i}.$$

For $k \in \mathbb{N}$ set

$$(2.2) \quad N_k := \max \{i \in \mathbb{N} : a_{k,1} \dots a_{k,i} = a_1 \dots a_i \text{ and } b_{k,1} \dots b_{k,i} = b_1 \dots b_i\}.$$

Note that $N_k \geq 1$ since $a_{k,1} = 0 = a_1$ and $b_{k,1} = \lfloor \beta \rfloor = b_1$ for all $\alpha_k \in (\alpha, 1 - \langle \beta \rangle)$. Observe that $\alpha_k > \alpha$ and $\lim_{k \rightarrow \infty} \alpha_k = \alpha$. Then we establish the following Claim.

CLAIM. *For every $N \in \mathbb{N}$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, we have*

$$(2.3) \quad \begin{aligned} 0 < \alpha_k - \alpha &< \frac{\beta - 1}{\beta^{N+1} - 1} \cdot \min \left\{ \frac{a_i + 1 - \alpha}{\beta} - T_{\beta, \alpha}^{i-1}(0) : 1 \leq i \leq N \right\}, \\ 0 < \alpha_k - \alpha &< \frac{\beta - 1}{\beta^{N+1} - 1} \cdot \min \left\{ \frac{b_i + 1 - \alpha}{\beta} - T_{\beta, \alpha}^{i-1}(1) : 1 \leq i \leq N \right\}, \end{aligned}$$

and consequently $N \leq N_k \leq \log_{\beta} \left(1 + \frac{\beta - 1}{\alpha_k - \alpha} \right)$.

By (1.5), we have $a_i = \lfloor \beta T_{\beta, \alpha}^{i-1}(0) + \alpha \rfloor > \beta T_{\beta, \alpha}^{i-1}(0) + \alpha - 1$ and $b_i > \beta T_{\beta, \alpha}^{i-1}(1) + \alpha - 1$ for all $i \geq 1$. This ensures the positivity of the minima of the right-hand terms in (2.3). Since $\alpha_k > \alpha$ and $\lim_{k \rightarrow \infty} \alpha_k = \alpha$, the inequalities in (2.3) hold for all sufficiently large k .

We now establish the bounds for N_k . First, consider the upper bound. From (1.6), for any $x \in [0, 1]$ we have

$$(2.4) \quad T_{\beta, \alpha}^n(x) = \beta^n \left(x - \sum_{i=1}^n \frac{c_i - \alpha}{\beta^i} \right) \quad \forall n \geq 1,$$

where $(c_i)_{i=1}^{\infty} \in \{0, 1, \dots, \lfloor \beta \rfloor\}^{\mathbb{N}}$ is the intermediate β -expansion of x with parameter α . Combining (2.2) and (2.4) yields

$$T_{\beta, \alpha_k}^{N_k}(0) - T_{\beta, \alpha}^{N_k}(0) = \beta^{N_k} \sum_{i=1}^{N_k} \frac{\alpha_k - \alpha}{\beta^i} = \frac{\beta^{N_k} - 1}{\beta - 1} (\alpha_k - \alpha).$$

As $T_{\beta, \alpha_k}^{N_k}(0), T_{\beta, \alpha}^{N_k}(0) \in [0, 1]$, it follows that $\frac{\beta^{N_k} - 1}{\beta - 1} (\alpha_k - \alpha) \leq 1$, implying

$$N_k \leq \log_{\beta} \left(1 + \frac{\beta - 1}{\alpha_k - \alpha} \right).$$

To establish the lower bound of N_k , we proceed by induction on N . The base case $N = 1$ holds trivially since $N_k \geq 1$ for all $\alpha_k \in (\alpha, 1 - \langle \beta \rangle)$.

Now assume $N \geq 2$, and suppose that

$$a_{k,1} \dots a_{k,n} = a_1 \dots a_n, \quad b_{k,1} \dots b_{k,n} = b_1 \dots b_n \quad \text{for some } n \leq N - 1.$$

We will show that $a_{k,n+1} = a_{n+1}$ and $b_{k,n+1} = b_{n+1}$ under assumption (2.3). By (2.4) and the condition $\alpha_k > \alpha$, we derive

$$\begin{aligned} T_{\beta, \alpha_k}^n(0) &= \beta^n \sum_{i=1}^n \frac{\alpha_k - a_i}{\beta^i} \\ &> \beta^n \sum_{i=1}^n \frac{\alpha - a_i}{\beta^i} = T_{\beta, \alpha}^n(0) \geq \frac{a_{n+1} - \alpha}{\beta} > \frac{a_{n+1} - \alpha_k}{\beta}, \end{aligned}$$

where the second inequality follows from $a_{n+1} = \lfloor \beta T_{\beta, \alpha}^n(0) + \alpha \rfloor \leq \beta T_{\beta, \alpha}^n(0) + \alpha$. This implies $a_{k,n+1} = \lfloor \beta T_{\beta, \alpha_k}^n(0) + \alpha_k \rfloor \geq a_{n+1}$.

Conversely, using (2.4) we obtain

$$\begin{aligned} T_{\beta, \alpha_k}^n(0) &= \beta^n \sum_{i=1}^n \frac{\alpha_k - a_i}{\beta^i} \\ &= T_{\beta, \alpha}^n(0) + \beta^n \sum_{i=1}^n \frac{\alpha_k - \alpha}{\beta^i} = T_{\beta, \alpha}^n(0) + \frac{\beta^n - 1}{\beta - 1} (\alpha_k - \alpha) \\ &< \frac{a_{n+1} + 1 - \alpha}{\beta} - \frac{\beta^{N+1} - 1}{\beta - 1} (\alpha_k - \alpha) + \frac{\beta^n - 1}{\beta - 1} (\alpha_k - \alpha) \\ &< \frac{a_{n+1} + 1 - \alpha}{\beta} - \frac{\alpha_k - \alpha}{\beta} = \frac{a_{n+1} + 1 - \alpha_k}{\beta}. \end{aligned}$$

The first inequality relies on (2.3), which gives

$$\alpha_k - \alpha < \frac{\beta - 1}{\beta^{N+1} - 1} \left(\frac{a_{n+1} + 1 - \alpha}{\beta} - T_{\beta, \alpha}^n(0) \right).$$

Consequently, $a_{k,n+1} = \lfloor \beta T_{\beta, \alpha_k}^n(0) + \alpha_k \rfloor < a_{n+1} + 1$, implying $a_{k,n+1} = a_{n+1}$.

By an identical reasoning applied to $b_{k,1} \dots b_{k,n} = b_1 \dots b_n$ and (2.3), we obtain $b_{k,n+1} = b_{n+1}$. Induction thus establishes $N_k \geq N$, completing the proof of the Claim.

By (1.4), we have

$$\begin{aligned} (2.5) \quad \|\tilde{g}_{\beta, \alpha_k} - \tilde{g}_{\beta, \alpha}\|_1 &\leq \int_{[0,1]} \left| \sum_{i=0}^{\infty} \frac{\mathbb{1}_{[0, T_{\beta, \alpha_k}^i(1)]}}{\beta^i} - \sum_{i=0}^{\infty} \frac{\mathbb{1}_{[0, T_{\beta, \alpha}^i(1)]}}{\beta^i} \right| d\lambda \\ &\quad + \int_{[0,1]} \left| \sum_{i=0}^{\infty} \frac{\mathbb{1}_{[0, T_{\beta, \alpha_k}^i(0)]}}{\beta^i} - \sum_{i=0}^{\infty} \frac{\mathbb{1}_{[0, T_{\beta, \alpha}^i(0)]}}{\beta^i} \right| d\lambda \end{aligned}$$

$$\begin{aligned} &\leq \int_{[0,1]} \left| \sum_{i=0}^{N_k} \frac{\mathbb{1}_{[0, T_{\beta, \alpha_k}^i(1)]}}{\beta^i} - \sum_{i=0}^{N_k} \frac{\mathbb{1}_{[0, T_{\beta, \alpha}^i(1)]}}{\beta^i} \right| d\lambda \\ &\quad + \int_{[0,1]} \left| \sum_{i=0}^{N_k} \frac{\mathbb{1}_{[0, T_{\beta, \alpha_k}^i(0)]}}{\beta^i} - \sum_{i=0}^{N_k} \frac{\mathbb{1}_{[0, T_{\beta, \alpha}^i(0)]}}{\beta^i} \right| d\lambda + \sum_{i=N_k+1}^{\infty} \frac{4}{\beta^i}. \end{aligned}$$

By (2.2) and (2.4), we have $T_{\beta, \alpha_k}^i(1) > T_{\beta, \alpha}^i(1)$ and $T_{\beta, \alpha_k}^i(0) > T_{\beta, \alpha}^i(0)$ for all $1 \leq i \leq N_k$. Moreover, the Claim implies $N_k \rightarrow \infty$ as $k \rightarrow \infty$. Combining (2.4), (2.5), and the Claim, we deduce that

$$\begin{aligned} &\|\tilde{g}_{\beta, \alpha_k} - \tilde{g}_{\beta, \alpha}\|_1 \\ &\leq \left\| \sum_{i=1}^{N_k} \frac{\mathbb{1}_{[T_{\beta, \alpha}^i(1), T_{\beta, \alpha_k}^i(1)]}}{\beta^i} \right\|_1 + \left\| \sum_{i=1}^{N_k} \frac{\mathbb{1}_{[T_{\beta, \alpha}^i(0), T_{\beta, \alpha_k}^i(0)]}}{\beta^i} \right\|_1 + \frac{4}{(\beta-1)\beta^{N_k}} \\ &= \sum_{i=1}^{N_k} \frac{T_{\beta, \alpha_k}^i(1) - T_{\beta, \alpha}^i(1)}{\beta^i} + \sum_{i=1}^{N_k} \frac{T_{\beta, \alpha_k}^i(0) - T_{\beta, \alpha}^i(0)}{\beta^i} + \frac{4}{(\beta-1)\beta^{N_k}} \\ &\leq 2 \sum_{i=1}^{N_k} \frac{1}{\beta^i} \cdot \frac{\beta^i - 1}{\beta - 1} (\alpha_k - \alpha) + \frac{4}{(\beta-1)\beta^{N_k}} \\ &\leq \frac{2}{\beta-1} N_k (\alpha_k - \alpha) + \frac{4}{(\beta-1)\beta^{N_k}} \\ &\leq \frac{2}{\beta-1} (\alpha_k - \alpha) \log_{\beta} \left(1 + \frac{\beta-1}{\alpha_k - \alpha} \right) + \frac{4}{(\beta-1)\beta^{N_k}} \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$, since $\lim_{k \rightarrow \infty} \alpha_k = \alpha$ and $\lim_{k \rightarrow \infty} N_k = +\infty$. This establishes (2.1), completing the proof of Lemma 2.1. ■

However, the method from the previous lemma cannot be directly applied to prove the left continuity of the mapping $\beta \mapsto g_{\beta, \alpha}$ in $L^1(\lambda)$. For example, for all $\beta > \sqrt{2}$, take $\alpha = 1 - \langle \beta \rangle$. The corresponding intermediate β -expansion of 1 has $b_1 = \lfloor \beta \rfloor + 1$. But for all $\alpha_k < \alpha$, the corresponding intermediate β -expansion of 1 has $b_{k,1} = \lfloor \beta \rfloor$, which does not satisfy the Claim.

To address this issue, for $\beta > 1$ and $\alpha \in (0, 1]$, we introduce an auxiliary expanding map $\tilde{T}_{\beta, \alpha} : [0, 1] \rightarrow [0, 1]$ defined by $\tilde{T}_{\beta, \alpha}(x) := \beta x + \alpha - \lceil \beta x + \alpha \rceil + 1$ for all $x \in [0, 1]$, which is left continuous. This map admits the following explicit expressions: for $\alpha \in (0, 1 - \langle \beta \rangle]$ we have

$$\tilde{T}_{\beta, \alpha}(x) = \begin{cases} \beta x + \alpha & \text{if } x \in [0, \frac{1-\alpha}{\beta}], \\ \beta x + \alpha - i & \text{if } x \in (\frac{i-\alpha}{\beta}, \frac{i+1-\alpha}{\beta}], i \in \{1, \dots, \lfloor \beta \rfloor - 1\}, \\ \beta x + \alpha - \lfloor \beta \rfloor & \text{if } x \in (\frac{\lfloor \beta \rfloor - \alpha}{\beta}, 1], \end{cases}$$

and for $\alpha \in (1 - \langle \beta \rangle, 1]$ we have

$$\tilde{T}_{\beta,\alpha}(x) = \begin{cases} \beta x + \alpha & \text{if } x \in [0, \frac{1-\alpha}{\beta}], \\ \beta x + \alpha - i & \text{if } x \in (\frac{i-\alpha}{\beta}, \frac{i+1-\alpha}{\beta}], i \in \{1, \dots, \lfloor \beta \rfloor\}, \\ \beta x + \alpha - \lfloor \beta \rfloor - 1 & \text{if } x \in (\frac{\lfloor \beta \rfloor + 1 - \alpha}{\beta}, 1]. \end{cases}$$

Comparison with (1.2) and (1.3) reveals that $\tilde{T}_{\beta,\alpha}$ coincides with $T_{\beta,\alpha}$ except at the finitely many points $\{\frac{i-\alpha}{\beta} : i = 1, \dots, \lfloor \beta \rfloor\}$. Adapting the proof of [23, Theorem 1.6] yields the following analogous result for $\tilde{T}_{\beta,\alpha}$.

THEOREM 2.2 (Parry, 1964). *Given $\beta > 1$ and $\alpha \in [0, 1)$, let $\tilde{T}_{\beta,\alpha}(x) = \beta x + \alpha - \lceil \beta x + \alpha \rceil + 1$ for $x \in [0, 1]$. There exists a unique $\tilde{T}_{\beta,\alpha}$ -invariant probability measure $\mu_{\beta,\alpha}$ on $[0, 1]$ satisfying*

$$\mu_{\beta,\alpha}(E) = \int_E h_{\beta,\alpha}(x) d\lambda(x)$$

for every Borel set $E \subset [0, 1]$, where the density function $h_{\beta,\alpha}$ is given by

$$(2.6) \quad h_{\beta,\alpha}(x) := \frac{\tilde{h}_{\beta,\alpha}(x)}{\int_{[0,1]} \tilde{h}_{\beta,\alpha} d\lambda} \quad \text{with} \quad \tilde{h}_{\beta,\alpha}(x) := \sum_{x < \tilde{T}_{\beta,\alpha}^n(1)} \frac{1}{\beta^n} - \sum_{x < \tilde{T}_{\beta,\alpha}^n(0)} \frac{1}{\beta^n}.$$

Furthermore:

- (i) $\tilde{h}_{\beta,\alpha}(x) \geq 0$ for λ -almost every $x \in [0, 1]$.
- (ii) If $\beta > \sqrt{2}$ and $\alpha \in (0, 1]$, then $\tilde{T}_{\beta,\alpha}$ is ergodic with respect to $\mu_{\beta,\alpha}$.

