INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

DISSERTATIONES MATHEMATICAE

Online First version

FAUSTIN ADICEAM

Homogeneous form inequalities (II): Lattice point counting

WARSZAWA 2026

Faustin Adiceam
Laboratoire d'Analyse et de Mathématiques Appliquées (LAMA)
Université Paris-Est Créteil
Créteil, France
E-mail: faustin.adiceam@u-pec.fr

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute

Abstracted/Indexed in: Mathematical Reviews, Zentralblatt MATH, Science Citation Index Expanded, Journal Citation Reports/Science Edition, Google Science, Scopus, EBSCO Discovery Service.

doi: 10.4064/dm250107-29-11

ISSN 0012-3862

Contents

Abstract

This is the second part of a work devoted to the study of the set of integer solutions to a system of inequalities determined by homogeneous forms. According to the heuristics that the number of such solutions should match the volume of the set of real solutions, and among other results, sharp estimates for the volume of the semialgebraic domain under consideration were established in the first part.

Here, these estimations are employed to establish (a set of) statements providing a global count on the number of solutions to the Diophantine inequalities under consideration. The results thus obtained go beyond the usual assumptions of smoothness and of nonvanishing of the Gaussian curvature. Specifically, the introduction of a semialgebraic level of flatness emerging from geometric tomography enables one to characterise the cases when an accurate counting estimate can be obtained in thin neighbourhoods of an algebraic variety. This is done with a specific view towards applications to Diophantine approximation.

Acknowledgements. This memoir is the conclusion of a long project during which the author exchanged ideas with many people. In this respect, he would like to thank Profs. Victor Beresnevich, Raf Cluckers, Pietro Corvaja, Roger Heath-Brown, Tobias Kaiser, Mircea Mustaţă and Mihnea Popa, and also Quentin Guignard and Gareth O. Jones.

The author would especially like to thank Dr. Detta Dickinson who quite unexpectedly initiated him to this area of research in 2011 when he started his PhD under her supervision: the problem she suggested him to work on somehow gave impetus to this project.

2020 *Mathematics Subject Classification*: Primary 11J13; Secondary 11J25, 11D75, 11K60, 11U09.
Key words and phrases: homogeneous form inequalities, Diophantine inequalities, lattice points counting, semialgebraic sets.
Received 7 January 2025; revised 19 August 2025.
Published online 25 May 2026.

Notations

Notations in Euclidean spaces:

- Vectors are always denoted with bold characters (e.g., $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots$).
- The Euclidean norm is denoted by $\|\cdot\|$.
- Given $r > 0$, $B_n(r)$ stands for the closed Euclidean ball centred at the origin with radius r in dimension $n \geq 1$, and B_n for $B_n(1)$.

Algebraic notations:

- $\mathbb{K}$ stands for the field of reals $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.
- The notation $\mathbb{N}$ refers to the set of (strictly) positive integers and $\mathbb{N}_0$ to the set of nonnegative integers (i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).
- Given a set of $p \geq 1$ homogeneous forms $\boldsymbol{F}(\boldsymbol{x}) = (F_1(\boldsymbol{x}), \ldots, F_p(\boldsymbol{x}))$, their common set of zeros over $\mathbb{K}$ is denoted by $\mathcal{Z}_{\mathbb{K}}(\boldsymbol{F})$; that is,
$$\mathcal{Z}_{\mathbb{K}}(\boldsymbol{F}) = \{\boldsymbol{y} \in \mathbb{K}^n : \forall i = 1, \ldots, p,\ F_i(\boldsymbol{x}) = 0\}.$$
- Given a subset $A \subset \mathbb{K}^n$, the above notation is extended by setting $\mathcal{Z}_A(\boldsymbol{F}) = \mathcal{Z}_{\mathbb{K}}(\boldsymbol{F}) \cap A$.

Analytic notations:

- The partial derivative with respect to the ith coordinate in dimension n (where $1 \leq i \leq n$) is denoted by ∂_i. The gradient operator is then $\nabla = (\partial_1, \ldots, \partial_n)$.
- When $\boldsymbol{k} = (k_1, \ldots, k_n)$ is an n-tuple of nonnegative integers, define its size by $|\boldsymbol{k}| = \sum_{i=1}^{n} k_i$ and the differential operator $\boldsymbol{\partial}^{\boldsymbol{k}}$ by $\boldsymbol{\partial}^{\boldsymbol{k}} = \prod_{i=1}^{n} \partial_i^{k_i}$.
- Similarly, given $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^{\boldsymbol{k}}$ is shorthand notation for $\prod_{i=1}^{n} x_i^{k_i}$.
- Given a subset $A \subset \mathbb{R}^n$, its characteristic function is denoted by χ_A. Similarly, given a property $\mathcal{P}$ depending on a vector $\boldsymbol{y} \in \mathbb{R}^n$, $\chi_{\{\mathcal{P}(\boldsymbol{y})\}}$ is the boolean function equal to 1 when $\mathcal{P}(\boldsymbol{y})$ holds and 0 otherwise.
- When $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{C}^k(\mathbb{K}^n)$ stands for the set of k times continuously differentiable functions over $\mathbb{K}^n$ and $\mathcal{C}_c^k(\mathbb{K}^n)$ for the subset of such functions with compact support; the support of an element ψ in $\mathcal{C}_c^k(\mathbb{K}^n)$ is denoted by $\operatorname{Supp}\psi$.
- Given two functions f and g depending on a variable $\boldsymbol{y} \in \mathbb{K}^n$, the notation $f(\boldsymbol{y}) \ll g(\boldsymbol{y})$ is used to mean the existence of a real $c > 0$, referred to as the *implicit constant*, such that $|f(\boldsymbol{y})| \leq c \cdot |g(\boldsymbol{y})|$ for all admissible values of $\boldsymbol{y}$. This is equivalent to the notation $f(\boldsymbol{y}) = O(g(\boldsymbol{y}))$. Also, $f(\boldsymbol{y}) \asymp g(\boldsymbol{y})$ means that $f(\boldsymbol{y}) \ll g(\boldsymbol{y})$ and $g(\boldsymbol{y}) \ll f(\boldsymbol{y})$ simultaneously.
- The Lebesgue measure in $\mathbb{R}^n$ is denoted by Vol_n.
- Given $x \in \mathbb{R}$, set $\mathrm{e}(x) = \exp(2i\pi x)$.

Miscellaneous notations:

- In Chapters 2 and 3, the hat symbol refers to the Fourier transform of a function.
- In Chapter 3, $\overline{X}$ stands for the topological closure of a set $X \subset \mathbb{R}^n$.

Au temps qui passe

1. Introduction

1.1. The general setup. Let $p \geq 1, n \geq 2$ and $d \geq 2$ be integers. Fix a p-tuple $\boldsymbol{F}(\boldsymbol{x}) = (F_1(\boldsymbol{x}), \ldots, F_p(\boldsymbol{x}))$ of nonconstant real homogeneous forms $F_i(\boldsymbol{x})$ of degree d in n variables. Let K be a subset of $\mathbb{R}^n$. Throughout, it is assumed that

($\mathcal{H}1$) *the set $K \subset \mathbb{R}^n$ is compact and is equal to the closure of its interior.*

Given reals $a, b > 0$, define the set

$$\mathcal{S}_{\boldsymbol{F}}(K, a, b) = \{\boldsymbol{x} \in a \cdot K : \|\boldsymbol{F}(\boldsymbol{x})\| \leq b\}. \tag{1.1}$$

Here, $a \cdot K = \{a\boldsymbol{x} : \boldsymbol{x} \in K\}$ and $\|\cdot\|$ denotes the Euclidean norm. Thus,

$$\|\boldsymbol{F}(\boldsymbol{x})\| = \sqrt{\sum_{i=1}^{p} F_i^2(\boldsymbol{x})}.$$

The overarching goal is the study of the behaviour of the lattice point counting function

$$\mathcal{N}_{\boldsymbol{F}}(K, a, b) = \#(\mathcal{S}_{\boldsymbol{F}}(K, a, b) \cap \mathbb{Z}^n) \tag{1.2}$$

under suitable assumptions on the parameters a and b and assuming mostly that the set K is semialgebraic (in the usual sense that it can be represented by finitely many equalities and inequalities involving polynomials).

Let $\boldsymbol{x} \in K$. The trivial inequalities

$$a^d \cdot \alpha_K(\boldsymbol{F}) \leq \|\boldsymbol{F}(a \cdot \boldsymbol{x})\| \leq a^d \cdot \beta_K(\boldsymbol{F}),$$

where

$$\alpha_K(\boldsymbol{F}) = \min_{\boldsymbol{y} \in K} \|\boldsymbol{F}(\boldsymbol{y})\| \quad \text{and} \quad \beta_K(\boldsymbol{F}) = \max_{\boldsymbol{y} \in K} \|\boldsymbol{F}(\boldsymbol{y})\|,$$

show that the set $\mathcal{S}_{\boldsymbol{F}}(K, a, b)$ is of interest only when

$$a^d \cdot \alpha_K(\boldsymbol{F}) \leq b < a^d \cdot \beta_K(\boldsymbol{F}). \tag{1.3}$$

Of opecial interest is the case when $a = T$ is a large parameter and when $b = b(T)$ is taken as a function of this parameter. Then for simplicity of notation, set

$$\mathcal{S}_{\boldsymbol{F}}(K, b(T)) = \mathcal{S}_{\boldsymbol{F}}(K, T, b(T)) \tag{1.4}$$

and

$$\mathcal{N}_{\boldsymbol{F}}(K, b(T)) = \mathcal{N}_{\boldsymbol{F}}(K, T, b(T)). \tag{1.5}$$

Here, in view of the right-hand inequality in (1.3), it is always assumed that the limit condition

$$c(T) := \frac{b(T)}{T^d} \xrightarrow[T \to \infty]{} 0 \tag{1.6}$$

holds. The left-hand inequality in (1.3) then shows that the problem is nontrivial only when

($\mathcal{H}2$) *the set* $K \subset \mathbb{R}^n$ *intersects nontrivially the algebraic variety* $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$.

Here,

$$\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F}) = \{\boldsymbol{x} \in \mathbb{R}^n : F_i(\boldsymbol{x}) = 0 \text{ for all } 1 \leq i \leq p\}. \tag{1.7}$$

Conventionally, the dependence on the set K in various notations is dropped when $K = B_n$ is the closed Euclidean ball centred at the origin.

The benchmark for most of the results is the generic case where the set of homogeneous forms $\boldsymbol{F}(\boldsymbol{x})$ has smooth complete intersection over K. This is understood in the sense that for all $\boldsymbol{x} \in K$, the gradient vectors $\nabla F_1(\boldsymbol{x}), \dots, \nabla F_p(\boldsymbol{x})$ are linearly independent. Equivalently, the map

$$K \ni \boldsymbol{x} \mapsto \bigwedge_{i=1}^{p} \nabla F_i(\boldsymbol{x}) \tag{1.8}$$

does not vanish.

1.2. Volume estimates. At a first approximation, it should be expected that the counting function $\underline{\mathcal{N}}_{\boldsymbol{F}}(K, b(T))$ should be comparable to the volume of the domain $\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T))$. Denoting by Vol_n the n-dimensional Lebesgue measure, the goal in this section is to recall the estimates obtained in the first part [3] of this work for the quantity $\mathrm{Vol}_n(\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T)))$ as the parameter T tends to infinity. These estimates are stated not in full generality but in a form tailored to the purposes of this paper.

Assume here that the set K is *globally semianalytic* (in the sense that it is defined by a finite number of inequalities involving analytic maps). The growth of the quantity $\mathrm{Vol}_n(\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T)))$ can then be estimated in three cases realising together a trade-off between, on the one hand, the accuracy of the bounds that can be obtained and, on the other, the generality of the assumptions under which they hold. Specifically, the asymptotic growth of $\mathrm{Vol}_n(\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T)))$ can be expressed as a function of the properties of the zeta distribution $\zeta_{\boldsymbol{F}}$ attached to the p-tuple $\boldsymbol{F}(\boldsymbol{x})$.

The distribution $\zeta_{\boldsymbol{F}}$ is defined for any complex number s with nonpositive real part and any element ψ lying in the space $\mathcal{C}_c^{\infty}(\mathbb{R}^n)$ of compactly supported smooth functions by setting

$$\langle \zeta_{\boldsymbol{F}}(s), \psi \rangle = \int_{\mathbb{R}^n} \frac{\psi(\boldsymbol{x})}{\|\boldsymbol{F}(\boldsymbol{x})\|^s} \, \mathrm{d}\boldsymbol{x}. \tag{1.9}$$

It can be extended meromorphically to the entire complex plane.

In **Case (1)**, the most general considered, an upper bound can be obtained for the quantity $\mathrm{Vol}_n(\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T)))$ as a function of the smallest real pole $r_{\boldsymbol{F}}(\psi)$ of the function $s \mapsto \langle \zeta_{\boldsymbol{F}}(s), \psi \rangle$, and of its order $m_{\boldsymbol{F}}(\psi)$. This is assuming only that the set K meets the above conditions ($\mathcal{H}_1$) and ($\mathcal{H}_2$) and that the smooth test function ψ is nonnegative and bounds from above the characteristic function of the set K.

In **Case (2)**, the set K is furthermore assumed to be generic enough so that the singularities of the algebraic variety $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$ should not be located on its boundary. This is formalised in the following assumption:

($\mathcal{H}3$) *There exist a real number $r_{\boldsymbol{F}}(K) \geq 0$, an integer $m_{\boldsymbol{F}}(K) \geq 1$, a compact set C contained in the interior of K and an open set U containing K satisfying the following property: the smallest real poles and the corresponding orders of the functions $s \mapsto \langle \zeta_{\boldsymbol{F}}(s), \psi \rangle$ remain constant, respectively equal to $r_{\boldsymbol{F}}(K)$ and $m_{\boldsymbol{F}}(K)$, for any nonnegative map ψ in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ positive over C with support contained in U.*

In particular, for any smooth test function ψ meeting this condition,

$$C \subset \operatorname{Supp} \psi \subset U. \tag{1.10}$$

As a matter of fact, Assumption ($\mathcal{H}_3$) turns out to be always satisfied in the generic case where the homogeneous forms defining $\boldsymbol{F}(\boldsymbol{x})$ have smooth complete intersection over any given set K (see [3, Chapter 2, Proposition 2.18]).

Under the three conditions ($\mathcal{H}_1$)–($\mathcal{H}_3$) assumed to hold in this second case, Theorem 1.1 stated below and proved as one of the main results in the prequel work [3] provides the exact asymptotic order of $\mathrm{Vol}_n(\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T)))$ as a function of the pair $(r_P(K), m_P(K))$; that is, its sharp asymptotic growth up to multiplicative constants.

In **Case (3)**, which is perhaps the most natural in view of the assumption of the homogeneity of the forms, the set K is assumed to be star-shaped with respect to the origin and to contain the origin in its interior. Then the precise asymptotic behaviour of the volume of the domain $\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T))$ is determined in Theorem 1.1 below. It is expressed as a function of the smallest pole $r_P > 0$ and of the corresponding order $m_P \geq 1$ of the above-defined zeta distribution $\zeta_{\boldsymbol{F}}$ attached to the p-tuple $\boldsymbol{F}(\boldsymbol{x})$.

From the above discussion, the assumptions of Case (2) clearly contain those of Case (1). Also, from the homogeneity of the polynomials under consideration, the condition ($\mathcal{H}_2$) required in the first two cases is immediately satisfied in Case (3) since the origin then lies in the set K. As established in [3, Chapter 2, Lemma 2.7], it also turns out that the condition ($\mathcal{H}_3$) assumed in Case (2) always holds under the assumptions of Case (3). This succession of observations shows that the various cases considered provide an increasing degree of refinement in the following sense:

$$(1) \Longleftarrow (2) \Longleftarrow (3).$$

Correspondingly, the volume estimates stated in Theorem 1.1 below become sharper as the cases are more refined.

The following is a specialisation of [3, Theorem 1.1] tailored to the present setup.

THEOREM 1.1 (Volume growth of domains determined by homogeneous form inequalities over globally semianalytic sets). *Let the set K be globally semianalytic and let it satisfy the assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_2$). Let the limit condition (1.6), whereby the quantity $c(T)$ is defined, hold.*

Then the volume of the set $\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T))$ can be estimated with an increasing degree of precision as follows:

(1) *Assume that $\psi \geq 0$ is a smooth test function bounding from above the characteristic function of the set K. Denote by $r_{\boldsymbol{F}}(\psi)$ the smallest real pole of the meromorphic*

map $s \mapsto \langle \zeta_{\boldsymbol{F}}(s), \psi \rangle$ *defined in* (1.9) *and by* $m_{\boldsymbol{F}}(\psi) \geq 1$ *its order. Then the pair* $(r_{\boldsymbol{F}}(\psi), m_{\boldsymbol{F}}(\psi))$ *is well-defined and for* $T \geq 1$,

$$\mathrm{Vol}_n(\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T))) \ll T^n \cdot c(T)^{r_{\boldsymbol{F}}(\psi)} \cdot |\log(c(T))|^{m_{\boldsymbol{F}}(\psi)-1}. \tag{1.11}$$

(2) *Assume that* $(\mathcal{H}_3)$ is satisfied, *and let* $(r_{\boldsymbol{F}}(K), m_{\boldsymbol{F}}(K))$ *be the pair introduced in* $(\mathcal{H}_3)$. *Then* $r_{\boldsymbol{F}}(K)$ *is a strictly positive rational number. Moreover, as the parameter* T *tends to infinity,*

$$\mathrm{Vol}_n(\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T))) \asymp T^n \cdot c(T)^{r_{\boldsymbol{F}}(K)} \cdot |\log(c(T))|^{m_{\boldsymbol{F}}(K)-1}. \tag{1.12}$$

Here, $(r_{\boldsymbol{F}}(K), m_{\boldsymbol{F}}(K)) = (p, 1)$ *when the set of homogeneous forms* $\boldsymbol{F}(\boldsymbol{x})$ *has smooth complete intersection over* K. *Furthermore, under this smoothness assumption, the condition* $(\mathcal{H}_3)$ *can be guaranteed under* $(\mathcal{H}_1)$ *provided that* $(\mathcal{H}_2)$ *is strenghtened as follows: the algebraic variety* $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$ *should intersect the* interior *of the set* K.

(3) *Assume that* the set K is star-shaped with respect to the origin and that it contains the origin in its interior. *Denote by* $r_{\boldsymbol{F}}$ *the smallest real pole of the meromorphic distribution* $\zeta_{\boldsymbol{F}}$ *and by* $m_{\boldsymbol{F}} \geq 1$ *its order. Then* $r_{\boldsymbol{F}}$ *is a well-defined rational number lying in the interval* $(0, n/d]$. *Furthermore, there exists a constant* $\gamma_{\boldsymbol{F}}(K) > 0$ *such that, as the parameter* T *tends to infinity,*

$$\mathrm{Vol}_n(\mathcal{S}_{\boldsymbol{F}}(K, b(T))) = (\gamma_{\boldsymbol{F}}(K) + o(1)) \cdot T^n \cdot c(T)^{r_{\boldsymbol{F}}} \cdot |\log(c(T))|^{m_{\boldsymbol{F}}-1}. \tag{1.13}$$

As detailed after the statement of [3, Theorem 1.1], the error term in the volume estimate (1.13) can be made explicit depending on invariants attached to the homogeneous form $\|\boldsymbol{F}(\boldsymbol{x})\|^2$.

1.3. The number of solutions to homogeneous form inequalities. Let $K \subset \mathbb{R}^n$ be a set meeting the assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ (recall here that $(\mathcal{H}_3)$ is necessarily satisfied when K is star-shaped with respect to the origin and contains the origin in its interior). Given a parameter $\alpha > 0$, consider the case where

$$b(T) = T^{d-\alpha} \tag{1.14}$$

in the definitions of the set $\underline{\mathcal{S}}_{\boldsymbol{F}}(K, b(T))$ in (1.4) and of the corresponding counting function $\underline{\mathcal{N}}_{\boldsymbol{F}}(K, b(T))$ in (1.5). The limit condition (1.6) then holds. In order to single out the rôle of the parameter α in the forthcoming statements, define

$$\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha) \underset{(1.4)}{=} \underline{\mathcal{S}}_{\boldsymbol{F}}(K, T^{d-\alpha}) = \{\boldsymbol{x} \in T \cdot K : \|\boldsymbol{F}(\boldsymbol{x})\| \leq T^{d-\alpha}\} \tag{1.15}$$

and

$$\mathcal{N}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha) \underset{(1.5)}{=} \underline{\mathcal{N}}_{\boldsymbol{F}}(K, T^{d-\alpha}) = \#(\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha) \cap \mathbb{Z}^n). \tag{1.16}$$

The focus on a power function as in (1.14) is motivated by the fundamental Diophantine problem of determining the best power saving $\delta \geq 0$ in an estimate such as

$$\mathcal{N}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha) \ll \mathrm{Vol}_n(\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)) + T^{\tau_{\boldsymbol{F}}(K)-\delta}. \tag{1.17}$$

Here,

$$\tau_{\boldsymbol{F}}(K) = \dim(\mathcal{Z}_K(\boldsymbol{F}))$$

stands for the dimension of the algebraic variety $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$ over the set K. It is defined (and well-defined) as the dimension of the tangent vector space at any nonsingular point of $\mathcal{Z}_K(\boldsymbol{F}) := \{\boldsymbol{x} \in K : \boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}\}$.

It is easy to see that if the zero set $\mathcal{Z}_K(\boldsymbol{F})$ is *degenerate* in the sense that it is contained in a linear subspace of dimension $\tau_{\boldsymbol{F}}(K)$, one cannot expect δ to be strictly positive. This may nevertheless also happen when $\mathcal{Z}_K(\boldsymbol{F})$ admits very high "orders of contact" with tangents and/or when it contains rational subspaces. An elementary example is provided by the case of the single homogeneous form

$$F_n(\boldsymbol{x}) := \prod_{i=1}^{n} x_i, \tag{1.18}$$

which has degree $d = n$. The zero set of $F_n(\boldsymbol{x})$ is here the union of the coordinate hyperplanes. Taking $K = K_n$ to be the unit cube $[0,1]^n$ and choosing any $\alpha > 0$, the counting function $\mathcal{N}^{\dagger}_{F_n}(K, T, \alpha)$ is closely related to the generalised divisor summatory function

$$\Delta_n(t) = \sum_{1 \leq F_n(\boldsymbol{x}) \leq t} 1$$

in the sense that, after elementary calculations,

$$\begin{aligned} 0 &\leq \Delta_n(T^{n-\alpha}) + n \cdot (T+1)^{n-1} - \mathcal{N}^{\dagger}_{F_n}(K, T, \alpha) \\ &\leq \frac{n(n+1)}{2} \cdot (T+1)^{n-2} + n \cdot \Delta_{n-1}(T^{n-1-\alpha}). \end{aligned} \tag{1.19}$$

Now, it is known that there exists a polynomial $Q_n(t)$ of degree $n-1$ such that

$$\Delta_n(t) = t \cdot Q_n(\log t) + O(t^{1-\varepsilon})$$

for some $\varepsilon > 0$. (Determining an optimal exponent for the error term is the Piltz Divisor Problem, which reduces to the better known Dirichlet Divisor Problem when $n = 2$.) More precisely, the coefficients of the polynomials $Q_n(t)$ can be evaluated from the formula

$$Q_n(\log t) = \mathrm{Res}_{s=1}(t^{s-1} \cdot \zeta^n(s) \cdot s^{-1})$$

and from the Laurent series expansion of the Riemann zeta function

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \gamma_k \cdot (s-1)^k$$

valid in a neighbourhood of $s = 1$. Here, $(\gamma_k)_{k \geq 0}$ denotes the sequence of the Stieltjes constants. Explicit calculations of the coefficients of $Q_n(t)$ are carried out, for instance, in [35, Theorem 1]. In particular,

$$Q_2(t) = t + (2\gamma - 1) \quad \text{and} \quad Q_3(t) = \frac{t^2}{2} + (3\gamma - 1) \cdot t(3\gamma^2 - 3\gamma + 3\gamma_1 + 1),$$

where $\gamma \approx 0.577$ is the Euler–Mascheroni constant and where $\gamma_1 \approx -0.073$. Since elementary calculations show that the volume of the region $\mathcal{S}^{\dagger}_{F_n}(K, T, \alpha)$ grows, up to a multiplicative constant, as $T^{n-\alpha} \cdot (\log T)^{n-1}$ (meaning that $(r_{F_n}, m_{F_n}) = (1, n)$ in the notations of Theorem 1.1), the estimates (1.19) imply that there can be no power saving in the error term when $n = 3$ and, more generally, in the generic case where the coefficients do not cancel out suitably.

