

On the inviscid limit of the viscous Boussinesq system with variable temperature diffusion

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Abstract. We prove the global-in-time well-posedness of the two-dimensional viscous Boussinesq system with temperature-dependent diffusivity under Yudovich-type initial data. Furthermore, we show that, as $\mu \rightarrow 0$, the viscous solutions (v_μ, θ_μ) converge to the inviscid solutions (v, θ) in $L_t^\infty L^p$ for $p \in [2, \infty[$, with an explicit rate of convergence.

1. Introduction

1.1. Introduction. The Boussinesq system is a fundamental model in geophysical fluid dynamics, widely used to describe large-scale atmospheric and oceanic flows. It also plays a central role in the analysis of Rayleigh–Bénard convection, a classical problem in fluid mechanics. This system captures convection phenomena in an incompressible viscous fluid driven by the buoyancy force induced by temperature; we refer for instance to [4, 6] for more details.

In this paper, we focus on the two-dimensional viscous Boussinesq system with variable dissipation, described by the following equations:

$$(B_\mu) \quad \begin{cases} \partial_t v_\mu + v_\mu \cdot \nabla v_\mu - \mu \Delta v_\mu + \nabla p_\mu = \theta_\mu \vec{e}_2 & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \theta_\mu + v_\mu \cdot \nabla \theta_\mu - \nabla \cdot (\kappa(\theta_\mu) \nabla \theta_\mu) = 0 & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \nabla \cdot v_\mu = 0, \\ (v_\mu, \theta_\mu)|_{t=0} = (v_\mu^0, \theta_\mu^0). \end{cases}$$

Here $v_\mu = (v_\mu^1, v_\mu^2)$ denotes the velocity field, p_μ the scalar pressure, and θ_μ the scalar temperature. The term $\theta_\mu \vec{e}_2$ represents the buoyancy force, with $\vec{e}_2 = (0, 1)^T$. The parameter μ denotes the viscosity of the fluid, while $\kappa(\cdot)$

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is a smooth function satisfying

$$(1.1) \quad \kappa(\theta) \geq \kappa_0,$$

where κ_0 is a fixed positive constant. If the initial temperature θ_μ^0 is identically zero, then the system (B_μ) reduces to the Navier–Stokes equations for $\mu > 0$, and to the Euler equations for $\mu = 0$. In both cases, global well-posedness has been established in various function spaces (see, for example, [1, 3, 5, 11, 12, 14–17, 21, 22, 35, 39] and the references therein).

The study of existence and uniqueness has also been extended to other models, including (B_μ) , which features several distinct cases. In the inviscid case, i.e. $\mu = \kappa(\theta) = \kappa_0 = 0$, the global existence problem remains open. However, D. Chae and K. Nam [10] established local well-posedness for initial data v_0, θ_0 in H^s with $s > 2$. Subsequently, the same authors obtained a similar result in [9] for initial data in the Hölder space C^s with $s > 1$. Recently, Z. Hassainia and T. Hmidi [19] studied the local theory for rough initial data of Yudovich type. In the case when system (B_μ) includes dissipation, the global-in-time regularity is well known. Specifically, three cases arise. In the fully viscous case ($\mu > 0$ and $\kappa(\theta) = \kappa_0 > 0$), J. R. Cannon and E. Dibenedetto [7], and independently B. Guo [18], proved the global existence of smooth solutions for arbitrarily large data by means of energy estimates.

For the cases of partial dissipation, i.e. either ($\mu > 0$, $\kappa(\theta) = \kappa_0 = 0$) or ($\mu = 0$, $\kappa(\theta) = \kappa_0 > 0$), D. Chae [8] established global well-posedness for initial data $v_0, \theta_0 \in H^s$ with $s > 2$. This result was subsequently improved by T. Hmidi and S. Keraani [23, 24]. In their first work, they considered initial data $(v_0, \theta_0) \in H^s \times \mathcal{B}_{2,1}^0 \cap \mathcal{B}_{p,\infty}^0$ with $(s, p) \in (0, 2] \times (2, \infty]$. In the second work, they obtained global well-posedness for initial data in the critical Besov spaces $\mathcal{B}_{p,1}^{1+2/p} \times L^r$, with (p, r) satisfying $2 < r \leq p < \infty$. An additional condition $\theta_0 \in \mathcal{B}^\infty$ is required in the endpoint case $p = \infty$. More recently, in the case of system (B_μ) with a general source term $(\theta \vec{e}_2 = F(\theta))$ for $F \in C^2(\mathbb{R}, \mathbb{R}^2)$ and zero viscosity, the second author and O. Melkemi [28] studied the global-in-time well-posedness in the framework of smooth vortex patches.