Since $\tilde{T}_{\beta,\alpha}$ coincides with $T_{\beta,\alpha}$ except at finitely many points, comparing (2.6) with (1.4) yields $\int_{[0,1]} \tilde{h}_{\beta,\alpha} d\lambda = \int_{[0,1]} \tilde{g}_{\beta,\alpha} d\lambda$ and $h_{\beta,\alpha} = g_{\beta,\alpha}$ for λ -a.e. $x \in [0, 1]$. Then, by (1.7), we have

$$M_\beta(\alpha) = \int_{[0,1]} x h_{\beta,\alpha}(x) d\lambda(x).$$

To establish the left continuity of M_β , it suffices to prove the left continuity of the mapping $\alpha \mapsto h_{\beta,\alpha}$ in $L^1(\lambda)$. Note that $T_{\beta,\alpha+n} = T_{\beta,\alpha}$ and $\tilde{T}_{\beta,\alpha+n} = \tilde{T}_{\beta,\alpha}$ for all $n \in \mathbb{Z}$, which implies $M_\beta(\alpha + n) = M_\beta(\alpha)$ for $\alpha \in [0, 1)$ and $n \in \mathbb{Z}$. We now prove the left continuity of M_β .

LEMMA 2.3. *Let $\beta > 1$ and $\alpha \in (0, 1]$. For any sequence $(\alpha_k)_{k \geq 1} \subset (0, \alpha)$ satisfying $\lim_{k \rightarrow \infty} \alpha_k = \alpha$, we have $\lim_{k \rightarrow \infty} \|h_{\beta,\alpha_k} - h_{\beta,\alpha}\|_1 = 0$.*

Proof. The argument proceeds analogously to Lemma 2.1 with $\tilde{g}_{\beta,\alpha}(x)$ replaced by $\tilde{h}_{\beta,\alpha}(x)$. ■

Lemmas 2.1 and 2.3 establish the continuity of M_β .

We now prove the central symmetry property of M_β .

LEMMA 2.4. *Let $\psi(x) := 1 - x$ for $x \in [0, 1]$. If $\beta > 1$ and $\alpha \in [0, 1]$, then*

$$\psi \circ T_{\beta, \alpha} = T_{\beta, 1 - \langle \beta - \alpha \rangle} \circ \psi$$

for Lebesgue almost every $x \in [0, 1]$. Consequently,

$$M_{\beta}(\alpha) + M_{\beta}(1 - \langle \beta - \alpha \rangle) = 1.$$

Proof. Since the case $\alpha \in [1 - \langle \beta \rangle, 1)$ follows similarly, we only consider $\alpha \in [0, 1 - \langle \beta \rangle)$. For $x \in [0, 1]$, we have $T_{\beta, \alpha}(x) = \beta x + \alpha - \lfloor \beta x + \alpha \rfloor$, which implies

$$\psi \circ T_{\beta, \alpha}(x) = 1 - T_{\beta, \alpha}(x) = 1 - \beta x - \alpha + \lfloor \beta x + \alpha \rfloor.$$

On the other hand,

$$\begin{aligned} T_{\beta, 1 - \langle \beta \rangle - \alpha} \circ \psi(x) &= T_{\beta, 1 - \langle \beta \rangle - \alpha}(1 - x) \\ &= \beta(1 - x) + 1 - \langle \beta \rangle - \alpha - \lfloor \beta(1 - x) + 1 - \langle \beta \rangle - \alpha \rfloor \\ &= \lfloor \beta \rfloor + 1 - \beta x - \alpha - \lfloor \lfloor \beta \rfloor + 1 - \beta x - \alpha \rfloor. \end{aligned}$$

For $x \in [0, 1]$ with $\beta x + \alpha \notin \mathbb{N}$,

$$\lfloor \lfloor \beta \rfloor + 1 - \beta x - \alpha \rfloor = \lfloor \beta \rfloor + \lfloor 1 - \beta x - \alpha \rfloor = \lfloor \beta \rfloor - \lfloor \beta x + \alpha \rfloor,$$

yielding

$$T_{\beta, 1 - \langle \beta \rangle - \alpha} \circ \psi(x) = 1 - \beta x - \alpha + \lfloor \beta x + \alpha \rfloor.$$

At points where $\beta x + \alpha \in \mathbb{N}$, $T_{\beta, 1 - \langle \beta \rangle - \alpha} \circ \psi(x) = 0$. Thus, $\psi \circ T_{\beta, \alpha}(x) = T_{\beta, 1 - \langle \beta \rangle - \alpha} \circ \psi(x)$ except at the finitely many points $\left\{ \frac{i - \alpha}{\beta} : i = 1, \dots, \lfloor \beta \rfloor \right\}$.

Iteratively, $\psi \circ T_{\beta, \alpha}^n = T_{\beta, 1 - \langle \beta \rangle - \alpha}^n \circ \psi$ except on a countable set. By (1.4), for Lebesgue almost every $x \in [0, 1]$,

$$\begin{aligned} \tilde{g}_{\beta, 1 - \langle \beta \rangle - \alpha}(x) &= \sum_{x < T_{\beta, 1 - \langle \beta \rangle - \alpha}^n(1)} \frac{1}{\beta^n} - \sum_{x < T_{\beta, 1 - \langle \beta \rangle - \alpha}^n(0)} \frac{1}{\beta^n} \\ &= \sum_{1 - x > 1 - T_{\beta, 1 - \langle \beta \rangle - \alpha}^n(1)} \frac{1}{\beta^n} - \sum_{1 - x > 1 - T_{\beta, 1 - \langle \beta \rangle - \alpha}^n(0)} \frac{1}{\beta^n} \\ &= \sum_{1 - x > T_{\beta, \alpha}^n(0)} \frac{1}{\beta^n} - \sum_{1 - x > T_{\beta, \alpha}^n(1)} \frac{1}{\beta^n} \\ &= \sum_{1 - x \leq T_{\beta, \alpha}^n(1)} \frac{1}{\beta^n} - \sum_{1 - x \leq T_{\beta, \alpha}^n(0)} \frac{1}{\beta^n}, \end{aligned}$$

where the third equality uses the identity

$$1 - T_{\beta, 1 - \langle \beta \rangle - \alpha}^n(z) = \psi \circ T_{\beta, 1 - \langle \beta \rangle - \alpha}^n(z) = T_{\beta, \alpha}^n \circ \psi(z) = T_{\beta, \alpha}^n(1 - z).$$

Therefore, $\tilde{g}_{\beta, 1 - \langle \beta \rangle - \alpha}(x) = \tilde{g}_{\beta, \alpha}(1 - x)$ in $L^1(\lambda)$, implying $g_{\beta, 1 - \langle \beta \rangle - \alpha}(x) =$

$g_{\beta,\alpha}(1-x)$ for Lebesgue a.e. $x \in [0, 1]$. By (1.7), we have

$$\begin{aligned} M_\beta(1 - \langle \beta \rangle - \alpha) &= \int_{[0,1]} x g_{\beta,1-\langle \beta \rangle-\alpha}(x) dx = \int_{[0,1]} x g_{\beta,\alpha}(1-x) dx \\ &= \int_{[0,1]} (1-y) g_{\beta,\alpha}(y) dy = 1 - M_\beta(\alpha). \blacksquare \end{aligned}$$

By Lemma 2.4, the function M_β is centrally symmetric about $(\frac{1-\langle \beta \rangle}{2}, \frac{1}{2})$ on $[0, 1 - \langle \beta \rangle]$ and about $(1 - \frac{\langle \beta \rangle}{2}, \frac{1}{2})$ on $(1 - \langle \beta \rangle, 1)$. Consequently, the range \mathcal{M}_β is symmetric with respect to the line $y = 1/2$. See Figure 2 for an example.

Proof of Theorem 1.2. The result follows directly from Lemmas 2.1, 2.3, and 2.4. \blacksquare

We conclude this section by noting that the continuity of M_β can alternatively be established using the Perron–Frobenius operator $P_{\beta,\alpha} = PT_{\beta,\alpha} : L^1([0, 1], \lambda) \rightarrow L^1([0, 1], \lambda)$, defined by (cf. [3])

$$(P_{\beta,\alpha}f)(x) := \frac{d}{dx} \int_{T_{\beta,\alpha}^{-1}([0,x])} f d\lambda, \quad f \in L^1([0, 1], \lambda).$$

3. Linearity property of $M_\beta(\cdot)$ on $[0, 1 - \langle \beta \rangle)$. In Theorem 1.2, we established the continuity of M_β . In this section, we prove the linearity property of M_β on $[0, 1 - \langle \beta \rangle)$ for a specific class of β . Combining (1.4) and (1.7), we have

$$\begin{aligned} (3.1) \quad M_\beta(\alpha) &= \int_{[0,1]} x g_{\beta,\alpha}(x) d\lambda(x) \\ &= \frac{1}{K_{\beta,\alpha}} \sum_{k=0}^{\infty} \frac{T_{\beta,\alpha}^k(1) - T_{\beta,\alpha}^k(0)}{\beta^k} \cdot \frac{T_{\beta,\alpha}^k(1) + T_{\beta,\alpha}^k(0)}{2}, \end{aligned}$$

where $K_{\beta,\alpha} := \sum_{k=0}^{\infty} \frac{T_{\beta,\alpha}^k(1) - T_{\beta,\alpha}^k(0)}{\beta^k}$. Thus, $M_\beta(\alpha)$ is the weighted average of the sequence $\left\{ \frac{T_{\beta,\alpha}^k(1) + T_{\beta,\alpha}^k(0)}{2} \right\}_{k \geq 0}$.

If the orbits of 0 and 1 coincide after finitely many iterations, i.e., $T_{\beta,\alpha}^k(0) = T_{\beta,\alpha}^k(1)$ for some $k \in \mathbb{N}$, then $M_\beta(\alpha)$ reduces to a finite sum. In this case, we say $T_{\beta,\alpha}$ has *matching* and the smallest integer k is referred to as the *matching time*.

Given $\beta > 1$ and $\alpha \in [0, 1)$, let (a_i) and (b_i) be the intermediate β -expansions of 0 and 1 under $T_{\beta,\alpha}$, respectively. Then for any $n \geq 1$, we

have

$$(3.2) \quad T_{\beta,\alpha}^n(0) = \beta^n \sum_{i=1}^n \frac{\alpha - a_i}{\beta^i} \quad \text{and} \quad T_{\beta,\alpha}^n(1) = \beta^n \left(1 + \sum_{i=1}^n \frac{\alpha - b_i}{\beta^i} \right).$$

If the matching time is $k \in \mathbb{N}$, then

$$1 = \sum_{i=1}^k \frac{b_i - a_i}{\beta^i}.$$

Thus, $T_{\beta,\alpha}$ has matching only if β is an algebraic integer. Given $q, m \in \mathbb{N}$, let $\beta_{q,m}$ be the unique root in $(q, q+1)$ of the equation $1 = \sum_{i=1}^m q/x^i$, i.e., $\beta_{q,m}$ satisfies

$$\beta_{q,m}^m - q\beta_{q,m}^{m-1} - q\beta_{q,m}^{m-2} - \dots - q\beta_{q,m} - q = 0.$$

THEOREM 3.1. *For all $q, m \in \mathbb{N}$, the function $M_{\beta_{q,m}}$ is linear and increasing on $[0, 1 - \langle \beta_{q,m} \rangle)$.*

If $m = 1$, then $\beta_{q,m} = q \in \mathbb{N}$ for any $q \in \mathbb{N}$, implying $M_\beta(\alpha) = 1/2$ for all $\alpha \in [0, 1)$. Hence, we assume $m \geq 2$. For $m = 2$, [4, Proposition 5.1] established that $T_{\beta_{q,2}}$ exhibits matching throughout $[0, 1 - \langle \beta_{q,2} \rangle)$ for every $q \in \mathbb{N}$. We now extend this result to $m \geq 3$.

LEMMA 3.2. *For all $q, m \in \mathbb{N}_{\geq 2}$, the transformation $T_{\beta_{q,m},\alpha}$ has matching at time m for every $\alpha \in [0, 1 - \langle \beta_{q,m} \rangle)$.*

Proof. For all $\beta > 1$, fix $\alpha \in [0, 1 - \langle \beta \rangle)$. In view of (1.2), we have

$$T_{\beta,\alpha}(x) = \beta x + \alpha - i \quad \text{for } x \in \Delta(i),$$

where the partition intervals are

$$\Delta(0) := \left[0, \frac{1-\alpha}{\beta} \right), \quad \Delta(q) := \left[\frac{q-\alpha}{\beta}, 1 \right],$$

and for $1 \leq i \leq q-1$,

$$\Delta(i) := \left[\frac{i-\alpha}{\beta}, \frac{i+1-\alpha}{\beta} \right).$$

Fix $q \in \mathbb{N}$, so $\lfloor \beta_{q,m} \rfloor = q$ for all $m \geq 1$. We prove by induction that

$$(3.3) \quad \begin{aligned} T_{\beta_{q,m},\alpha}^k(0) &\in \Delta(0), & T_{\beta_{q,m},\alpha}^k(1) &\in \Delta(q), \\ T_{\beta_{q,m},\alpha}^k(1) - T_{\beta_{q,m},\alpha}^k(0) &= \sum_{i=1}^{m-k} \frac{q}{\beta_{q,m}^i} \end{aligned}$$

for all $1 \leq k \leq m-1$. For $k=1$, by (3.2) and $1 = \sum_{i=1}^m \frac{q}{\beta_{q,m}^i}$, we have

$$T_{\beta_{q,m},\alpha}(1) - T_{\beta_{q,m},\alpha}(0) = (\beta_{q,m} + \alpha - q) - \alpha = \beta_{q,m} - q = \sum_{i=1}^{m-1} \frac{q}{\beta_{q,m}^i} \geq \frac{q}{\beta_{q,m}}.$$

The partition structure implies $T_{\beta_{q,m},\alpha}(0) \in \Delta(0)$ and $T_{\beta_{q,m},\alpha}(1) \in \Delta(q)$, establishing the base case. This means (3.3) holds for $m = 2$.

For $m \geq 3$, assume (3.3) holds for some $k \in \{1, \dots, m-2\}$. Then

$$\begin{aligned} T_{\beta_{q,m},\alpha}^{k+1}(1) - T_{\beta_{q,m},\alpha}^{k+1}(0) &= (\beta_{q,m} T_{\beta_{q,m},\alpha}^k(1) + \alpha - q) - (\beta_{q,m} T_{\beta_{q,m},\alpha}^k(0) + \alpha) \\ &= \beta_{q,m} (T_{\beta_{q,m},\alpha}^k(1) - T_{\beta_{q,m},\alpha}^k(0)) - q \\ &= \beta_{q,m} \sum_{i=1}^{m-k} \frac{q}{\beta_{q,m}^i} - q = \sum_{i=1}^{m-k-1} \frac{q}{\beta_{q,m}^i} \geq \frac{q}{\beta_{q,m}}. \end{aligned}$$

Thus, $T_{\beta_{q,m},\alpha}^{k+1}(0) \in \Delta(0)$ and $T_{\beta_{q,m},\alpha}^{k+1}(1) \in \Delta(q)$, completing the induction.

Finally, we verify matching at time m :

$$T_{\beta_{q,m},\alpha}^m(1) - T_{\beta_{q,m},\alpha}^m(0) = \beta_{q,m} (T_{\beta_{q,m},\alpha}^{m-1}(1) - T_{\beta_{q,m},\alpha}^{m-1}(0)) - q = 0. \quad \blacksquare$$

Motivated by the preceding proof, we introduce the definition of matching intervals.