Given that the assumption of nondegeneracy is thus not enough to guarantee the existence of a positive power saving δ in the error term in (1.17), much work has been devoted to the understanding of the cases when such a power saving holds and, when it does, the largest value that $\delta > 0$ can then take. Indeed, an observation made in [44, p. 177] (with the choice of $\alpha = 1/2$ in the above notations) shows that the right-hand side of (1.17) then bounds nontrivially the number of solutions to the system of Diophantine *equations* $\boldsymbol{F}(\boldsymbol{m}) = \boldsymbol{0}$ when $\boldsymbol{m} \in T \cdot K$. Much more importantly, and this is the main motivation of the work undertaken here, determining an optimal value for $\delta > 0$ is the cornerstone of the theory of metric Diophantine approximation on manifolds: as observed by Sprindžuk [45] as part of his work on the extension of Khinchin's theorem to manifolds and as later formalised in the seminal work by Budarina, Dickinson and Levesley [10], this is indeed the key ingredient in the determination of the Hausdorff dimension of the set of well approximable points lying on manifolds.

Thus far, a power saving in the error term in the counting bound in (1.14) has been obtained in particular cases assuming either the smoothness of the variety and/or various curvature conditions. More precisely, assume first that the zero set $\mathcal{Z}_K(\boldsymbol{F})$ intersects nontrivially the interior of K and that it is nondegenerate. Let also the set of homogeneous forms $\boldsymbol{F}(\boldsymbol{x})$ meet the smooth complete intersection condition defined by the nonvanishing of the map (1.8) over the set K. In this case, $\tau_{\boldsymbol{F}}(K) = n - p$ and, from Case (2) in Theorem 1.1, it follows that $\mathrm{Vol}_n(\mathcal{S}^\dagger_{\boldsymbol{F}}(K, T, \alpha)) \asymp T^{n-p\alpha}$. Sarnak [40, Appendix 1] then proves that

$$\mathcal{N}^\dagger_{\boldsymbol{F}}(K, T, \alpha) \ll T^{n-p\alpha} + T^{n-p-\delta} = \mathrm{Vol}_n(\mathcal{S}^\dagger_{\boldsymbol{F}}(K, T, \alpha)) + T^{\tau_{\boldsymbol{F}}(K)-\delta} \tag{1.20}$$

for some $\delta > 0$ provided that the set K is "nice" enough, for instance when it is convex and has piecewise smooth boundary. Moreover, the proof shows that one can then take $\delta = 1/(nd)$.

Even when assuming further additional strong curvature conditions, the theory is satisfactory essentially only in the case of planar curves. This follows from the remarkable work by Vaughan and Velani [48] based on harmonic analysis. They indeed obtain the best possible strengthening of Huxley's earlier bound [31]: translated into the present setting, it implies that in the case of $p = 1$ and a homogeneous form $F(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$ in $n = 3$ variables defining a smooth projective curve over a compact domain K where the Gaussian curvature of the algebraic variety $\mathcal{Z}_K(\boldsymbol{F})$ does not vanish, the upper bound (1.14) holds for any $\delta \in (0, 1)$. The complementary lower bound had been established by Beresnevich, Dickinson and Velani [4]. Various improvements on these results, including asymptotic formulae, have been established for planar curves – see [8, 11, 25, 28] for further details. Obtaining estimates with $\delta \geq 1$ (i.e. in "ultra thin" neighbourhoods of a curve) represents a significant challenge as it requires taking into account not only the analytic properties of the curve, but also its algebraic properties. The first result of this kind was obtained only very recently in [2] with the help of the determinant method for smooth *algebraic* curves (without assuming any curvature condition).

In higher dimensions, following earlier contributions [7, 30] valid under various regularity assumptions, Huang [29] obtained an essentially best possible bound on the error term for a fairly large class of hypersurfaces assuming the nonvanishing of the Gaussian curvature. His proof is again based on harmonic analysis and shows that the bound (1.17) then holds for any $\delta \in (0,1)$. The curvature assumption in this result has recently been slightly relaxed by Schindler and Yamagishi [42]. Even more recently, Beresnevich and Yang [5] developed an approach based on quantitative nondivergence from dynamics to count rational points avoiding regions of a neighbourhood of a manifold where they abnormally cluster.

The main contribution of the present work is to develop a fairly general framework guaranteeing the existence of an explicit nontrivial power saving $\delta > 0$ in the counting bound (1.17). This is obtained without making any *a priori* assumption on the smoothness and/or the curvature of the algebraic variety under consideration. Rather, this follows upon defining a geometric quantity attached to any such variety, the *semialgebraic level of flatness*, which can be computed explicitly. It captures both the algebraic and the analytic properties needed for one to be able to count rational points in thin neighbourhoods of the zero set.

With this in mind, let the assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ hold. As the above-stated goals are achieved working in the context of semialgebraic geometry, it is natural to assume furthermore that the set K is semialgebraic. The statement of the results furthermore depends on the choice of a compact semialgebraic set $\mathcal{K}$ containing K in its interior and contained in the open set U introduced in (1.10). This is because to make the values of the power savings effective whenever they hold, one indeed needs to consider the behaviour of the set of homogeneous forms $\boldsymbol{F}(\boldsymbol{x})$ over a compact neighbourhood of K.

The definition of the semialgebraic level of flatness over the zero set $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ relies on the principle of geometric tomography claiming that the data of the volumes of all lower-dimensional sections of a set captures the properties of this set. To formalise this idea, given a unit vector $\boldsymbol{v} \in \mathbb{S}^{n-1}$ and a real σ, define the affine space

$$\boldsymbol{v}^{\perp}(\sigma) := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{v} \cdot \boldsymbol{x} = \sigma\}.$$

The flatness of the zero set $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ along this set is measured by the volume section

$$\mu_{\boldsymbol{F},\mathcal{K}}(\boldsymbol{v},\sigma,\varepsilon) = \mathrm{Vol}_{n-1}(\{\boldsymbol{x} \in \mathcal{K} \cap \boldsymbol{v}^{\perp}(\sigma) : \|\boldsymbol{F}(\boldsymbol{x})\| \leq \varepsilon\}),$$

where $\varepsilon > 0$. The "biggest" such volume section is then defined as

$$M_{\boldsymbol{F}}(\mathcal{K},\varepsilon) = \sup_{\boldsymbol{v} \in \mathbb{S}^{n-1}} \sup_{\sigma \in \mathbb{R}} \mu_{\boldsymbol{F},\mathcal{K}}(\boldsymbol{v},\sigma,\varepsilon).$$

From this, the relevant semialgebraic level of flatness is defined as

$$q_{\boldsymbol{F}}(\mathcal{K}) = \liminf_{\varepsilon \to 0^+} \left(\frac{\log M_{\boldsymbol{F}}(\mathcal{K},\varepsilon)}{\log \varepsilon} \right). \tag{1.21}$$

This is a measure of the "biggest level of flatness" that can be achieved when intersecting the sublevel set $\{\boldsymbol{x} \in \mathcal{K} : \|\boldsymbol{F}(\boldsymbol{x})\| \leq \varepsilon\}$ with affine hyperplanes in the following sense: the smaller it is, the more flat the intersection of the sublevel set in some direction. This measure of flatness is a well-defined real number under the assumption that $\mathcal{Z}_K(\boldsymbol{F}) \neq \emptyset$. Crucially, the last section of Chapter 3 shows that $q_{\boldsymbol{F}}(\mathcal{K})$ can then be estimated effectively in a finite number of steps. The statement of the main result in this section (Theorem 1.2

below) relies on a comparison between $q_{\boldsymbol{F}}(\mathcal{K})$ and the volume growth exponent $r_{\boldsymbol{F}}(K)$ present in the statement of Theorem 1.1 and the assumption $(\mathcal{H}_3)$.

Specifically, for the coherence of exposition and as detailed further below, we provide a simplified illustration of the class of statements established in Chapters 2 and 3 in order to count effectively rational points in thin neighbourhoods of an algebraic variety. Set

$$\widehat{\tau}_{\boldsymbol{F}}(\mathcal{K}) = \operatorname{codim}(\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})) = n - \dim(\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})) = n - \tau_{\boldsymbol{F}}(\mathcal{K}).$$

Theorem 1.2 (Counting solutions to homogeneous form inequalities). *Assume that the conditions* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold and the above defined sets* $K \subset \mathcal{K}$ *are semialgebraic. Fix a parameter* $\alpha > 0$ *and assume that the level of semialgebraic flatness* (1.21) *is high enough in the sense that*

$$q_{\boldsymbol{F}}(\mathcal{K}) > \max\left\{ r_{\boldsymbol{F}}(K) - 1, n - 1 - r_{\boldsymbol{F}}(K) \cdot \frac{\tau_{\boldsymbol{F}}(\mathcal{K})}{\widehat{\tau}_{\boldsymbol{F}}(\mathcal{K})} \right\}. \tag{1.22}$$

Then there exists an effectively computable real $\delta(K, \mathcal{K}) > 0$ *depending on the semialgebraic sets* K *and* $\mathcal{K}$ *such that the following estimate with power saving error term holds:*

$$\mathcal{N}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha) \ll \operatorname{Vol}_n(\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)) + T^{\tau_{\boldsymbol{F}}(\mathcal{K}) - \delta_{\boldsymbol{F}}(\mathcal{K}, K)}. \tag{1.23}$$

Furthermore, the inequality (1.22) *is satisfied when the following two conditions are simultaneously met: the set of polynomials* $\boldsymbol{F}(\boldsymbol{x})$ *has smooth complete intersection over* $\mathcal{K}$ *and the zero set* $\mathcal{Z}_K(\boldsymbol{F})$ *is nondegenerate in the sense that it is not contained in a linear subspace of dimension* $\tau_{\boldsymbol{F}}(\mathcal{K}) = n - p$.

Note that for the homogeneous form $F_n(\boldsymbol{x})$ defined in (1.18), the inequality (1.22) is indeed not met since for any semialgebraic compact neighbourhood $\mathcal{K}_n^+$ of $K_n^+ = [0,1]^n$, one easily checks that $\widehat{\tau}_{F_n}(\mathcal{K}_n^+) = 1$, $r_{F_n}(\mathcal{K}_n^+) = n - 1$ and $q_{F_n}(\mathcal{K}_n^+) = 0$.

As mentioned above, Theorem 1.2 is a very abridged version of a class of statements providing a much more general picture of the behaviour of the counting function $\mathcal{N}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)$. Thus, it is established in Chapter 2 that the volume $\operatorname{Vol}_n(\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha))$ determines the *exact* asymptotic order of this counting function as T tends to infinity when $\alpha \in (0,1)$ (i.e. when the algebraic set $\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)$ is "large enough") and also that the counting function is, up to a multiplicative constant, bounded above by this volume for $\alpha = 1$. These estimates are further refined when $\alpha \in (0,1)$ under the stronger assumptions of Case (3) stated in Section 1.2: an asymptotic expansion with power saving error term is then established for $\mathcal{N}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)$ as T tends to infinity.

In the complementary case when $\alpha > 1$ (i.e. when the algebraic set $\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)$ is "small"), it is proved in Chapter 3 that an estimate of the form (1.23) can still hold in the nongeneric case when $q_{\boldsymbol{F}}(\mathcal{K}) \leq r_{\boldsymbol{F}}(K) - 1$ under an explicit condition (which is, of course, still not satisfied by the homogeneous form $F_n(\boldsymbol{x})$ defined in (1.18)). Also, in all instances where an estimate with power saving error term holds, a range of admissible values of the exponent $\delta_{\boldsymbol{F}}(K, \mathcal{K}) > 0$ is explicitly determined.

When seeing Theorem 1.2 as a lattice point counting statement, it complements and extends a number of previously known results. Thus, for a single *positive* homogeneous function $F(\boldsymbol{x})$ (not necessarily assumed to be a polynomial), Randol [39] establishes a counting bound with power saving error term depending on the decay of a Fourier operator acting on the boundary surface $\{F = 1\}$, which is supposed to be regular enough (in a suitable sense). This has been slightly generalised by Colin de Verdière [16] who considers the behaviour of the counting bound when rotating randomly the domain $\{F \leq 1\}$. For a domain with analytic boundary, Greenblatt [26] provides an estimate related to Theorem 1.2 by the use of suitable resolutions of singularities. This nevertheless comes with a quite strong restriction not present in our Theorem 1.2: in his work, the boundary of the domain must be locally the graph of an analytic function which separates the interior of the domain from its complement into exactly two pieces.

Finally, although Theorem 1.2 holds in the case of $n = 2$ variables, explicit and sharper results are known if one considers a single homogeneous form with integer coefficents (one is then dealing with a Thue inequality). To see recent progress in this well-studied topic, see the works by Fouvry and Waldschmidt [24] and by Stewart and Xiao [46], and the references therein.

1.4. Structure of the paper. In the range $0 < \alpha \leq 1$, Theorem 1.2 is proved in the more general form of Theorem 2.1. In the range $\alpha > 1$, it is established in the more general form of Theorem 3.1.

2. Counting in large domains

Let K be a globally semianalytic subset of $\mathbb{R}^n$ meeting the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ stated in the Introduction. The goal of this section is to establish a more precise version of Theorem 1.2 when the domain $\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)$ defined in (1.15) is "large enough" in the sense that $0 < \alpha \leq 1$. Indeed, if moreover the assumption $(\mathcal{H}_3)$ holds (one then deals with what is referred to as Case (2) in Section 1.2), it is possible to determine the precise asymptotic order for the counting function $\mathcal{N}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)$ defined in (1.16) when $0 < \alpha < 1$, and also to obtain an upper bound for this function when $\alpha = 1$. When one assumes instead the stronger condition that the set K is star-shaped with respect to the origin and that it contains the origin in its interior (one then deals with Case (3) of Section 1.2), it becomes possible to get an asymptotic expansion for $\mathcal{N}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)$ when $0 < \alpha < 1$.

To state the main result proved in this chapter, let $P(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ denote a homogeneous form of degree $q \geq 2$. By analogy with the above notations, let

$$\mathcal{S}^{\dagger}_{P}(K, T, \alpha) = \{\boldsymbol{x} \in T \cdot K : |P(\boldsymbol{x})| \leq T^{q-\alpha}\} \tag{2.1}$$

and

$$\mathcal{N}^{\dagger}_{P}(K, T, \alpha) = \#(\mathbb{Z}^n \cap \mathcal{S}^{\dagger}_{P}(K, T, \alpha)). \tag{2.2}$$

Theorem 1.2 for $0 < \alpha \leq 1$ is then a consequence of Case $(1')$ in the statement below: it indeed suffices to apply it to the polynomial $P(\boldsymbol{x}) = \|\boldsymbol{F}(\boldsymbol{x})\|^2$ after substituting the parameters $(2d, 2\alpha)$ for (q, α) in (2.1). The resulting conclusion generalises that of Theorem 1.2 in as much as it is valid *without assuming the flatness condition* (1.22).

Theorem 2.1 (Asymptotic behaviour of the counting function of homogeneous form inequalities in large domains). *Assume that the set $K \subset \mathbb{R}^n$ is globally semianalytic and satisfies the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$.*

Then the counting function $\mathcal{N}_P^{\dagger}(K,T,\alpha)$ *can be estimated as follows:*

(1′) *Assume that the condition* $(\mathcal{H}_3)$ *holds. Then as* $T\to\infty$, *one has*
$$\mathcal{N}_P^{\dagger}(K,T,\alpha)\asymp \mathrm{Vol}_n(\mathcal{S}_P^{\dagger}(K,T,\alpha))$$
provided that $\alpha\in(0,1)$. *When* $\alpha=1$,
$$\mathcal{N}_P^{\dagger}(K,T,1)\ll \mathrm{Vol}_n(\mathcal{S}_P^{\dagger}(K,T,1)). \tag{2.3}$$

(2′) *Assume the stronger condition that* K *is star-shaped with respect to the origin and contains the origin in its interior. Then provided that* $\alpha\in(0,1)$, *one has, as* $T\to\infty$,
$$\mathcal{N}_P^{\dagger}(K,T,\alpha)=\left(1+O\left(\frac{1}{T^{\delta}}\right)\right)\cdot \mathrm{Vol}_n(\mathcal{S}_P^{\dagger}(K,T,\alpha))$$
for some $\delta>0$.

In this statement, the volumes of the sets $\mathcal{S}_P^{\dagger}(K,T,\alpha)$ (where $0<\alpha\le 1$) are respectively given by Cases (2) and (3) of Theorem 1.1.

The rest of this chapter is devoted to the proof of Theorem 2.1, starting with the inequality (2.3).

Proof of the volume estimate (2.3). An elementary packing argument, a variant of which appears in [40, Appendix 1], shows that the sought estimate holds for any value of
$$\alpha\in[0,1]. \tag{2.4}$$
Indeed, fix $\eta\in(0,1/4)$ and assume that $\boldsymbol{m}\in\mathbb{Z}^n\cap\mathcal{S}_P^{\dagger}(K,T,\alpha)$. Let $\boldsymbol{u}\in\mathbb{R}^n$ be a vector with Euclidean norm at most η. Consider then the Taylor expansion
$$P(\boldsymbol{m}+\boldsymbol{u})=P(\boldsymbol{m})+\sum_{k=1}^{q}\frac{1}{k!}\cdot((\boldsymbol{u}\cdot\nabla)^kP)(\boldsymbol{m}),$$
where $\boldsymbol{u}\cdot\nabla$ is shorthand for the differential operator $\sum_{i=1}^n u_i\cdot\partial_i$ when $\boldsymbol{u}=(u_1,\dots,u_n)$ and $\nabla=(\partial_1,\dots,\partial_n)$, and where the power applied to it must be understood as a formal multiplication process.

For any given integer $1\le k\le q$, the map $\boldsymbol{x}\mapsto((\boldsymbol{u}\cdot\nabla)^kP)(\boldsymbol{x})$ is homogeneous of degree $q-k$. As a consequence, upon decomposing the integer vector $\boldsymbol{m}$ as $\boldsymbol{m}=T\cdot\boldsymbol{y}$, where $\boldsymbol{y}\in K$, one obtains
$$|P(\boldsymbol{m}+\boldsymbol{u})|\le T^{q-\alpha}+\eta\cdot c_P(K)\cdot T^{q-1}\underset{(2.4)}{\le}(1+\eta\cdot c_P(K))\cdot T^{q-\alpha}$$
with a constant $c_P(K)>0$ determined by the sup norms of $P(\boldsymbol{x})$ and of its partial derivatives over K. This shows that the pairwise disjoint Euclidean balls with radius η centred at the integer points in $\mathcal{S}_P^{\dagger}(K,T,\alpha)$ are contained in, say, the set $\underline{\mathcal{S}}_P(K^{(\eta)},(1+\eta\cdot c_P(K))\cdot T^{q-\alpha})$. Here, $K^{(\eta)}$ denotes the η-neighbourhood of K and, given a map $T\mapsto b(T)\ge 0$,
$$\underline{\mathcal{S}}_P(K^{(\eta)},b(T))=\{\boldsymbol{x}\in T\cdot K^{(\eta)}:|P(\boldsymbol{x})|\le b(T)\}.$$
Thus,
$$\mathcal{N}_P^{\dagger}(K,T,\alpha)\le \mathrm{Vol}_n\big(\underline{\mathcal{S}}_P\big(K^{(\eta)},(1+\eta\cdot c_P(K))\cdot T^{q-\alpha}\big)\big). \tag{2.5}$$

If $\eta > 0$ is chosen small enough, it is immediate that the set $K^{(\eta)}$ also satisfies the assumption $(\mathcal{H}_3)$ with the same compact set C and the same open set U as those introduced in the support restriction condition (1.10) for K. Consequently, with the notations of $(\mathcal{H}_3)$, the pair $(r_P(K^{(\eta)}), m_P(K^{(\eta)}))$ equals $(r_P(K), m_P(K))$. It then follows from Case (2) of Theorem 1.1 that

$$\begin{aligned}\mathrm{Vol}_n\big(\underline{\mathcal{S}}_P\big(K^{(\eta)}, (1+\eta\cdot c_P(K))\cdot T^{q-\alpha}\big)\big) &\ll \mathrm{Vol}_n(\mathcal{S}_P(K, T^{q-\alpha}))\\ &= \mathrm{Vol}_n(\underline{\mathcal{S}}_P^{\dagger}(K,T,\alpha)).\end{aligned}$$

In view of the inequality (2.5), this completes the proof. ■

Establishing the remaining statements part of Theorem 2.1 requires more work. To do so, assume from now on and until the end of the chapter that $\alpha \in (0,1)$.

2.1. Smooth counting. It is convenient to work first with a smooth version of the counting function $\mathcal{N}_P^{\dagger}(K,T,\alpha)$. To this end, fix a smooth map $\psi \geq 0$ in $\mathcal{C}_c^{\infty}(\mathbb{R}^n)$ whose support will be specified later to approximate the set K suitably. The proof of the remaining cases in Theorem 2.1 also requires approximating the characteristic function of an interval by a function whose Fourier transform behaves "nicely". This is a well-studied topic where the extremal functions introduced by Beurling and Selberg are often used (see the discussion and the references in [17] for more details). In the present situation however, it is more relevant to work with the map obtained from the following (particular case of a) result due to Colzani, Gigante and Travaglini [17, Theorem 1.1].