In recent years, much attention has been devoted to the Boussinesq system with temperature-dependent diffusivity κ in (B_μ) . For example, S. Wang and Z. Zhang [36] established global well-posedness of (B_μ) with temperature-dependent viscosity, assuming $v_0, \theta_0 \in H^s$ with $s > 2$. Their proof relies on the De Giorgi method, Littlewood–Paley decomposition, and paradifferential calculus. The same result was later obtained independently by D. Li and X. Xu [26], and by Q. Chen and L. Jiang [13], for the inviscid case $\mu = 0$. More recently, Z. He and X. Liao [20] proved the propagation of global regularity under the assumption $(\theta_0, v_0) \in H^{s_\theta}(\mathbb{R}^2) \times H^{s_v}(\mathbb{R}^2)$ with

$(s_\theta, s_v) \in [1, \infty) \times [0, \infty) \mid s_v - 1 \leq s_\theta \leq s_v + 2\} \setminus \{(0, 2), (1, 2)\}$. In addition, the functions $\kappa(\theta)$ and $\mu(\theta)$ satisfy $(\kappa(\theta), \mu(\theta)) \in C_b^2 \cap C^{[s_\theta]+1} \times C_b^2 \cap C^{[s_v]+1}$, in the setting where the classical dissipation (i.e., $\mu \Delta v_\mu$) is replaced by the nonlinear dissipation (i.e., $\nabla \cdot (\mu(\theta) \nabla v)$). We now turn to weak solutions of Yudovich type. This framework allows for less regular initial data. Roughly speaking, it requires that (v_0, θ_0) has finite energy and that the corresponding initial vorticity $\omega_0 = \nabla^\perp \cdot v_0$ is bounded and integrable. In this setting, the velocity field is not Lipschitz, but belongs to the class of log-Lipschitz functions. In [32] M. Paicu and N. Zhu proved the global existence of weak solutions under Yudovich-type initial data, namely bounded initial vorticity ω_0 and temperature $\theta_0 \in L^2 \cap \mathcal{B}_{p,r}^{2-2/r}$ with $(p, r) \in (2, \infty) \times (1, \infty)$. In addition, they imposed a smallness condition on $\kappa(\theta_0)$. The second author and M. Zerguine [30] obtained similar results for (B_μ) , establishing uniform estimates with respect to viscosity and proving convergence of the viscous solutions to those of the inviscid Boussinesq system with variable dissipation in $L_t^\infty L^p$ for $p \in [2, \infty]$ as $\mu \rightarrow 0$. In more recent work, Z. Ye [38] established global existence and uniqueness of weak solutions for initial data in Yudovich's class without any smallness assumption on $\kappa(\theta_0)$. Moreover, he proved global regularity under different assumptions on the initial data, namely $(v_0, \theta_0) \in H^s \times H^\rho \cap L^\infty$ with $(s, \rho) \in (2, \infty) \times [1, \infty)$ satisfying $|s - \rho| < 1$.

Motivated by the works of [29, 30] and [37], we study the global well-posedness and the inviscid limit for the two-dimensional viscous Boussinesq system with variable dissipation.

1.2. Main results. The main results of this paper are twofold. First, we extend the global-in-time well-posedness result for Yudovich-type initial data obtained in [38] to system (B_μ) , and establish uniform estimates for the corresponding viscous solutions with respect to the viscosity parameter. Second, we address the inviscid limit of system (B_μ) toward the corresponding inviscid system

$$(B_0) \quad \begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \theta \vec{e}_2, \\ \partial_t \theta + v \cdot \nabla \theta - \nabla \cdot (\kappa(\theta) \nabla \theta) = 0, \\ \nabla \cdot v = 0, \\ (v, \theta)|_{t=0} = (v_0, \theta_0), \end{cases}$$

in the L^p -spaces for $p \in [2, \infty[$, as the viscosity tends to zero. In particular, the rate of convergence of the velocities and densities is quantified. To formulate the problem, we adopt the vorticity formulation. The vorticity ω is defined by

$$\omega = \partial_1 v^2 - \partial_2 v^1,$$

which, in two dimensions, is a scalar function. Applying the curl operator to the momentum equation in (B_μ) , we obtain the vorticity-temperature system

$$(VD_\mu) \quad \begin{cases} \partial_t \omega_\mu + v_\mu \cdot \nabla \omega_\mu - \mu \Delta \omega_\mu = \partial_1 \theta_\mu, \\ \partial_t \theta_\mu + v_\mu \cdot \nabla \theta_\mu - \nabla \cdot (\kappa(\theta_\mu) \nabla \theta_\mu) = 0, \\ v_\mu = \nabla^\perp \Delta^{-1} \omega_\mu, \\ (\theta_\mu, \omega_\mu)|_{t=0} = (\theta_0^\mu, \omega_0^\mu). \end{cases}$$

The first main result is as follows.

THEOREM 1.1. *Let $v_\mu^0 \in L^2$ be a divergence-free initial velocity field whose initial vorticity ω_μ^0 belongs to $L^2 \cap L^\infty$. Let $\theta_\mu^0 \in H^\rho$ with $\rho \in (1, 2]$. If (1.1) holds, then the system (B_μ) admits a global-in-time solution (v_μ, θ_μ) such that*

$$(v_\mu, \theta_\mu) \in L_{\text{loc}}^\infty(\mathbb{R}_+; L^2) \times \mathcal{C}(\mathbb{R}_+; H^\rho) \cap L_{\text{loc}}^2(\mathbb{R}_+; H^{\rho+1}).$$

In addition,

$$\omega_\mu \in L_{\text{loc}}^\infty(\mathbb{R}_+; L^2 \cap L^\infty).$$

Moreover,

$$\begin{aligned} \|v_\mu\|_{L_t^\infty L^2 \cap L_t^2 H^1}^2 + \|\theta_\mu\|_{L_t^\infty L^\infty \cap H^\rho}^2 + \|\theta_\mu\|_{L_t^2 H^{\rho+1}}^2 + \|\omega_\mu\|_{L_t^\infty L^2 \cap L^\infty}^2 \\ \leq C_0 \exp(\exp(C_0 t^2)), \end{aligned}$$

where C_0 depends only on the norms of the initial data and not on the viscosity μ .