DEFINITION 3.3. Given $\beta > 1$ and two blocks

$$a_1 \dots a_{k+1}, b_1 \dots b_{k+1} \in \{0, 1, \dots, \lfloor \beta \rfloor + 1\}^{k+1},$$

define the *matching interval* associated with $(a_1 \dots a_{k+1}, b_1 \dots b_{k+1})$ as

$$\begin{aligned} I_\beta(a_1 \dots a_{k+1}, b_1 \dots b_{k+1}) &:= \{\alpha \in [0, 1) : T_{\beta,\alpha} \text{ has matching at time } k+1 \\ &\text{and } T_{\beta,\alpha}^{i-1}(0) \in \Delta(a_i), T_{\beta,\alpha}^{i-1}(1) \in \Delta(b_i) \forall 1 \leq i \leq k+1\}. \end{aligned}$$

LEMMA 3.4. Let $\beta > 1$. For any blocks $a_1 \dots a_{k+1}, b_1 \dots b_{k+1} \in \{0, 1, \dots, \lfloor \beta \rfloor + 1\}^{k+1}$, the function M_β is linear on $I_\beta(a_1 \dots a_{k+1}, b_1 \dots b_{k+1})$.

Proof. For all $\alpha \in I_\beta(a_1 \dots a_{k+1}, b_1 \dots b_{k+1})$ and $1 \leq i \leq k$, we have

$$T_{\beta,\alpha}^i(0) = \beta^i \left(0 - \sum_{j=1}^i \frac{a_j - \alpha}{\beta^j} \right), \quad T_{\beta,\alpha}^i(1) = \beta^i \left(1 - \sum_{j=1}^i \frac{b_j - \alpha}{\beta^j} \right),$$

and $T_{\beta,\alpha}^i(0) = T_{\beta,\alpha}^i(1)$ for all $i \geq k+1$. Adopt the convention $\sum_{j=1}^0 c_j = 0$ for any sequence (c_j) . Then for $0 \leq i \leq k$,

$$(3.4) \quad \frac{T_{\beta,\alpha}^i(1) - T_{\beta,\alpha}^i(0)}{\beta^i} = 1 + \sum_{j=1}^i \frac{a_j - b_j}{\beta^j} =: \lambda_i$$

is independent of α , and

$$(3.5) \quad \frac{T_{\beta,\alpha}^i(1) + T_{\beta,\alpha}^i(0)}{2} = \frac{\beta^i \left(1 - \sum_{j=1}^i \frac{a_j + b_j}{\beta^j} + \sum_{j=1}^i \frac{2\alpha}{\beta^j} \right)}{2} =: g_i(\alpha)$$

is affine in α . Therefore,

$$\begin{aligned}
M_\beta(\alpha) &= \frac{1}{\sum_{i=0}^{\infty} \frac{T_{\beta,\alpha}^i(1) - T_{\beta,\alpha}^i(0)}{\beta^i}} \sum_{i=0}^{\infty} \frac{(T_{\beta,\alpha}^i(1))^2 - (T_{\beta,\alpha}^i(0))^2}{2\beta^i} \\
&= \frac{1}{\sum_{i=0}^k \frac{T_{\beta,\alpha}^i(1) - T_{\beta,\alpha}^i(0)}{\beta^i}} \sum_{i=0}^k \frac{T_{\beta,\alpha}^i(1) - T_{\beta,\alpha}^i(0)}{\beta^i} \cdot \frac{T_{\beta,\alpha}^i(1) + T_{\beta,\alpha}^i(0)}{2} \\
&= \frac{1}{\sum_{i=0}^k \lambda_i} \sum_{i=0}^k \lambda_i g_i(\alpha),
\end{aligned}$$

which is linear in α on $I_\beta(a_1 \dots a_{k+1}, b_1 \dots b_{k+1})$. ■

Proof of Theorem 3.1. For all integers $q \geq 1$ and $m \geq 2$, from the proof of Lemma 3.2 we have $[0, 1 - \langle \beta_{q,m} \rangle) \subset I_{\beta_{q,m}}(0^m, q^m)$. By Lemma 3.4, $M_{\beta_{q,m}}$ is linear on $[0, 1 - \langle \beta_{q,m} \rangle)$.

Observe that $\lambda_0 = 1$, and for $1 \leq i \leq m-1$,

$$\lambda_i = 1 + \sum_{j=1}^i \frac{0-q}{\beta_{q,m}^j} = \sum_{j=1}^m \frac{q}{\beta_{q,m}^j} - \sum_{j=1}^i \frac{q}{\beta_{q,m}^j} > 0,$$

where the second equality follows from the fact that $\beta_{q,m} = \beta_{q,m_{q,m}}$ satisfies $1 = \sum_{j=1}^m q/\beta_{q,m}^j$. Consequently, the coefficient of α in $M_{\beta_{q,m}}(\alpha)$ is positive, implying that $M_{\beta_{q,m}}$ is strictly increasing and linear on $[0, 1 - \langle \beta_{q,m} \rangle)$. ■

Given $q, m \in \mathbb{N}$, recall that $\beta_{q,m}$ is the unique root in $(q, q+1)$ of $1 = \sum_{i=1}^m q/x^i$ which satisfies

$$\beta_{q,m}^m - q\beta_{q,m}^{m-1} - q\beta_{q,m}^{m-2} - \dots - q\beta_{q,m} - q = 0.$$

For $q \in \mathbb{N}$ fixed, the sequence $(\beta_{q,m})_{m \geq 2}$ increases to $q+1$. By Theorem 3.1,

$$M_{\beta_{q,m}}(\alpha) = k_{q,m}\alpha + b_{q,m} \quad \text{for } \alpha \in [0, 1 - \langle \beta_{q,m} \rangle).$$

At the end of this section, we establish the strict monotonicity of $(k_{q,m})_m$ and $(b_{q,m})_m$. First, we derive explicit formulas for $k_{q,m}$ and $b_{q,m}$.

For $\alpha \in [0, 1 - \langle \beta_{q,m} \rangle) = [0, q+1 - \beta_{q,m})$, the proof of Lemma 3.4 gives

$$(3.6) \quad M_{\beta_{q,m}}(\alpha) = \frac{1}{\sum_{i=0}^{m-1} \lambda_i} \sum_{i=0}^{m-1} \lambda_i g_i(\alpha),$$

where λ_i is constant for $\alpha \in [0, q+1 - \beta_{q,m})$ and $0 \leq i \leq m-1$. Denote

$$K_{q,m} := K_{\beta_{q,m},\alpha} = \frac{1}{\sum_{i=0}^{m-1} \lambda_i}.$$

Since $[0, 1 - \langle \beta_{q,m} \rangle) \subset I_{\beta_{q,m}}(0^m, q^m)$, equation (3.4) yields

$$K_{q,m} = \frac{1}{m-S} \quad \text{with} \quad S := \frac{(m-1)q}{\beta_{q,m}} + \frac{(m-2)q}{\beta_{q,m}^2} + \dots + \frac{q}{\beta_{q,m}^{m-1}}.$$

Then

$$(\beta_{q,m} - 1)S = q(m-1) - \frac{q}{\beta_{q,m}} - \dots - \frac{q}{\beta_{q,m}^{m-1}} = q(m-1) - \frac{q(\beta_{q,m}^{m-1} - 1)}{\beta_{q,m}^{m-1}(\beta_{q,m} - 1)},$$

implying

$$S = \frac{q(m-1)}{\beta_{q,m} - 1} - \frac{q(\beta_{q,m}^{m-1} - 1)}{(\beta_{q,m} - 1)^2 \beta_{q,m}^{m-1}}.$$

Combining this with

$$\beta_{q,m} - q = \sum_{k=1}^{m-1} \frac{q}{\beta_{q,m}^k} = \frac{q(\beta_{q,m}^{m-1} - 1)}{(\beta_{q,m} - 1)\beta_{q,m}^{m-1}},$$

we obtain

$$K_{q,m} = \frac{1}{m - \frac{(m-1)q}{\beta_{q,m} - 1} + \frac{\beta_{q,m} - q}{\beta_{q,m} - 1}}.$$

Using $\beta_{q,m} - 1 = q(1 - 1/\beta_{q,m}^m)$, we simplify

$$m - \frac{(m-1)q}{\beta_{q,m} - 1} + \frac{\beta_{q,m} - q}{\beta_{q,m} - 1} = \frac{\beta_{q,m} - \frac{mq}{\beta_{q,m}^m}}{\beta_{q,m} - 1},$$

yielding the closed-form expression

$$(3.7) \quad K_{q,m} = \frac{\beta_{q,m} - 1}{\beta_{q,m} - \frac{mq}{\beta_{q,m}^m}}.$$

Since $[0, 1 - \langle \beta_{q,m} \rangle) \subset I_{\beta_{q,m}}(0^m, q^m)$, equations (3.4) and (3.5) yield

$$(3.8) \quad \sum_{i=0}^{m-1} \lambda_i g_i(\alpha) = \sum_{i=0}^{m-1} \frac{\beta_{q,m}^i (1 - \sum_{j=1}^i \frac{q}{\beta_{q,m}^j}) (1 - \sum_{j=1}^i \frac{q}{\beta_{q,m}^j} + \sum_{j=1}^i \frac{2}{\beta_{q,m}^j} \alpha)}{2} = \sum_{i=1}^{m-1} \left(\sum_{j=1}^i \frac{1}{\beta_{q,m}^j} \right) \left(\sum_{j=1}^{m-i} \frac{q}{\beta_{q,m}^j} \right) \alpha + \frac{1}{2} + \sum_{i=1}^{m-1} \frac{\beta_{q,m}^i (\sum_{j=i+1}^m \frac{q}{\beta_{q,m}^j})^2}{2},$$

where $\sum_{j=1}^0 q/\beta_{q,m}^j = 0$ by convention, and the term $1/2$ arises from the case $i = 0$.

Now we give a formula for $k_{q,m}$. From (3.6) and (3.8), we expand $k_{q,m}$ as

$$\begin{aligned} k_{q,m} &= K_{q,m} \sum_{i=1}^{m-1} \left(\frac{1}{\beta_{q,m}} + \cdots + \frac{1}{\beta_{q,m}^i} \right) \left(\frac{q}{\beta_{q,m}} + \cdots + \frac{q}{\beta_{q,m}^{m-i}} \right) \\ &= \frac{qK_{q,m}}{(\beta_{q,m} - 1)^2} \sum_{i=1}^{m-1} \left(1 - \frac{1}{\beta_{q,m}^i} - \frac{1}{\beta_{q,m}^{m-i}} + \frac{1}{\beta_{q,m}^m} \right) \\ &= \frac{qK_{q,m} \left((m-1) + \frac{m+1}{\beta_{q,m}^m} - \frac{2}{q} \right)}{(\beta_{q,m} - 1)^2}, \end{aligned}$$

where the last equality follows from $1 = \sum_{i=1}^m q/\beta_{q,m}^i$. Combining this with (3.7), we obtain

$$(3.9) \quad k_{q,m} = \frac{q \left((m-1) + \frac{m+1}{\beta_{q,m}^m} - \frac{2}{q} \right)}{(\beta_{q,m} - 1) \left(\beta_{q,m} - \frac{mq}{\beta_{q,m}^m} \right)}.$$

Analogous derivations for $m+1$ yield

$$(3.10) \quad \begin{aligned} K_{q,m+1} &= \frac{1}{m+1 - \frac{mq}{\beta_{q,m+1}-1} + \frac{\beta_{q,m+1}-q}{\beta_{q,m+1}-1}} = \frac{\beta_{q,m+1} - 1}{\beta_{q,m+1} - \frac{(m+1)q}{\beta_{q,m+1}^{m+1}}}, \\ k_{q,m+1} &= \frac{q \left(m + \frac{m+2}{\beta_{q,m+1}^{m+1}} - \frac{2}{q} \right)}{(\beta_{q,m+1} - 1) \left(\beta_{q,m+1} - \frac{(m+1)q}{\beta_{q,m+1}^{m+1}} \right)}. \end{aligned}$$

We now systematically derive the closed-form expression for $d_{q,m}$. By the foundational relations in (3.6) and (3.8), we develop the expansion through successive transformations:

$$\begin{aligned} d_{q,m} &= \frac{K_{q,m}}{2} \left(1 + \sum_{i=1}^{m-1} \beta_{q,m}^i \left(\sum_{j=i+1}^m \frac{q}{\beta_{q,m}^j} \right)^2 \right) \\ &= \frac{K_{q,m}}{2} \left(1 + \frac{q^2}{(\beta_{q,m} - 1)^2} \sum_{i=1}^{m-1} \frac{1 - \frac{2}{\beta_{q,m}^i} + \frac{1}{\beta_{q,m}^{2i}}}{\beta_{q,m}^{m-i}} \right) \\ &= \frac{K_{q,m}}{2} \left(1 + \frac{q^2}{(\beta_{q,m} - 1)^2} \left(\sum_{i=1}^{2m-1} \frac{1}{\beta_{q,m}^i} - \frac{2m-1}{\beta_{q,m}^m} \right) \right) \\ &= \frac{K_{q,m}}{2} \left(1 + \frac{q^2}{(\beta_{q,m} - 1)^2} \left(\frac{1}{\beta_{q,m} - 1} - \frac{1}{\beta_{q,m}^{2m-1}(\beta_{q,m} - 1)} - \frac{2m-1}{\beta_{q,m}^m} \right) \right) \\ &= \frac{(\beta_{q,m} - 1) + \frac{q^2}{(\beta_{q,m} - 1)^2} - \frac{q^2}{\beta_{q,m}^{2m-1}(\beta_{q,m} - 1)^2} - \frac{q^2(2m-1)}{\beta_{q,m}^m}}{2 \left(\beta_{q,m} - \frac{mq}{\beta_{q,m}^m} \right)}, \end{aligned}$$

where the final simplification employs the definition of $K_{q,m}$ from (3.7).

To further reduce the expression, we simplify the following term:

$$\frac{q^2}{(\beta_{q,m}-1)^2} \left(1 - \frac{1}{\beta_{q,m}^{2m-1}}\right) = \frac{\beta_{q,m}^{2m}-1}{(\beta_{q,m}^m-1)^2 \beta_{q,m}^{2m-1}} = 1 + \frac{2}{\beta_{q,m}^m-1} - \frac{q}{\beta_{q,m}^m(\beta_{q,m}^m-1)}.$$

Finally, we employ the characteristic equation $\beta_{q,m}-1 = q - \frac{q}{\beta_{q,m}^m}$ to achieve the critical term simplification:

$$(3.11) \quad d_{q,m} = \frac{q+1 + \frac{2}{\beta_{q,m}^m-1} - \frac{q}{\beta_{q,m}^m(\beta_{q,m}^m-1)} - \frac{2q^2(m-1)+q}{\beta_{q,m}^m}}{2\left(q+1 - \frac{q(m+1)}{\beta_{q,m}^m}\right)} \\ = \frac{1}{2} + \frac{\frac{2}{\beta_{q,m}^m-1} - \frac{q}{\beta_{q,m}^m(\beta_{q,m}^m-1)} - \frac{2q^2(m-1)-qm}{\beta_{q,m}^m}}{2\left(q+1 - \frac{q(m+1)}{\beta_{q,m}^m}\right)}$$

The recursive structure becomes apparent when considering the $(m+1)$ -dimensional case:

$$(3.12) \quad d_{q,m+1} = \frac{1}{2} + \frac{\frac{2}{\beta_{q,m+1}^{m+1}-1} - \frac{q}{\beta_{q,m+1}^{m+1}(\beta_{q,m+1}^{m+1}-1)} - \frac{2q^2m-q(m+1)}{\beta_{q,m+1}^{m+1}}}{2\left(q+1 - \frac{q(m+2)}{\beta_{q,m+1}^{m+1}}\right)}.$$

Now we establish the relationship between $\beta_{q,m}$ and $\beta_{q,m+1}$, which is crucial for proving the inequalities $k_{q,m+1} > k_{q,m}$ and $d_{q,m+1} > d_{q,m}$.