Before stating it, recall that a function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ has *fast decay at infinity* if for any $\beta > 0$, there exists a constant $c_\beta > 0$ such that for all $t \geq 0$,

$$0 \leq \theta(t) \leq \frac{c_\beta}{(1+t)^{\beta}}\cdot \tag{2.6}$$

Also, a holomorphic function F is said to be *of exponential type* $R > 0$ if

$$R = \limsup_{r\to\infty} \max_{|z|=r} \frac{\log|F(z)|}{|z|}\cdot$$

Equivalently, for any $\varepsilon > 0$, there exists a constant $\kappa(\varepsilon) > 0$ such that $|F(z)| \leq \kappa(\varepsilon)\cdot e^{(R+\varepsilon)\cdot|z|}$.

Theorem 2.2 (Colzani, Gigante & Travaglini, 2011). *Let $c \in (0,1)$. Set $I(c) = [-c,c]$. Then there exists a positive function θ with fast decay at infinity such that for any $R > 0$, there exist integrable entire functions $A_{(R,c)}$ and $B_{(R,c)}$ satisfying the following properties: these two functions are of exponential type R and have their Fourier transforms supported in the interval $[-R,R]$; furthermore, for all $x \in \mathbb{R}$,*

$$A_{(R,c)}(x) \leq \chi_{I(c)}(x) \leq B_{(R,c)}(x) \tag{2.7}$$

and

$$0 \leq B_{(R,c)}(x) - A_{(R,c)}(x) \leq \theta\big(R\cdot \big||x|-c\big|\big). \tag{2.8}$$

The main point of this statement is that it allows easy control of the gap $|B_{(R,c)}(x) - A_{(R,c)}(x)|$ by the distance from $x \in \mathbb{R}$ to the *boundary* of the interval $I(c)$: roughly speaking, the quantity $|B_{(R,c)}(x) - A_{(R,c)}(x)|$ is approximately equal to 1 at points x within distance $1/R$ of the interval $I(c)$ and essentially zero at larger distances.

Let $T \geq 1$. The proof of Theorem 2.1 mainly reduces to the estimate for $\widetilde{\mathcal{N}}_P(\psi, T, \alpha)$ obtained in Proposition 2.4 below, which takes into account the support restriction condition (1.10) induced by the assumption $(\mathcal{H}_3)$. To state it, fix a compact set $\mathcal{K}$ contained in the open set U and containing the compact set K in its interior, where U is defined as part of the condition (1.10). Assume then that the map ψ is such that

$$C \subset \operatorname{Supp}\psi \subset \mathcal{K}, \quad 0 \leq \psi \leq 1 \quad \text{and} \quad \max_{\boldsymbol{x}\in\mathcal{K}} \psi(\boldsymbol{x}) = 1, \tag{2.9}$$

where C is also defined in (1.10). Take

$$c = T^{-\alpha} \quad \text{and} \quad R = \frac{T}{2\cdot \Delta_P(\mathcal{K})}, \quad \text{where} \quad \Delta_P(\mathcal{K}) = \max_{\boldsymbol{x}\in\mathcal{K}} \|(\nabla P)(\boldsymbol{x})\|, \tag{2.10}$$

and define $E_{(T,\alpha)}$ as being either $A_{(T/(2\Delta_P(\mathcal{K})),T^{-\alpha})}$ or $B_{(T/(2\Delta_P(\mathcal{K})),T^{-\alpha})}$ in Theorem 2.2 (i.e. the maps obtained with the above choice for the parameters R and c).

Let

$$\widetilde{\mathcal{N}}_P(\psi, T, \alpha) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} G_{T,\alpha}^{(P,\psi)}(\boldsymbol{k}), \tag{2.11}$$

where, given $\boldsymbol{x} \in \mathbb{R}^n$,

$$G_{T,\alpha}^{(P,\psi)}(\boldsymbol{x}) = \psi\left(\frac{\boldsymbol{x}}{T}\right) \cdot E_{(T,\alpha)}\left(P\left(\frac{\boldsymbol{x}}{T}\right)\right). \tag{2.12}$$

Given $c \in (0,1)$, set furthermore

$$\mu_P(\psi, c) = \int_{\mathbb{R}^n} \chi_{\{|P(\boldsymbol{x})|<c\}} \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{2.13}$$

It is useful to record right here the asymptotic behaviour at the origin of the map $0 < c \mapsto \mu_P(\psi, c)$. This can be inferred from [3, Chapter 2, Lemma 2.5 & Proposition 2.16(X1)], which are reproduced here in a synthetic form tailored to the present goal.

LEMMA 2.3. *Let $c \in (0,1)$ and let ψ be a nonnegative map in $\mathcal{C}_c^\infty(\mathbb{R}^n)$. Let $(r_P(\psi), m_P(\psi))$ be the pair involving the smallest real pole $r_P(\psi) \geq 0$ of the meromorphic function $s \mapsto \langle \zeta_P(s), \psi\rangle = \int_{\mathbb{R}^n} |P(\boldsymbol{x})|^{-s} \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.*

Then $r_P(\psi) \in (0, \infty)$ (and thus $m_P(\psi) \geq 1$) and there exists a univariate monic polynomial $R_\psi(x) \in \mathbb{R}[x]$ of degree $m_P(\psi) - 1$ and some $\varepsilon > 0$ such that, as c tends to zero,

$$\mu_P(\psi, c) = \Theta_P(\psi) \cdot c^{r_P(\psi)} \cdot R_\psi(|\log c|) + O(c^{r_P(\psi)+\varepsilon}). \tag{2.14}$$

Here,

$$\Theta_P(\psi) = \frac{1}{m_P(\psi)!} \lim_{\sigma\to r_P(\psi)} (\sigma - r_P(\psi))^{m_P(\psi)} \langle \zeta_P(s), \psi\rangle > 0.$$

Assume now that $\mathcal{F}$ is a family of nonnegative maps in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfying the following two properties:

(A1) [uniform boundedness of the support] *there exists a compact set $\mathcal{K} \subset \mathbb{R}^n$ such that* $\operatorname{Supp}\psi \subset \mathcal{K}$ *for all $\psi \in \mathcal{F}$;*

(A2) [uniform boundedness of the range] *there exists* $\varphi_{\mathcal{F}}$ *in* $\mathcal{C}_c^\infty(\mathbb{R}^n)$ *such that for all* $\psi \in \mathcal{F}$*, one has* $0 \leq \psi \leq \varphi_{\mathcal{F}}$ *and, with the above notation,* $(r_P(\psi), m_P(\psi)) = (r_P(\varphi_{\mathcal{F}}), m_P(\varphi_{\mathcal{F}}))$ *(in particular, the pair* $(r_P(\psi), m_P(\psi))$ *remains constant over the choice of* $\psi \in \mathcal{F}$*).*

Then the following uniformity claims on the parameters appearing in the expansion (2.14) *hold when the map* ψ *varies in* $\mathcal{F}$*:*

(V1) *the leading coefficient* $\Theta_P(\psi)$ *is uniformly bounded above by a constant depending on the smooth map* $\varphi_{\mathcal{F}}$*;*

(V2) *there exist integers* $M, N \geq 1$ *depending on the polynomial* $P(\boldsymbol{x})$ *and on the compact set* $\mathcal{K}$ *and, given* $\varepsilon > 0$ *and* $\varphi_{\mathcal{F}}$ *as above, there exists a constant* $\Gamma(\mathcal{F}, \varphi_F, \varepsilon) > 0$ *such that the following holds: the implicit constant in the error term can be taken as* $2\max\{\Gamma(\mathcal{F}, \varphi_{\mathcal{F}}, \varepsilon), (C_N(\mathcal{F}, \mathcal{K}))^M\}$ *provided that the quantity*

$$C_N(\mathcal{F}, \mathcal{K}) := \sup_{\psi\in\mathcal{F}} \max_{0\leq|\boldsymbol{k}|\leq N} \max_{\boldsymbol{x}\in\mathcal{K}} \left|\frac{\partial^{|\boldsymbol{k}|}\psi}{\partial \boldsymbol{x}^{\boldsymbol{k}}}(\boldsymbol{x})\right| \tag{2.15}$$

is finite. In particular, as long as $\max_{\boldsymbol{x}\in\mathcal{K}} \psi(\boldsymbol{x}) = 1$ *for some* $\psi \in \mathcal{F}$*, this implicit constant can be taken as* $(C_N(\mathcal{F}, \mathcal{K}))^M$ *up to a multiplicative constant depending on* $P(\boldsymbol{x})$*,* $\mathcal{F}$*,* $\mathcal{K}$*,* $\varepsilon > 0$ *and on the choice of* $\varphi_{\mathcal{F}}$*;*

(V3) *there exist integers* $M, N \geq 1$ *depending on the polynomial* $P(\boldsymbol{x})$ *and on the compact set* $\mathcal{K}$ *such that the coefficients of the univariate polynomial* $\Theta_P(\psi)\cdot R_\psi(x)$ *are, up to a multiplicative constant depending on* $\mathcal{K}$*, upper bounded by the above defined quantity* $(C_N(\mathcal{F}, \mathcal{K}))^M$*, provided it is finite.*

To obtain an asymptotic expansion for the map $0 < c \mapsto \mu_P(\psi, c)$, apply this lemma to the compact set $\mathcal{K}$ introduced in (2.9) and to the collection of maps $\mathcal{F}$ reduced to the singleton $\{\psi\}$. One then obtains the existence of a real $\rho > 0$ and of an integer $m \geq 0$, both independent of ψ as chosen in (2.9), satisfying the estimate

$$\mu_P(\psi, c) = (\Theta_P(\psi) + h_\psi(c))\cdot c^\rho \cdot |\log c\,|^m, \text{ where } |h_\psi(c)| \leq (m+1)\cdot (C_N(\psi, \mathcal{K}))^M \cdot \eta(c). \tag{2.16}$$

In this relation,

- the integers $M, N \geq 1$ depend only on $\mathcal{K}$ and on the polynomial $P(\boldsymbol{x})$;
- $C_N(\psi, \mathcal{K})$ is the constant defined in (2.15), namely

$$C_N(\psi, \mathcal{K}) = \max_{0\leq|\boldsymbol{k}|\leq N} \max_{\boldsymbol{x}\in\mathcal{K}} \left|\frac{\partial^{|\boldsymbol{k}|}\psi}{\partial \boldsymbol{x}^{\boldsymbol{k}}}(\boldsymbol{x})\right|, \tag{2.17}$$

  and the last relation in (2.9) implies that $C_N(\psi, \mathcal{K}) \geq 1$;
- from the uniformity claim (V3) in Lemma 2.3,

$$0 < \Theta_P(\psi) \leq (C_N(\psi, \mathcal{K}))^M; \tag{2.18}$$

- the quantity $\eta(c)$ tends to 0 as $c \to 0^+$ uniformly in ψ meeting (2.9) under the assumption $(\mathcal{H}_3)$. Indeed, the factor $(m+1)\cdot(C_N(\psi, \mathcal{K}))^M \cdot \eta(c)$ in (2.16) is obtained upon adding upper bounds for two maps depending on c:
  - the map

$$c \mapsto \Theta_P(\psi)\cdot(R_\psi(|\log c\,|) - |\log c\,|^m)/|\log c\,|^m, \tag{2.19}$$

where R_ψ is, with the present notations, the degree m monic polynomial appearing in (2.14); furthermore, from (V3), each of the m coefficients obtained when expanding this expression is, up to a multiplicative constant depending on $\mathcal{K}$, bounded above by $(C_N(\psi,\mathcal{K}))^M$;

– the map determined by the error term present in (2.14), which is of the form $O(c^\varepsilon)$ for some fixed $\varepsilon > 0$. Here, from the uniformity claim (V2) in Lemma 2.3 and under $(\mathcal{H}_3)$, the implicit constant can be chosen as $C_N(\psi,\mathcal{K})$ up to a multiplicative constant depending on $\mathcal{K}$ and on ε, and uniformly in any ψ meeting the conditions (2.9).

Upon adding the resulting form of the error term in the latter case with the upper bound obtained for the map (2.19), this yields the required shape $(m+1)\cdot(C_N(\psi,\mathcal{K}))^M\cdot\eta(c)$ with the sought uniformity claim on η.

Proposition 2.4. *Assume that $\psi\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfies the relations* (2.9) *and takes strictly positive values over the set C defined therein. Let $\alpha\in(0,1)$. Then under the assumption of the homogeneity of the polynomial $P(\boldsymbol{x})$, there exist integers $M,N\geq 1$ such that*

$$|\widetilde{\mathcal{N}}_P(\psi,T,\alpha)-T^n\cdot\mu_P(\psi,T^{-\alpha})|\ll C_N(\psi,\mathcal{K})^M\cdot T^{n-\delta}\cdot\mu_P(\psi,T^{-\alpha}) \tag{2.20}$$

for any $\delta\in(0,1-\alpha)$. In this inequality, the constant $C_N(\psi,\mathcal{K})$ is defined in (2.17)*. Furthermore, the implicit constant, the integers M and N and the exponent δ do not depend on ψ.*

Deduction of Theorem 2.1 from Proposition 2.4 when $0<\alpha<1$. Consider first Case $(1')$ of Theorem 2.1. Fix $0<\alpha<1$ and let ψ_1,ψ_2 be two maps in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfying the conditions (2.9) and the inequalities

$$0\leq\psi_1\leq\chi_K\leq\psi_2.$$

As a consequence of the homogeneity of the polynomial $P(\boldsymbol{x})$,

$$\mathrm{Vol}_n(\mathcal{S}_P^\dagger(K,T,\alpha))=\int_{\mathbb{R}^n}\chi_{\{|P(\boldsymbol{x})|\leq T^{q-\alpha}\}}\cdot\chi_{\{\boldsymbol{x}\in T\cdot K\}}\,\mathrm{d}\boldsymbol{x}=T^n\int_K\chi_{\{|P(\boldsymbol{x})|\leq T^{-\alpha}\}}\,\mathrm{d}\boldsymbol{x}.$$

Therefore, under the assumption $(\mathcal{H}_3)$, the relation (2.14) implies that

$$T^n\cdot\mu_P(\psi_1,T^{-\alpha})\asymp\mathrm{Vol}_n\big(\mathcal{S}_P^\dagger(K,T,\alpha)\big)\asymp T^n\cdot\mu_P(\psi_2,T^{-\alpha}).$$

It then follows from Proposition 2.4 that

$$\begin{aligned}\mathrm{Vol}_n(\mathcal{S}_P^\dagger(K,T,\alpha))\ll\widetilde{\mathcal{N}}_P(\psi_1,T,\alpha)&\leq\mathcal{N}_P^\dagger(K,T,\alpha)\\&\leq\widetilde{\mathcal{N}}_P(\psi_2,T,\alpha)\ll\mathrm{Vol}_n(\mathcal{S}_P^\dagger(K,T,\alpha)).\end{aligned}$$

Here, the leftmost inequality relies on the fact that the exponent δ in (2.20) is, by assumption, strictly positive. This establishes the conclusion of Theorem 2.1 in Case $(1')$ when $0<\alpha<1$.

As for Case $(2')$, fix an integer $k\geq1$. Let ψ_k^- and ψ_k^+ be smooth maps in $\mathcal{C}_c^\infty(\mathbb{R}^n)$, pointwise monotonic in k for k large, such that

$$\chi_{K(1-1/k)}\leq\psi_k^-\leq\chi_K\leq\psi_k^+\leq\chi_{K(1+1/k)}. \tag{2.21}$$

Here, given $\lambda>0$, one sets $K(\lambda)=\lambda\cdot K$; also, the inequalities between the characteristic functions are indeed valid under the assumptions that K is star-shaped with respect to the origin. Upon choosing k larger than some integer k_0, assume furthermore that the maps $\psi_k^{\pm}$ also meet the conditions (2.9). From [27, Theorem 1.4.1, p. 25], they can be chosen so that the constant $C_N(\psi_k^{\pm},\mathcal{K})$ defined in (2.17) satisfies

$$C_N(\psi_k^{\pm},\mathcal{K}) \ll k^N \tag{2.22}$$

for some implicit constant depending only on $N\geq 1$. Given $k\geq k_0$, set

$$\widetilde{\mathcal{N}}_{P,k}^{-}(T,\alpha)=\widetilde{\mathcal{N}}_P(\psi_k^{-},T,\alpha) \quad \text{when} \quad E_{(T,\alpha)}=A_{(T/(2\Delta_P(\mathcal{K})),T^{-\alpha})}$$

and

$$\widetilde{\mathcal{N}}_{P,k}^{+}(T,\alpha)=\widetilde{\mathcal{N}}_P(\psi_k^{+},T,\alpha) \quad \text{when} \quad E_{(T,\alpha)}=B_{(T/(2\Delta_P(\mathcal{K})),T^{-\alpha})}$$

so that

$$\widetilde{\mathcal{N}}_{P,k}^{-}(T,\alpha)\leq \mathcal{N}_P^{\dagger}(T,\alpha)\leq \widetilde{\mathcal{N}}_{P,k}^{+}(T,\alpha). \tag{2.23}$$

Given $c\in(0,1)$, let furthermore

$$\nu_P(K,c)=\int_K \chi_{\{|P(\boldsymbol{x})|\leq c\}}\,\mathrm{d}\boldsymbol{x} \tag{2.24}$$

and

$$\nu_P^{\pm}(k,c)=\mu_P(\psi_k^{\pm},c)=\int_{\mathbb{R}^n}\chi_{\{|P(\boldsymbol{x}|\leq c\}}\cdot\psi_k^{\pm}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Then for any $k\geq 1$,

$$\begin{aligned}|\widetilde{\mathcal{N}}_{P,k}^{\pm}(T,\alpha)-\mathrm{Vol}_n(\mathcal{S}_P^{\dagger}(K,T,\alpha))| &= |\widetilde{\mathcal{N}}_{P,k}^{\pm}(T,\alpha)-T^n\cdot\nu_P(K,T^{-\alpha})|\\ &\underset{(2.21)}{\leq} |\widetilde{\mathcal{N}}_{P,k}^{\pm}(T,\alpha)-T^n\cdot\nu_P^{\pm}(k,T^{-\alpha})|+T^n\cdot|\nu_P^{+}(k,T^{-\alpha})-\nu_P^{-}(k,T^{-\alpha})|.\end{aligned} \tag{2.25}$$

The goal is now to show that for any $c>0$ and any $k\geq k_0$,

$$|\nu_P^{+}(k,c)-\nu_P^{-}(k,c)|\ll\frac{\nu_P^{+}(k_0,c)}{k} \tag{2.26}$$

with an implicit constant independent of k and c. To this end, note that from the inequalities (2.21) and the homogeneity of $P(\boldsymbol{x})$, an elementary change of variables leads to

$$\begin{aligned}|\nu_P^{+}(k,c)-\nu_P^{-}(k,c)| &\leq \int_{K(1+1/k)\setminus K(1-1/k)}\chi_{\{|P(\boldsymbol{x})|\leq c\}}\,\mathrm{d}\boldsymbol{x}\\ &=\left(1+\frac{1}{k}\right)^n\cdot\int_K\chi_{\{|P(\boldsymbol{y})|\leq c\cdot(1+1/k)^{-q}\}}\,\mathrm{d}\boldsymbol{y}\\ &\quad-\left(1-\frac{1}{k}\right)^n\cdot\int_K\chi_{\{|P(\boldsymbol{y})|\leq c\cdot(1-1/k)^{-q}\}}\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

Expanding the nth powers multiplying the integrals, rewriting the whole expression as a polynomial in $1/k$ and isolating the constant term in this polynomial, one obtains

$$\begin{aligned}|\nu_P^{+}(k,c)-\nu_P^{-}(k,c)| &\ll \int_K\chi_{\{c\cdot(1+1/k)^{-q}<|P(\boldsymbol{x})|\leq c\cdot(1-1/k)^{-q}\}}\,\mathrm{d}\boldsymbol{x}\\ &\quad+\frac{1}{k}\cdot\int_K\chi_{\{|P(\boldsymbol{x})|\leq 2^q\cdot c\}}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

To bound this quantity from above, note that the inequality $\chi_K \leq \psi^+_{k_0}$ implies on the one hand that

$$\int_K \chi_{\{c\cdot(1+1/k)^{-q}<|P(\boldsymbol{x})|\leq c\cdot(1-1/k)^{-q}\}}\,\mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{R}^n} \chi_{\{c\cdot(1+1/k)^{-q}<|P(\boldsymbol{x})|\leq c\cdot(1-1/k)^{-q}\}}\cdot\psi^+_{k_0}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

and on the other hand that

$$\int_K \chi_{\{|P(\boldsymbol{x})|\leq 2^q\cdot c\}}\,\mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{R}^n} \chi_{\{|P(\boldsymbol{x})|\leq 2^q\cdot c\}}\cdot\psi^+_{k_0}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Consequently, it follows from the definition of $\nu^+_P(k_0,c)$ that

$$|\nu^+_P(k,c)-\nu^-_P(k,c)| \ll \nu^+_P\left(k_0, c\cdot\left(1-\frac{1}{k}\right)^{-q}\right) - \nu^+_P\left(k_0, c\cdot\left(1+\frac{1}{k}\right)^{-q}\right) + \frac{\nu^+_P(k_0, c\cdot 2^{-q})}{k}.$$

Since the set K contains the origin in its interior, one infers from (2.21) combined with [3, Chapter 2, Lemma 2.7] that the pair $(r_P(\psi^\pm_k), m_P(\psi^\pm_k))$ is independent of K and $k\geq 1$ — denote this common value by (r_P, m_P). One then finds from the asymptotic expansion (2.14) that

$$|\nu^+_P(k,c)-\nu^-_P(k,c)| \ll \nu^+_P(k_0,c)\cdot\left(\left(1-\frac{1}{k}\right)^{-qr_P} - \left(1+\frac{1}{k}\right)^{-qr_P} + \frac{1}{k}\right) \ll \frac{\nu^+_P(k_0,c)}{k},$$

where the implicit constant depends on the polynomial $P(\boldsymbol{x})$ and on $\varepsilon>0$ appearing in (2.14). This establishes (2.26).