Before giving the details of the proof, we present a few remarks.

REMARK 1.2. In this work, we do not impose any smallness assumption on either the initial data or the thermal diffusivity coefficient $\kappa(\theta)$, and we only assume that $\kappa(\theta)$ is bounded from below.

REMARK 1.3. Very recently, Z. Ye [38] proved the global existence and uniqueness of a Yudovich-type solution for (B_μ) when the viscosity μ vanishes (i.e., in the nonviscous case).

The main point in the proof of Theorem 1.1 is to bound the norm of the vorticity ω in $L_t^\infty L^\infty$. Indeed, from the maximum principle for the vorticity equation, we have

$$\|\omega_\mu\|_{L_t^\infty L^\infty} \leq \|\omega_\mu^0\|_{L^\infty} + \|\nabla \theta_\mu\|_{L_t^1 L^\infty},$$

To bound the quantity $\|\nabla \theta_\mu\|_{L_t^1 L^\infty}$, we use the Sobolev embedding $H^{\rho+1}(\mathbb{R}^2) \hookrightarrow \text{Lip}(\mathbb{R}^2)$ with $1 < \rho \leq 2$, which yields

$$\|\omega_\mu\|_{L_t^\infty L^\infty} \leq \|\omega_\mu^0\|_{L^\infty} + \int_0^t \|\theta_\mu(\tau)\|_{H^{\rho+1}} d\tau.$$

To estimate $\|\theta_\mu\|_{L_t^1 H^{\rho+1}}$, we use the equation

$$(1.2) \quad (\partial_t + v \cdot \nabla)\phi(\theta) - \kappa(\theta)\Delta\phi(\theta) = 0,$$

where

$$\phi(\theta) = \int_0^\theta \kappa(\sigma) d\sigma,$$

as introduced in [33]. First, we control the temperature θ in the mixed homogeneous Sobolev space $L_t^\infty \dot{H}^1$ (see Section 3.2) by taking the L^2 inner product of (1.2) with $-\Delta(\phi(\theta))$. This estimate is then used to bound θ in $L_t^\infty H^\rho$ for $\rho \in (1, 2]$ (see Section 3.3). Finally, we apply the operator Λ^ρ with $\rho \in (1, 2]$ to the θ -equation and take the L^2 inner product with $\Lambda^\rho \theta$.

The second main result addresses the inviscid limit for the velocities and densities and establishes a rate of convergence. More precisely, it reads as follows.

THEOREM 1.4. *Let (v_μ, θ_μ) and (v, θ) be the solutions of (B_μ) and (B_0) , respectively, corresponding to the same initial data satisfying the conditions of Theorem 1.1. Then for all $t \in \mathbb{R}_+$, $\mu \in]0, 1[$, we have*

$$(1.3) \quad \|v_\mu - v\|_{L_t^\infty L^2} + \|\theta_\mu - \theta\|_{L_t^\infty L^2} \leq (\mu t)^{\frac{1}{2}} \exp(-\exp(\exp(C_0 t^2))t) \Phi(t),$$

where

$$\Phi(t) = C_0(1+t)^{\frac{1}{2}} \exp(-\exp(\exp(C_0 t^2))t) e^{\frac{1}{2}[1-\exp(-\exp(\exp(C_0 t^2)))t]}$$

REMARK 1.5. We point out that we can obtain the convergence for the velocity in the L^p spaces with $p \in]2, \infty[$. More precisely, applying the interpolation inequality

$$\|f\|_{L^p} \lesssim \|f\|_{L^2}^{2/p} \|f\|_{L^\infty}^{1-2/p}, \quad p \in]2, \infty[,$$

together with (3.1), (3.28), and the inequality (1.3), we have the following rate of convergence:

$$\begin{aligned} \|v_\mu - v\|_{L_t^\infty L^p} + \|\theta_\mu - \theta\|_{L_t^\infty L^p} &\leq C_0^{1-2/p} (\mu t)^{\frac{1}{p}} \exp(-\exp(\exp(C_0 t^2))t) \\ &\quad \times \exp((1-2/p) \exp(C_0 t^2)) \Phi^{1/p}(t). \end{aligned}$$

REMARK 1.6. The rate of convergence in the inviscid limit is also of interest. In our case, it has an exponential form, which differs from the convergence rate in the L^2 setting, where the rate is of order μt (see [27, 31]), due to the lack of Lipschitz regularity of the velocity field.

The paper is organized as follows. In Section 2, we introduce the function spaces and preliminary results. Section 3 is devoted to the proof of Theorem 1.1, where we obtain uniform estimates for the viscous solutions with respect to the viscosity parameter. Section 4 addresses the inviscid limit of system (B_μ) , namely the convergence of its solutions to those of (B_0) as

$\mu \rightarrow 0$ in Lebesgue spaces. Finally, the last section presents some technical lemmas used in the proof of the main result.