LEMMA 3.5. *For all $q \geq 1$ and $m \geq 2$, we have*

$$\frac{q}{\beta_{q,m+1}^m} > \beta_{q,m+1} - \beta_{q,m} > \frac{q\left(1 - \frac{1}{\beta_{q,m}}\right)}{\beta_{q,m+1}^m}.$$

Proof. The left inequality follows directly from the monotonicity $\beta_{q,m} < \beta_{q,m+1}$ and the definition of $\beta_{q,m}$. From those, we have

$$\beta_{q,m+1} - \beta_{q,m} \\ = q + \frac{q}{\beta_{q,m+1}} + \dots + \frac{q}{\beta_{q,m+1}^m} - \left(q + \frac{q}{\beta_{q,m}} + \dots + \frac{q}{\beta_{q,m}^{m-1}}\right) < \frac{q}{\beta_{q,m+1}^m}.$$

For the right inequality, consider the difference

$$(3.13) \quad \beta_{q,m+1} - \beta_{q,m} \\ = \frac{q}{\beta_{q,m+1}^m} - q(\beta_{q,m+1} - \beta_{q,m}) \left(\frac{1}{\beta_{q,m+1}\beta_{q,m}} + \frac{\beta_{q,m+1} + \beta_{q,m}}{\beta_{q,m+1}^2\beta_{q,m}^2} + \dots \right. \\ \left. + \frac{\beta_{q,m+1}^{m-1} + \beta_{q,m+1}^{m-2}\beta_{q,m} + \dots + \beta_{q,m}^{m-1}}{\beta_{q,m+1}^{m-1}\beta_{q,m}^{m-1}} \right)$$

$$\begin{aligned}
&> \frac{q}{\beta_{q,m+1}^m} \\
&\quad - q(\beta_{q,m+1} - \beta_{q,m}) \left(\frac{1}{\beta_{q,m+1}\beta_{q,m}} + \frac{2\beta_{q,m+1}}{\beta_{q,m+1}^2\beta_{q,m}^2} + \cdots + \frac{(m-1)\beta_{q,m+1}^{m-2}}{\beta_{q,m+1}^{m-1}\beta_{q,m}^{m-1}} \right) \\
&= \frac{q}{\beta_{q,m+1}^m} - \frac{q(\beta_{q,m+1} - \beta_{q,m})}{\beta_{q,m+1}} \left(\frac{1}{\beta_{q,m}} + \frac{2}{\beta_{q,m}^2} + \cdots + \frac{m-1}{\beta_{q,m}^{m-1}} \right).
\end{aligned}$$

Let $S_1 := \sum_{k=1}^{m-1} k/\beta_{q,m}^k$. Through geometric series manipulation, we have

$$\begin{aligned}
(\beta_{q,m} - 1)S_1 &= 1 + \frac{1}{\beta_{q,m}} + \frac{1}{\beta_{q,m}^2} + \cdots + \frac{1}{\beta_{q,m}^{m-2}} - \frac{m-1}{\beta_{q,m}^{m-1}} \\
&= \frac{\beta_{q,m} - \frac{1}{\beta_{q,m}^{m-1}}}{\beta_{q,m} - 1} - \frac{m}{\beta_{q,m}}.
\end{aligned}$$

This yields $S_1 = \frac{\beta_{q,m} - \beta_{q,m}^{-(m-1)}}{(\beta_{q,m} - 1)^2} - \frac{m}{\beta_{q,m}(\beta_{q,m} - 1)}$. Substituting back into (3.13), we obtain

$$\beta_{q,m+1} - \beta_{q,m} > \frac{q}{\beta_{q,m+1}^m} \left[1 + \frac{q}{\beta_{q,m+1}} \left(\frac{\beta_{q,m} - \beta_{q,m}^{-(m-1)}}{(\beta_{q,m} - 1)^2} - \frac{m}{\beta_{q,m}(\beta_{q,m} - 1)} \right) \right]^{-1}.$$

Through successive approximations, we conclude that

$$\begin{aligned}
(3.14) \quad &\beta_{q,m+1} - \beta_{q,m} \\
&> \frac{q}{\beta_{q,m+1}^m \left(1 + \frac{q(\beta_{q,m}^{m-1} - 1)}{(\beta_{q,m} - 1)^2 \beta_{q,m}^m} \right)} = \frac{q}{\beta_{q,m+1}^m \left(1 + \frac{q(\beta_{q,m} - 1)(\beta_{q,m}^{m-1} + \cdots + 1)}{(\beta_{q,m} - 1)^2 \beta_{q,m}^m} \right)} \\
&= \frac{q}{\beta_{q,m+1}^m} \cdot \frac{\beta_{q,m} - 1}{\beta_{q,m}} = \frac{q \left(1 - \frac{1}{\beta_{q,m}} \right)}{\beta_{q,m+1}^m},
\end{aligned}$$

which completes the proof of Lemma 3.5. ■

Proof of Theorem 1.5. Fix $q > 1$. To prove that the coefficient sequence $(k_{q,m})$ is increasing, we need to show $k_{q,m+1} > k_{q,m}$. By (3.9) and (3.10), this reduces to establishing that

$$\begin{aligned}
(3.15) \quad &(\beta_{q,m} - 1) \left(\beta_{q,m} - \frac{mq}{\beta_{q,m}^m} \right) \left(m + \frac{m+2}{\beta_{q,m+1}^{m+1}} - \frac{2}{q} \right) \\
&\quad - (\beta_{q,m+1} - 1) \left(\beta_{q,m+1} - \frac{(m+1)q}{\beta_{q,m+1}^{m+1}} \right) \left((m-1) + \frac{m+1}{\beta_{q,m}^m} - \frac{2}{q} \right) > 0.
\end{aligned}$$

Applying Lemma 3.5, we transform the left-hand side as follows:

$$\begin{aligned}
 & \frac{(\beta_{q,m} - 1)(\beta_{q,m} - \frac{mq}{\beta_{q,m}^m})}{\beta_{q,m+1}^{m+1}} + \beta_{q,m+1}(\beta_{q,m+1} - 1) - \frac{q(m+1)(\beta_{q,m+1} - 1)}{\beta_{q,m+1}^{m+1}} \\
 & - \left(m - \frac{2}{q}\right)(\beta_{q,m+1} - \beta_{q,m})(\beta_{q,m} + \beta_{q,m+1} - 1) \\
 & - \frac{mq(m - \frac{2}{q})(\beta_{q,m} - 1)}{\beta_{q,m}^m} \\
 & + \frac{(m+1)q(m - \frac{2}{q})(\beta_{q,m+1} - 1)}{\beta_{q,m+1}^{m+1}} + \frac{(m+1)(\beta_{q,m} - 1)(\beta_{q,m} - \frac{mq}{\beta_{q,m}^m})}{\beta_{q,m+1}^{m+1}} \\
 & - \frac{(m+1)(\beta_{q,m+1} - 1)(\beta_{q,m+1} - \frac{(m+1)q}{\beta_{q,m+1}^{m+1}})}{\beta_{q,m}^m} \\
 & > \beta_{q,m+1}(\beta_{q,m+1} - 1) - \frac{mq(m - \frac{2}{q})(\beta_{q,m} - 1) + (m+1)\beta_{q,m+1}(\beta_{q,m+1} - 1)}{\beta_{q,m}^m} \\
 & + \frac{q(m+1)(m - 1 - \frac{2}{q})(\beta_{q,m+1} - 1) + (m+2)(\beta_{q,m} - 1)(\beta_{q,m} - \frac{mq}{\beta_{q,m}^m})}{\beta_{q,m+1}^{m+1}} \\
 & - \frac{q\beta_{q,m+1}(m - \frac{2}{q})(\beta_{q,m} + \beta_{q,m+1} - 1)}{\beta_{q,m+1}^{m+1}} \\
 & > q^2 - \frac{(m^2 + m + 1)q^2}{\beta_{q,m}^m}.
 \end{aligned}$$

It is easy to check that the final expression $q^2 - (m^2 + m + 1)q^2/\beta_{q,m}^m$ is positive for the following cases:

- all $q \geq 3$;
- $q = 2$ with $m \geq 3$;
- $q = 1$ with $m \geq 6$.

For the remaining exceptional cases, the positivity can be verified by direct computation of specific values.

We now prove that $(d_{q,m})$ is increasing, or equivalently $d_{q,m+1} - d_{q,m} > 0$. By (3.11) and (3.12), this reduces to establishing that

$$\begin{aligned}
 & \left[q + 1 - \frac{q(m+1)}{\beta_{q,m}^m} \right] \left[\frac{2}{\beta_{q,m+1}^{m+1} - 1} - \frac{q}{\beta_{q,m+1}^{m+1}(\beta_{q,m+1}^{m+1} - 1)} - \frac{2q^2m - q(m+1)}{\beta_{q,m+1}^{m+1}} \right] \\
 & - \left[q + 1 - \frac{q(m+2)}{\beta_{q,m+1}^{m+1}} \right] \left[\frac{2}{\beta_{q,m}^m - 1} - \frac{q}{\beta_{q,m}^m(\beta_{q,m}^m - 1)} - \frac{2q^2(m-1) - qm}{\beta_{q,m}^m} \right] > 0.
 \end{aligned}$$

Set $x := \beta_{q,m}^m$ and $y := \beta_{q,m+1}^{m+1}$ to simplify notation. The above expression transforms to

$$\begin{aligned}
& \left[q+1 - \frac{q(m+1)}{x} \right] \left[\frac{2}{y-1} - \frac{q}{y(y-1)} - \frac{2q^2m - q(m+1)}{y} \right] \\
& - \left[q+1 - \frac{q(m+2)}{y} \right] \left[\frac{2}{x-1} - \frac{q}{x(x-1)} - \frac{2q^2(m-1) - qm}{x} \right] \\
& = \frac{2(q+1)}{y-1} - \frac{q(q+1)}{y(y-1)} - \frac{(q+1)(2q^2m - q(m+1))}{y} \\
& - \left[\frac{2(q+1)}{x-1} - \frac{q(q+1)}{x(x-1)} - \frac{(q+1)(2q^2(m-1) - qm)}{x} \right] \\
& + \frac{2q(m+2)}{y(x-1)} - \frac{q^2(m+2)}{yx(x-1)} - \frac{q(m+2)(2q^2(m-1) + qm)}{yx} \\
& - \left[\frac{2q(m+1)}{x(y-1)} - \frac{q^2(m+1)}{xy(y-1)} - \frac{q(m+1)(2q^2m + q(m+1))}{yx} \right] \\
& = \frac{(q+1)(x-y)(2xy - q(x+y-1))}{xy(x-1)(y-1)} - \frac{qm(q+1)(2q-1)(x-y)}{xy} \\
& + \frac{q(q+1)}{y} - \frac{2q^2(q+1)}{x} + \frac{(2-q)q(m+1)(y-x)}{xy(x-1)(y-1)} \\
& + \frac{2q}{y(x-1)} - \frac{q^2}{yx(x-1)} + \frac{q^2(3q - mq + 1)}{xy}.
\end{aligned}$$

For x and y , we have the inequality chain $y-1 > y-x > q\beta_{q,m+1}^m$. We analyze each component of the above target expression by using this chain:

$$\begin{aligned}
0 &> \frac{(q+1)(x-y)(2xy - q(x+y-1))}{xy(x-1)(y-1)} \\
&> -\frac{(q+1)(2xy - q(x+y-1))}{xy(x-1)} > -\frac{2(q+1)}{x}
\end{aligned}$$

and

$$-\frac{qm(q+1)(2q-1)(x-y)}{xy} > \frac{qm(q+1)(2q-1)q\beta_{q,m+1}^m}{xy} > \frac{q^2m(2q-1)}{x}.$$

Additionally, direct verification shows that

$$\begin{aligned}
& \frac{(2-q)q(m+1)(y-x)}{xy(x-1)(y-1)} + \frac{q^2(3q - mq + 1)}{xy} > 0, \\
& \frac{2q}{y(x-1)} - \frac{q^2}{yx(x-1)} > 0.
\end{aligned}$$

Combining these bounds, we obtain the critical lower bound of the target

expression:

$$\begin{aligned}
 & \frac{(q+1)(x-y)(2xy - q(x+y-1))}{xy(x-1)(y-1)} - \frac{qm(q+1)(2q-1)(x-y)}{xy} \\
 & + \frac{q(q+1)}{y} - \frac{2q^2(q+1)}{x} + \frac{(2-q)q(m+1)(y-x)}{xy(x-1)(y-1)} \\
 & + \frac{2q}{y(x-1)} - \frac{q^2}{yx(x-1)} + \frac{q^2(3q-mq+1)}{xy} \\
 & > \frac{q^2m(2q-1)}{x} - \frac{2(q+1)}{x} - \frac{2q^2(q+1)}{x} > 0
 \end{aligned}$$

for all $m \geq 2$ and $q \geq 1$. Therefore, $d_{q,m+1} - d_{q,m} > 0$ holds universally. ■

4. Local linearity of $M_\beta(\alpha)$ on $[1 - \langle \beta \rangle, 1)$. This section examines the case $\alpha \in [1 - \langle \beta \rangle, 1)$ for $\beta > \sqrt{2}$. Following the methodology of Section 3, we focus on multinacci numbers $\beta_{q,m} \in (q, q+1)$ defined as the unique real roots of the polynomials

$$P_{q,m}(x) = x^m - qx^{m-1} - \dots - q$$

for integers $m \geq 2$ and $q \geq 1$. We establish that $M_{\beta_{q,m}}(\alpha)$ exhibits local linearity for Lebesgue almost every $\alpha \in [1 - \langle \beta_{q,m} \rangle, 1)$. This case presents greater analytical complexity than $\alpha \in [0, 1 - \langle \beta_{q,m} \rangle)$.

THEOREM 4.1. *For integers $m \geq 2$ and $q \geq 1$, the function $M_{\beta_{q,m}}(\alpha)$ is locally linear for Lebesgue almost every $\alpha \in [1 - \langle \beta_{q,m} \rangle, 1)$.*

To prove this theorem, we introduce an essential lemmas and some notation. Recall from (1.3) that for $\alpha \in [1 - \langle \beta \rangle, 1)$, the transformation $T_{\beta,\alpha}$ has exactly $\lfloor \beta \rfloor + 2$ branches corresponding to the digits $\{0, 1, \dots, \lfloor \beta \rfloor + 1\}$. The fundamental digit partition of $[0, 1]$ is given by

$$\begin{aligned}
 \Delta(0) & := \left[0, \frac{1-\alpha}{\beta} \right), & \Delta(\lfloor \beta \rfloor + 1) & := \left[\frac{\lfloor \beta \rfloor + 1 - \alpha}{\beta}, 1 \right], \\
 \Delta(i) & := \left[\frac{i-\alpha}{\beta}, \frac{i+1-\alpha}{\beta} \right) \quad \text{for } 1 \leq i \leq \lfloor \beta \rfloor.
 \end{aligned}$$

Each subinterval $\Delta(j)$ consists of the points $x \in [0, 1)$ where the digit j is selected in the next iteration of $T_{\beta,\alpha}$.