Going back to the inequalities (2.25), it then follows from Proposition 2.4 that

$$|\widetilde{\mathcal{N}}^\pm_{P,k}(T,\alpha) - \mathrm{Vol}_n(\mathcal{S}^\dagger_P(K,T,\alpha))| \ll T^n\cdot\left(C_N(\psi^\pm_k,\mathcal{K})^M\cdot\frac{\nu^+_P(k_0,T^{-\alpha})}{T^\delta} + \frac{\nu^+_P(k_0,T^{-\alpha})}{k}\right) \tag{2.27}$$

for some $\delta>0$ and some integers $M,N\geq 1$. Here, the inclusions $\mathrm{Supp}\,\psi^+_{k_0}\subset K(1+1/k_0)\subset K(2)$ yield

$$\nu^+_P(k_0,T^{-\alpha}) \underset{(2.24)}{\leq} \frac{\nu_P(K, 2^{-q}\cdot T^{-\alpha})}{2^n} = \frac{\mathrm{Vol}_n(\mathcal{S}^\dagger_P(K, 2^{q/\alpha}T,\alpha))}{2^n\cdot(2^{q/\alpha}T)^n} \ll \frac{\mathrm{Vol}_n(\mathcal{S}^\dagger_P(K,T,\alpha))}{T^n} = \nu_P(K,T^{-\alpha}),$$

where the second last relation is easily deduced from the explicit volume estimate obtained in Case (3) of Theorem 1.1. It is then a consequence of (2.22) and (2.27) that

$$|\widetilde{\mathcal{N}}^\pm_{P,k}(T,\alpha) - \mathrm{Vol}_n(\mathcal{S}^\dagger_P(K,T,\alpha))| \ll \mathrm{Vol}_n(\mathcal{S}^\dagger_P(K,T,\alpha))\cdot\left(\frac{k^{MN}}{T^\delta}+\frac{1}{k}\right).$$

In view of the inequalities (2.23), specialising k to be the integer part of $T^{\delta/(1+MN)}$ provides the sought power saving. This completes the deduction of Theorem 2.1 from Proposition 2.4 when $\alpha \in (0,1)$. ∎

The proof of Proposition 2.4 relies on the Poisson summation formula, which implies that the counting function (2.11) can be expanded as

$$\widetilde{\mathcal{N}}_P(\psi, T, \alpha) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \widehat{G}_{T,\alpha}^{(P,\psi)}(\boldsymbol{k}). \tag{2.28}$$

Here, $\widehat{G}_{T,\alpha}^{(P,\psi)}$ denotes the Fourier transform of the map $G_{T,\alpha}^{(P,\psi)}$ defined in (2.12). Explicitly, an elementary change of variables shows that, given $\boldsymbol{\xi} \in \mathbb{R}^n$,

$$\widehat{G}_{T,\alpha}^{(P,\psi)}(\boldsymbol{\xi}) = T^n \cdot \widehat{H}_{T,\alpha}^{(P,\psi)}(\boldsymbol{\xi}), \tag{2.29}$$

where

$$\widehat{H}_{T,\alpha}^{(P,\psi)}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} \mathrm{e}(-T\boldsymbol{\xi}\cdot\boldsymbol{x}) \cdot E_{(T,\alpha)}(P(\boldsymbol{x})) \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{2.30}$$

The term $\widehat{G}_{T,\alpha}^{(P,\psi)}(\mathbf{0})$ obtained when $\boldsymbol{k} = \mathbf{0}$ will be referred to as the *leading term* in the sum (2.28), and the remaining sum as the *error term*. These two quantities are analysed separately, starting with the latter.

2.2. Nonstationary phase analysis and estimate of the error term. The following claim essentially establishes that the error term has fast decay in T.

LEMMA 2.5 (Asymptotic behaviour of the error term). *Let $\alpha \in (0,1)$ and $\eta \in (0,1)$. Fix an integer $j \geq 1$ such that $j\eta > n$. Assume that ψ is a smooth map supported in the compact set $\mathcal{K}$ introduced in* (2.9) *and that $T > 2^{1/(1-\eta)}$. Then*

$$\Bigg|\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^n\\ \boldsymbol{k}\neq\mathbf{0}}} \widehat{G}_{T,\alpha}^{(P,\psi)}(\boldsymbol{k})\Bigg| \ll C_j(\psi,\mathcal{K}) \cdot T^{n-j\eta} \cdot \log T,$$

where the constant $C_j(\psi,\mathcal{K})$ is defined in (2.17) *and where the implicit constant does not depend on ψ.*

The proof of this statement relies on the following effective nonstationary phase estimate which can be found in [27, Theorem 7.7.1].

PROPOSITION 2.6 (Effective nonstationary phase estimate). *Let $j \geq 1$ be an integer and let ψ be in $\mathcal{C}_c^j(\mathbb{R}^n)$ with support contained in a compact set $\mathcal{K}$. Assume that the real-valued function f is $j+1$ times continuously differentiable in an open neighbourhood $\mathcal{V}$ of $\mathcal{K}$. Then for any real parameter $\lambda > 0$,*

$$\left|\int_{\mathbb{R}^n} e(\lambda f(\boldsymbol{x})) \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| \ll \frac{C_j(\psi,\mathcal{K})}{\lambda^j \cdot \delta_\psi(f)^j \cdot \min\{1, \delta_\psi(f)^j\}}$$

assuming that the quantity

$$\delta_\psi(f) = \inf_{\boldsymbol{x}\in\mathcal{K}} \|\nabla f(\boldsymbol{x})\|$$

is strictly positive. In the above inequality, the implicit constant is independent of ψ and λ; furthermore, it remains bounded as long as the function f remains in a bounded subset of the set of $j+1$ times continuously differentiable functions on $\mathcal{V}$.

Proof of Lemma 2.5. Fix $\boldsymbol{k}\neq\mathbf{0}$ in $\mathbb{Z}^n$. Then by Fourier inversion,

$$\widehat{H}_{T,\alpha}^{(P,\psi)}(\boldsymbol{k})=\int_{-T/(2\Delta_P(\mathcal{K}))}^{T/(2\Delta_P(\mathcal{K}))}\widehat{E}_{(T,\alpha)}(t)\cdot\left(\int_{\mathbb{R}^n}e(tP(\boldsymbol{x})-T\boldsymbol{k}\cdot\boldsymbol{x})\cdot\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t, \tag{2.31}$$

where the real $\Delta_P(\mathcal{K})$ is defined in (2.10) and $\widehat{H}_{T,\alpha}^{(P,\psi)}(\boldsymbol{k})$ in (2.30). Fix $t\in[-T/(2\Delta_P(\mathcal{K})),T/(2\Delta_P(\mathcal{K}))]$. Then given $\boldsymbol{x}\in\operatorname{Supp}\psi\subset\mathcal{K}$,

$$\|t\cdot\nabla P(\boldsymbol{x})-T\boldsymbol{k}\|\geq T\cdot\|\boldsymbol{k}\|-|t|\cdot\|\nabla P(\boldsymbol{x})\|\underset{(2.10)}{\geq}T\cdot\|\boldsymbol{k}\|-|t|\cdot\Delta_P(\mathcal{K}). \tag{2.32}$$

As a consequence, $\|t\cdot\nabla P(\boldsymbol{x})-T\boldsymbol{k}\|\geq(T\cdot\|\boldsymbol{k}\|)^\eta$ as soon as

$$\|\boldsymbol{k}\|\geq\frac{\|\boldsymbol{k}\|^\eta}{T^{1-\eta}}+\frac{1}{2}\cdot$$

Since $T\geq 2^{1/(1-\eta)}$, this condition is fulfilled for all $\boldsymbol{k}\in\mathbb{Z}^n\backslash\{\mathbf{0}\}$. Proposition 2.6 (applied with $\lambda=1$) then implies that for any integer $j\geq1$,

$$\left|\int_{\mathbb{R}^n}e(tP(\boldsymbol{x})-T\boldsymbol{k}\cdot\boldsymbol{x})\cdot\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right|\ll\frac{C_j(\psi,\mathcal{K})}{(T\cdot\|\boldsymbol{k}\|)^{j\eta}}\cdot \tag{2.33}$$

Under the assumption $j\eta>n$, one thus obtains

$$\begin{aligned}\Big|\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^n\\ \boldsymbol{k}\neq\mathbf{0}}}\widehat{G}_{T,\alpha}^{(P,\psi)}(\boldsymbol{k})\Big| &\underset{(2.29)}{\leq} T^n\cdot\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^n\\ \boldsymbol{k}\neq\mathbf{0}}}|\widehat{H}_{T,\alpha}^{(P,\psi)}(\boldsymbol{k})|\\ &\underset{(2.31)\,\&\,(2.33)}{\ll} C_j(\psi,\mathcal{K})\cdot T^{n-j\eta}\cdot\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^n\\ \boldsymbol{k}\neq\mathbf{0}}}\|\boldsymbol{k}\|^{-j\eta}\Big)\\ &\qquad\times\left(\int_{|t|\leq T/(2\Delta_P(\mathcal{K}))}|\widehat{E}_{(T,\alpha)}(t)|\,\mathrm{d}t\right)\\ &\ll C_j(\psi,\mathcal{K})\cdot T^{n-j\eta}\cdot\left(\int_{|t|\leq T/(2\Delta_P(\mathcal{K}))}|\widehat{E}_{(T,\alpha)}(t)|\,\mathrm{d}t\right).\end{aligned} \tag{2.34}$$

To evaluate this last factor, let $t\in[-T/(2\Delta_P(\mathcal{K})),T/(2\Delta_P(\mathcal{K}))]$. From the elementary estimate

$$|\widehat{\chi}_{I(T^{-\alpha})}(t)|=\left|\frac{\sin(2\pi T^{-\alpha}t)}{\pi t}\right|\leq\min\left\{T^{-\alpha},\frac{1}{|\pi t|}\right\},$$

one deduces that

$$\begin{aligned}|\widehat{E}_{(T,\alpha)}(t)|&\leq|\widehat{\chi}_{I(T^{-\alpha})}(t)|+|(\widehat{E}_{(T,\alpha)}-\widehat{\chi}_{I(T^{-\alpha})})(t)|\\ &\leq|\widehat{\chi}_{I(T^{-\alpha})}(t)|+\int_{\mathbb{R}}|E_{(T,\alpha)}(x)-\chi_{I(T^{-\alpha})}(x)|\,\mathrm{d}x\\ &\underset{(2.7)\&(2.8)}{\leq}\min\left\{2,\frac{1}{|t|}\right\}+\int_{\mathbb{R}}\theta\left(\frac{T}{2\Delta_P(\mathcal{K})}\cdot\big||x|-T^{-\alpha}\big|\right)\mathrm{d}x\\ &\ll\min\left\{2,\frac{1}{|t|}\right\}+\frac{1}{T}\end{aligned}$$

with an implicit constant depending on the integral of θ over its domain of definition. Inserting this estimate in (2.34) then concludes the proof. ■

2.3. Analysis of the leading term. The estimation of the leading term $\widehat{G}_{T,\alpha}^{(P,\psi)}(\mathbf{0})$ in the sum (2.28) reduces to the following claim:

LEMMA 2.7 (Asymptotic behaviour of the leading term). *Keep the assumptions and the notations of Proposition 2.4, assuming in particular that $\alpha \in (0,1)$. Then there exist integers $M, N \geq 1$ such that for any $\delta \in (0, 1-\alpha)$,*

$$\left|\widehat{H}_{T,\alpha}^{(P,\psi)}(\mathbf{0}) - \mu_P(\psi, T^{-\alpha})\right| \ll (C_N(\psi, \mathcal{K}))^M \cdot \frac{\mu_P(\psi, T^{-\alpha})}{T^{\delta}}.$$

In this inequality, the quantity $\widehat{H}_{T,\alpha}^{(P,\psi)}(\mathbf{0})$ is defined in (2.30)*, the map $\mu_P(\psi, \cdot\,)$ in* (2.13) *and the constant $C_N(\psi, \mathcal{K})$ in* (2.17)*. Also, the implicit constant is independent of ψ.*

Proof. It follows from Theorem 2.2 that

$$\begin{aligned}
&\left|\widehat{H}_{T,\alpha}^{(P,\psi)}(\mathbf{0}) - \mu_P(\psi, T^{-\alpha})\right| \\
&\quad \leq \int_{\mathbb{R}^n} \left|B_{(T/(2\Delta_P(\mathcal{K})),T^{-\alpha})}(P(\boldsymbol{x})) - A_{(T/(2\Delta_P(\mathcal{K})),T^{-\alpha})}(P(\boldsymbol{x}))\right| \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&\quad \leq \int_{\mathbb{R}^n} \theta\left(\frac{T}{2\Delta_P(\mathcal{K})} \cdot \left||P(\boldsymbol{x})| - T^{-\alpha}\right|\right) \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \qquad (2.35)
\end{aligned}$$

where the constant $\Delta_P(\mathcal{K})$ is defined in (2.10).

Let $\delta > 0$ be a small exponent, the range of which is to be determined. Define

$$\mathfrak{S}_P(T, \alpha) = \left\{\boldsymbol{x} \in \mathbb{R}^n : \left||P(\boldsymbol{x})| - T^{-\alpha}\right| \leq \frac{2 \cdot \Delta_P(\mathcal{K})}{T^{1-\delta}}\right\}$$

and decompose the integral in (2.35) as

$$\begin{aligned}
&\int_{\mathbb{R}^n} \theta\left(\frac{T}{2\Delta_P(\mathcal{K})} \cdot \left||P(\boldsymbol{x})| - T^{-\alpha}\right|\right) \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&\quad = \int_{\mathbb{R}^n} \chi_{\mathfrak{S}_P(T,\alpha)}(\boldsymbol{x}) \cdot \theta\left(\frac{T}{2\Delta_P(\mathcal{K})} \cdot \left||P(\boldsymbol{x})| - T^{-\alpha}\right|\right) \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&\qquad + \int_{\mathbb{R}^n} (1 - \chi_{\mathfrak{S}_P(T,\alpha)}(\boldsymbol{x})) \cdot \theta\left(\frac{T}{2\Delta_P(\mathcal{K})} \cdot \left||P(\boldsymbol{x})| - T^{-\alpha}\right|\right) \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \qquad (2.36)
\end{aligned}$$

The first integral on the right-hand side of this equation can be bounded as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \chi_{\mathfrak{S}_P(T,\alpha)}(\boldsymbol{x}) \cdot \theta\left(\frac{T}{2\Delta_P(\mathcal{K})} \cdot \left||P(\boldsymbol{x})| - T^{-\alpha}\right|\right) \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&\qquad \leq \left(\max_{x \geq 0} \theta(x)\right) \cdot \int_{\mathbb{R}^n} \chi_{\mathfrak{S}_P(T,\alpha)}(\boldsymbol{x}) \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \qquad (2.37)
\end{aligned}$$

Introducing the map $\mu_P(\psi, \cdot\,)$ defined in (2.13), the last integral becomes

$$\begin{aligned}
&\int_{\mathbb{R}^n} \chi_{\mathfrak{S}_P(T,\alpha)}(\boldsymbol{x}) \cdot \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&\qquad = \mu_P\left(\psi, \frac{1}{T^{\alpha}} + \frac{2\Delta_P(\mathcal{K})}{T^{1-\delta}}\right) - \mu_P\left(\psi, \frac{1}{T^{\alpha}} - \frac{2\Delta_P(\mathcal{K})}{T^{1-\delta}}\right) \\
&\qquad \underset{(2.16)\&(2.18)}{\ll} (C_N(\psi, \mathcal{K}))^M \cdot \frac{1}{T^{1-\delta}} \cdot \frac{(\log T)^m}{T^{\alpha(\rho-1)}} \qquad (2.38)
\end{aligned}$$

with an implicit constant depending on the parameters α, ρ, m and on the polynomial $P(\boldsymbol{x})$. Here, the inequality (2.38) holds as long as

$$0<\delta<1-\alpha \tag{2.39}$$

since this condition guarantees that $T^{-1+\delta}=o(T^{-\alpha})$ as T tends to infinity. Under this assumption, the upper bound (2.37) implies that

$$\int_{\mathbb{R}^n}\chi_{\mathfrak{S}_P(T,\alpha)}(\boldsymbol{x})\cdot\theta\left(\frac{T}{2\Delta_P(\mathcal{K})}\cdot\big||P(\boldsymbol{x})|-T^{-\alpha}\big|\right)\cdot\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \ll (C_N(\psi,\mathcal{K}))^M\cdot\frac{(\log T)^m}{T^{1-\delta+\alpha(\rho-1)}}\cdot \tag{2.40}$$

As for the second term on the right-hand side of (2.36), one infers from the definition of the set $\mathfrak{S}_P(T,\alpha)$ and from the fast decay of the function θ that for any large $\beta>0$,

$$\int_{\mathbb{R}^n}(1-\chi_{\mathfrak{S}_P(T,\alpha)}(\boldsymbol{x}))\cdot\theta\left(\frac{T}{2\Delta_P(\mathcal{K})}\cdot\big||P(\boldsymbol{x})|-T^{-\alpha}\big|\right)\cdot\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\ll T^{-\beta\delta} \tag{2.41}$$

with an implicit constant depending only on β and on the volume of the compact set $\mathcal{K}$ containing the support of ψ (recall that ψ is assumed to meet the conditions (2.9)).

Under the restriction (2.39), one can choose $\beta>0$ large enough so that inequalities (2.40) and (2.41) imply that

$$|\widehat{H}_{T,\alpha}^{(P,\psi)}(\mathbf{0})-\mu_P(\psi,T^\alpha)|\ll\frac{(C_N(\psi,\mathcal{K}))^M}{T^{1-\delta+\alpha(\rho-1)-\eta}}.$$

Since (2.39) also guarantees that $1+\alpha(\rho-1)-\delta>\alpha\rho$ when $\alpha<1$, in view of the asymptotic expansion (2.16), this provides the required power saving in the error term upon choosing $\eta>0$ small enough. This completes the proof of the lemma. ∎

Completion of the proof of Proposition 2.4. In view of the Poisson summation formula (2.28), the sought estimate (2.20) is an immediate consequence of Lemmata 2.5 and 2.7 when $\alpha\in(0,1)$. ∎

3. Counting in thin domains

Throughout this final chapter, let $\alpha>1$ be a real and let $\boldsymbol{F}(\boldsymbol{x})=(F_1(\boldsymbol{x}),\dots,F_p(\boldsymbol{x}))$ be a p-tuple of homogeneous forms in $n\ge 2$ variables, each of degree $d\ge 2$. Assume that the set $K\subset\mathbb{R}^n$ meets the conditions $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ stated in the Introduction and that it is furthermore semialgebraic (recall that this means that it is a finite union of sets that can be defined by finitely many equalities and inequalities involving polynomial maps). Given a large parameter $T\ge 1$, recall also the definitions of the set $\mathcal{S}_{\boldsymbol{F}}^\dagger(K,T,\alpha)$ and of the corresponding counting function $\mathcal{N}_{\boldsymbol{F}}^\dagger(K,T,\alpha)$ given in Chapter 1:

$$\mathcal{S}_{\boldsymbol{F}}^\dagger(K,T,\alpha)=\{\boldsymbol{x}\in T\cdot K\,:\,\|\boldsymbol{F}(\boldsymbol{x})\|\le T^{d-\alpha}\}, \tag{3.1}$$

$$\mathcal{N}_{\boldsymbol{F}}^\dagger(K,T,\alpha)=\#(\mathcal{S}_{\boldsymbol{F}}^\dagger(K,T,\alpha)\cap\mathbb{Z}^n). \tag{3.2}$$

The main goal of this final chapter is to establish Theorem 3.1 below, which includes Theorem 1.2 stated in the Introduction as a particular case when $\alpha>1$ (the case $\alpha\le 1$ being settled in the previous chapter).

To this end, first are recalled further notations and definitions introduced in Chapter 1. When $\boldsymbol{v} \in \mathbb{S}^{n-1}$ and $\sigma \in \mathbb{R}$, we consider the affine hyperplane

$$\boldsymbol{v}^{\perp}(\sigma) := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{v} \cdot \boldsymbol{x} = \sigma\}. \tag{3.3}$$

Fixing a compact set $\mathcal{K}$ containing K in its interior and contained in the open set U defined as part of the support restriction condition (1.10), let us set, for any $\varepsilon > 0$,

$$\mu_{\boldsymbol{F},\mathcal{K}}(\boldsymbol{v}, \sigma, \varepsilon) = \mathrm{Vol}_{n-1}\left(\{\boldsymbol{x} \in \mathcal{K} \cap \boldsymbol{v}^{\perp}(\sigma) : \|\boldsymbol{F}(\boldsymbol{x})\| \leq \varepsilon\}\right)$$

and

$$M_{\boldsymbol{F}}(\mathcal{K}, \varepsilon) = \sup_{\boldsymbol{v} \in \mathbb{S}^{n-1}} \sup_{\sigma \in \mathbb{R}} \mu_{\boldsymbol{F},\mathcal{K}}(\boldsymbol{v}, \sigma, \varepsilon). \tag{3.4}$$

The "biggest level of flatness" that can be achieved when intersecting the sublevel set $\{\boldsymbol{x} \in \mathcal{K} : |\boldsymbol{F}(\boldsymbol{x})| \leq \varepsilon\}$ with affine hyperplanes is quantified by

$$q_{\boldsymbol{F}}(\mathcal{K}) = \liminf_{\varepsilon \to 0^+} \left(\frac{\log M_{\boldsymbol{F}}(\mathcal{K}, \varepsilon)}{\log \varepsilon}\right). \tag{3.5}$$

This is a well-defined real number under the assumption $(\mathcal{H}_2)$.

The dimension $\dim(\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F}))$ of the algebraic variety $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$ over the set $\mathcal{K}$ is denoted by $\tau_{\boldsymbol{F}}(\mathcal{K})$ and its codimension by $\widehat{\tau}_{\boldsymbol{F}}(\mathcal{K})$, that is, $\widehat{\tau}_{\boldsymbol{F}}(\mathcal{K}) = n - \dim(\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F}))$. Recall that if the variety $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ has smooth complete intersection, one has $\widehat{\tau}_{\boldsymbol{F}}(\mathcal{K}) = p$.

The main result in this chapter can now be stated as follows:

THEOREM 3.1. *Let $\alpha > 1$ be a real. Assume that K is a semialgebraic set meeting the assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ stated in the Introduction. Let it be contained in the interior of a compact set $\mathcal{K} \subset U$, where the open set U is defined in the support restriction condition* (1.10)*. Also, let $r_{\boldsymbol{F}}(K)$ be the nonnegative real introduced in $(\mathcal{H}_3)$. Then the following claims are satisfied:*

(Z1) *The upper bound*

$$\mathcal{N}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha) \ll \mathrm{Vol}_n(\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)) + T^{\tau_{\boldsymbol{F}}(\mathcal{K}) - \delta_{\boldsymbol{F}}(\mathcal{K}, K)} \tag{3.6}$$

holds for some exponent $\delta_{\boldsymbol{F}}(\mathcal{K}, K) > 0$ depending on the polynomial $P(\boldsymbol{x})$ and on the compact sets $K \subset \mathcal{K}$ if either of the following two mutually exclusive assumptions is satisfied:

(i) *In the generic case when*

$$q_{\boldsymbol{F}}(\mathcal{K}) > r_{\boldsymbol{F}}(K) - 1, \tag{3.7}$$

the semialgebraic level of flatness $q_{\boldsymbol{F}}(\mathcal{K})$ is bounded below in this sense:

$$q_{\boldsymbol{F}}(\mathcal{K}) > n - 1 - r_{\boldsymbol{F}}(K) \cdot \frac{\tau_{\boldsymbol{F}}(\mathcal{K})}{\widehat{\tau}_{\boldsymbol{F}}(\mathcal{K})}. \tag{3.8}$$

One can then choose any exponent $\delta_{\boldsymbol{F}}(K, \mathcal{K})$ such that

$$0 < \delta_{\boldsymbol{F}}(K, \mathcal{K}) < \tau_{\boldsymbol{F}}(\mathcal{K}) - \frac{n \cdot (n - 1 - q_{\boldsymbol{F}}(\mathcal{K}))}{n - 1 - q_{\boldsymbol{F}}(\mathcal{K}) + r_{\boldsymbol{F}}(K)}. \tag{3.9}$$

(ii) *In the degenerate case when*

$$q_{\boldsymbol{F}}(\mathcal{K}) \leq r_{\boldsymbol{F}}(K) - 1, \tag{3.10}$$

$q_{\boldsymbol{F}}(\mathcal{K})$ is bounded below in this sense:

$$q_{\boldsymbol{F}}(\mathcal{K}) > \widehat{\tau}_{\boldsymbol{F}}(\mathcal{K}) - 1. \tag{3.11}$$

One can then choose any exponent $\delta_{\boldsymbol{F}}(K, \mathcal{K})$ such that

$$0 < \delta_{\boldsymbol{F}}(\mathcal{K}, K) < q_{\boldsymbol{F}}(\mathcal{K}) - \widehat{\tau}_{\boldsymbol{F}}(\mathcal{K}) + 1. \tag{3.12}$$

In (3.6)*, the volume term on the right-hand side is explicitly determined by Case* (2) *in Theorem* 1.1*.*

(Z2) *When the algebraic variety $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$ is not contained in a linear subspace of dimension $n-p$ and furthermore the set of homogeneous forms $\boldsymbol{F}(\boldsymbol{x})$ has smooth complete intersection over $\mathcal{K}$ (meaning that the map* (1.8) *does not vanish over $\mathcal{K}$), then the conditions* (3.7) *and* (3.8) *are both met. As a consequence, an estimate of the form* (3.6) *indeed holds.*

Claim (Z2) is established in Section 3.3.2 below. Most of the work in proving the theorem is in establishing Claim (Z1). To this end, it is enough to consider the case of $p = 1$ homogeneous form, say $P(\boldsymbol{x})$, of degree denoted by $q \geq 1$. The general case stated in (Z1) then follows upon applying the results obtained in this particular situation to the polynomial $P_{\boldsymbol{F}}(\boldsymbol{x}) = \|\boldsymbol{F}(\boldsymbol{x})\|^2$ and upon squaring all relations involving this polynomial. Explicitly, it suffices to replace in what follows the polynomial $P(\boldsymbol{x})$ with $P_{\boldsymbol{F}}(\boldsymbol{x})$, the real $\varepsilon > 0$ with ε^2, the integer q with $2d$ and the real $\alpha > 1$ with 2α. The details of this elementary verification are left to the reader.