2. Setup and technical tool box. Throughout this paper, we employ the following notations:

- We denote by κ_1 and κ_2 the quantities

$$(2.1) \quad \kappa_1 = \sup_{|\sigma| \leq \|\theta_0\|_{L^\infty}} \kappa(\sigma), \quad \kappa_2 = \sup_{|\sigma| \leq \|\theta_0\|_{L^\infty}} \kappa'(\sigma).$$

- We designate by C a positive constant which may be different at each occurrence, but it does not depend on the initial data.
- We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of the type $X \leq CY$ with C independent of X and Y .
- The notation C_0 denotes a constant depending only on the norms of the initial data and independent of the viscosity parameter.
- We write L^p instead of $L^p(\mathbb{R}^2)$.

2.1. Function spaces. In this section, we introduce the definitions of Hölder spaces $C^{m+\alpha}$ and Sobolev spaces of type $W^{1,p}$ which will be used in our analysis. For $\alpha \in]0, 1[$, we define C^α to be the set of $u \in L^\infty$ such that

$$(2.2) \quad \|u\|_{C^\alpha} = \|u\|_{L^\infty} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty.$$

We will also make use of the space $C^{1+\alpha}$ which is the set of continuously differentiable functions u such that

$$\|u\|_{C^{1+\alpha}} = \|u\|_{L^\infty} + \|\nabla u\|_{C^\alpha} < \infty.$$

In the same way, we can generally define the space $C^{m+\alpha}$ with $n \in \mathbb{N}$ and $\alpha \in]0, 1[$.

The homogeneous Sobolev space $\dot{W}^{1,p}$ for $p \in [1, \infty]$ is the set of tempered distributions $u \in \mathcal{S}'$ whose gradient belongs to L^p , endowed with the norm

$$\|u\|_{\dot{W}^{1,p}} = \|\nabla u\|_{L^p}.$$

Next, we outline some elements of Littlewood–Paley theory which will be required in several steps. Let $(\chi, \varphi) \in \mathcal{D}(\mathbb{R}^2) \times \mathcal{D}(\mathbb{R}^2)$ be radial cut-off functions such that $\text{supp } \chi \subset \{\xi \in \mathbb{R}^2 : \|\xi\| \leq 1\}$ and $\text{supp } \varphi \subset \{1/2 \leq \|\xi\| \leq 2\}$, so that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1.$$

The *Littlewood–Paley* or *frequency cut-off operators* $(\Delta_q)_{q \geq -1}$ and $(\dot{\Delta}_q)_{q \geq -1}$ are defined for u in $\mathcal{S}'(\mathbb{R}^2)$, the class of tempered distributions, by

$$\Delta_{-1}u = \chi(D)u, \quad \Delta_q u = \varphi(2^{-q}D)u \text{ for } q \in \mathbb{N}, \quad \dot{\Delta}_q u = \varphi(2^{-q}D)u \text{ for } q \in \mathbb{Z},$$

where, in the general case, $f(D)$ stands for the pseudo-differential operator

$$u \mapsto \mathcal{F}^{-1}(f\mathcal{F}u),$$

with constant symbol. The lower frequencies sequence $(S_q)_{q \geq 0}$ is defined for $q \geq 0$ by

$$S_q u := \sum_{j \leq q-1} \Delta_j u.$$

In accordance with the previous properties, we have the well-known decomposition of unity

$$u = \sum_{q \geq -1} \Delta_q u, \quad u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u.$$

We now define the inhomogeneous Besov spaces $\mathcal{B}_{p,r}^s$ and the homogeneous Besov spaces $\dot{\mathcal{B}}_{p,r}^s$.

DEFINITION 2.1. For $(p, r, s) \in [1, \infty]^2 \times \mathbb{R}$, the spaces $\mathcal{B}_{p,r}^s$ and $\dot{\mathcal{B}}_{p,r}^s$ are defined by

$$\begin{aligned} \mathcal{B}_{p,r}^s &= \{u \in \mathcal{S}'(\mathbb{R}^2) : \|u\|_{\mathcal{B}_{p,r}^s} < \infty\}, \\ \dot{\mathcal{B}}_{p,r}^s &= \{u \in \mathcal{S}'(\mathbb{R}^2)_{|\mathbb{P}} : \|u\|_{\dot{\mathcal{B}}_{p,r}^s} < \infty\}, \end{aligned}$$

where $\mathbb{P}$ refers to the set of polynomial functions in $\mathbb{R}^2$, and we have

$$\|u\|_{\mathcal{B}_{p,r}^s} := \begin{cases} (\sum_{q \geq -1} 2^{rqs} \|\Delta_q u\|_{L^p}^r)^{1/r} & \text{if } r \in [1, \infty[, \\ \sup_{q \geq -1} 2^{qs} \|\Delta_q u\|_{L^p} & \text{if } r = \infty \end{cases}$$

and

$$\|u\|_{\dot{\mathcal{B}}_{p,r}^s} := \begin{cases} (\sum_{q \in \mathbb{Z}} 2^{rqs} \|\dot{\Delta}_q u\|_{L^p}^r)^{1/r} & \text{if } r \in [1, \infty[, \\ \sup_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q u\|_{L^p} & \text{if } r = \infty. \end{cases}$$

The celebrated *Bernstein inequalities* are listed in the following lemma (see for example [2, Lemma 2.1]).