The following lemma establishes a finite-variation property for critical orbit differences under $T_{\beta,\alpha}$ at multinacci parameters. For notational clarity in subsequent analysis, we denote

$$(4.1) \quad \delta^k(\beta, \alpha) := T_{\beta,\alpha}^k(1) - T_{\beta,\alpha}^k(0) \quad \text{for } k \geq 1.$$

LEMMA 4.2. *For all integers $m \geq 2$, $q \geq 1$, and every $\alpha \in [1 - \langle \beta_{q,m} \rangle, 1)$, the critical orbit differences exhibit the following algebraic structure:*

(4.2)

$$\delta^k(\beta_{q,m}, \alpha) = \pm \left(\frac{e_1^k(\beta_{q,m}, \alpha)}{\beta_{q,m}} + \frac{e_2^k(\beta_{q,m}, \alpha)}{\beta_{q,m}^2} + \dots + \frac{e_m^k(\beta_{q,m}, \alpha)}{\beta_{q,m}^m} \right) \quad \forall k \in \mathbb{N},$$

where $e_j^k(\beta_{q,m}, \alpha) \in \{0, q\}$, $(e_1^k(\beta_{q,m}, \alpha), \dots, e_m^k(\beta_{q,m}, \alpha)) \neq (q, \dots, q)$. Moreover, the cardinality of the set of difference sequences satisfies

$$\#\{\delta^k(\beta_{q,m}, \alpha) : k \in \mathbb{N}, \alpha \in [1 - \langle \beta_{q,m} \rangle, 1)\} \leq 2^{m+1} - 3.$$

Proof. The bound $|\delta^k(\beta_{q,m}, \alpha)| < 1$ combined with the identity

$$\pm \left(\frac{q}{\beta_{q,m}} + \dots + \frac{q}{\beta_{q,m}^m} \right) = \pm 1$$

ensures $(e_1^k(\beta_{q,m}, \alpha), \dots, e_m^k(\beta_{q,m}, \alpha)) \neq (q, \dots, q)$ for all $k \geq 1$.

The proof is by induction on k . For the base case $k = 1$, direct computation yields

$$\delta^1(\beta_{q,m}, \alpha) = \beta_{q,m} + \alpha - (q + 1) - \alpha = -\frac{q}{\beta_{q,m}^m}.$$

Assume the inductive hypothesis holds for $k - 1 \geq 1$, i.e.,

$$\delta^{k-1}(\beta_{q,m}, \alpha) = \pm \left(\frac{e_1^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}} + \dots + \frac{e_m^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}^m} \right)$$

with $e_j^{k-1}(\beta_{q,m}, \alpha) \in \{0, q\}$, $(e_1^{k-1}(\beta_{q,m}, \alpha), \dots, e_m^{k-1}(\beta_{q,m}, \alpha)) \neq (q, \dots, q)$.

If $(e_1^{k-1}(\beta_{q,m}, \alpha), \dots, e_m^{k-1}(\beta_{q,m}, \alpha)) = (0, \dots, 0)$, we have $\delta^k(\beta_{q,m}, \alpha) = 0$. Otherwise, assume $\delta^{k-1}(\beta_{q,m}, \alpha) > 0$ (the case $\delta^{k-1}(\beta_{q,m}, \alpha) < 0$ follows in a similar way). Denote by $\ell \in \{0, 1, \dots, q + 1\}$ the unique integer satisfying $T_{\beta_{q,m}, \alpha}^{k-1}(0) \in \Delta(\ell)$. We analyze two cases of the orbit differences based on the leading coefficient.

CASE 1: $e_1^{k-1}(\beta_{q,m}, \alpha) = 0$ with $(e_1^{k-1}(\beta_{q,m}, \alpha), \dots, e_m^{k-1}(\beta_{q,m}, \alpha)) \neq (0, \dots, 0)$. Note that

$$0 < \delta^{k-1}(\beta_{q,m}, \alpha) \leq \sum_{j=2}^m \frac{q}{\beta_{q,m}^j} < \frac{1}{\beta_{q,m}}.$$

So, $T_{\beta_{q,m}, \alpha}^{k-1}(1)$ lies in either $\Delta(\ell)$ or $\Delta(\ell + 1)$. If $T_{\beta_{q,m}, \alpha}^{k-1}(1) \in \Delta(\ell)$, we have

$$\begin{aligned} \delta^k(\beta_{q,m}, \alpha) &= \beta_{q,m} \delta^{k-1}(\beta_{q,m}, \alpha) \\ &= \frac{e_2^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}} + \frac{e_3^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}^2} + \dots + \frac{e_m^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}^{m-1}} \quad (\geq 0). \end{aligned}$$

Otherwise, $T_{\beta_{q,m},\alpha}^{k-1}(1) \in \Delta(\ell + 1)$ implies that

$$\begin{aligned} \delta^k(\beta_{q,m}, \alpha) &= \frac{e_2^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}} + \dots + \frac{e_m^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}^{m-1}} - 1 \\ &= - \left(\frac{q - e_2^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}} + \dots + \frac{q - e_m^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}^{m-1}} + \frac{q}{\beta_{q,m}^m} \right) \quad (\leq 0). \end{aligned}$$

CASE 2: $e_1^{k-1}(\beta_{q,m}, \alpha) = q$ with $(e_1^{k-1}(\beta_{q,m}, \alpha), \dots, e_m^{k-1}(\beta_{q,m}, \alpha)) \neq (q, \dots, q)$. From the inequality $q/\beta_{q,m} < \delta^{k-1}(\beta_{q,m}, \alpha) < 1$, we see that $T_{\beta_{q,m},\alpha}^{k-1}(1)$ lies in either $\Delta(\ell + q)$ (requiring $\ell \leq 1$) or $\Delta(\ell + q + 1)$ (requiring $\ell = 0$). If $T_{\beta_{q,m},\alpha}^{k-1}(1) \in \Delta(\ell + q)$, by simple computation we have

$$\delta^k(\beta_{q,m}, \alpha) = \frac{e_2^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}} + \dots + \frac{e_m^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}^{m-1}} \quad (\geq 0).$$

However, if $T_{\beta_{q,m},\alpha}^{k-1}(1) \in \Delta(\ell + q + 1)$, we have

$$\begin{aligned} \delta^k(\beta_{q,m}, \alpha) &= - \left(\frac{q - e_2^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}} + \dots + \frac{q - e_m^{k-1}(\beta_{q,m}, \alpha)}{\beta_{q,m}^{m-1}} + \frac{q}{\beta_{q,m}^m} \right) \quad (\leq 0). \end{aligned}$$

In both cases, the form (4.2) holds for $\delta^k(\beta_{q,m}, \alpha)$. The cardinality bound $2^{m+1} - 3$ follows from the binary choices (sign and coefficients) while excluding the forbidden states (q, \dots, q) and $(0, \dots, 0) = -(0, \dots, 0)$. ■

Fix the parameter pair $(\beta_{q,m}, \alpha)$. To simplify notation, for all $k \geq 1$, we express the orbit difference (4.2) as

$$(4.3) \quad \delta^k(\beta_{q,m}, \alpha) =: \pm e_1^k(\beta_{q,m}, \alpha) \dots e_m^k(\beta_{q,m}, \alpha).$$

When no ambiguity arises, we may omit $(\beta_{q,m}, \alpha)$.

From Lemma 4.2, we derive the following evolution rules. For $\delta^k(\beta_{q,m}, \alpha) = \pm 0e_2^k \dots e_m^k$ such that $T_{\beta_{q,m},\alpha}^k(0) \in \Delta(\ell)$ for some $\ell \in \{0, 1, \dots, q + 1\}$,

$$(4.4) \quad \delta^{k+1}(\beta_{q,m}, \alpha) = \begin{cases} \pm e_2^k \dots e_m^k 0 & \text{if } T_{\beta_{q,m},\alpha}^k(1) \in \Delta(\ell), \\ \mp (q - e_2^k) \dots (q - e_m^k) q & \text{if } T_{\beta_{q,m},\alpha}^k(1) \in \Delta(\ell \pm 1). \end{cases}$$

For $\delta^k(\beta_{q,m}, \alpha) = \pm qe_2^k \dots e_m^k$ such that $T_{\beta_{q,m},\alpha}^k(0) \in \Delta(\ell)$ for some $\ell \in \{0, 1, \dots, q + 1\}$, we have

$$(4.5) \quad \delta^{k+1}(\beta_{q,m}, \alpha) = \begin{cases} \pm e_2^k \dots e_m^k 0 & \text{if } T_{\beta_{q,m},\alpha}^k(1) \in \Delta(\ell \pm q), \\ \mp (q - e_2^k) \dots (q - e_m^k) q & \text{if } T_{\beta_{q,m},\alpha}^k(1) \in \Delta(\ell \pm (q + 1)). \end{cases}$$

The sign of δ^{k+1} remains unchanged when both orbits stay in $\Delta(\ell)$ or shift by $\pm q$ intervals, but reverses for ± 1 or $\pm(q + 1)$ interval shifts.

We now characterize matching propagation. When $\delta^k(\beta_{q,m}, \alpha) = \pm e_1^k \dots e_m^k$ with $e_m^k = q$, the first matching occurs at step $k + m$ (here $\delta^{k+m}(\beta_{q,m}, \alpha) = 0$) if and only if the subsequent differences maintain sign consistency, i.e.,

$$\delta^{k+i}(\beta_{q,m}, \alpha) = \pm e_{i+1}^k \dots e_m^k 0^i \quad \text{for all } 1 \leq i \leq m-1.$$

More generally, for $\delta^k(\beta_{q,m}, \alpha) = \pm e_1^k \dots e_i^k 0^{m-i}$ with $e_i^k = q$, matching occurs at step $k + i$ if and only if $\delta^{k+1}(\beta_{q,m}, \alpha), \dots, \delta^{k+i}(\beta_{q,m}, \alpha)$ share the same sign. Then we can deduce that no matching occurs before step $m + 1$ due to the initial difference structure

$$\delta^1(\beta_{q,m}, \alpha) = -\frac{q}{\beta_{q,m}^m} \quad \forall \alpha \in [q+1 - \beta_{q,m}, 1)$$

and the first potential matching arises at step $m + 1$.

Following the approach in [5], we prove Theorem 4.1 by establishing the existence of an interval $I \subset [q+1 - \beta_{q,m}, 1)$ such that when $T_{\beta_{q,m}, \alpha}^k(0) \in I$ for some $k \in \mathbb{N}$, matching occurs within finite time $k' \geq k$ for all $\alpha \in [q+1 - \beta_{q,m}, 1)$. This constitutes the core technical result.

PROPOSITION 4.3. *For all integers $q \geq 1$, $m \geq 2$, let $\alpha \in [q+1 - \beta_{q,m}, 1)$. Denote*

$$P := P(\beta_{q,m}, \alpha) = \frac{q - \alpha}{\beta_{q,m} - 1}.$$

Suppose there exists a minimal $k \in \mathbb{N}$ such that $T_{\beta_{q,m}, \alpha}^k(0) = P$. Then:

- (i) *For $q \geq 2$, matching occurs at some finite time $k' \geq k$.*
- (ii) *For $q = 1$, matching occurs at some finite time $k' \geq k$ except for at most countably many $\alpha \in [2 - \beta_{1,m}, 1)$.*

Proof. Let k be the minimal integer with $T_{\beta_{q,m}, \alpha}^k(0) = P$. Since P is a fixed point satisfying $T_{\beta_{q,m}, \alpha}(P) = P$, we denote the discrepancy by $\delta^i := T_{\beta_{q,m}, \alpha}^i(1) - T_{\beta_{q,m}, \alpha}^i(0) = T_{\beta_{q,m}, \alpha}^i(1) - P$ for all $i \geq k$. If $\delta^k = 0$, matching occurs immediately at $k' = k$. Otherwise, Lemma 4.2 implies $|\delta^k| \geq q/\beta_{q,m}^m > 0$.

For $\alpha \in [q/\beta_{q,m}^m, q/\beta_{q,m}^{m-1})$, we have $P + q/\beta_{q,m}^m > 1$. The orbit position implies

$$T_{\beta_{q,m}, \alpha}^{k+i}(1) \leq P = T_{\beta_{q,m}, \alpha}^{k+i}(0) \quad \text{for all } i \geq 0.$$

Then matching propagation rules (4.4) and (4.5) ensure matching time $k' \leq k + m$.

For $\alpha \geq q/\beta_{q,m}^{m-1}$, the symmetry property established in Lemma 2.4 allows restriction to $\alpha \in [\frac{q}{\beta_{q,m}^{m-1}}, \frac{q+2-\beta_{q,m}}{2})$. Note that

$$\frac{q}{\beta_{q,m}^{m-1}} > \frac{q+2-\beta_{q,m}}{2}$$

for $m = 2$, $q \geq 1$. Then we only need to consider $m \geq 3$. Without loss of generality, assume $\delta^k < 0$. Otherwise, consider the smallest $j \in \{k + 1, \dots, k + m - 1\}$ with $\delta^j < 0$. If $\delta^{k+i} \geq 0$ for all $1 \leq i \leq m - 1$, then matching occurs by step $k' \leq k + m$. The analysis is now divided into four cases.

CASE 1: $\delta^k = -q/\beta_{q,m}$. Direct calculation shows that matching occurs at step $k + 1$.

CASE 2: $\delta^k < -q/\beta_{q,m}$. The bounding inequality

$$\frac{q - \alpha}{\beta_{q,m}} < P = T_{\beta_{q,m}, \alpha}^k(0) < \frac{q + 1 - \alpha}{\beta_{q,m}}$$

implies that

$$T_{\beta_{q,m}, \alpha}^k(1) = T_{\beta_{q,m}, \alpha}^k(0) + \delta_k < P - \frac{q}{\beta_{q,m}} < \frac{1 - \alpha}{\beta_{q,m}}.$$

The recurrence relation yields

$$(4.6) \quad T_{\beta_{q,m}, \alpha}^{k+1}(1) < T_{\beta_{q,m}, \alpha}^{k+1}(0) = P \quad \text{and} \quad \delta^{k+1} = \beta_{q,m} \delta^k + q < 0.$$

Through an analysis of consecutive iterations, the following dynamical principles emerge. If the inequality $\delta^{k+i} \leq -q/\beta_{q,m}$ persists for all indices $0 \leq i \leq j \leq m - 1$ and $\delta^{k+j} = -q/\beta_{q,m}$, it triggers matching at step $k + j + 1$. Conversely, failure of this persistent inequality implies the existence of at least one iteration $2 \leq j \leq m - 1$ satisfying $0 > \delta^{k+i} > -q/\beta_{q,m}$.