In view of this reduction, it is natural to establish (Z1) employing the notations corresponding to the analogues in the case $p = 1$ of the set $\mathcal{S}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)$ and of the counting function $\mathcal{N}^{\dagger}_{\boldsymbol{F}}(K, T, \alpha)$. These are respectively the set $\mathcal{S}^{\dagger}_{P}(K, T, \alpha)$ and the counting function $\mathcal{N}^{\dagger}_{P}(K, T, \alpha)$ introduced in Chapter 2 – see (2.1) and (2.2).

In order to establish (Z1), let $j \geq 1$ be an integer and let $\varphi, \psi \in \mathcal{C}^{j}_{c}(\mathbb{R}^n)$ be such that

$$\chi_{[-1,1]} \leq \varphi \leq \chi_{[-2,2]} \quad \text{and} \quad \chi_K \leq \psi \leq \chi_{\mathcal{K}}. \tag{3.13}$$

As the proof relies on semialgebraic geometry, the maps φ and ψ are also assumed to be semialgebraic (meaning that their graphs are semialgebraic sets). The existence of such maps is guaranteed, for instance, by [34, Proposition 4.8].

Fix

$$\varepsilon \geq T^{-\alpha}. \tag{3.14}$$

It then follows from the Poisson summation formula and from the homogeneity of the polynomial $P(\boldsymbol{x})$ that

$$\begin{aligned}
\mathcal{N}^{\dagger}_{P}(K, T, \alpha) &\leq \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \psi\left(\frac{\boldsymbol{k}}{T}\right) \cdot \varphi\left(\frac{P(\boldsymbol{k})}{\varepsilon \cdot T^q}\right) \\
&= T^n \cdot \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(-T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\
&= T^n \cdot \int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x} \\
&\quad + T^n \cdot \sum_{\boldsymbol{k} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}} \int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(-T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.15}$$

Upon specialising the real $\varepsilon > 0$ and the maps φ and ψ, the integral in the former term in this last sum is related in Section 3.3 to the volume of the set of points $\boldsymbol{x} \in K$ such that $|P(\boldsymbol{x})| < T^{-\alpha}$. The analysis of the oscillatory integrals and of the resulting sum in the latter term in (3.15) constitutes the main substance of the proof and will be carried out along these lines: by a nonstationary phase analysis developed in Section 3.1 below, the sum is first reduced to a finite one up to an error term with rapid decay in $\|T\boldsymbol{k}\|$, provided that the integer $j \geq 1$ defining the regularity of φ and ψ is large enough. To deal with this finite sum, upon decomposing in polar coordinates the nonzero vector $T\boldsymbol{k}$ as

$$-T\boldsymbol{k} = \lambda \boldsymbol{v}, \quad \text{where} \quad \lambda > 0 \text{ and } \boldsymbol{v} \in \mathbb{S}^{n-1}, \tag{3.16}$$

the goal is to obtain, under suitable assumptions, a uniform decay estimate for the Fourier coefficient appearing in (3.15), namely an estimate of the form

$$\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(\lambda \boldsymbol{v} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \ll \frac{\varepsilon^{\nu}}{\lambda^{\gamma}} \tag{3.17}$$

for some positive exponents γ and ν. The crucial requirement here, which prevents the reduction of the problem to a (more or less standard) local harmonic analysis estimate of oscillatory integrals, is that the implicit constant in the inequality above must be independent of $\boldsymbol{v} \in \mathbb{S}^{n-1}$. This issue is overcome in Section 3.2 with the help of o-minimal theory. The sum appearing in (3.15), when taken over finitely many nonzero integer vectors $\boldsymbol{k}$, can then be estimated upon specialising λ to the value $\|T\boldsymbol{k}\|$ in (3.17) and upon optimising the value of ε under the constraint (3.14). This will be seen to imply Theorem 3.1.

3.1. Nonstationary phase analysis and reduction to a finite sum. This section deals with the problem of reducing the infinite sum appearing in (3.15) to a finite one through a nonstationary phase analysis.

LEMMA 3.2. *Let $\beta > 0$ be a large parameter and let $\kappa \in (0,1)$ be a small one. Fix $T \geq 2$ and $\varepsilon \in (0, 1/T)$ and assume that $\varphi \geq 0$ is in $\mathcal{C}_c^j(\mathbb{R})$ and $\psi \geq 0$ in $\mathcal{C}_c^j(\mathbb{R}^n)$, where*

$$j \geq 1 + \frac{n + \beta \cdot (1 - \kappa)}{\kappa}. \tag{3.18}$$

Then there exists a constant $a_P(\psi, \kappa) > 0$ depending on κ and on the sup norm of the polynomial $P(\boldsymbol{x})$ over the support of ψ such that

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}} \int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(-T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$
$$\ll (\varepsilon T)^{\beta} + \sum_{1 \leq \|\boldsymbol{k}\| \leq a_P(\psi,\kappa) \cdot (T\varepsilon)^{-1/(1-\kappa)}} \left| \int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(-T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right|.$$

Here, the implicit constant depends on j, β, κ and on the weight functions ψ and φ.

Proof. Let $\boldsymbol{k} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and $L \geq 1$. By Fourier inversion,

$$\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(-T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

$$= \underbrace{\int_{|t| \leq L} \hat{\varphi}(t) \cdot \left(\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot e(t\varepsilon^{-1} \cdot P(\boldsymbol{x}) - T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t}_{=I_1(L)}$$

$$+ \underbrace{\int_{|t| > L} \hat{\varphi}(t) \cdot \left(\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot e(t\varepsilon^{-1} \cdot P(\boldsymbol{x}) - T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t}_{=I_2(L)}.$$

Since φ is in $\mathcal{C}_c^j(\mathbb{R})$, repeated integration by parts shows that its Fourier transform $t \mapsto \hat{\varphi}(t)$ decays as $O(t^{-j})$ when $t \to \infty$. As a consequence,

$$|I_2(L)| \leq \left(\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right) \cdot \left(\int_{|t|>L} |\hat{\varphi}(t)| \mathrm{d}t \right) \ll L^{-j+1}. \tag{3.19}$$

As for the integral $I_1(L)$, it can be estimated with the help of the effective nonstationary phase result stated in Proposition 2.6 (upon letting $\lambda = 1$ and $f(\boldsymbol{x}) = t\varepsilon^{-1} \cdot P(\boldsymbol{x}) - T\boldsymbol{k} \cdot \boldsymbol{x}$ therein). To this end, given $t \in [-L, L]$ and $\boldsymbol{x} \in \operatorname{Supp} \psi$, note first that, upon setting $\Delta_P(\mathcal{K}) = \max_{\boldsymbol{x} \in \mathcal{K}} \|\nabla P(\boldsymbol{x})\|$,

$$\|t\varepsilon^{-1} \cdot \nabla P(\boldsymbol{x}) - T\boldsymbol{k}\| \underset{(3.13)}{\geq} T\|\boldsymbol{k}\| - L\varepsilon^{-1} \cdot \Delta_P(\mathcal{K}). \tag{3.20}$$

Under the assumption that $T \geq 2$, this quantity is larger than $(T\|\boldsymbol{k}\|)^\kappa$ whenever

$$\|\boldsymbol{k}\| \geq \frac{L \cdot \Delta_P(\mathcal{K})}{T\varepsilon \cdot (1 - 2^{-(1-\kappa)})}. \tag{3.21}$$

Proposition 2.6 then yields

$$|I_1(L)| \ll (T\|\boldsymbol{k}\|)^{-j\kappa} \tag{3.22}$$

with an implicit constant depending on ψ and j. Specialise the inequalities (3.19) and (3.20) to $L = \|\boldsymbol{k}\|^\kappa$ and note that the condition (3.21) then amounts to

$$\|\boldsymbol{k}\| \geq a_P(\psi, \kappa) \cdot (T\varepsilon)^{-1/(1-\kappa)}, \quad \text{where} \quad a_P(\psi, \kappa) = \left(\frac{\Delta_P(\mathcal{K})}{1 - 2^{-(1-\kappa)}} \right)^{1/(1-\kappa)}. \tag{3.23}$$

One then obtains

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}} \int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(-T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

$$\leq \sum_{1 \leq \|\boldsymbol{k}\| \leq a_P(\psi,\kappa) \cdot (T\varepsilon)^{-1/(1-\kappa)}} \left| \int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(-T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right|$$

$$+ \sum_{\|\boldsymbol{k}\| > a_P(\psi,\kappa) \cdot (T\varepsilon)^{-1/(1-\kappa)}} (I_1(\|\boldsymbol{k}\|^\kappa) + I_2(\|\boldsymbol{k}\|^\kappa)).$$

Here, given any integer j satisfying the bound (3.18),

$$\begin{aligned}\sum_{\|\boldsymbol{k}\|>a_P(\psi,\kappa)\cdot(T\varepsilon)^{-1/(1-\kappa)}} & \left(I_1(\|\boldsymbol{k}\|^{\kappa})+I_2(\|\boldsymbol{k}\|^{\kappa})\right)\\ & \underset{(3.19)\&(3.22)}{\ll} \sum_{\|\boldsymbol{k}\|>a_P(\psi,\kappa)\cdot(T\varepsilon)^{-1/(1-\kappa)}} \|\boldsymbol{k}\|^{-\kappa(j-1)}\\ & \ll (T\varepsilon)^{(\kappa\cdot(j-1)-n)/(1-\kappa)}\\ & \underset{(3.18)}{\ll} (T\varepsilon)^{\beta}\end{aligned}$$

since $T\varepsilon<1$ by assumption. This concludes the proof of the lemma. ■

3.2. Geometric tomography on semialgebraic sets. In view of Lemma 3.2 and of the discussion in the introduction to this chapter, the problem is reduced to obtaining uniform decay estimates for the oscillatory integral appearing in (3.15) when the integer vector $\boldsymbol{k}$ varies in a bounded domain. To this end, the integral is first expressed as the Fourier transform of the so-called Gel'fand–Leray function in Section 3.2.1 below. The properties of this function relevant to the proof of Theorem 3.1 are then established in Section 3.2.2.

3.2.1. Uniform Fourier decay of the Gel'fand–Leray function. Define the *Gel'fand–Leray function* g_P depending on $(\boldsymbol{v},\varepsilon,\sigma)\in\mathbb{S}^{n-1}\times\mathbb{R}^2$ and on the weight functions ψ and φ as the map

$$\mathbb{S}^{n-1}\times\mathbb{R}^2\ni(\boldsymbol{v},\varepsilon,\sigma)\mapsto\langle g_P(\boldsymbol{v},\varepsilon,\sigma),(\varphi,\psi)\rangle=\int_{\boldsymbol{v}^{\perp}(\sigma)}\psi(\boldsymbol{x})\cdot\varphi(\varepsilon^{-1}\cdot P(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x},\quad(3.24)$$

where the affine subspace $\boldsymbol{v}^{\perp}(\sigma)$ is defined in (3.3) (this slightly unusual notation for the function g_P takes into account the fact that it depends on both φ and ψ). It satisfies the property that, given $\varepsilon>0$, $\lambda\geq 1$ and $\boldsymbol{v}\in\mathbb{S}^{n-1}$, the oscillatory integral in (3.15) can be decomposed after an elementary change of variables as the Fourier transform of the Gel'fand–Leray function as follows:

$$\int_{\mathbb{R}^n}\psi(\boldsymbol{x})\cdot\varphi(\varepsilon^{-1}\cdot P(\boldsymbol{x}))\cdot e(\lambda\boldsymbol{v}\bullet\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{\mathbb{R}}\langle g_P(\boldsymbol{v},\varepsilon,\sigma),(\varphi,\psi)\rangle\cdot e(\lambda\sigma)\,\mathrm{d}\sigma.\qquad(3.25)$$

This can be seen as a formalisation of the principle of geometric tomography according to which the properties of a sufficiently nice set can be understood from the knowledge of the volumes of its slices with lower-dimensional families of sets.

The main property of interest satisfied by the Gel'fand–Leray function is contained in the following lemma, which is proved in the next section.

LEMMA 3.3. *Assume the weight functions ψ and φ introduced in* (3.13) *are semialgebraic. Then there exists an integer $b_P(\varphi,\psi)\geq 1$ depending on these weights and on the polynomial $P(\boldsymbol{x})$ only such that for any given $\varepsilon>0$ and any $\boldsymbol{v}\in\mathbb{S}^{n-1}$, the real line can be partitioned into at most $b_P(\varphi,\psi)$ intervals on the interior of which, whenever nonempty, the map*

$$\mathbb{R}\ni\sigma\mapsto\langle g_P(\boldsymbol{v},\varepsilon,\sigma),(\varphi,\psi)\rangle\qquad(3.26)$$

is continuously differentiable and monotonic.

This immediately implies the main ingredient in the proof of the counting estimate in Theorem 3.1:

COROLLARY 3.4. *Under the assumptions of the previous lemma, given* $\varepsilon > 0$, *set*

$$M_P(\psi, \varphi, \varepsilon) = \sup_{\boldsymbol{v} \in \mathbb{S}^{n-1}} \sup_{\sigma \in \mathbb{R}} \langle g_P(\boldsymbol{v}, \varepsilon, \sigma), (\varphi, \psi) \rangle.$$

Then

$$\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(\lambda \boldsymbol{v} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \ll \frac{M_P(\psi, \varphi, \varepsilon)}{\lambda} \tag{3.27}$$

for some implicit constant independent of $\boldsymbol{v} \in \mathbb{S}^{n-1}$ *and* $\lambda > 0$.

Proof. Express the oscillatory integral on the left-hand side of (3.27) as the Fourier transform of the Gel'fand–Leray function as in (3.25). Then decompose this Fourier transform as

$$\int_{\mathbb{R}} \langle g_P(\boldsymbol{v}, \varepsilon, \sigma), (\varphi, \psi) \rangle \cdot e(\lambda \sigma) \, \mathrm{d}\sigma = \sum_{j=1}^{c} \left(\int_{I_j} \langle g_P(\boldsymbol{v}, \varepsilon, \sigma), (\varphi, \psi) \rangle \cdot e(\lambda \sigma) \, \mathrm{d}\sigma \right). \tag{3.28}$$

Here, $c = c_P(\varphi, \psi, \boldsymbol{v}, \varepsilon)$ is at most equal to the integer $b_P(\varphi, \psi)$ appearing in the statement of Lemma 3.3 and, for each $1 \le j \le c = c_P(\varphi, \psi, \boldsymbol{v}, \varepsilon)$, the set $I_j = I_j^{(P)}(\varphi, \psi, \boldsymbol{v}, \varepsilon)$ is an interval with nonempty interior where the restriction of the Gel'fand–Leray function seen as a map in σ is continuously differentiable and monotonic.

Fixing j and denoting by $d_j = d_j^{(P)}(\varphi, \psi, \boldsymbol{v}, \varepsilon)$ and $e_j = e_j^{(P)}(\varphi, \psi, \boldsymbol{v}, \varepsilon)$ the endpoints of the interval I_j, one finds by integration by parts that

$$\begin{aligned} &\left| \int_{I_j} \langle g_P(\boldsymbol{v}, \varepsilon, \sigma), (\varphi, \psi) \rangle \cdot e(\lambda \sigma) \, \mathrm{d}\sigma \right| \\ &= \left| \frac{[\langle g_P(\boldsymbol{v}, \varepsilon, \sigma), (\varphi, \psi) \rangle \cdot e(\lambda \sigma)]_{\sigma = d_j}^{\sigma = e_j}}{2 i \pi \lambda} - \frac{1}{2 i \pi \lambda} \cdot \int_{d_j}^{e_j} \frac{\mathrm{d}}{\mathrm{d}\sigma} (\langle g_P(\boldsymbol{v}, \varepsilon, \sigma), (\varphi, \psi) \rangle) \cdot e(\lambda \sigma) \, \mathrm{d}\sigma \right| \\ &\le \frac{\sup_{\sigma \in I_j} |\langle g_P(\boldsymbol{v}, \varepsilon, \sigma), (\varphi, \psi) \rangle|}{\lambda}, \end{aligned}$$

where the last inequality follows from the triangle inequality and the monotonicity of $\sigma \mapsto \langle g_P(\boldsymbol{v}, \varepsilon, \sigma), (\varphi, \psi) \rangle$ restricted to I_j. The definition of $M_P(\psi, \varphi, \varepsilon)$ and the decomposition (3.28) then yield

$$\begin{aligned} \left| \int_{\mathbb{R}} \langle g_P(\boldsymbol{v}, \varepsilon, \sigma), (\varphi, \psi) \rangle \cdot e(\lambda \sigma) \, \mathrm{d}\sigma \right| &\le \frac{c_P(\varphi, \psi, \boldsymbol{v}, \varepsilon) \cdot M_P(\psi, \varphi, \varepsilon)}{\lambda} \\ &\le \frac{b_P(\varphi, \psi) \cdot M_P(\psi, \varphi, \varepsilon)}{\lambda}, \end{aligned}$$

which completes the proof of the statement. ■

3.2.2. Counting intervals of monotonicity through o-minimality. The goal in this section is to establish Lemma 3.3. To this end, recall that an *o-minimal structure* is a collection $\mathfrak{O} = \{\mathcal{O}_k\}_{k \ge 0}$ of families of sets satisfying the following properties:

(i) for each $k \ge 0$, the family $\mathcal{O}_k$ is made up of subsets of $\mathbb{R}^k$ stable under complementation and union and containing the empty set;
(ii) for any integer $k \ge 0$, if $A \in \mathcal{O}_k$, then $A \times \mathbb{R} \in \mathcal{O}_{k+1}$ and $\mathbb{R} \times A \in \mathcal{O}_{k+1}$;

(iii) for any integer $k \geq 0$ and any indices $1 \leq i < j \leq k$, the set $\{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_i = x_j\}$ lies in $\mathcal{O}_k$;
(iv) given an integer $k \geq 0$ and a set $A \in \mathcal{O}_{k+1}$, denoting by $\pi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ the canonical projection map, one has $\pi(A) \in \mathcal{O}_k$;
(v) any $x \in \mathbb{R}$ satisfied $\{x\} \in \mathcal{O}_1$; furthermore, $\{(x, y) \in \mathbb{R}^2 : x < y\} \in \mathcal{O}_2$;
(vi) the family $\mathcal{O}_1$ consists precisely of the finite unions of points and open intervals in the real line.

A set is *definable* in the structure $\mathfrak{O}$ if it belongs to $\mathcal{O}_k$ for some $k \geq 0$, and a map is definable if so is its graph.

The following generic process to generate o-minimal structures is described by Scanlon [41]: given $k \geq 0$, fix a class of so-called *distinguished functions* $\mathcal{F}_k = \{f : \mathbb{R}^k \to \mathbb{R}\}$. An *atomic set* is a set of the form

$$\{\boldsymbol{x} \in \mathbb{R}^k : a(\boldsymbol{x}) < b(\boldsymbol{x})\} \quad \text{or} \quad \{\boldsymbol{x} \in \mathbb{R}^k : a(\boldsymbol{x}) = b(\boldsymbol{x})\},$$

where $a, b : \mathbb{R}^k \to \mathbb{R}$ are any well-defined functions taken as compositions of coordinate maps, constant maps and distinguished maps. Letting

$$\mathcal{F} = \bigcup_{k \geq 0} \mathcal{F}_k,$$

denote by $\mathcal{O}_{\mathcal{F}}$ the smallest collection of subsets of $\mathbb{R}^k$ (as $k \geq 0$ varies) containing the atomic sets and stable under the images of the canonical projections $\pi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ and under taking complements and finite unions. Then $\mathcal{O}_{\mathcal{F}}$ defines an o-minimal structure.

The following examples are of particular importance in the proof of Lemma 3.3:

(a) When $\mathcal{F} := \mathcal{F}_{\mathrm{alg}}$ is the set of all polynomials defined over $\mathbb{R}^k$ for some (varying) $k \geq 0$, the collection $\mathcal{O}_{\mathcal{F}}$ is the set of all *semialgebraic sets*. That the above condition (iv) is satisfied is the content of the Tarski–Seidenberg Theorem [20, §2.10], the other properties being easily verified.
(b) When $\mathcal{F} := \mathcal{F}_{\mathrm{an}}$ is defined as the union of $\mathcal{F}_{\mathrm{alg}}$ and of all restricted analytic functions (these are maps $f : \mathbb{R}^k \to \mathbb{R}$ vanishing outside $[-1, 1]^k$ which are the restrictions to $[-1, 1]^k$ of a function which is real analytic in a neighbourhood of $[-1, 1]^k$), one obtains the o-minimal structure of *restricted analytic functions*. The proof in this case is due to Denef and van den Dries [19]. A detailed description of the sets in $\mathcal{F}_{\mathrm{an}}$ is provided in [21]: these are precisely the *globally subanalytic* sets, that is, sets $V \subset \mathbb{R}^k$ such that for each $\boldsymbol{x} \in \mathbb{R}^k$, there exists a relatively compact semianalytic set X contained in $\mathbb{R}^{k+l}$ for some integer $l \geq 1$ and a neighbourhood U of $\boldsymbol{x}$ such that $V \cap U$ is the projection of X onto the first k coordinates (recall here that a set is *semianalytic* if it is defined locally around any of its points by a finite sequence of equations or inequalities involving analytic maps, or a finite union of such sets).
(c) When $\mathcal{F} := \mathcal{F}_{\mathrm{an,exp}}$ is defined as the union of $\mathcal{F}_{\mathrm{an}}$ and of the singleton consisting of the exponential function $\exp : \mathbb{R} \to \mathbb{R}$, one obtains the o-minimal structure of *the expansion of restricted analytic functions by the exponential map*. The proof in this case is due to van den Dries and Miller [22]. It should be noted that the logarithm function $\log : \mathbb{R}_{>0} \to \mathbb{R}$ is definable in $\mathcal{F}_{\mathrm{an,exp}}$ since its graph is the set $\{(x, y) \in \mathbb{R}^2 : (x > 0) \wedge (x = \exp(y))\}$.