LEMMA 2.2. *There exists a constant $C > 0$ such that for $q, k \in \mathbb{N}$, $1 \leq a \leq b \leq \infty$, and for $u \in L^p$,*

- (i) $\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q(k+2(1/a-1/b))} \|S_q u\|_{L^a},$
- (ii) $C^{-k} 2^{qk} \|\Delta_q u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \leq C^k 2^{qk} \|\Delta_q u\|_{L^a}.$

Moreover,

- (iii) (a consequence of (i)) *we have the embedding $\mathcal{B}_{p,r}^s \hookrightarrow \mathcal{B}_{\tilde{p},\tilde{r}}^{\tilde{s}}$ whenever $\tilde{p} \geq p$ and either*

$$\tilde{s} < s - 2(1/p - 1/\tilde{p})$$

or

$$\tilde{s} = s - 2(1/p - 1/\tilde{p}) \quad \text{and} \quad \tilde{r} \leq r.$$

We define the mixed space-time spaces as follows.

DEFINITION 2.3. Let $T > 0$ and $(s, \beta, p, r) \in \mathbb{R} \times [1, \infty]^3$. We define

$$L_T^\beta \mathcal{B}_{p,r}^s := \{u : [0, T] \rightarrow \mathcal{S}'; \|u\|_{L_T^\beta \mathcal{B}_{p,r}^s} = \|(2^{qs} \|\Delta_q u\|_{L^p})_{\ell^r}\|_{L_T^\beta} < \infty\}.$$

Since $\nabla \cdot v = 0$, we can express v in terms of the stream function via the Biot–Savart law $v = \nabla^\perp \Delta^{-1} \omega$. Combining this with Calderón–Zygmund theory, we deduce the following.

PROPOSITION 2.4. *Let $p \in]1, \infty[$ and v be a divergence-free vector field with vorticity $\omega \in L^p$. Then $\nabla v \in L^p$ and*

$$\|\nabla v\|_{L^p} \leq C \frac{p^2}{p-1} \|\omega\|_{L^p}$$

with C being a universal constant.

3. A priori estimates. The existence part of Theorem 1.1 is classical and can be proved for example by using the standard recursive method: see, e.g., [32]. We will focus on a priori estimates for the velocity, temperature and vorticity in various function spaces.

3.1. Energy estimates. We start with elementary energy estimates.

PROPOSITION 3.1. *Let (v, θ) be a smooth solution of (B_μ) .*

(i) *Let $\theta_0 \in L^p$ for some $p \in [2, \infty]$. Then for $t \in \mathbb{R}_+$, we have*

$$(3.1) \quad \|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}.$$

(ii) *Let $(v_0, \theta_0) \in L^2 \times L^2$. Then for $t \in \mathbb{R}_+$, we have*

$$\begin{aligned} \|v(t)\|_{L^2} &\leq \|v_0\|_{L^2} + t \|\theta_0\|_{L^2}, \\ \|v(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla v(\tau)\|_{L^2}^2 d\tau &\leq C_0(1 + t^2). \end{aligned}$$

(iii) *Let $(\omega_0, \theta_0) \in L^2 \times L^2$. Then for $t \in \mathbb{R}_+$, we have*

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \sqrt{\kappa_0 t} \|\theta_0\|_{L^2}.$$

Proof. (i) *First case:* $p \in [2, \infty[$. Multiplying the θ -equation by $|\theta|^{p-2}\theta$, integrating by parts over $\mathbb{R}^2$ and using $\nabla \cdot v = 0$, we obtain

$$\frac{1}{p} \frac{d}{dt} \|\theta(t)\|_{L^p}^p + \int_{\mathbb{R}^2} \kappa(\theta) \nabla \theta \nabla (|\theta|^{p-2}\theta) dx = 0.$$

In view of (1.1) we may write

$$(3.2) \quad \frac{1}{p} \frac{d}{dt} \|\theta(t)\|_{L^p}^p + \kappa_0^{-1}(p-1) \int_{\mathbb{R}^2} |\nabla \theta|^2 |\theta|^{p-2} dx \leq 0.$$

Integrating in time over $[0, t]$ we infer that $\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}$.

Second case: $p = \infty$. Taking $p \rightarrow \infty$ we find the desired result.

(ii) The classical L^2 scalar product for the v -equation allows us to achieve

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2}^2 + \mu \int_{\mathbb{R}^2} |\nabla v|^2 dx \leq \int_{\mathbb{R}^2} |\theta v| dx.$$

By the Cauchy–Schwarz inequality, we readily get

$$(3.3) \quad \frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2}^2 + \mu \|\nabla v(t)\|_{L^2}^2 \leq \|v(t)\|_{L^2} \|\theta(t)\|_{L^2}.$$

Thanks to (i) for $p = 2$, that is, $\|\theta(t)\|_{L^2} \leq \|\theta_0\|_{L^2}$, it follows after integration with respect to time that

$$\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + t \|\theta_0\|_{L^2}.$$