The residual scenario corresponds to $0 > \delta^k > -q/\beta_{q,m}$. By Lemma 4.2 we have

$$\delta^k = -\left(\frac{e_2^k}{\beta_{q,m}^2} + \dots + \frac{e_m^k}{\beta_{q,m}^m}\right) > -\frac{1}{\beta_{q,m}} \geq -\frac{q}{\beta_{q,m}},$$

where $e_2^k \dots e_m^k \in \{0, q\}^{m-1} \setminus \{0^{m-1}\}$. Direct inspection reveals the following two critical criteria (see Figure 4):

$$(4.7) \quad \begin{cases} \delta^{k+1} = -\left(\frac{e_2^k}{\beta_{q,m}} + \dots + \frac{e_m^k}{\beta_{q,m}^{m-1}}\right) < 0 & \text{if } \frac{q-\alpha}{\beta_{q,m}} \leq T_{\beta_{q,m}, \alpha}^k(1) < P, \\ \delta^{k+1} = \frac{q-e_2^k}{\beta_{q,m}} + \dots + \frac{q-e_m^k}{\beta_{q,m}^{m-1}} + \frac{q}{\beta_{q,m}^m} > 0 & \text{if } P - \frac{1}{\beta_{q,m}} < T_{\beta_{q,m}, \alpha}^k(1) < \frac{q-\alpha}{\beta_{q,m}}. \end{cases}$$

CASE 3: $\frac{q-\alpha}{\beta_{q,m}} < T_{\beta_{q,m}, \alpha}^k(1) < P$. In this case, $\delta^{k+1} = \beta_{q,m} \delta^k < 0$. We now state the bound for δ^k :

$$(4.8) \quad \delta^k > \frac{q - \alpha}{\beta_{q,m}} - \frac{q - \alpha}{\beta_{q,m} - 1} = \frac{\alpha - \frac{q}{\beta_{q,m}^m}}{\beta_{q,m}(\beta_{q,m} - 1)} - \frac{1}{\beta_{q,m}}.$$

If the inequality $\delta^{k+i} < -q/\beta_{q,m}$ or (4.8) persists for all indices $2 \leq i \leq j \leq m - 1$, then $\delta^{k+j} = -q/\beta_{q,m}$, triggering matching at step $k + j + 1$. Conversely, failure of these persistent inequalities implies the existence of at

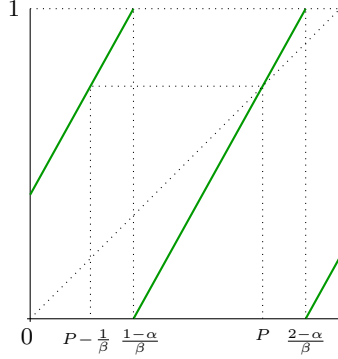


Fig. 4. Fix $\beta = 1.8$ and $\alpha \in [1 - \langle \beta \rangle, 1)$.

least one iteration $j' \leq m - 1$ satisfying

$$\frac{\alpha - \frac{q}{\beta_{q,m}^m}}{\beta_{q,m}(\beta_{q,m} - 1)} - \frac{1}{\beta_{q,m}} > \delta^{k+j'} > -\frac{1}{\beta_{q,m}},$$

which falls into Case 4.

CASE 4: $P - \frac{1}{\beta_{q,m}} < T_{\beta_{q,m},\alpha}^k(1) < \frac{q-\alpha}{\beta_{q,m}}$. By the analysis in Case 3, the bound of δ^k is

$$\frac{\alpha - \frac{q}{\beta_{q,m}^m}}{\beta_{q,m}(\beta_{q,m} - 1)} - \frac{1}{\beta_{q,m}} > \delta^k > -\frac{1}{\beta_{q,m}}.$$

Here $T_{\beta_{q,m},\alpha}^{k+1}(1) > P$ and

$$\delta^{k+1} = \beta_{q,m}\delta^k + 1 = \frac{q - e_2^k}{\beta_{q,m}} + \dots + \frac{q - e_m^k}{\beta_{q,m}^{m-1}} + \frac{q}{\beta_{q,m}} > 0.$$

Note that $T_{\beta_{q,m},\alpha}(1) = \beta_{q,m} + \alpha - (q+1) < P$ for all $\alpha \in [0, 1/2 + q/\beta_{q,m}^m)$. Then there are two things that can happen for δ^{k+2} (see Figure 4 for an illustration):

- If $T_{\beta_{q,m},\alpha}^{k+1}(1) > \frac{q+1-\alpha}{\beta_{q,m}}$, then we have $T_{\beta_{q,m},\alpha}^{k+2}(1) < P$ and $\delta^{k+2}(\beta_{q,m}, \alpha) = \beta_{q,m}\delta^{k+1}(\beta_{q,m}, \alpha) - 1 < 0$.
- Otherwise, $\delta^{k+2}(\beta_{q,m}, \alpha) = \beta_{q,m}\delta^{k+1}(\beta_{q,m}, \alpha) > 0$.

To consider the situation of δ^{k+2} , we need more analysis of this dynamical system. Using the definition of $\beta_{q,m}$ and $q - q/\beta_{q,m}^m = \beta_{q,m} - 1$, we derive

$$(4.9) \quad \frac{1}{\beta_{q,m}} - \frac{\alpha - \frac{q}{\beta_{q,m}^m}}{\beta_{q,m}(\beta_{q,m} - 1)} > \frac{q}{\beta_{q,m}^2} + \dots + \frac{q}{\beta_{q,m}^{m+1}} - \frac{1 - \frac{q}{\beta_{q,m}^m}}{2\beta_{q,m}(\beta_{q,m} - 1)}$$

$$\begin{aligned}
 &= \frac{q}{\beta_{q,m}^2} + \cdots + \frac{q}{\beta_{q,m}^{m+1}} - \left(\frac{1}{2\beta_{q,m}^2} + \frac{1}{2\beta_{q,m}^3} + \cdots \right) + \left(\frac{q}{2\beta_{q,m}^{m+2}} + \frac{q}{2\beta_{q,m}^{m+3}} + \cdots \right) \\
 &> \frac{q}{\beta_{q,m}^2} + \frac{1}{2\beta_{q,m}^2} + \frac{q}{2\beta_{q,m}^3} \cdots + \frac{q}{2\beta_{q,m}^{m+1}} - \frac{q}{2\beta_{q,m}^{m+2}} - \left(\frac{1}{2\beta_{q,m}^2} + \cdots + \frac{1}{2\beta_{q,m}^{m+1}} \right) \\
 &= \frac{q}{\beta_{q,m}^2} + \frac{q-1}{2\beta_{q,m}^3} + \cdots + \frac{q-1}{2\beta_{q,m}^m} + \frac{q-1}{2\beta_{q,m}^{m+1}} - \frac{q}{2\beta_{q,m}^{m+2}}.
 \end{aligned}$$

For $b_1 \dots b_m \in \{0, q\}^m$, recall from (4.3) that we simply denote

$$\frac{b_1}{\beta_{q,m}} + \cdots + \frac{b_m}{\beta_{q,m}^m} = b_1 \dots b_m.$$

Define

$$b_1 \dots b_m^- := \max \{a_1 \dots a_m \in \{0, q\}^m : a_1 \dots a_m < b_1 \dots b_m\}.$$

Then for any $\alpha \in \left[\frac{q}{\beta_{q,m}^{m-1}}, \frac{q+2-\beta_{q,m}}{2} \right)$, there exists a word $b^* := 0b_2 \dots b_m \in \{0, q\}^m$ such that

$$\begin{aligned}
 (4.10) \quad &0b_2 \dots b_m^- \leq P - \frac{p-\alpha}{\beta_{q,m}} < 0b_2 \dots b_m, \\
 &\frac{1}{\beta_{q,m}} - 0b_2 \dots b_m < \frac{p+1-\alpha}{\beta_{q,m}} - P \leq \frac{1}{\beta_{q,m}} - 0b_2 \dots b_m^-.
 \end{aligned}$$

For $q \geq 2$, inequality (4.9) implies that $0b_2 \dots b_m \geq 0q^2 0^{m-3}$. Therefore, we rewrite $\delta^k \leq -b^*$ as

$$(4.11) \quad \delta^k = - \left(\frac{q}{\beta_{q,m}^2} + \cdots + \frac{q}{\beta_{q,m}^{l_1}} + \frac{q}{\beta_{q,m}^{l_2}} + \cdots + \frac{q}{\beta_{q,m}^{l_3}} + \cdots + \frac{q}{\beta_{q,m}^{l_{j-1}}} + \cdots + \frac{q}{\beta_{q,m}^{l_j}} \right)$$

for some odd $j \geq 1$ with $3 \leq l_1 < l_2 \leq l_3 < \cdots \leq l_{j-2} < l_{j-1} \leq l_j \leq m$. This gives

$$\delta^{k+1} = \beta_{q,m} \delta^k + 1 = \frac{q}{\beta_{q,m}^{l_1}} + \cdots + \frac{q}{\beta_{q,m}^{l_2-2}} + \frac{q}{\beta_{q,m}^{l_3}} + \cdots + \frac{q}{\beta_{q,m}^{l_j}} + \cdots + \frac{q}{\beta_{q,m}^m} > 0.$$

From (4.9) and (4.10), we find that $(p+1-\alpha)/\beta_{q,m} - P < 0^2 q 0^{m-3}$. Note that if $\delta^{k+1} > (p+1-\alpha)/\beta_{q,m} - P$, then $T_{\beta_{q,m}, \alpha}^{k+2}(1) < P$ and $\delta^{k+2} = \beta_{q,m} \delta^{k+1} - 1 < 0$. Otherwise, $T_{\beta_{q,m}, \alpha}^{k+2}(1) > P$ and $\delta^{k+2} = \beta_{q,m} \delta^{k+1} > 0$. By direct computation, we have

$$\beta_{q,m}^{l_1-2} \delta^{k+1} = \frac{q}{\beta_{q,m}^2} + \cdots + \frac{q}{\beta_{q,m}^{m-l_1+2}} > \frac{p+1-\alpha}{\beta_{q,m}} - P.$$

Thus, there exists $i \leq l_1 - 2$ such that

$$\begin{aligned} \beta_{q,m}^i \delta^{k+1} &= \frac{q}{\beta_{q,m}^{l_1-i}} + \cdots + \frac{q}{\beta_{q,m}^{l_2-2-i}} + \frac{q}{\beta_{q,m}^{l_3-i}} + \cdots + \frac{q}{\beta_{q,m}^{l_j-i}} + \cdots + \frac{q}{\beta_{q,m}^{m-i}} \\ &> \frac{p+1-\alpha}{\beta_{q,m}} - P. \end{aligned}$$

Consequently, there exists a unique minimal integer $1 \leq l'_1 \leq l_1 - 1$ satisfying $T_{\beta_{q,m},\alpha}^{k+l'_1}(1) < P$ and

$$\begin{aligned} \delta^{k+l'_1} &= \beta_{q,m}^{l'_1-1} \delta^{k+1} - 1 \\ &= - \left(\frac{q}{\beta_{q,m}} + \cdots + \frac{q}{\beta_{q,m}^{l_1-l'_1}} + \frac{q}{\beta_{q,m}^{l_2-l'_1}} + \cdots + \frac{q}{\beta_{q,m}^{l_3-l'_1}} + \cdots \right. \\ &\quad \left. + \frac{q}{\beta_{q,m}^{l_j-1-l'_1}} + \cdots + \frac{q}{\beta_{q,m}^{l_j-l'_1}} + \frac{q}{\beta_{q,m}^{m-l'_1+2}} + \cdots + \frac{q}{\beta_{q,m}^m} \right) < 0. \end{aligned}$$

Steps $k+l'_1+1$ to $k+l_1$ follow Case 2, while steps $k+l_1$ to $k+l_2-2$ follow Case 3 (if $l_1 > l_2 - 2$, no Case 3 occurs). Thus, we have

$$\begin{aligned} \delta^{k+l_2-2} &= - \left(\frac{q}{\beta_{q,m}^2} + \cdots + \frac{q}{\beta_{q,m}^{l_3-(l_2-2)}} + \cdots + \frac{q}{\beta_{q,m}^{l_j-1-(l_2-2)}} + \cdots + \frac{q}{\beta_{q,m}^{l_j-(l_2-2)}} \right. \\ &\quad \left. + \frac{q}{\beta_{q,m}^{m-l_2+3}} + \cdots + \frac{q}{\beta_{q,m}^{m+l'_1-l_2+2}} \right). \end{aligned}$$

If δ^{k+l_2-2} enters Case 3, then we have

$$\begin{aligned} \delta^{k+l_2-1} &= - \left(\frac{q}{\beta_{q,m}} + \cdots + \frac{q}{\beta_{q,m}^{l_3-(l_2-2)-1}} + \cdots + \frac{q}{\beta_{q,m}^{l_j-1-(l_2-2)-1}} + \cdots \right. \\ &\quad \left. + \frac{q}{\beta_{q,m}^{l_j-(l_2-2)-1}} + \frac{q}{\beta_{q,m}^{m-l_2+3-1}} + \cdots + \frac{q}{\beta_{q,m}^{m+l'_1-l_2+2-1}} \right), \end{aligned}$$

This remains in Case 2 until step $k+l_3$. Then if it enters Case 3 until step $k+l_4-2$, we set $l'_2 = l_4$ and conclude that

$$\begin{aligned} \delta^{k+l_4-2} &= - \left(\frac{q}{\beta_{q,m}^2} + \cdots + \frac{q}{\beta_{q,m}^{l_5-(l_4-2)}} + \cdots + \frac{q}{\beta_{q,m}^{l_j-1-(l_4-2)}} + \cdots + \frac{q}{\beta_{q,m}^{l_j-(l_4-2)}} \right. \\ &\quad \left. + \frac{q}{\beta_{q,m}^{m-l_4+3}} + \cdots + \frac{q}{\beta_{q,m}^{m+l'_1-l_4+2}} \right). \end{aligned}$$

We take $l'_2 = l_4$.

Otherwise, δ^{k+l_2-2} enters Case 4, we have $\delta^{k+l_2-1} = \beta_{q,m} \delta^{k+l_2-2} + 1 > 0$, and we set $l'_2 = l_2$. Then there exists $l'_3 \leq l'_3 \leq l_3 - 1$ such that $T_{\beta_{q,m}}^{k+l'_3-1} > (q+1-\alpha)/\beta_{q,m}$ and $\delta^{k+l'_3} < 0$. This enters Case 3 until step $k+l_3$, and

then Case 2 until step $k + l_4 - 2$. So we have

$$\delta^{k+l_4-2} = - \left(\frac{q}{\beta_{q,m}^2} + \cdots + \frac{q}{\beta_{q,m}^{l_5-(l_4-2)}} + \cdots + \frac{q}{\beta_{q,m}^{l_{j-1}-(l_4-2)}} + \cdots + \frac{q}{\beta_{q,m}^{l_j-(l_4-2)}} \right. \\ \left. + \frac{q}{\beta_{q,m}^{m-l_4+3}} + \cdots + \frac{q}{\beta_{q,m}^{m+l'_1-l_4+2}} + \frac{q}{\beta_{q,m}^{m+l_2-l_4+3}} + \cdots + \frac{q}{\beta_{q,m}^{m+l'_3-l_4+2}} \right).$$

After m steps, there is an increasing sequence $\{i_2, i_4, \dots, i_{j_1-1}\} \subset \{2, 4, \dots, j-1\}$ such that $l'_0 = 2$ and $l'_n = l_{i_n}$ for all $n = 2, \dots, j_1 - 1$ and

$$\delta^{k+m} = - \left(\frac{q}{\beta_{q,m}^2} + \cdots + \frac{q}{\beta_{q,m}^{l'_1}} + \frac{q}{\beta_{q,m}^{l'_2}} + \cdots + \frac{q}{\beta_{q,m}^{l'_3}} + \cdots + \frac{q}{\beta_{q,m}^{l'_{j_1-1}}} + \cdots + \frac{q}{\beta_{q,m}^{l'_{j_1}}} \right).$$

By the above process, we have $l_{i_n} \leq l'_{n+1} \leq l_{i_{n+1}} - 1$ for $n = 0, 2, \dots, j_1$ and $j_1 \leq j$. Iterating this process, we deduce that matching occurs within $k + m(m-1)$ steps.