The proof of Lemma 3.3 requires working with the parametric integral of a semialgebraic map (see the definition of the Gel'fand–Leray function in (3.24) and the equation (3.26)). It turns out that such a parametric integral map need not remain in the structure of semialgebraic sets. To see this, it is enough to consider the semialgebraic function

$$g(x,y) = \begin{cases} 1/y & \text{if } (x>0)\wedge(1<y<x), \\ 0 & \text{otherwise.} \end{cases}$$

Integrating with respect to y then defines a parametric integral expressed as the logarithm function, which is not semialgebraic. This is the reason for the introduction of the larger structure $\mathcal{F}_{\mathrm{an,exp}}$ (which in particular includes the logarithm function).

Proof of Lemma 3.3. Under the assumption that the weights ψ and φ are semialgebraic, the Gel'fand–Leray map $\mathbb{S}^{n-1}\times\mathbb{R}^2 \ni (\boldsymbol{v},\varepsilon,\sigma)\mapsto \langle g_P(\boldsymbol{v},\varepsilon,\sigma),(\varphi,\psi)\rangle$ defined in (3.24) is a parametric integral of a globally subanalytic function (since semialgebraic functions are globally subanalytic). The seminal work [37] by Lion and Rolin proves that a parametric integral of a globally subanalytic function, and therefore the Gel'fand–Leray map itself, is definable in the o-minimal structure $\mathcal{F}_{\mathrm{an,exp}}$. (Kaiser [33] exhibits a smaller o-minimal structure containing all parametric integrals of semialgebraic functions. The so-called *constructible functions* form a class of functions containing all such integrals which is optimal in a sense detailed in Section 3.4 below.)

The proof relies on the existence of a $\mathcal{C}^1$-cell decomposition of the domain of definition of the Gel'fand–Leray map. For the present considerations, the actual (technical) definition of a cell is irrelevant so that the reader is referred to [20, Chap. 3] for further details. It is enough to mention here that these are, in some suitable sense, the "building blocks" of an o-minimal structure (here $\mathcal{F}_{\mathrm{an,exp}}$). They are definable and stable under the same operations as this structure (in particular under canonical projections as in (iv)) and can be used to partition any definable set into finitely many pieces. Furthermore, and this is the above mentioned existence claim, it follows from [20, Theorem 7.3.2] that the domain of definition of any definable function can be partitioned into finitely many cells where the function is $\mathcal{C}^1$: in the case under consideration, this means that there exists a finite collection $\mathcal{D}$ of cells partitioning $\mathbb{S}^{n-1}\times\mathbb{R}^2$ such that the restriction of the Gel'fand–Leray map to each cell D in $\mathcal{D}$ is continuously differentiable. In other words, for each $D\in\mathcal{D}$, there exists a definable open set U_D and a $\mathcal{C}^1$ map $\Gamma_D : U_D\to\mathbb{R}$ such that the restrictions to D of Γ_D and of the Gel'fand–Leray map $\langle g_P,(\varphi,\psi)\rangle$ coincide.

Theorem 7.3.2 in [20] establishes that any definable set can be partitioned into finitely many cells which are $\mathcal{C}^1$ (this is imposing the regularity of the boundary of the cells, see *loc. cit.* for further details). Consider then a partition $\mathcal{A}$ of the domain $\mathbb{S}^{n-1}\times\mathbb{R}^2$ into $\mathcal{C}^1$-cells inducing for each $D\in\mathcal{D}$ partitions of the sets

$$A_D^+ = \left\{(\boldsymbol{v},\varepsilon,\sigma)\in D : \frac{\partial\Gamma_D}{\partial\sigma}\geq 0\right\} \quad\text{and}\quad A_D^- = \left\{(\boldsymbol{v},\varepsilon,\sigma)\in D : \frac{\partial\Gamma_D}{\partial\sigma}< 0\right\}.$$

These sets are definable as the partial derivatives of a definable function are definable on their domain of definition (see, e.g., [49, point (3), §2.3] for a justification). Since the union over $D \in \mathcal{D}$ of the sets A_D^- and A_D^+ covers $\mathbb{S}^{n-1} \times \mathbb{R}^2$, each cell in the family $\mathcal{A}$ lies in some $A_D^\pm$.

Fix $(\boldsymbol{v}_0, \varepsilon_0) \in \mathbb{S}^{n-1} \times \mathbb{R}$ and denote by $\pi : \mathbb{S}^{n-1} \times \mathbb{R}^2 \ni (\boldsymbol{v}, \varepsilon, \sigma) \mapsto (\boldsymbol{v}, \varepsilon) \in \mathbb{S}^{n-1} \times \mathbb{R}$ the canonical projection. Consider the various sets

$$A_{(\boldsymbol{v}_0,\varepsilon_0)} = \{\sigma \in \mathbb{R} : (\boldsymbol{v}_0, \varepsilon_0, \sigma) \in A\} \tag{3.29}$$

obtained as the projection over $\mathbb{R}$ of the π-preimages of all cells $A \in \mathcal{A}$ such that $(\boldsymbol{v}_0, \varepsilon_0) \in \pi(A)$. From [20, Proposition 3.3.5], they form a cell decomposition of $\mathbb{R}$ denoted by $\mathcal{A}_{(\boldsymbol{v}_0,\varepsilon_0)}$.

Consider a set of the form (3.29) not reduced to a finite union of points and choose $D \in \mathcal{D}$ such that $A \subset A_D^\pm$. Then over each of the finitely many open intervals making up its interior, the map $\mathbb{R} \ni \sigma \mapsto \langle g_P(\boldsymbol{v}_0, \varepsilon_0, \sigma), (\varphi, \psi)\rangle$ is $\mathcal{C}^1$ and monotonic as it coincides therein with the continuously differentiable map $\sigma \mapsto \Gamma_D(\boldsymbol{v}_0, \varepsilon_0, \sigma)$ whose derivative keeps a constant sign. Lemma 3.3 is then an immediate consequence of the following three observations:

- $\mathcal{A}_{(\boldsymbol{v}_0,\varepsilon_0)}$ is contained in $\mathcal{A}$;
- $\mathcal{A}$ is finite;
- given a set in $\mathcal{A}$, the number of connected components of its projection over $\mathbb{R}$ (with respect to the third coordinate $\sigma \in \mathbb{R}$) is finite by property (vi). ■

3.3. Completion of the proof of the main theorem. The first part of this section is devoted to the completion of the proof of the claim (Z1) in Theorem 3.1, and the second one to the proof of (Z2).

3.3.1. The counting bound. The goal is here to establish the counting bound (3.6) of (Z1). Recall that the discussion following the statement of the theorem makes it clear that it suffices to establish this claim for a single polynomial $P(\boldsymbol{x})$.

Proof of Claim (Z1) *in Theorem 3.1.* Let

$$T^{-\alpha} \leq \varepsilon < T^{-1}, \tag{3.30}$$

where, by assumption, $\alpha > 1$. Let $\beta > 0$ be a large parameter and let $\kappa \in (0, 1)$ be a small parameter. Fix for the time being two nonnegative compactly supported semialgebraic maps φ and ψ which satisfy the inequalities (3.13). These maps are assumed to be j times continuously differentiable, where $j \geq 1$ is any integer large enough (depending on n, β and κ) so that the assumption (3.18) of the non-stationary phase Lemma 3.2 holds. Recall that the existence of such maps is guaranteed, e.g., by [34, Proposition 4.8].

From the Poisson summation estimate (3.15) and from Lemma 3.2, one has

$$\begin{aligned}\mathcal{N}_P^\dagger(K, T, \alpha) \ll T^n \cdot \left(\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}\right) + T^n \cdot (\varepsilon T)^\beta \\ + T^n \cdot \left(\sum_{1 \leq \|\boldsymbol{k}\| \leq a_P(\psi,\kappa)\cdot(T\varepsilon)^{-1/(1-\kappa)}} \left|\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(-T\boldsymbol{k} \cdot \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right|\right)\end{aligned}$$

for some implicit constant depending on j, β, κ, ψ and φ. In this inequality, Corollary 3.4 applied to the polar decomposition (3.16) implies that for any $\boldsymbol{k} \in \mathbb{Z}^n \backslash \{\boldsymbol{0}\}$,

$$\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \cdot e(-T\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \ll \frac{M_P(\psi, \varphi, \varepsilon)}{T \cdot \|\boldsymbol{k}\|}$$

with an implicit constant depending only on the polynomial $P(\boldsymbol{x})$ and on the weights ψ and φ. As a consequence,

$$\mathcal{N}_P^{\dagger}(K, T, \alpha) \ll T^n \cdot \left(\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x} \right) + T^n \cdot (\varepsilon T)^{\beta} + T^{n-1} \cdot (T\varepsilon)^{-(n-1)/(1-\kappa)} \cdot M_P(\psi, \varphi, \varepsilon)$$

in such a way that

$$\mathcal{N}_P^{\dagger}(K, T, \alpha) \ll T^n \cdot \left(\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x} \right) + T^n \cdot (\varepsilon T)^{\beta} + T^{-\frac{\kappa \cdot (n-1)}{1-\kappa}} \cdot \varepsilon^{-\frac{n-1}{1-\kappa}} \cdot M_P(\psi, \varphi, \varepsilon). \tag{3.31}$$

Recall that K and $\mathcal{K}$ are two compact sets such that the former is contained in the interior of the latter, and the latter is contained in the open set U defined as part of the support restriction condition (1.10). Given any $c \in (0,1)$ and any compactly supported smooth map $\widetilde{\psi}$, recall also the definition of $\mu_P(\widetilde{\psi}, c)$ in (2.13). Choose $\widetilde{\psi}$ such that

$$\chi_K \leq \psi \leq \widetilde{\psi} \leq \chi_{\mathcal{K}}. \tag{3.32}$$

Under the assumption $(\mathcal{H}_3)$ applied to $P(\boldsymbol{x})$, and with the notation of Lemma 2.3, the pair $(r_P(\widetilde{\psi}), m_P(\widetilde{\psi}))$ only depends on the set K – denote it $(r_P(K), m_P(K))$. Furthermore, Lemma 2.3 implies that $r_P(K) > 0$ and

$$\int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot \varphi(\varepsilon^{-1} \cdot P(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x} \underset{(3.13)}{\leq} \int_{\mathbb{R}^n} \widetilde{\psi}(\boldsymbol{x}) \cdot \chi_{\{|P(\boldsymbol{x})| \leq 2\varepsilon\}} \, \mathrm{d}\boldsymbol{x} \underset{(2.13)}{=} \mu_P(\widetilde{\psi}, 2\varepsilon) \underset{(2.14)}{\ll} \varepsilon^{r_P(K)} \cdot |\log \varepsilon|^{m_P(K)-1}.$$

As a consequence,

$$\mathcal{N}_P^{\dagger}(K, T, \alpha) \underset{(3.31)}{\ll} T^n \cdot \varepsilon^{r_P(K)} \cdot |\log \varepsilon|^{m_P(K)-1} + T^n \cdot (\varepsilon T)^{\beta} + T^{-\frac{\kappa \cdot (n-1)}{1-\kappa}} \cdot \varepsilon^{-\frac{n-1}{1-\kappa}} \cdot M_P(\psi, \varphi, \varepsilon).$$

Let $\eta > 0$. Provided that $\varepsilon > 0$ is small enough, it follows from the definition of the semialgebraic level of flatness $q_P(\mathcal{K})$ in (3.5) that

$$\mathcal{N}_P^{\dagger}(K, T, \alpha) \ll T^n \cdot \varepsilon^{r_P(K)} \cdot |\log \varepsilon|^{m_P(K)-1} + T^n \cdot (\varepsilon T)^{\beta} + \varepsilon^{-\frac{n-1}{1-\kappa} + q_P(\mathcal{K}) - \eta}. \tag{3.33}$$

The goal is first to prove the counting bound (3.6) in the generic case (i) of the statement of the theorem. To this end, note that the condition (3.7) implies that, provided that η and κ are small enough,

$$\rho_P(K, \mathcal{K}, \kappa, \eta) := \frac{n}{\frac{n-1}{1-\kappa} - q_P(\mathcal{K}) + r_P(K) + \eta} > 1. \tag{3.34}$$

The counting bound is then established by a distinction of cases:

- Assume that

$$\alpha \leq \rho_P(K, \mathcal{K}, \kappa, \eta) \tag{3.35}$$

and set $\varepsilon = T^{-\alpha}$. Then (3.33) yields

$$\mathcal{N}_P^\dagger(K, T, \alpha) \ll T^{n - r_P(K)\cdot\alpha} \cdot |\log T|^{m_P(K)-1} + T^{n+\beta(1-\alpha)}. \tag{3.36}$$

Under the assumption that $\alpha > 1$, it is enough to choose $\beta > \alpha \cdot r_P(K)/(\alpha - 1)$ so that this sum should be bounded by a constant multiple of the first term which, from Case (2) in Theorem 1.1, is the volume of the set $\mathcal{S}_P^\dagger(K, T, \alpha)$. In other words,

$$\mathcal{N}_P^\dagger(K, T, \alpha) \ll \mathrm{Vol}_n(\mathcal{S}_P^\dagger(K, T, \alpha)). \tag{3.37}$$

- Assume now that the converse to (3.35) holds, that is,

$$\alpha > \rho_P(K, \mathcal{K}, \kappa, \eta). \tag{3.38}$$

In this case, choose

$$\varepsilon = T^{-\rho_P(K, \mathcal{K}, \kappa, \eta)} \tag{3.39}$$

in such a way that the inequality (3.33) yields

$$\mathcal{N}_P^\dagger(K, T, \alpha) \ll T^{\widehat{\rho}_P(K, \mathcal{K}, \kappa, \eta) + \gamma} + T^{n+\beta(1-\alpha)}$$

with

$$\widehat{\rho}_P(K, \mathcal{K}, \kappa, \eta) := n \cdot \left(1 - \frac{r_P(K)}{\frac{n-1}{1-\kappa} - q_P(\mathcal{K}) + r_P(K) + \eta}\right)$$

for any $\gamma > 0$ (this extra factor is to absorb the possible logarithmic contribution in (3.33)). The condition (3.8) ensures that one can choose $\delta_P(K, \mathcal{K}) > 0$ lying in the interval determined by (3.9). Provided that γ, η and κ are chosen small enough, we then have

$$\widehat{\rho}_P(\mathcal{K}, K, \kappa, \eta) + \gamma < \tau_P(\mathcal{K}) - \delta_P(K, \mathcal{K}).$$

Under the assumption $\alpha > 1$ and upon choosing β large enough, this says that

$$\mathcal{N}_P^\dagger(K, T, \alpha) \ll T^{\tau_P(\mathcal{K}) - \delta_P(K, \mathcal{K})}. \tag{3.40}$$

Thus, under the assumptions (3.7) and (3.8), the inequalities (3.37) and (3.40) establish the counting bound (3.6) for any $\alpha > 1$ and for any exponent $\delta_P(K, \mathcal{K}) > 0$ in the range determined by (3.9), namely,

$$\mathcal{N}_P^\dagger(K, T, \alpha) \ll \mathrm{Vol}_n(\mathcal{S}_P^\dagger(K, T, \alpha)) + T^{\tau_P(\mathcal{K}) - \delta_P(K, \mathcal{K})}. \tag{3.41}$$

It remains to prove this bound in the degenerate case (ii) of (Z1), when it is assumed that (3.10) holds. The reason why the above argument is not applicable anymore is that (3.34) need not hold any longer when κ and η are small enough. As a consequence, the condition (3.35) need not hold either for $\alpha > 1$, nor can one necessarily set ε as in (3.39) when the complementary assumption (3.38) is satisfied because of the constraints imposed in (3.30).

To deal with this situation, set $\varepsilon = T^{-(1+\eta)}$ for the same $\eta > 0$ as above. Then (3.33) yields

$$\mathcal{N}_P^\dagger(K, T, \alpha) \ll T^{n - r_P(K)\cdot(1-\eta/2)} + T^{n - \eta\cdot\beta} + T^{(1+\eta)\cdot(\frac{n-1}{1-\kappa} - q_P(\mathcal{K}) + \eta)}.$$

If $\gamma > 0$ is given and if η and κ are chosen suitably small, the assumption of degeneracy (3.10) implies that

$$\mathcal{N}_P^{\dagger}(K, T, \alpha) \ll T^{n-1-q_P(\mathcal{K})+\gamma}.$$

Under the assumption (3.11), one can fix $\delta_P(K, \mathcal{K}) > 0$ in the range determined by (3.12). It is then enough to choose $\gamma > 0$ small enough that

$$n - 1 - q_P(\mathcal{K}) + \gamma \leq \tau_P(\mathcal{K}) - \delta_P(K, \mathcal{K}).$$

This establishes the bound (3.41) in this degenerate case also and thus completes the proof of (Z1). ■

3.3.2. The case of smooth varieties. The goal is now to establish Claim (Z2) of Theorem 3.1. To this end, fix $p \geq 1$ homogeneous forms $F_1(\boldsymbol{x}), \dots, F_p(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ such that the smoothness condition given by the nonvanishing of the map (1.8) over $\mathcal{K}$ holds. In this case, the dimension of the algebraic variety $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$ over $\mathcal{K}$ equals $\tau_{\boldsymbol{F}}(\mathcal{K}) = n - \widehat{\tau}_{\boldsymbol{F}}(\mathcal{K}) = n - p$. Furthermore, from Theorem 1.1(2), one then has $r_P(K) = p$ (recall here that K is a compact set contained in the interior of $\mathcal{K} \subset U$ and that it contains the set C in its interior, where C and U are defined in (1.10)). Under these assumptions, the inequalities (3.7) and (3.8) are the same: showing that they hold amounts to establishing that

$$q_{\boldsymbol{F}}(\mathcal{K}) > \widehat{\tau}_{\boldsymbol{F}}(\mathcal{K}) - 1 = p - 1. \tag{3.42}$$

Proof that (3.42) *holds under the assumptions of* (Z2). Given $\rho > 0$, denote by $\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, \rho)$ the ρ-tubular neighbourhood of $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$; it is the set of all points in $\mathbb{R}^n$ lying at distance less than ρ from $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$.

Given open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ contained in $\mathbb{R}^n$, the set $\mathcal{O}_1$ satisfies the *unique nearest point property* with respect to $\mathcal{Z}_{\mathcal{O}_2}(\boldsymbol{F}) := \mathcal{Z}_{\mathbb{R}}(\boldsymbol{F}) \cap \mathcal{O}_2$ if for every $\boldsymbol{y} \in \mathcal{O}_1$, there exists a unique point $\pi_{\boldsymbol{F}}(\boldsymbol{y}) \in \mathcal{Z}_{\mathcal{O}_2}(\boldsymbol{F})$ minimising the distance from $\boldsymbol{y}$ to $\mathcal{Z}_{\mathcal{O}_2}(\boldsymbol{F})$. The *reach* $\rho_{\boldsymbol{F}}(\boldsymbol{y})$ of a point $\boldsymbol{y} \in \mathcal{Z}_{\mathcal{O}_2}(\boldsymbol{F})$ is then defined as the supremum of $\rho > 0$ such that the open ball with radius ρ centred at $\boldsymbol{y}$ has the unique nearest point property with respect to $\mathcal{Z}_{\mathcal{O}_2}(\boldsymbol{F})$.

Under the assumption that the map (1.8) does not vanish over $\mathcal{K}$, and therefore over a small enough neighbourhood $\mathcal{O}$ of it, a well-known result by Federer [23, Theorem 4.12] implies that $\rho_{\boldsymbol{F}}(\boldsymbol{y}) > 0$ for any $\boldsymbol{y} \in \mathcal{Z}_{\mathcal{O}}(\boldsymbol{F})$. Since, from [23, Remark 4.2], the reach map $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F}) \ni \boldsymbol{y} \mapsto \rho_{\boldsymbol{F}}(\boldsymbol{y})$ is continuous and since the set $\mathcal{K}$ is assumed to be compact, one infers the existence of $\rho_{\boldsymbol{F}}(\mathcal{K}) > 0$ such that the projection map

$$\pi_{\boldsymbol{F}} : \mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, \rho_{\boldsymbol{F}}(\mathcal{K})) \ni \boldsymbol{y} \mapsto \pi_{\boldsymbol{F}}(\boldsymbol{y}) \in \mathcal{Z}_{\mathbb{R}}(\mathcal{K})$$

is well-defined. Now, establishing the inequality (3.42) requires an elementary lemma:

LEMMA 3.5 (Tubular neigbourhoods are comparable to the set of small solutions to inequalities under the assumption of smoothness). *Let $\rho_{\boldsymbol{F}}(\mathcal{K}) > 0$ be as above. Define*

$$\alpha_{\boldsymbol{F}}(\mathcal{K}) = \min_{1 \leq i \leq p} \alpha_i(\mathcal{K})^{-1} \quad \textit{and} \quad \beta_{\boldsymbol{F}}(\mathcal{K}) = \max_{1 \leq i \leq p} \{1, \beta_i(\mathcal{K})\},$$

where for $i = 1, \dots, p$,

$$\alpha_i(\mathcal{K}) = \max\{\|\nabla F_i(\boldsymbol{x})\| : \boldsymbol{x} \in \overline{\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, \rho_{\boldsymbol{F}}(\mathcal{K}))}\},$$
$$\beta_i(\mathcal{K}) = \max\{|||\nabla^2 F_i(\boldsymbol{x})||| : \boldsymbol{x} \in \overline{\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, \rho_{\boldsymbol{F}}(\mathcal{K}))}\}.$$

Here, $\nabla^2 F_i(\boldsymbol{x})$ denotes the $n \times n$ Hessian matrix associated to the homogeneous form F_i at $\boldsymbol{x}$ and $|||\cdot|||$ denotes the matrix norm induced by the Euclidean norm. Set furthermore

$$\theta_{\boldsymbol{F}}(\mathcal{K}) = \max\{1, (\max\{|||\mathrm{Gr}_{\boldsymbol{F}}(\boldsymbol{x})^{-1}||| : \boldsymbol{x} \in \overline{\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, \rho_{\boldsymbol{F}}(\mathcal{K}))}\})^{1/2}\},$$

where $\mathrm{Gr}_{\boldsymbol{F}}(\boldsymbol{x})$ is the Gram $p \times p$ matrix whose (i, j)th entry is the scalar product $\nabla F_i(\pi_{\boldsymbol{F}}(\boldsymbol{x})) \cdot \nabla F_j(\pi_{\boldsymbol{F}}(\boldsymbol{x}))$, and let

$$\rho^*_{\boldsymbol{F}}(\mathcal{K}) = \min\left\{\rho_{\boldsymbol{F}}(\mathcal{K}), \frac{1}{2\sqrt{n} \cdot \theta_{\boldsymbol{F}}(\mathcal{K}) \cdot \beta_{\boldsymbol{F}}(\mathcal{K})}\right\}. \tag{3.43}$$

Then all these constants are well-defined under the smoothness assumption that the map (1.8) does not vanish. Moreover, whenever $0 < \varepsilon < \rho^*_{\boldsymbol{F}}(\mathcal{K})$,

$$\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, \alpha_{\boldsymbol{F}}(\mathcal{K}) \cdot \varepsilon) \subset \mathcal{M}_{\boldsymbol{F}}(\mathcal{K}, \varepsilon) \subset \mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, 2 \cdot \theta_{\boldsymbol{F}}(\mathcal{K}) \cdot \varepsilon), \tag{3.44}$$

where

$$\mathcal{M}_{\boldsymbol{F}}(\mathcal{K}, \varepsilon) = \{\boldsymbol{x} \in \mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, \rho^*_{\boldsymbol{F}}(\mathcal{K})) : |F_i(\boldsymbol{x})| \le \varepsilon \text{ for all } 1 \le i \le p\}.$$

Proof. Let $\boldsymbol{x} \in \mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, \rho^*_{\boldsymbol{F}}(\mathcal{K}))$. Given $1 \le i \le p$, Taylor–Lagrange's Theorem provides the existence of a point $\boldsymbol{y}_{\boldsymbol{x}}$ in the line segment $[\pi_{\boldsymbol{F}}(\boldsymbol{x}), \boldsymbol{x}]$ such that

$$|F_i(\boldsymbol{x})| = |\nabla F_i(\boldsymbol{y}_{\boldsymbol{x}}) \cdot (\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x}))| \le \|\nabla F_i(\boldsymbol{y}_{\boldsymbol{x}})\| \cdot \|\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x})\|,$$

where the last relation follows from the Cauchy–Schwarz inequality. This is easily seen to imply the first inclusion in (3.44).