Finally, integrating (3.3) in time and using the last estimate gives

$$\begin{aligned} \|v(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla v(\tau)\|_{L^2}^2 d\tau &\leq \|v_0\|_{L^2}^2 + 2t \|\theta_0\|_{L^2} \|v_0\|_{L^2} + t^2 \|\theta_0\|_{L^2}^2 \\ &\leq 2(\|\theta_0\|_{L^2} + \|v_0\|_{L^2})^2 (1 + t^2). \end{aligned}$$

(iii) The classical L^2 -estimate for the ω -equation gives

$$\frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{L^2}^2 = \int_{\mathbb{R}^2} \partial_1 \theta \omega dx.$$

Using the Cauchy–Schwarz inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{L^2}^2 \leq \|\nabla \theta(t)\|_{L^2} \|\omega(t)\|_{L^2}.$$

Therefore,

$$\frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{L^2} \leq \|\nabla \theta(t)\|_{L^2}.$$

Integrating in time and using the Cauchy–Schwarz inequality with respect to time, we find

$$(3.4) \quad \|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + t^{1/2} \|\nabla \theta\|_{L_t^2 L^2}.$$

On the other hand, by exploring (3.2) for $p = 2$, we have

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{L^2}^2 + \kappa_0^{-1} \|\nabla \theta(t)\|_{L^2}^2 \leq 0.$$

It follows from integrating in time that

$$(3.5) \quad \|\nabla \theta\|_{L_t^2 L^2}^2 \leq \kappa_0 \|\theta_0\|_{L^2}^2.$$

Substituting the last estimate into (3.4), we deduce that

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \sqrt{\kappa_0 t} \|\theta_0\|_{L^2}.$$

This completes the proof of Proposition 3.1. ■

3.2. The H^1 -bound of the temperature. In the next step, we will bound the temperature θ in the mixed homogeneous Sobolev space $L_t^\infty \dot{H}^1$. For this purpose, we define

$$\phi(\theta) = \int_0^\theta \kappa(\sigma) d\sigma,$$

which is inspired by [33]. Then $\phi(\theta)$ satisfies

$$\begin{aligned} \partial_t(\phi(\theta)) &= \kappa(\theta)\partial_t\theta, \\ v \cdot \nabla(\phi(\theta)) &= \kappa(\theta)v \cdot \nabla\theta, \end{aligned}$$

and

$$(3.6) \quad \Delta\phi(\theta) = \kappa'(\theta)|\nabla\theta|^2 + \kappa(\theta)\Delta\theta.$$

Combining these with the second equation of (B_μ) , we infer that

$$(3.7) \quad (\partial_t + v \cdot \nabla)\phi(\theta) - \kappa(\theta)\Delta\phi(\theta) = 0.$$

In view of Proposition 3.1(i) we have $\|\theta(t)\|_{L^m} \leq \|\theta_0\|_{L^m}$. Therefore we get

$$\forall m \in [2, \infty], \quad \|\phi(\theta)(t)\|_{L^m} = \left\| \int_0^\theta \kappa(\sigma) d\sigma \right\|_{L^m} \leq \kappa_1 \|\theta(t)\|_{L^m} \leq \kappa_1 \|\theta_0\|_{L^m}.$$

PROPOSITION 3.2. *Let $(\omega^0, \theta^0) \in L^2 \times H^1 \cap L^\infty$. Then the solution of the second equation of (B_μ) satisfies*

$$\|\theta\|_{L_t^\infty \dot{H}^1}^2 + \|\theta\|_{L_t^2 \dot{H}^2}^2 \leq C_0 \exp(C_0 t^2).$$

Proof. Taking the L^2 inner product of (3.7) with $-\Delta(\phi(\theta))$, we find that

$$(3.8) \quad \frac{1}{2} \frac{d}{dt} \|\nabla\phi(\theta)(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} \kappa(\theta)(\Delta(\phi(\theta)))^2 dx = \int_{\mathbb{R}^2} v \cdot \nabla(\phi(\theta))\Delta(\phi(\theta)) dx.$$

Now, integrating by parts and using the incompressibility condition $\nabla \cdot u = 0$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} v \cdot \nabla(\phi(\theta))\Delta(\phi(\theta)) dx &= - \int_{\mathbb{R}^2} \partial_2 v^1 \partial_1 \phi(\theta) \partial_2 \phi(\theta) dx \\ &\quad - \int_{\mathbb{R}^2} \partial_1 v^2 \partial_2 \phi(\theta) \partial_1 \phi(\theta) dx. \end{aligned}$$

Hence, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla\phi(\theta)(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} \kappa(\theta)(\Delta(\phi(\theta)))^2 dx &= - \int_{\mathbb{R}^2} \partial_2 v^1 \partial_1 \phi(\theta) \partial_2 \phi(\theta) dx \\ &\quad - \int_{\mathbb{R}^2} \partial_1 v^2 \partial_2 \phi(\theta) \partial_1 \phi(\theta) dx. \end{aligned}$$

We successively use (1.1) and Hölder's inequality to get

$$\frac{1}{2} \frac{d}{dt} \|\nabla \phi(\theta)(t)\|_{L^2}^2 + \kappa_0 \|\Delta(\phi(\theta))(t)\|_{L^2}^2 \leq \|\nabla v(t)\|_{L^2} \|\nabla \phi(\theta)(t)\|_{L^4}^2.$$