For $q = 1$, we consider the matching properties in three cases based on the value of $P - \frac{1-\alpha}{\beta_{1,m}}$.

For $P - \frac{1-\alpha}{\beta_{1,m}} \geq \frac{1}{\beta_{1,m}^2} + \frac{1}{\beta_{1,m}^3}$, rewrite δ^k as in (4.11). Using the same iterative argument as in the $q \geq 2$ case, we prove that matching occurs within finitely many steps.

If $\frac{1}{\beta_{1,m}^2} + \sum_{i=4}^m \frac{1}{\beta_{1,m}^i} < P - \frac{1-\alpha}{\beta_{1,m}} < \frac{1}{\beta_{1,m}^2} + \frac{1}{\beta_{1,m}^3}$, we have $b^* = \frac{1}{\beta_{1,m}^2} + \frac{1}{\beta_{1,m}^3}$, which satisfies (4.10). For $\delta^k \leq -b^*$ with $m \geq 4$, the $q \geq 2$ argument applies. The case $m = 3$ and $\delta^k = -b^*$ requires separate consideration. If $\frac{2-\alpha}{\beta_{1,3}} - P < \frac{1}{\beta_{1,3}^3}$, then

$$\delta^{k+1} = \frac{1}{\beta_{1,3}^3}, \quad \delta^{k+2} = - \left(\frac{1}{\beta_{1,3}} + \frac{1}{\beta_{1,3}^3} \right), \quad \delta^{k+3} = -\frac{1}{\beta_{1,3}^2}, \quad \delta^{k+4} = -\frac{1}{\beta_{1,3}}$$

and matching occurs at step $k + 5$. But if $\frac{1}{\beta_{1,3}^3} < \frac{2-\alpha}{\beta_{1,3}} - P < \frac{1}{\beta_{1,3}^3}$, then

$$\delta^{k+1} = \frac{1}{\beta_{1,3}^3}, \quad \delta^{k+2} = \frac{1}{\beta_{1,3}^2}, \quad \delta^{k+3} = - \left(\frac{1}{\beta_{1,3}^2} + \frac{1}{\beta_{1,3}^3} \right) = \delta^k,$$

resulting in a 3-periodic orbit. In this case, the intermediate β -expansions of 0 and 1 develop periodic tails:

- the coding of 0 is $b_1 \dots b_{k-1} 1^\infty$;
- the coding of 1 is $a_1 \dots a_{k-1} (012)^\infty$.

By the Claim in the proof of Lemma 2.1, for each fixed prefix pair $(b_1 \dots b_{k-1}, a_1 \dots a_{k-1})$, at most one α satisfies both the inequality condition and the periodic coding requirement. Consequently, the set of such exceptional α is at most countable.

For $\frac{1}{\beta_{1,m}^2} \leq P - \frac{1-\alpha}{\beta_{1,m}} \leq \frac{1}{\beta_{1,m}^2} + \frac{1}{\beta_{1,m}^3}$, we find that $b^* = 010e_4 \dots e_m$ with $e_i \in \{0, 1\}$ and $\frac{2-\alpha}{\beta_{1,m}} - P > 001(1 - e_4) \dots (1 - e_m)1$. Express $\delta^k < -b^*$ as

$$(4.12) \quad \delta^k = -\left(\frac{1}{\beta_{1,m}^2} + \dots + \frac{1}{\beta_{1,m}^{l_1}} + \frac{1}{\beta_{1,m}^{l_2}} + \dots + \frac{1}{\beta_{1,m}^{l_j}} \right)$$

for odd $j \geq 1$ and $2 \leq l_1 < l_2 \leq \dots \leq l_j \leq m$. Then

$$\delta^{k+1} = \beta_{1,m} \delta^k + 1 = \frac{1}{\beta_{1,m}^{l_1}} + \dots + \frac{1}{\beta_{1,m}^{l_2-2}} + \frac{1}{\beta_{1,m}^{l_3}} + \dots + \frac{1}{\beta_{1,m}^{l_j}} > 0.$$

Compare $\beta_{1,m}^{l_1-3} \delta^{k+1}$ to $\frac{2-\alpha}{\beta_{1,m}} - P$. If $\beta_{1,m}^{l_1-3} \delta^{k+1} > \frac{2-\alpha}{\beta_{1,m}} - P$, then

$$\delta^{k+l_1-2} = -\left(\frac{1}{\beta_{1,m}} + \frac{1}{\beta_{1,m}^2} + \frac{1}{\beta_{1,m}^{l_2-l_1+2}} + \dots + \frac{1}{\beta_{1,m}^{l_j}} \right).$$

Otherwise, $\beta_{1,m}^{l_1-2} \delta^{k+1} > \frac{2-\alpha}{\beta_{1,m}} - P$ implies that

$$\delta^{k+l_1-1} = -\left(\frac{1}{\beta_{1,m}} + \frac{1}{\beta_{1,m}^{l_2-l_1+1}} + \dots + \frac{1}{\beta_{1,m}^{l_j}} \right).$$

After m steps, there exist indices $\{i_2, \dots, i_{j_1-1}\} \subset \{2, 4, \dots, j-1\}$ with $l'_0 = 2$, $l'_n = l_{i_n}$, and $l_{i_n} \leq l'_{n+1} \leq l_{i_{n+1}} - 1$ for $n = 0, 2, \dots, j_1$ ($j_1 \leq j$) such that

$$\delta^{k+m} = -\left(\frac{1}{\beta_{1,m}^2} + \dots + \frac{1}{\beta_{1,m}^{l'_1}} + \frac{1}{\beta_{1,m}^{l'_2}} + \dots + \frac{1}{\beta_{1,m}^{l'_{j_1}}} \right).$$

The iteration process leads to two distinct outcomes. The sequence $\{\delta^n\}$ eventually satisfies the matching condition within finitely many steps. Otherwise, after κm iterations, the difference term becomes

$$\delta^{k+\kappa m} = -\left(\frac{1}{\beta_{1,m}^2} + \frac{1}{\beta_{1,m}^{l'_1}} + \dots + \frac{1}{\beta_{1,m}^{l'_{j'}}} \right) < \frac{1-\alpha}{\beta_{1,m}} - P,$$

with $\delta^{k+\kappa m+l_i-1} < \frac{1-\alpha}{\beta_{1,m}} - P$ for all $1 \leq i \leq j'$. After m additional steps, it returns to its previous state:

$$\delta^{k+(\kappa+1)m} = -\left(\frac{1}{\beta_{1,m}^2} + \frac{1}{\beta_{1,m}^{l'_1}} + \dots + \frac{1}{\beta_{1,m}^{l'_{j'}}} \right) = \delta^{k+\kappa m},$$

establishing a periodic orbit with period m . For instance, when $m \geq 4$ is even and $b^* = (01)^{m/2}$, $\delta^k = -b^*$ induces such periodicity.

Each periodic pattern corresponds to at most one α (determined by the β -expansion condition). Since these patterns are characterized by eventually periodic expansions of 0 and 1, and the set of eventually periodic sequences is countable, the set of $\alpha \in [1 - \langle \beta_{1,m} \rangle, 1)$ without matching is at most

countable. Consequently, for $q = 1$, matching occurs within finitely many steps except for a countable set of α . ■

Proof of Theorem 4.1. By Lemma 3.4, it suffices to show that, for any $m \geq 2$ and $q \geq 1$, the transformation $T_{\beta_{q,m},\alpha}$ exhibits matching for Lebesgue almost every $\alpha \in [1 - \langle \beta_{q,m} \rangle, 1)$.

Let $P = P_{\beta_{q,m},\alpha} = \frac{q-\alpha}{\beta_{q,m}-1}$. For $q \geq 2$, Proposition 4.3 implies that matching occurs at step $\ell > k$ when $T_{\beta_{q,m},\alpha}^k(0) = P$. Fix $\varepsilon > 0$ such that for $I = (P-\varepsilon, P)$, if $T_{\beta_{q,m},\alpha}^k(0) \in I$, the sign sequence $(T_{\beta_{q,m},\alpha}^i(1) - T_{\beta_{q,m},\alpha}^i(0))_{i=k}^\ell$ coincides with $(T_{\beta_{q,m},\alpha}^i(1) - P)_{i=k}^\ell$, ensuring matching.

For $q = 1$, the preceding proof establishes that matching fails for at most countably many $\alpha \in [1 - \langle \beta_{1,m} \rangle, 1)$.

By [5, Theorem 3.1], the orbit $\{T_{\beta_{q,m},\alpha}^k(0)\}$ is dense in $[0, 1)$ for Lebesgue almost every $\alpha \in [q+1 - \beta_{q,m}, 1)$. Therefore, for $q \geq 2$, $T_{\beta_{q,m},\alpha}^k(0) \in I$ occurs almost surely, triggering matching. When $q = 1$, the countable exclusion set has Lebesgue measure zero. Since $[q+1 - \beta_{q,m}, 1) = [1 - \langle \beta_{q,m} \rangle, 1)$, matching holds for Lebesgue almost every α in this interval. ■

REMARK 4.4. In general, it is not easy to find an α such that $T_{\beta_{q,m},\alpha}$ has no matching. But for $q = 1$, the proof of Proposition 4.3 provides a way to find such an α .

Given $m \geq 2$ and $q \geq 1$, we have shown that $[0, 1 - \langle \beta_{q,m} \rangle)$ is a matching interval where $M_{\beta_{q,m}}(\alpha)$ increases linearly. For Lebesgue almost every $\alpha \in [1 - \langle \beta_{q,m} \rangle, 1)$, $M_{\beta_{q,m}}(\alpha)$ is locally linear (see Figure 5). These observations raise a natural question: on which matching intervals is $M_{\beta_{q,m}}(\alpha)$ increasing, and on which is it decreasing?

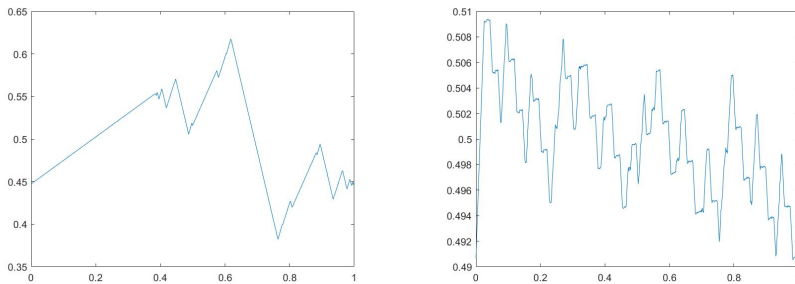


Fig. 5. Left: $M_{\beta_{1,2}}(\alpha)$ for $\alpha \in [0, 1]$ where $\beta_{1,2}$ is the gold mean. Right: $M_{\beta_{2,4}}(\alpha)$ for $\alpha \in [0, 1]$ where $\beta_{2,4} = 2.974449244\dots$

Let $I_{\beta_{q,m}}(a_1 \dots a_{k+1}, b_1 \dots b_{k+1}) \subset [1 - \langle \beta_{q,m} \rangle, 1)$ be a matching interval for $k \geq m$. By (3.4) in the proof of Lemma 3.4, we see that $\delta^i(\beta_{q,m}, \alpha)$ is constant on the interval for $1 \leq i \leq k$ and $\delta^{k+1}(\beta_{q,m}, \alpha) = 0$. The monotonicity of $M_{\beta_{q,m}}$ depends on the sign of $\delta^k(\beta_{q,m}, \alpha)$.

PROPOSITION 4.5. *Given $m \geq 2$ and $q \geq 1$, let $I_{\beta_{q,m}}(a_1 \dots a_{k+1}, b_1 \dots b_{k+1}) \subset [1 - \langle \beta_{q,m} \rangle, 1)$ be a matching interval for $k \geq m$. Then $M_{\beta_{q,m}}(\alpha)$ is increasing on this interval if $\delta^k(\beta_{q,m}, \alpha) > 0$. Otherwise, $M_{\beta_{q,m}}(\alpha)$ is decreasing.*

Proof. Take $\alpha \in I_{\beta_{q,m}}(a_1 \dots a_{k+1}, b_1 \dots b_{k+1})$. We previously established in Lemma 3.4 that

$$M_{\beta_{q,m}}(\alpha) = \frac{1}{\sum_{i=0}^k \lambda_i} \sum_{i=0}^k \frac{(T_{\beta_{q,m},\alpha}^i(1) - T_{\beta_{q,m},\alpha}^i(0))g_i(\alpha)}{\beta_{q,m}^i},$$

where

$$g_i(\alpha) = \frac{\beta_{q,m}^i}{2} \left(1 - \sum_{j=1}^i \frac{a_j + b_j}{\beta_{q,m}^j} + 2\alpha \sum_{j=1}^i \frac{1}{\beta_{q,m}^j} \right).$$

Since $\sum_{i=0}^k \lambda_i \geq 0$, to prove that $M_{\beta_{q,m}}$ is increasing it suffices to show

$$(4.13) \quad C^k(\beta_{q,m}, \alpha) := \sum_{i=1}^k \frac{(\beta_{q,m}^{i-1} + \dots + 1)(T_{\beta_{q,m},\alpha}^i(1) - T_{\beta_{q,m},\alpha}^i(0))}{\beta_{q,m}^i}$$

is positive when $\delta^k(\beta_{q,m}, \alpha) > 0$, and negative when $\delta^k < 0$ for the decreasing of $M_{\beta_{q,m}}$.

We establish the sign relationships for matching at step $k+1$, for which $\delta^{k-m+1}(\beta_{q,m}, \alpha)$ through $\delta^k(\beta_{q,m}, \alpha)$ share the same sign, while $\delta^{k-m}(\beta_{q,m}, \alpha)$ exhibits the opposite sign. Note this sign alternation is necessary; otherwise, matching would occur earlier at step $k+1$.

For the specific case $m=2$, if matching occurs in even steps $k+1 \geq 4$, the difference sequence follows the alternating pattern

$$\delta^i(\beta_{q,2}, \alpha) = \begin{cases} -q/\beta_{q,2}^2 & \text{for odd } i \leq k-1, \\ q/\beta_{q,2}^2 & \text{for even } i \leq k-1, \end{cases}$$

and then $\delta^k(\beta_{q,2}, \alpha) = q/\beta_{q,2}$. This yields the positive coefficient expression

$$\begin{aligned} C^k(\beta_{q,2}, \alpha) &= \sum_{i=1}^k \frac{(\beta_{q,2}^{i-1} + \dots + 1)\delta^i(\beta_{q,2}, \alpha)}{\beta_{q,2}^i} \\ &= \sum_{i=2}^{(k+1)/2} \frac{q}{\beta_{q,2}^{2i}} + \frac{(\beta_{q,2}^{k-1} + \dots + 1)}{\beta_{q,2}^k} \frac{q}{\beta_{q,2}}. \end{aligned}$$

When matching occurs at odd steps $k+1 \geq 3$, the difference sequences are $\delta^k(\beta_{q,2}, \alpha) = -q/\beta_{q,2}$ and

$$\delta^i(\beta_{q,2}, \alpha) = \begin{cases} -q/\beta_{q,2}^2 & \text{for odd } i \leq k-1, \\ q/\beta_{q,2}^2 & \text{for even } i \leq k-1, \end{cases}$$

producing the negative coefficient

$$C^k(\beta_{q,2}, \alpha) = \sum_{i=2}^{k/2} \frac{q}{\beta_{q,2}^{2i}} - \frac{\beta_{q,2}^{k-2} + \cdots + 1}{\beta_{q,2}^{k-1}} \cdot \frac{q}{\beta_{q,2}^2} - \frac{\beta_{q,2}^{k-1} + \cdots + 1}{\beta_{q,2}^k} \cdot \frac{q}{\beta_{q,2}}.$$

Now consider $m \geq 3$. For $k = m$, we have $\delta^1(\beta_{q,m}, \alpha) = -q/\beta_{q,m}^{-m}$, and all differences $\delta^i(\beta_{q,m}, \alpha)$ for $1 \leq i \leq m$ remain negative. Consequently, by (4.13),

$$C^m(\beta_{q,m}, \alpha) = \sum_{i=1}^m \frac{(\beta_{q,m}^{i-1} + \cdots + 1)\delta^i(\beta_{q,m}, \alpha)}{\beta_{q,m}^i} < 0,$$

establishing that $M_{\beta_{q,m}}(\alpha)$ decreases strictly on those matching intervals with matching at step $m + 1$.