As for the second one, fix again $\boldsymbol{x} \in \mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, \rho^*_{\boldsymbol{F}}(\mathcal{K}))$ and $1 \le i \le p$. The Taylor–Lagrange Theorem at order 2 guarantees the existence of $\boldsymbol{y}_{\boldsymbol{x}} \in [\pi_{\boldsymbol{F}}(\boldsymbol{x}), \boldsymbol{x}]$ such that

$$\begin{aligned} |F_i(\boldsymbol{x})| &= \left|\nabla F_i(\pi_{\boldsymbol{F}}(\boldsymbol{x})) \cdot (\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x}) + \tfrac{1}{2} \cdot (\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x}))^T \cdot \nabla^2 F_i(\boldsymbol{y}_{\boldsymbol{x}}) \cdot (\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x}))\right| \\ &\ge |\nabla F_i(\pi_{\boldsymbol{F}}(\boldsymbol{x})) \cdot (\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x}))| - \beta_i(\mathcal{K}) \cdot \|\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x})\|^2, \end{aligned} \tag{3.45}$$

where $(\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x}))^T$ denotes the transpose of the (column) vector $\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x})$. This vector lies in the subspace normal to the manifold $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ at $\pi_{\boldsymbol{F}}(\boldsymbol{x})$. Under the smoothness assumption that the map (1.8) does not vanish, a basis for this normal space is the set $(\nabla F_i(\pi_{\boldsymbol{F}}(\boldsymbol{x})))_{1 \le i \le p}$ of linearly independent vectors. The smoothness assumption also guarantees that the $p \times p$ Gram matrix $\mathrm{Gr}_{\boldsymbol{F}}(\boldsymbol{x})$ formed by these vectors is invertible.

The norm of the vector $\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x})$ lying in the normal subspace under consideration then satisfies the relations

$$\|\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x})\|^2 = \boldsymbol{v}_{\boldsymbol{F}}(\boldsymbol{x})^T \cdot \mathrm{Gr}_{\boldsymbol{F}}(\boldsymbol{x})^{-1} \cdot \boldsymbol{v}_{\boldsymbol{F}}(\boldsymbol{x}) \le \theta_{\boldsymbol{F}}(\mathcal{K})^2 \cdot \|\boldsymbol{v}_{\boldsymbol{F}}(\boldsymbol{x})\|^2,$$

where $\boldsymbol{v}_{\boldsymbol{F}}(\boldsymbol{x})$ is the p-dimensional (column) vector whose coordinates are the scalars $\nabla F_i(\pi_{\boldsymbol{F}}(\boldsymbol{x})) \cdot (\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x}))$ for $1 \le i \le p$. If one assumes that $|F_i(\boldsymbol{x})| < \varepsilon$ for some $\varepsilon > 0$ and for all $1 \le i \le p$, the inequality (3.45) then yields

$$\varepsilon > \frac{\|\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x})\|}{\theta_{\boldsymbol{F}}(\mathcal{K})} - \sqrt{n} \cdot \beta_{\boldsymbol{F}}(\mathcal{K}) \cdot \|\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x})\|^2 \underset{(3.43)}{\ge} \frac{\|\boldsymbol{x} - \pi_{\boldsymbol{F}}(\boldsymbol{x})\|}{2 \cdot \theta_{\boldsymbol{F}}(\mathcal{K})}.$$

This establishes the second inclusion in (3.44) and thus completes the proof. ∎

To resume the proof of (3.42), note that the Łojasiewicz inequality (as stated, e.g., in [38, Theorem 4.1]) guarantees the existence of constants $c>0$ and $\lambda>0$ depending on the compact set $\mathcal{K}$ and on the set of homogeneous forms $\boldsymbol{F}(\boldsymbol{x})=(F_1(\boldsymbol{x}),\dots,F_p(\boldsymbol{x}))$ such that

$$\|\boldsymbol{F}(\boldsymbol{x})\|\geq c\cdot \operatorname{dist}(\boldsymbol{x},\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F}))^{\lambda}.$$

Here, diststands for the Euclidean distance from a point $\boldsymbol{x}\in\mathcal{K}$ to the set $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$. This implies that for any $\rho>0$, there exists $\varepsilon_{\boldsymbol{F}}(\mathcal{K},\rho)>0$ such that for all $0<\varepsilon<\varepsilon_{\boldsymbol{F}}(\mathcal{K},\rho)$, the inequality $\|\boldsymbol{F}(\boldsymbol{x})\|<\varepsilon$ forces the point $\boldsymbol{x}\in\mathcal{K}$ to lie in the ρ-neighbourhood of the variety $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$. Combined with Lemma 3.5, one obtains the existence of $\widetilde{\varepsilon}_{\boldsymbol{F}}(\mathcal{K})>0$ and $c_{\boldsymbol{F}}(\mathcal{K})>0$ such that for any $0<\varepsilon<\widetilde{\varepsilon}_{\boldsymbol{F}}(\mathcal{K})$ and any $\boldsymbol{v}\in\mathbb{S}^{n-1}$ and $\sigma\in\mathbb{R}$,

$$\{\boldsymbol{x}\in\mathcal{K}\cap\boldsymbol{v}^{\perp}(\sigma):\|\boldsymbol{F}(\boldsymbol{x})\|<\varepsilon\}\subset\boldsymbol{v}^{\perp}(\sigma)\cap\mathcal{M}_{\boldsymbol{F}}(\mathcal{K},\varepsilon)\subset\boldsymbol{v}^{\perp}(\sigma)\cap\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K},c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon).$$

As a consequence, for $\varepsilon>0$ small enough,

$$\operatorname{Vol}_{n-1}(\{\boldsymbol{x}\in\mathcal{K}\cap\boldsymbol{v}^{\perp}(\sigma):\|\boldsymbol{F}(\boldsymbol{x})\|\leq\varepsilon\})\leq\operatorname{Vol}_{n-1}\left(\boldsymbol{v}^{\perp}(\sigma)\cap\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K},c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon)\right).\quad(3.46)$$

To bound from above the volume on the right-hand side of this inequality, denote by $NM_{\boldsymbol{F}}(\mathcal{K})$ the normal bundle of $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$: it is the union over $\boldsymbol{x}\in\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ of the normal spaces $NM_{\boldsymbol{F}}(\mathcal{K},\boldsymbol{x})$ where $(\boldsymbol{x},\boldsymbol{y})\in NM_{\boldsymbol{F}}(\mathcal{K},\boldsymbol{x})$ if $\boldsymbol{x}\in\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ and if $\boldsymbol{y}$ lies in the subspace orthogonal to the tangent space to $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ at $\boldsymbol{x}$. The *tubular neighbourhood theorem* states that there exists a continous mapping $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})\ni\boldsymbol{x}\mapsto\delta(\boldsymbol{x})>0$ such that the map $(\boldsymbol{x},\boldsymbol{y})\mapsto\boldsymbol{x}+\boldsymbol{y}$ realises a diffeomorphism between the open subset of $NM_{\boldsymbol{F}}(\mathcal{K})$ defined as $\{(\boldsymbol{x},\boldsymbol{y})\in NM_{\boldsymbol{F}}(\mathcal{K}):\|\boldsymbol{y}\|<\delta(\boldsymbol{x})\}$ and its image. The proof of this statement in [36, Theorem 6.24] makes it also clear that the mapping δ is uniformly bounded below by a strictly positive constant under the assumption that the reach of the manifold $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ is strictly positive.

One thus obtains for some $\widetilde{\rho}=\widetilde{\rho}_{\boldsymbol{F}}(\mathcal{K})>0$ a diffeomorphism

$$NM_{\boldsymbol{F}}(\mathcal{K})[\widetilde{\rho}]\ni(\boldsymbol{x},\boldsymbol{y})\mapsto\boldsymbol{t}=\boldsymbol{x}+\boldsymbol{y}\in\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K},\widetilde{\rho})$$

with inverse

$$\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K},\widetilde{\rho})\ni\boldsymbol{t}\mapsto(\boldsymbol{x},\boldsymbol{y})=(\pi_{\boldsymbol{F}}(\boldsymbol{t}),\boldsymbol{t}-\pi_{\boldsymbol{F}}(\boldsymbol{t}))\in NM_{\boldsymbol{F}}(\mathcal{K})[\widetilde{\rho}].\quad(3.47)$$

Here, one has set $NM_{\boldsymbol{F}}(\mathcal{K})[\widetilde{\rho}]=\{(\boldsymbol{x},\boldsymbol{y})\in NM_{\boldsymbol{F}}(\mathcal{K}):\boldsymbol{x}\in\mathcal{K}\text{ and }\|\boldsymbol{y}\|<\widetilde{\rho}\}$, where the vector $\boldsymbol{y}$ lies in a p-dimensional subspace (namely, the subspace normal to the $(n-p)$-dimensional tangent subspace to $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ at a given point).

Given $\varepsilon>0$ small enough, up to a multiplicative constant absorbing the sup-norm of the Jacobian of the change of variables (3.47), the volume on the right-hand side of (3.46) is thus at most the product of two nonnegative quantities: on the one hand, the (surface) measure of the set of $\boldsymbol{x}\in\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ lying at distance less than $c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon$ from a point in the affine space $\boldsymbol{v}^{\perp}(\tau)$ and, on the other hand, the maximal volume of the intersection of a p-dimensional ball of radius $c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon$ with $\boldsymbol{v}^{\perp}(\sigma)$. In other words,

$$\operatorname{Vol}_{n-1}(\boldsymbol{v}^{\perp}(\sigma)\cap\mathfrak{N}_{\boldsymbol{F}}(\mathcal{K},c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon))\ll\tau_{\boldsymbol{F}}(\{\boldsymbol{x}\in\mathcal{K}:|\boldsymbol{v}\cdot\boldsymbol{x}-\sigma|\leq c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon\})\cdot\varepsilon^{p-1},\quad(3.48)$$

where $\tau_{\boldsymbol{F}}$ denotes the surface measure of the restriction of $\mathcal{Z}_{\mathbb{R}}(\boldsymbol{F})$ to a small enough neighbourhood of $\mathcal{K}$ where the gradient map (1.8) does not vanish, and where the implicit constant in (3.48) is independent of $\boldsymbol{v}$ and σ. To complete the proof, it suffices to show that the surface measure of the set under consideration decays as a positive power of ε.

From the compactness of $\mathcal{K}$, the problem can be reduced to finitely many similar local estimates which can be proved using the theory of (C,α)-good functions along the lines of [1, Corollary 3]. As this analytic approach requires rather tedious calculations and a distinction of cases to obtain estimates uniform in $\boldsymbol{v}$ and σ, the claim is established hereafter with a more global approach.

To this end, let ψ in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ be constant equal to 1 on $\mathcal{K}$ and such that its support is contained in the interior of the neighbourhood of $\mathcal{K}$ used to define $\tau_{\boldsymbol{F}}$. Define then the volume element

$$\mathrm{d}\mu_{\boldsymbol{F}} = \psi\,\mathrm{d}\tau_{\boldsymbol{F}}. \tag{3.49}$$

In particular, the induced measure $\mu_{\boldsymbol{F}}$ is absolutely continuous with respect to the measure $\tau_{\boldsymbol{F}}$.

Under the assumption that the variety $\mathcal{Z}_{\mathcal{K}}(\boldsymbol{F})$ is not contained in a linear subspace of dimension $n-p$, it follows from [43, p. 351, point (iii) & Theorem 2] that there exists $\kappa > 0$ such that the Fourier transform of the measure $\mu_{\boldsymbol{F}}$ satisfies the decay property

$$|\widehat{\mu}_{\boldsymbol{F}}(\boldsymbol{\xi})| \ll \|\boldsymbol{\xi}\|^{-\kappa} \tag{3.50}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}$ with implicit constant depending on ψ. From the Esséen concentration inequality [47, §2.2.11], it then follows that for any $\boldsymbol{v} \in \mathbb{S}^{n-1}$ and $\sigma \in \mathbb{R}$,

$$\mu_{\boldsymbol{F}}(\{\boldsymbol{x} \in \mathcal{K} : |\boldsymbol{v}\cdot\boldsymbol{x} - \sigma| \le c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon\}) \ll (c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon)\cdot\left(\int_{|t|\le(c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon)^{-1}} |\widehat{\mu}_{\boldsymbol{F}}(t\boldsymbol{v})|\,\mathrm{d}t\right)$$

with implicit constant independent of $\boldsymbol{v}$, σ and ε. Upon combining this relation with (3.49) and (3.50), one deduces from (3.48) that for some $\kappa > 0$,

$$\mathrm{Vol}_{n-1}\left(\boldsymbol{v}^\perp(\sigma) \cap \mathfrak{N}_{\boldsymbol{F}}(\mathcal{K}, c_{\boldsymbol{F}}(\mathcal{K})\cdot\varepsilon)\right) \ll \varepsilon^{p-1+\kappa}.$$

As a consequence, the definition of $M_{\boldsymbol{F}}(\mathcal{K},\varepsilon)$ in (3.4) implies that

$$M_{\boldsymbol{F}}(\mathcal{K},\varepsilon) \ll \varepsilon^{p-1+\kappa}$$

so that

$$q_{\boldsymbol{F}}(\mathcal{K}) := \liminf_{\varepsilon\to0^+}\left(\frac{\log M_{\boldsymbol{F}}(\mathcal{K},\varepsilon)}{\log\varepsilon}\right) \ge p-1+\kappa > p-1.$$

This completes the proof of the inequality (3.42). ∎

3.4. Determining the semialgebraic level of flatness. The practical use of Theorem 3.1 relies on the determination of the semialgebraic level of flatness introduced in (3.5). This final section is devoted to the development of the theory enabling one to determine this quantity effectively. To this end, following the discussion after the statement of Theorem 3.1, consider, without loss of generality, the case of a single real homogeneous polynomial $P(\boldsymbol{x})$ of degree $q \ge 2$. One is thus interested in finding effectively the value of

$$q_P(\mathcal{K}) = \liminf_{\varepsilon\to0^+}\left(\frac{\log M_P(\mathcal{K},\varepsilon)}{\log\varepsilon}\right), \tag{3.51}$$

where

$$M_P(\mathcal{K},\varepsilon) = \sup_{\boldsymbol{v}\in\mathbb{S}^{n-1}} \sup_{\sigma\in\mathbb{R}} \mu_{P,\mathcal{K}}(\boldsymbol{v},\sigma,\varepsilon),$$
$$\mu_{P,\mathcal{K}}(\boldsymbol{v},\sigma,\varepsilon) = \mathrm{Vol}_{n-1}(\{\boldsymbol{x}\in\mathcal{K}\cap\boldsymbol{v}^{\perp}(\sigma) : |P(\boldsymbol{x})|\le\varepsilon\}).$$

Here, $\mathcal{K}$ is assumed to be any compact semialgebraic set with nonempty interior not intersecting trivially the algebraic variety $\mathcal{Z}_{\mathbb{R}}(P)$. Under these assumptions, the real $q_P(\mathcal{K})$ is well-defined.

Fix $\varepsilon > 0$, $\boldsymbol{v}\in\mathbb{S}^{n-1}$ and $\sigma\in\mathbb{R}$. The first step is to make explicit the above-defined volume $\mu_{P,\mathcal{K}}(\boldsymbol{v},\sigma,\varepsilon)$ of the slice of the sublevel set $\{\boldsymbol{x}\in\mathcal{K} : |P(\boldsymbol{x})|\le\varepsilon\}$ with the affine space $\boldsymbol{v}^{\perp}(\sigma)$. To this end, denote by $R_{\boldsymbol{v}}$ the rotation matrix mapping the last element $\boldsymbol{e}_n\in\mathbb{S}^{n-1}$ of the canonical basis to $\boldsymbol{v}$ and leaving the orthogonal of the plane spanned by $\boldsymbol{e}_n$ and $\boldsymbol{v}$ unchanged (in the degenerate case when $\boldsymbol{v}=\boldsymbol{e}_n$, the matrix $R_{\boldsymbol{v}}$ is taken to be the identity). Then one can decompose the vector $\boldsymbol{x}\in\mathcal{K}\cap\boldsymbol{v}^{\perp}(\sigma)$ as $\boldsymbol{x}=R_{\boldsymbol{v}}(\boldsymbol{y},\sigma)$, where $\boldsymbol{y}\in\mathbb{R}^{n-1}$ lies in the projection onto the first $n-1$ coordinates of the preimage of the semialgebraic set $\mathcal{K}$ by $R_{\boldsymbol{v}}^{-1}$. One is thus reduced to computing the $n-1$-volume of the set of vectors $\boldsymbol{y}$ such that

$$|(P\circ R_{\boldsymbol{v}})(\boldsymbol{y},\sigma)|\le\varepsilon \quad\text{with}\quad (\boldsymbol{y},\sigma)\in R_{\boldsymbol{v}}^{-1}(\mathcal{K}). \tag{3.52}$$

This is achieved in the following statement.

PROPOSITION 3.6. *Keep the above notations and assume that the set $\mathcal{K}\subset\mathbb{R}^n$ is bounded and semialgebraic. Then there exists a semialgebraic set*

$$S_n(\mathcal{K})\subset\mathbb{R}^{n-2}\times\mathbb{R}\times\mathbb{S}^{n-1}\times(0,1)$$

depending on $\mathcal{K}$ which can be partitioned into $N\ge 1$ semialgebraic subsets $\mathcal{D}_j$ (where $1\le j\le N$) such that the following property holds: over each of these subsets are defined $l(j)\ge 0$ analytic semialgebraic maps $\xi_1<\dots<\xi_{l(j)} : \mathcal{D}_j\to\mathbb{R}$ such that, defining for any given $(\sigma,\boldsymbol{v},\varepsilon)\in\mathbb{R}\times\mathbb{S}^{n-1}\times(0,1)$ the projected sets

$$\mathcal{D}_j(\boldsymbol{v},\sigma,\varepsilon)=\{\boldsymbol{t}\in\mathbb{R}^{n-2} : (\boldsymbol{t},\boldsymbol{v},\sigma,\varepsilon)\in\mathcal{D}_j\},$$

the volume $\mu_{P,\mathcal{K}}(\boldsymbol{v},\sigma,\varepsilon)$ is the sum of finitely many integrals of the form

$$\int_{\mathcal{D}_j(\boldsymbol{v},\sigma,\varepsilon)} (\xi_{i+1}(\boldsymbol{t},\boldsymbol{v},\sigma,\varepsilon)-\xi_i(\boldsymbol{t},\boldsymbol{v},\sigma,\varepsilon))\,\mathrm{d}\boldsymbol{t}.$$

Here, $1\le j\le N$ and $1\le i\le l(j)$ whenever $l(j)\ge 1$.

The definition of $S_n(\mathcal{K})$ is explicitly given in the proof of the proposition, which makes it also clear that it can be constructed effectively. This proof can be inferred from the following result essentially due to Coste [18]:

LEMMA 3.7. *Let $Q(\boldsymbol{x})\in\mathbb{R}[\boldsymbol{x}]$ be a polynomial in $n\ge 1$ variables. Assume that $B\subset\mathbb{R}^{n-1}$ is a connected semialgebraic set and that k and d are integers such that $k\le d$ and such that for every point $\widehat{x}\in B$, the polynomial $Q(x_1,\widehat{x})$ has degree d in x_1 and exactly k distinct roots in $\mathbb{C}$. Then there exist $l\le k$ analytic semialgebraic maps $\zeta_1<\dots<\zeta_l : B\to\mathbb{R}$ which can be effectively determined such that, for every $\widehat{x}\in B$, the set of real roots of the polynomial $Q(x_1,\widehat{x})$ is exactly $\{\zeta_1(\widehat{x}),\dots,\zeta_l(\widehat{x})\}$. Furthermore, for any $i=1,\dots,l$, the multiplicity of the root $\zeta_i(\widehat{x})$ is constant for $\widehat{x}\in B$.*

Proof. This is [18, Proposition 2.6] with the exception that it is shown there that the maps ξ_i are just continuous rather than analytic. This additional feature is obtained as a consequence of the continuity of the roots of a polynomial as functions of the coefficients. A stronger statement established in [9] holds: the distinct roots of a polynomial are analytic functions of the coefficients in the open set where the roots retain their multiplicities. This yields the above statement. ∎

Deduction of Proposition 3.6 from Lemma 3.7. Consider the real polynomial

$$Q(\boldsymbol{y}, \sigma, \boldsymbol{v}, \varepsilon) = ((P \circ R_{\boldsymbol{v}})(\boldsymbol{y}, \sigma))^2 - \varepsilon^2.$$

Even if it means reindexing the coordinates of the vector $\boldsymbol{y} = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$, assume without loss of generality that the variable y_1 appears therein explicitly. Denote by $S_n(\mathcal{K})$ the image of the set

$$\{(\boldsymbol{y}, \sigma, \boldsymbol{v}, \varepsilon) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{S}^{n-1} \times (0,1) : (\boldsymbol{y}, \sigma) \in R_{\boldsymbol{v}}^{-1}(\mathcal{K})\} \tag{3.53}$$

under the projection map

$$\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{S}^{n-1} \times (0,1) \ni (\boldsymbol{y}, \sigma, \boldsymbol{v}, \varepsilon) \mapsto (\widehat{\boldsymbol{y}}, \sigma, \boldsymbol{v}, \varepsilon) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{S}^{n-1} \times (0,1),$$

where $\widehat{\boldsymbol{y}} = (y_2, \dots, y_{n-1}) \in \mathbb{R}^{n-2}$ when $\boldsymbol{y} = (y_1, y_2, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$. From the Tarski–Seidenberg Theorem (see point (a) in §3.2.2), the set $S_n(\mathcal{K})$ is also semialgebraic.