Thanks to the Calderón–Zygmund estimate (2.4) and the Gagliardo–Nirenberg inequality $\|f\|_{L^4} \leq C \|\nabla f\|_{L^2}^{1/2} \|f\|_{L^2}^{1/2}$, we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \phi(\theta)(t)\|_{L^2}^2 + \kappa_0 \|\Delta(\phi(\theta))(t)\|_{L^2}^2 \\ \leq C \|\omega(t)\|_{L^2} \|\nabla \phi(\theta)(t)\|_{L^2} \|\Delta \phi(\theta)(t)\|_{L^2}. \end{aligned}$$

Therefore, by Young's inequality, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \phi(\theta)(t)\|_{L^2}^2 + \kappa_0 \|\Delta(\phi(\theta))(t)\|_{L^2}^2 \\ \leq C \left(\frac{1}{2\kappa_0} \|\omega(t)\|_{L^2}^2 \|\nabla \phi(\theta)(t)\|_{L^2}^2 + \frac{\kappa_0}{2} \|\Delta \phi(\theta)(t)\|_{L^2}^2 \right). \end{aligned}$$

It follows from integrating in time that

$$\begin{aligned} \|\nabla \phi(\theta)\|_{L_t^\infty L^2}^2 + \kappa_0 \|\Delta(\phi(\theta))\|_{L_t^2 L^2}^2 \\ \lesssim \|\nabla \phi(\theta)(0)\|_{L^2}^2 + \frac{1}{\kappa_0} \int_0^t \|\omega(\tau)\|_{L^2}^2 \|\nabla \phi(\theta)(\tau)\|_{L^2}^2 d\tau. \end{aligned}$$

Gronwall's inequality gives

$$\begin{aligned} \|\nabla \phi(\theta)\|_{L_t^\infty L^2}^2 + \kappa_0 \|\Delta(\phi(\theta))\|_{L_t^2 L^2}^2 \\ \lesssim \frac{1}{\kappa_0} \|\nabla \phi(\theta)(0)\|_{L^2}^2 \exp \left(\int_0^t \|\omega(\tau)\|_{L^2}^2 d\tau \right). \end{aligned}$$

The fact that $\nabla \phi(\theta_0) = \kappa'(\theta_0) \nabla \theta_0$ and Proposition 3.1 (iii) give

$$\begin{aligned} (3.9) \quad \|\nabla \phi(\theta)\|_{L_t^\infty L^2}^2 + \kappa_0 \|\Delta(\phi(\theta))\|_{L_t^2 L^2}^2 \\ \leq C \left(\frac{\kappa_2}{\kappa_0} \right)^2 \|\nabla \theta_0\|_{L^2}^2 \exp((\|\omega_0\|_{L^2}^2 + \kappa_0 \|\theta_0\|_{L^2}^2 t)). \end{aligned}$$

We now use (3.9) to derive bounds for $\nabla \theta$ in $L^\infty(0, t; L^2)$ and for $\Delta \theta$ in $L^2(0, t; L^2)$. On the one hand, we exploit the fact $\partial_i \theta = (\kappa(\theta))^{-1} \partial_i \phi(\theta)$ and (1.1) to get

$$(3.10) \quad \|\nabla \theta(t)\|_{L^2} \leq \frac{1}{\kappa_0} \|\nabla \phi(\theta)(t)\|_{L^2},$$

On the other hand, from (3.6), we have

$$\Delta \theta = (\kappa(\theta))^{-1} (\Delta \phi(\theta) - \kappa'(\theta) |\nabla \theta|^2).$$

It follows that

$$\int_0^t \|\Delta\theta(\tau)\|_{L^2}^2 d\tau \leq \frac{1}{\kappa_0} \|\Delta\phi(\theta)\|_{L_t^2 L^2}^2 + \frac{\kappa_2}{\kappa_0^5} \int_0^t \|\nabla\phi(\theta)(\tau)\|_{L^4}^4 d\tau.$$

We use the interpolation inequality

$$(3.11) \quad \|\nabla f\|_{L^4} \lesssim \|f\|_{L^\infty}^{1/2} \|\Delta f\|_{L^2}^{1/2}$$

and (3.8) to deduce that

$$\begin{aligned} \int_0^t \|\Delta\theta(\tau)\|_{L^2}^2 d\tau &\lesssim \frac{1}{\kappa_0} \|\Delta\phi(\theta)\|_{L_t^2 L^2}^2 \\ &\quad + \frac{\kappa_2}{\kappa_0} \int_0^t \|\phi(\theta)(\tau)\|_{L^\infty}^2 \|\Delta\phi(\theta)(\tau)\|_{L^2}^2 d\tau \\ &\lesssim \frac{1}{\kappa_0} \|\Delta\phi(\theta)\|_{L_t^2 L^2}^2 + \frac{\kappa_1^2 \kappa_2}{\kappa_0} \|\theta_0\|_{L^\infty}^2 \|\Delta\phi(\theta)\|_{L_t^2 L^2}^2 \\ &\leq C_0 \|\Delta\phi(\theta)\|_{L_t^2 L^2}^2. \end{aligned}$$

Combining this with (3.9) and (3.10), we conclude that

$$\|\nabla\theta\|_{L_t^\infty L^2}^2 + \|\Delta\theta\|_{L_t^2 L^2}^2 \leq C_0 \exp(C_0 t^2).$$

This completes the proof of Proposition 3.2. ■

The following corollary is a consequence of the previous proposition.