For $k = m + 1$, we have $\delta^1(\beta_{q,m}, \alpha) = -q/\beta_{q,m}^m$ and

$$\delta^2(\beta_{q,m}, \alpha) = q^{m-2}0q, \quad \delta^3(\beta_{q,m}, \alpha) = q^{m-3}0q0, \quad \dots, \quad \delta^{m-1}(\beta_{q,m}, \alpha) = q0^{m-1},$$

all of which are positive. Therefore,

$$C^{m+1}(\beta_{q,m}, \alpha) = \sum_{i=1}^{m+1} \frac{(\beta_{q,m}^{i-1} + \cdots + 1)\delta^i(\beta_{q,m}, \alpha)}{\beta_{q,m}^i} > 0.$$

Thus, $M_{\beta_{q,m}}(\alpha)$ is strictly increasing on intervals where matching occurs at step $m + 2$.

For $k = m + 2$, the monotonicity of $M_{\beta_{q,m}}(\alpha)$ depends on sequences $\{\delta^i(\beta_{q,m}, \alpha)\}_{i=1}^{m+2}$, which can be either

$$\text{Case 1: } -0^{m-1}q, -0^{m-2}q0, q^{m-3}0q^2, \dots, q0^{m-1}, \quad \text{or}$$

$$\text{Case 2: } -0^{m-1}q, q^{m-2}0q, -0^{m-3}q0q, \dots, -q0^{m-1}.$$

The corresponding coefficient sum is given by

$$C^{m+2}(\beta_{q,m}, \alpha) = \sum_{i=1}^{m+2} \frac{(\beta_{q,m}^i - 1)\delta^i(\beta_{q,m}, \alpha)}{\beta_{q,m}^i(\beta_{q,m} - 1)}.$$

A direct calculation shows that $C^{m+2} > 0$ in Case 1 and $C^{m+2} < 0$ in Case 2.

For $k = m + 3$, the sequences $\delta^1(\beta_{q,m}, \alpha), \dots, \delta^{m+3}(\beta_{q,m}, \alpha)$ admit one of four cases:

$$\text{Case A: } -0^{m-1}q, -0^{m-2}q0, -0^{m-3}q0^2, q^{m-4}0q^3, \dots, q0^{m-1};$$

$$\text{Case B: } -0^{m-1}q, -0^{m-2}q0, q^{m-3}0q^2, -0^{m-4}q0^2q, \dots, -q0^{m-1};$$

$$\text{Case C: } -0^{m-1}q, q^{m-2}0q, q^{m-3}0q0, -0^{m-4}q0q^2, \dots, -q0^{m-1};$$

$$\text{Case D: } -0^{m-1}q, q^{m-2}0q, -0^{m-3}q0q, q^{m-4}0q0q, \dots, q0^{m-1}.$$

Using the same coefficient sum formula

$$C^{m+3}(\beta_{q,m}, \alpha) = \sum_{i=1}^{m+3} \frac{(\beta_{q,m}^i - 1)\delta^i(\beta_{q,m}, \alpha)}{\beta_{q,m}^i(\beta_{q,m} - 1)},$$

we find $C^{m+3} > 0$ in Cases B and C, while $C^{m+3} < 0$ in Cases A and D. Note that Case A cannot occur when $m = 3$.

Now suppose that for all matching intervals $I_{\beta_{q,m}}(a_1 \dots a_j, b_1 \dots b_j) \subset [1 - \langle \beta_{q,m} \rangle, 1)$ with $m + 3 \leq j \leq k$, the function $M_{\beta_{q,m}}(\alpha)$ is increasing whenever $\delta^{j-1}(\beta_{q,m}, \alpha) > 0$ and decreasing whenever $\delta^{j-1}(\beta_{q,m}, \alpha) < 0$.

Consider a matching interval $I_{\beta_{q,m}}(a_1 \dots a_{k+1}, b_1 \dots b_{k+1}) \subset [1 - \langle \beta_{q,m} \rangle, 1)$. For the initial segment, assume

$$\delta^{k-(m+1)}(\beta_{q,m}, \alpha) = -\left(\frac{e_1}{\beta_{q,m}} + \dots + \frac{e_m}{\beta_{q,m}^m}\right),$$

where $e_1 \dots e_m \in \{0, q\}^m \setminus \{0^m, q^m\}$.

The sequence $\{\delta^\ell(\beta_{q,m}, \alpha)\}_{k-(m+1) \leq \ell \leq k}$ must be one of the following cases:

Case I: $-e_1 \dots e_m, -e_2 \dots e_m 0, (q - e_3) \dots (q - e_m)q^2, \dots, q0^{m-1}$;

Case II: $-e_1 \dots e_m, (q - e_2) \dots (q - e_m)q, -e_3 \dots e_m 0q, \dots, -q0^{m-1}$.

Let I_k be a matching interval with matching at step k . For all $\alpha' \in I_k$, the first $k - (m + 1)$ terms match, and $\{\delta^\ell(\beta_{q,m}, \alpha')\}_{k-(m+1) \leq \ell \leq k-1}$ is

$$-e_1 \dots e_m, (q - e_2) \dots (q - e_m)q, (q - e_3) \dots (q - e_m)q0, \dots, q0^{m-1}.$$

The coefficient sum satisfies

$$\begin{aligned} C^k(\beta_{q,m}, \alpha') &= \sum_{\ell=1}^{k-m-1} \frac{(\beta_{q,m}^{\ell-1} + \dots + 1)\delta^\ell(\beta_{q,m}, \alpha')}{\beta_{q,m}^\ell} \\ &\quad + \sum_{\ell=k-m}^{k-1} \frac{(\beta_{q,m}^{\ell-1} + \dots + 1)\delta^\ell(\beta_{q,m}, \alpha')}{\beta_{q,m}^\ell} > 0. \end{aligned}$$

For Case I, since the initial $k - (m + 1)$ terms of $C^{k+1}(\beta_{q,m}, \alpha)$ and $C^k(\beta_{q,m}, \alpha')$ are identical, it suffices to compare the remaining segments. By direct termwise comparison of the sequences, we obtain

$$\begin{aligned} &\sum_{\ell=k-m}^k \frac{(\beta_{q,m}^{\ell-1} + \dots + 1)\delta^\ell(\beta_{q,m}, \alpha)}{\beta_{q,m}^\ell} - \sum_{\ell=k-m}^{k-1} \frac{(\beta_{q,m}^{\ell-1} + \dots + 1)\delta^\ell(\beta_{q,m}, \alpha')}{\beta_{q,m}^\ell} \\ &= -\frac{\beta_{q,m}^{k-m-1} + \dots + 1}{\beta_{q,m}^{k-m}} + \sum_{\ell=k-m+1}^k \frac{(\beta_{q,m}^{\ell-1} + \dots + 1)q}{\beta_{q,m}^{k+1}} \\ &> -\frac{\beta_{q,m}^{k-m-1} + \dots + 1}{\beta_{q,m}^{k-m}} + \frac{\beta_{q,m}^{k-m} + \dots + 1}{\beta_{q,m}^{k-m+1}} \sum_{r=1}^m \frac{q}{\beta_{q,m}^r} = \frac{1}{\beta_{q,m}^{k-m+1}} > 0, \end{aligned}$$

where the first equality results from term-by-term comparison of the sequences. Thus, $M_{\beta_{q,m}}$ is increasing in Case I.

For Case II, there exists a matching interval I_κ (with matching at step κ between $k - m$ and $k - 1$) sharing the first $k - (m + 1) - 2$ terms and the coefficient sum is smaller than 0. Its last m terms are

$$-e_0 e_1 \dots e_m, -e_1 \dots e_{m-1} 0, \dots, -e_m 0^{m-1},$$

where if $e_\ell \dots e_m 0^{\ell-1} = 0^m$ for some ℓ , then the subsequent terms vanish. A similar calculation shows that the coefficient sum of $C^{k+1}(\beta_{q,m}, \alpha)$ is smaller, confirming the decreasing property. ■

5. Further question. We conclude this paper with an open problem. For fixed $\beta = \beta_{q,m}$ with $m \geq 2$ and $q \geq 1$, we proved that $M_\beta(\alpha)$ exhibits matching for Lebesgue almost every $\alpha \in [0, 1]$. Define

$$\Lambda(\beta_{q,m}) := \{\alpha \in [0, 1] : T_{\beta_{q,m}, \alpha} \text{ has no matching}\}.$$

What is the Hausdorff dimension of $\Lambda(\beta_{q,m})$? While

$$\dim_{\text{H}}(\Lambda(\beta_{q,2})) = \frac{\log q}{\log \beta_{q,2}}$$

was established for $m = 2$ and any $q \geq 1$ in [4], we conjecture that for general $m \geq 2$,

$$\dim_{\text{H}}(\Lambda(\beta_{q,m})) = \frac{\log q}{\log \beta_{q,m}}.$$

Acknowledgements. Part of this work was completed during a visit by the second author to Leiden University. The authors thank the Mathematical Institute of Leiden University for its hospitality, and particularly express gratitude to Charlene Kalle. The authors also thank Derong Kong for valuable suggestions that improved the exposition.

References

- [1] S. Baker, *Approximation properties of β -expansions*, Acta Arith. 168 (2015), 269–287.
- [2] S. Baker, *Approximation properties of β -expansions II*, Ergodic Theory Dynam. Systems 38 (2018), 1627–1641.
- [3] A. Boyarsky and P. Góra, *Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimension*, Probab. Appl., Birkhäuser Boston, Boston, MA, 1997.
- [4] H. Bruin, C. Carminati, and C. Kalle, *Matching for generalised β -transformations*, Indag. Math. (N.S.) 28 (2017), 55–73.
- [5] H. Bruin and G. Keszthelyi, *On co- σ -porosity of the parameters with dense critical orbits for skew tent maps and matching on generalised β -transformations*, Indag. Math. (N.S.) 33 (2022), 533–545.
- [6] R. Cabane, *Sur les transformations linéaires modulo un*, Ann. Inst. H. Poincaré Sect. B (N.S.) 15 (1979), 187–193.

- [7] C. Carminati, S. Marmi, A. Profeti, and G. Tiozzo, *The entropy of α -continued fractions: numerical results*, Nonlinearity 23 (2010), 2429–2456.
- [8] C. Carminati and G. Tiozzo, *A canonical thickening of \mathbb{Q} and the entropy of α -continued fraction transformations*, Ergodic Theory Dynam. Systems 32 (2012), 1249–1269.
- [9] K. Dajani, M. de Vries, V. Komornik, and P. Loreti, *Optimal expansions in non-integer bases*, Proc. Amer. Math. Soc. 140 (2012), 437–447.
- [10] K. Dajani and C. Kraaikamp, *From greedy to lazy expansions and their driving dynamics*, Expo. Math. 20 (2002), 315–327.
- [11] P. Erdős, M. Horváth, and I. Joó, *On the uniqueness of the expansions $1 = \sum q^{-n_i}$* , Acta Math. Hungar. 58 (1991), 333–342.
- [12] P. Erdős and I. Joó, *On the number of expansions $1 = \sum q^{-n_i}$* , Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 35 (1992), 129–132.
- [13] L. Flatto and J. C. Lagarias, *The lap-counting function for linear mod one transformations. I. Explicit formulas and renormalizability*, Ergodic Theory Dynam. Systems 16 (1996), 451–491.
- [14] S. Halfin, *Explicit construction of invariant measures for a class of continuous state Markov processes*, Ann. Probab. 3 (1975), 859–864.
- [15] F. Hofbauer, *Maximal measures for piecewise monotonically increasing transformations on $[0, 1]$* , in: Ergodic Theory (Oberwolfach, 1978), Lecture Notes in Math. 729, Springer, Berlin, 1979, 66–77.
- [16] F. Hofbauer, *Maximal measures for simple piecewise monotonic transformations*, Z. Wahrsch. Verw. Gebiete 52 (1980), 289–300.
- [17] F. Hofbauer, *The structure of piecewise monotonic transformations*, Ergodic Theory Dynam. Systems 1 (1981), 159–178.
- [18] C. Kalle, N. Langeveld, M. Maggioni, and S. Munday, *Matching for a family of infinite measure continued fraction transformations*, Discrete Contin. Dynam. Systems 40 (2020), 6309–6330.
- [19] M. Lü, *Diophantine approximation and beta-expansions over the field of formal Laurent series*, J. Approx. Theory 174 (2013), 140–147.
- [20] C. Ma and S. Wang, *Dynamical Diophantine approximation of beta expansions of formal Laurent series*, Finite Fields Appl. 34 (2015), 176–191.
- [21] H. Nakada and R. Natsui, *The non-monotonicity of the entropy of α -continued fraction transformations*, Nonlinearity 21 (2008), 1207–1225.
- [22] W. Parry, *On the β -expansions of real numbers*, Acta Math. Acad. Sci. Hungar. 11 (1960), 401–416.
- [23] W. Parry, *Representations for real numbers*, Acta Math. Acad. Sci. Hungar. 15 (1964), 95–105.
- [24] W. Parry, *The Lorenz attractor and a related population model*, in: Ergodic Theory (Oberwolfach, 1978), Lecture Notes in Math. 729, Springer, Berlin, 1979, 169–187.
- [25] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. 8 (1957), 477–493.
- [26] N. Sidorov, *Almost every number has a continuum of β -expansions*, Amer. Math. Monthly 110 (2003), 838–842.
- [27] Y. Sun, B. Li, and Y. Ding, *Fiber denseness of intermediate β -shifts of finite type*, Nonlinearity 36 (2023), 5973–5997.
- [28] W. Wang and L. Li, *Simultaneous dynamical Diophantine approximation in beta expansions*, Bull. Austral. Math. Soc. 102 (2020), 186–195.
- [29] K. M. Wilkinson, *Ergodic properties of certain linear mod one transformations*, Adv. Math. 14 (1974), 64–72.

- [30] K. M. Wilkinson, *Ergodic properties of a class of piecewise linear transformations*,
Z. Wahrsch. Verw. Gebiete 31 (1975), 303–328.

Karma Dajani
Department of Mathematics
Utrecht University
3508 TA Utrecht, The Netherlands
E-mail: k.dajani@uu.nl

Yan Huang
School of Mathematics and Statistics
Wuhan University
Wuhan 430072, Hubei, P. R. China
E-mail: yanhuangyh@126.com