Partition $S_n(\mathcal{K})$ into finitely many semialgebraic subsets such that over each of them, the polynomial Q seen as a function of y_1 alone with coefficients in $S_n(\mathcal{K})$ keeps a constant degree with its complex roots retaining their multiplicity. This is indeed possible since the degree of Q in y_1 is determined by mutually exclusive semialgebraic conditions (namely, the vanishing of a suitable set of coefficients) and so is the multiplicity of its set of roots (namely, the vanishing of discriminant polynomials whose variables are the set of coefficients).

From [18, Theorem 2.23], each of the elements of the semialgebraic partition thus obtained has finitely many semialgebraic connected components. Lemma 3.7 then implies that over each such connected component, there exist a finite number (possibly zero) of semialgebraic analytic maps describing the roots of the polynomial Q (in y_1) as functions of the coefficients $(\widehat{\boldsymbol{y}}, \sigma, \boldsymbol{v}, \varepsilon)$. Denote these maps by $\zeta_2, \dots, \zeta_{l-1}$ and define the corresponding maps $\xi_2, \dots, \xi_{l-1}$ in the following way: when $2 \le i \le l-1$, ξ_i is the restriction of ζ_i to $\zeta_i^{-1}(\pi_1(R_v^{-1}(\mathcal{K})))$, where $\pi_1(R_v^{-1}(\mathcal{K}))$ is the semialgebraic image of $R_v^{-1}(\mathcal{K})$ under the projection map $(\boldsymbol{y}, \sigma) \mapsto y_1$. Since from [18, Corollary 2.9], the preimage of a semialgebraic set under a semialgebraic map is also semialgebraic, the maps ξ_i, where $1 \le i \le l-1$, remain semialgebraic. Then by definition, for all $2 \le i \le l-1$, the point $((\xi_i(\widehat{\boldsymbol{y}}, \sigma, \boldsymbol{v}, \varepsilon), \widehat{\boldsymbol{y}}), \sigma, \boldsymbol{v}, \varepsilon)$ lies in the set (3.53). Even if it means further considering separately the restrictions of the maps ξ_i $(2 \le i \le l-1)$ to the (finite number of) connected components of their common domain of definition, they may be assumed to be defined over connected semialgebraic sets.

Let then ξ_1 and ξ_l be the infimum and the supremum of the admissible values of y_1, respectively, when $(\widehat{\boldsymbol{y}}, \sigma, \boldsymbol{v}, \varepsilon)$ belongs to the connected semialgebraic set, say $\mathcal{D}$, defining the domain of definition of the maps ξ_i, where $2 \le i \le l-1$ (the infimum and supremum are finite under the assumption that the set $\mathcal{K}$ is bounded). The mean value theorem implies that if $Q(y_1, \widehat{\boldsymbol{y}}, \boldsymbol{v}, \sigma, \varepsilon) \leq 0$ for some $((y_1, \widehat{\boldsymbol{y}}), \sigma, \boldsymbol{v}, \varepsilon)$ lying in the set (3.53), there exists an index $1 \leq i \leq l$ such that $\xi_i(\widehat{\boldsymbol{y}}, \sigma, \boldsymbol{v}, \varepsilon) \leq y_1 \leq \xi_{i+1}(\widehat{\boldsymbol{y}}, \sigma, \boldsymbol{v}, \varepsilon)$.

Fixing the components $\boldsymbol{v}$, σ and ε, it follows from the Tarski–Seidenberg Theorem [20, §2.10] that the projected set $\mathcal{D}(\boldsymbol{v}, \sigma, \varepsilon) := \{\widehat{\boldsymbol{y}} \in \mathbb{R}^{n-2} : (\widehat{\boldsymbol{y}}, \sigma, \boldsymbol{v}, \varepsilon) \in \mathcal{D}\}$ is semialgebraic. Assuming that its $(n-2)$-dimensional Lebesgue measure is positive, the definition of the polynomial Q implies that the measure of the set of points $\boldsymbol{y} \in \mathbb{R}^{n-1}$ such that $|(P \circ R_{\boldsymbol{v}})(\boldsymbol{y}, \sigma)| \leq \varepsilon$ with $(\boldsymbol{y}, \sigma) \in R_{\boldsymbol{v}}^{-1}(\mathcal{K})$ is a finite sum of integrals of the form

$$\int_{\mathcal{D}(\boldsymbol{v},\sigma,\varepsilon)} (\xi_{i+1}(\widehat{\boldsymbol{y}}, \boldsymbol{v}, \sigma, \varepsilon) - \xi_i(\widehat{\boldsymbol{y}}, \boldsymbol{v}, \sigma, \varepsilon))\, \mathrm{d}\widehat{\boldsymbol{y}}.$$

The proof is complete upon indexing the (finitely many) semialgebraic sets $\mathcal{D}(\boldsymbol{v}, \sigma, \varepsilon)$ which have positive Lebesgue measure. ∎

The next step is to show that, given $\varepsilon > 0$, the suprema defining the real $M_P(\mathcal{K}, \varepsilon)$ in (3.4) are attained at some point $(\boldsymbol{v}, \sigma) = (\boldsymbol{v}(\varepsilon), \sigma(\varepsilon)) \in \mathbb{S}^{n-1} \times \mathbb{R}$. Combined with Proposition 3.6, this implies that $M_P(\mathcal{K}, \varepsilon)$ can be determined in a finite number of steps.

PROPOSITION 3.8. *Assume that $\mathcal{K} \subset \mathbb{R}^n$ is compact. Fix $\varepsilon > 0$. Then there exists $(\boldsymbol{v}(\varepsilon), \sigma(\varepsilon)) \in \mathbb{S}^{n-1} \times \mathbb{R}$ such that*

$$M_P(\mathcal{K}, \varepsilon) = \mu_{P,\mathcal{K}}(\boldsymbol{v}(\varepsilon), \sigma(\varepsilon), \varepsilon).$$

To prove the statement, it is convenient to introduce, given $\varepsilon > 0$, the map

$$\mu_{P,\mathcal{K}}(\cdot, \cdot, \varepsilon) : (\boldsymbol{v}, \sigma) \mapsto \mu_{P,\mathcal{K}}(\boldsymbol{v}, \sigma, \varepsilon) = \int_{\mathbb{R}^{n-1}} \chi_{\{|(P \circ R_{\boldsymbol{v}})(\boldsymbol{y},\sigma)| \leq \varepsilon\}} \cdot \chi_{R_{\boldsymbol{v}}^{-1}(\mathcal{K})}(\boldsymbol{y}, \sigma)\, \mathrm{d}\boldsymbol{y}. \quad (3.54)$$

Proof of Proposition 3.8. Since $\mathcal{K}$ is bounded, the support of the map $\mu_{P,\mathcal{K}}(\cdot, \cdot, \varepsilon)$ is compact. To establish the claim, it is therefore enough to show that this map is upper semicontinuous (so that it then assumes a maximal value over its support).

Let then $(\boldsymbol{v}_k, \sigma_k)_{k \geq 1}$ be a sequence in the support of $\mu_{P,\mathcal{K}}(\cdot, \cdot, \varepsilon)$ converging to a point $(\boldsymbol{v}_0, \sigma_0)$ in this support. The goal is to prove that

$$\limsup_{k \to \infty} \mu_{P,\mathcal{K}}(\boldsymbol{v}_k, \sigma_k, \varepsilon) \leq \mu_{P,\mathcal{K}}(\boldsymbol{v}_0, \sigma_0, \varepsilon). \quad (3.55)$$

To this end, one infers first from Fatou's Lemma (which is applicable since $\mathcal{K}$ is bounded) that

$$\limsup_{k \to \infty} \mu_{P,\mathcal{K}}(\boldsymbol{v}_k, \sigma_k, \varepsilon) \leq \int_{\mathbb{R}^{n-1}} \limsup_{k \to \infty} \left(\chi_{\{|(P \circ R_{\boldsymbol{v}_k})(\boldsymbol{y},\sigma_k)| \leq \varepsilon\}} \cdot \chi_{R_{\boldsymbol{v}_k}^{-1}(\mathcal{K})}(\boldsymbol{y}, \sigma_k)\right) \mathrm{d}\boldsymbol{y}.$$

Fix $\boldsymbol{y} \in \mathbb{R}^{n-1}$ and let $(\boldsymbol{v}_{k_l}, \sigma_{k_l})_{l \geq 1}$ be a subsequence (depending on $\boldsymbol{y} \in \mathbb{R}^{n-1}$) realising the upper limit in the integrand. By the continuity of the polynomial map defined by $P(\boldsymbol{x})$ and from the closedness of $\mathcal{K}$, upon letting $l \to \infty$ in $|(P \circ R_{\boldsymbol{v}_{k_l}})(\boldsymbol{y}, \sigma_{k_l})| \leq \varepsilon$ and $R_{\boldsymbol{v}_{k_l}}(\boldsymbol{y}, \sigma_{k_l}) \in \mathcal{K}$, it follows that $|(P \circ R_{\boldsymbol{v}_0})(\boldsymbol{y}, \sigma_0)| \leq \varepsilon$ and $R_{\boldsymbol{v}_0}(\boldsymbol{y}, \sigma_0) \in \mathcal{K}$. As a consequence,

$$\limsup_{k \to \infty} \left(\chi_{\{|(P \circ R_{\boldsymbol{v}_k})(\boldsymbol{y},\sigma_k)| \leq \varepsilon\}} \cdot \chi_{R_{\boldsymbol{v}_k}^{-1}(\mathcal{K})}(\boldsymbol{y}, \sigma_k)\right) \leq \chi_{\{|(P \circ R_{\boldsymbol{v}_0})(\boldsymbol{y},\sigma_k)| \leq \varepsilon\}} \cdot \chi_{R_{\boldsymbol{v}_0}^{-1}(\mathcal{K})}(\boldsymbol{y}, \sigma_0),$$

so that (3.55) is satisfied. This completes the proof. ∎

The conclusive step in the theory developed in this final section is to show that a conjecture in o-minimality (communicated to the author by Raf Cluckers) enables one to make the calculation of the measure of flatness $q_P(\mathcal{K})$ completely effective in the following sense: the lower limit of the ratio $(\log M_P(\mathcal{K},\varepsilon))/(\log\varepsilon)$ defining it in (3.51) is an actual limit which differs from this ratio up to an explicit error term as $\varepsilon \to 0^+$.

To this end, one needs to consider a relevant class of functions to analyse the properties of the map $\mu_{P,\mathcal{K}}(\cdot,\cdot,\varepsilon)$ introduced in (3.54) (for a fixed $\varepsilon > 0$) and then of the map $0 < \varepsilon \mapsto M_P(\mathcal{K},\varepsilon)$. This is the class of *constructible functions* over a given globally subanalytic set $X \subset \mathbb{R}^k$ defined in [12, §2] as the ring of real-valued functions over X generated by functions which are either globally subanalytic or the logarithms of positive globally subanalytic functions. In other words, a function f is constructible over X if it can be expressed as

$$f(\boldsymbol{x}) = \sum_{i=1}^{a} f_j(\boldsymbol{x}) \cdot \prod_{j=1}^{b} \log f_{ij}(\boldsymbol{x})$$

for some integers $a, b \geq 0$, where the f_i's and the f_{ij}'s are subanalytic with $f_{ij} > 0$ for all i and all j. More generally, a map is said to be constructible if it is constructible over some subanalytic set.

The main property satisfied by constructible functions, which is of fundamental importance in the present considerations, is the stability under parametric integration established in [15, Theorem 1.3]. This means that for any globally subanalytic subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ and for any map $f : A \times B \to \mathbb{R}$ which is constructible (over $A \times B$), the map $A \ni \boldsymbol{y} \mapsto \int_B f(\boldsymbol{y},\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ is also constructible (over A). As a consequence, the map $\mu_{P,\mathcal{K}}(\cdot,\cdot,\varepsilon)$ is constructible as a function defined over a semialgebraic set as the parametric integral over a semialgebraic domain of a semialgebraic function.

The function

$$M_P(\mathcal{K},\cdot) : 0 > \varepsilon \mapsto M_P(\mathcal{K},\varepsilon) = \max_{\boldsymbol{v}\in\mathbb{S}^{n-1}} \max_{\sigma\in\mathbb{R}} \mu_{P,\mathcal{K}}(\boldsymbol{v},\sigma,\varepsilon)$$

thus turns out to be the parametric maximum of a constructible function. The class of constructible functions is nevertheless not stable under taking parametric maxima: an explicit counterexample is in [32]. A weaker form of this stability property is nevertheless expected to hold:

CONJECTURE 3.9 (Adiceam & Cluckers). *Given a constructible, nonnegative function $F : X \ni (\varepsilon,\boldsymbol{x}) \mapsto F(\varepsilon,\boldsymbol{x})$ defined over some subanalytic set $X \subset (0,\infty) \times \mathbb{R}^k$, there exist a constant $\delta > 0$ and a constructible function $G : (0,\infty) \to \mathbb{R}$ of the form*

$$G(\varepsilon) = c \cdot |\log\varepsilon|^l \cdot \varepsilon^a \tag{3.56}$$

for some real $c \geq 0$, some rational number a and some integer $l \geq 0$ such that

$$\delta \cdot G(\varepsilon) \leq \sup_{\boldsymbol{x}\in\mathbb{R}^k} F(\varepsilon,\boldsymbol{x}) \leq G(\varepsilon) \quad \text{for all } \varepsilon < \delta.$$

A more general form of this conjecture would assert that, under suitable assumptions, the parametric supremum of a constructible function defined over any subanalytic set is "sandwiched" between constant multiples of a constructible function depending on the remaining variables. This is inspired by analogy to the p-adic case where the corresponding result holds (even uniformly in p) – see [13, Theorem B] and [14, Theorem 2.1.3].

If Conjecture 3.9 holds, keeping the notation therein, the function $M_P(\mathcal{K},\cdot)$ satisfies the inequalities

$$\delta\cdot c\cdot|\log\varepsilon|^l\cdot\varepsilon^a\leq M_P(\mathcal{K},\varepsilon)\leq c\cdot|\log\varepsilon|^l\cdot\varepsilon^a$$

when $0<\varepsilon<\delta$. As a consequence, for such ε,

$$\frac{\log(c\cdot l)}{\log\varepsilon}+\frac{\log|\log\varepsilon|}{\log\varepsilon}\leq\left|\frac{\log M_P(\mathcal{K},\varepsilon)}{\log\varepsilon}-a\right|\leq\frac{\log(c\cdot l\cdot\delta)}{\log\varepsilon}+\frac{\log|\log\varepsilon|}{\log\varepsilon},$$

implying in particular that

$$\frac{\log M_P(\mathcal{K},\varepsilon)}{\log\varepsilon}=q_P(\mathcal{K})+O\left(\frac{\log|\log\varepsilon|}{\log\varepsilon}\right),\quad\text{where}\quad q_P(\mathcal{K})=a.$$

With the help of Propositions 3.6 and 3.8, this relation shows that the measure of flatness $q_P(\mathcal{K})$ can be determined in a finite number of steps assuming the validity of Conjecture 3.9. This can furthermore be done effectively based on the theory developed in [6].

References

[1] F. Adiceam, V. Beresnevich, J. Levesley, S. Velani and E. Zorin, *Diophantine approximation and applications in interference alignment*, Adv. Math. 302 (2016), 231–279.

[2] F. Adiceam and O. Marmon, *Rational points close to non-singluar algebraic curves*, Mathematika 69 (2023), 957–987.

[3] F. Adiceam and O. Marmon, *Homogeneous form inequalities (I): volume estimates and the generalised metric oppenheim conjecture*, Trans. Amer. Math. Soc. 378 (2025), 2643–2693.

[4] V. Beresnevich, D. Dickinson and S. Velani, *Diophantine approximation on planar curves and the distribution of rational points* (with Appendix II by R. C. Vaughan), Ann. of Math. (2) 166 (2007), 367–426.

[5] V. Beresnevich and L. Yang, *Khintchine's theorem and diophantine approximation on manifolds*, Acta Math. 231 (2023), 1–30.

[6] S. Basu, R. Pollack and M.-F. Roy, *Algorithms in Real Algebraic Geometry*, 2nd ed., Algorithms Comput. Math. 10, Springer, Berlin, 2006.

[7] V. Beresnevich, R. Vaughan, S. Velani and E. Zorin, *Diophantine approximation on manifolds and the distribution of rational points: contributions to the convergence theory*, Int. Math. Res. Notices 2017, no. 10, 2885–2908.

[8] V. Beresnevich and E. Zorin, *Explicit bounds for rational points near planar curves and metric Diophantine approximation*, Adv. Math. 225 (2010), 3064–3087.

[9] D. Brillinger, *The analyticity of the roots of a polynomial as functions of the coefficients*, Math. Mag. 39 (1966), 145–147.

[10] N. Budarina, D. Dickinson and J. Levesley, *Simultaneous Diophantine approximation on polynomial curves*, Mathematika 56 (2010), 77–85.

[11] S. Chow, *A note on rational points near planar curves*, Acta Arith. 177 (2017), 393–396.

[12] R. Cluckers, G. Comte, D. Miller, J.-P. Rolin and T. Servi, *Integration of oscillatory and subanalytic functions*, Duke Math. J. 167 (2018), 1239–1309.

[13] R. Cluckers, I. Glazer and Y. Hendel, *A number theoretic characterization of E-smooth and (FRS) morphisms: estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points*, Algebra Number Theory, to appear; arXiv:2103.00282v2 (2021).

[14] R. Cluckers, J. Gordon and I. Halupczok, *Uniform analysis on local fields and applications to orbital integrals*, Trans. Amer. Math. Soc. Ser. B 5 (2018), 125–166.

[15] R. Cluckers and D. Miller, *Stability under integration of sums of products of real globally subanalytic functions and their logarithms*, Duke Math. J. 156 (2011), 311–348.

[16] Y. Colin de Verdière, *Nombre de points entiers dans une famille homothétique de domaines de* $\mathbb{R}$, Ann. Sci. École Norm. Sup. (4) 10 (1977), 559–575.

[17] L. Colzani, G. Gigante and G. Travaglini, *Trigonometric approximation and a general form of the Erdős Turán inequality*, Trans. Amer. Math. Soc. 363 (2011), 1101–1123.

[18] M. Coste, *An introduction to semialgebraic geometry*, https://gcomte.perso.math.cnrs.fr/M2/CosteIntroToSemialGeo.pdf (2002).

[19] J. Denef and L. van den Dries, *p-adic and real subanalytic sets*, Ann. of Math. (2) 128 (1988), 79–138.

[20] L. van den Dries, *Tame Topology and o-Minimal Structures*, London Math. Soc. Lecture Note Ser. 248, Cambridge Univ. Press, Cambridge, 1998.

[21] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. 84 (1996), 497–540.

[22] L. van den Dries and C. Miller, *On the real exponential field with restricted analytic functions*, Israel J. Math. 85 (1994), 19–56.

[23] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. 93 (1959), 418–491.

[24] E. Fouvry and M. Waldschmidt, *Number of integers represented by families of binary forms*, Acta Arith. 209 (2023), 219–267.

[25] A. Gafni, *Counting rational points near planar curves*, Acta Arith. 165 (2014), 91–100.

[26] M. Greenblatt, *Fourier transforms of indicator functions, lattice point discrepancy, and the stability of integrals*, Math. Ann. 380 (2021), 1959–1990.

[27] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I, Distribution Theory and Fourier Analysis*, 2nd ed., Grundlehren Math. Wiss. 256, Springer, Berlin, 1990.

[28] J. Huang, *Rational points near planar curves and Diophantine approximation*, Adv. Math. 274 (2015), 490–515.

[29] J. Huang, *The density of rational points near hypersurfaces*, Duke Math. J. 169 (2020), 2045–2077.

[30] J. Huang and J. Liu, *Simultaneous approximation on affine subspaces*, Int. Math. Res. Notices 2021, no. 19, 14905–14921.

[31] M. N. Huxley, *The rational points close to a curve*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 21 (1994), 357–375.

[32] T. Kaiser, *Growth of log-analytic functions*, arXiv:2301.02187v2 (2023).

[33] T. Kaiser, *Integration of semialgebraic functions and integrated Nash functions*, Math. Z. 275 (2013), 349–366.

[34] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology Appl. 123 (2002), 323–349.

[35] A. Lavrik, *The principal term of the divisor problem and the power series of the Riemann zeta-function in a neighbourhood of a pole*, Trudy Mat. Inst. Steklov. 142 (1976), 165–173, 269 (in Russian).

[36] J. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Grad. Texts in Math. 218, Springer, New York, 2013.

[37] J.-M. Lion et J.-P. Rolin, *Intégration des fonctions sous-analytiques et volumes des sous-ensembles sous-analytiques*, Ann. Inst. Fourier (Grenoble) 48 (1998), 755–767.

[38] B. Malgrange, *Ideals of Differentiable Functions*, Tata Inst. Fund. Res. Stud. Math. 3, Tata Institute of Fundamental Research, Bombay, and Oxford Univ. Press, London, 1967.

[39] B. Randol, *A lattice-point problem*, Trans. Amer. Math. Soc. 121 (1966), 257–268.

[40] P. Sarnak, *Values at integers of binary quadratic forms*, in: Harmonic Analysis and Number Theory (Montreal, PQ, 1996), CMS Conf. Proc. 21, Amer. Math. Soc, Providence, RI, 1997, 181–203.

[41] T. Scanlon, *A proof of the André–Oort conjecture using mathematical logic [after Pila, Wilkie and Zannier]*, in: Séminaire Bourbaki: Vol. 2010/2011, Astérisque 348 (2012), exp. 1037, ix, 299–315.

[42] D. Schindler and S. Yamagishi, *Density of rational points near/on compact manifolds with certain curvature conditions*, Adv. Math. 403 (2022), art. 108358, 36 pp.

[43] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. 43, Princeton Univ. Press, Princeton, NJ, 1993.

[44] J.-P. Serre, *Lectures on the Mordell–Weil Theorem*, Aspects Math. E15, Vieweg, Braunschweig, 1989.

[45] V. G. Sprindžuk, *Metric Theory of Diophantine Approximations*, Halsted Press, 1979.

[46] C. Stewart and S. Xiao, *On the representation of integers by binary forms*, Math. Ann. 375 (2019), 133–163.

[47] T. Tao, *Topics in Random Matrix Theory*, Grad. Stud. Math. 132, Amer. Math. Soc., Providence, RI, 2012.

[48] R. C. Vaughan and S. Velani, *Diophantine approximation on planar curves: the convergence theory*, Invent. Math. 166 (2006), 103–124.

[49] A. Wilkie, *O-minimal structures*, in: Séminaire Bourbaki: Vol. 2007/2008, Astérisque 326 (2009), exp. 985, vii, 131–142.