COROLLARY 3.3. *Let v be the divergence-free velocity vector field, and $q > 2$. Then for any smooth solution (θ, ω) of the system (VD_μ) we have, for $t > 0$,*

$$\|\omega(t)\|_{L_t^\infty L^q} \leq C_0 e^{C_0 t^2}.$$

Proof. Multiplying the ω -equation by $|\omega|^{q-2}\omega$ for $q > 2$, integrating by parts over $\mathbb{R}^2$, and using $\operatorname{div} v = 0$, we get

$$\frac{1}{q} \frac{d}{dt} \|\omega(t)\|_{L^q}^q = \int_{\mathbb{R}^2} \partial_1 \theta |\omega|^{q-2} \omega dx.$$

Hölder's inequality implies that

$$\frac{1}{q} \frac{d}{dt} \|\omega(t)\|_{L^q}^q \lesssim \|\nabla\theta\|_{L^q} \|\omega\|_{L^q}^{q-1}.$$

From the embedding

$$\|f\|_{L^q} \leq C\sqrt{q}\|f\|_{H^1},$$

it follows that

$$\frac{1}{q} \frac{d}{dt} \|\omega(t)\|_{L^q}^q \leq C\sqrt{q}\|\theta\|_{H^2} \|\omega\|_{L^q}^{q-1}.$$

So, integrating in time and using the Cauchy–Schwarz inequality with respect to time one gets

$$\|\omega(t)\|_{L^q} \leq \|\omega_0\|_{L^q} + C\sqrt{q}\|\theta\|_{L_t^2 H^2} t^{1/2}.$$

According to Proposition 3.2, we deduce that

$$\|\omega\|_{L_t^\infty L^q} \leq C_0 e^{C_0 t^2},$$

which is the desired estimate. ■

3.3. The H^ρ -bound of the temperature. Now, we will control the temperature θ in $L_t^\infty H^\rho$ where $\rho \in (1, 2]$.

PROPOSITION 3.4. *Let $\theta^0 \in \dot{H}^\rho$ with $\rho \in (1, 2]$. Then the solution of the second equation of (B_μ) satisfies*

$$(3.12) \quad \|\theta\|_{L_t^\infty H^\rho}^2 + \|\theta\|_{L_t^2 H^{\rho+1}}^2 \leq C_0 \exp(\exp(C_0 t^2)).$$

Proof. First, apply the operator Λ^ρ with $\rho > 1$ to the second equation in (B_μ) . Then, taking the L^2 inner product with $\Lambda^\rho \theta$ yields

$$(3.13) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda^\rho \theta(t)\|_{L^2}^2 - \int_{\mathbb{R}^2} \Lambda^\rho \nabla \cdot (\kappa(\theta) \nabla \theta) \Lambda^\rho \theta \, dx = - \int_{\mathbb{R}^2} \Lambda^\rho (v \cdot \nabla \theta) \Lambda^\rho \theta \, dx.$$

Integrating by parts and using (1.1), we obtain

$$(3.14) \quad \begin{aligned} & - \int_{\mathbb{R}^2} \Lambda^\rho \nabla \cdot (\kappa(\theta) \nabla \theta) \Lambda^\rho \theta \, dx \\ &= - \sum_{i=1}^2 \int_{\mathbb{R}^2} \Lambda^\rho \partial_i (\kappa(\theta) \partial_i \theta) \Lambda^\rho \theta \, dx = \sum_{i=1}^2 \int_{\mathbb{R}^2} \Lambda^\rho (\kappa(\theta) \partial_i \theta) \partial_i \Lambda^\rho \theta \, dx \\ &= \sum_{i=1}^2 \int_{\mathbb{R}^2} \kappa(\theta) \Lambda^\rho \partial_i \theta \Lambda^\rho \partial_i \theta \, dx + \sum_{i=1}^2 \int_{\mathbb{R}^2} [\Lambda^\rho, \kappa(\theta)] \partial_i \theta \Lambda^\rho \partial_i \theta \, dx \\ &\geq \kappa_0 \|\Lambda^\rho \nabla \theta\|_{L^2}^2 + \sum_{i=1}^2 \int_{\mathbb{R}^2} [\Lambda^\rho, \kappa(\theta)] \partial_i \theta \Lambda^\rho \partial_i \theta \, dx. \end{aligned}$$

Plugging (3.14) into (3.13), we find that

$$(3.15) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^\rho \theta(t)\|_{L^2}^2 + \kappa_0 \|\Lambda^\rho \nabla \theta\|_{L^2}^2 &\leq - \int_{\mathbb{R}^2} \Lambda^\rho (v \cdot \nabla \theta) \Lambda^\rho \theta \, dx \\ &\quad - \sum_{i=1}^2 \int_{\mathbb{R}^2} [\Lambda^\rho, \kappa(\theta)] \partial_i \theta \Lambda^\rho \partial_i \theta \, dx \\ &=: \text{I} + \sum_{i=1}^2 \text{II}_i. \end{aligned}$$

Let $\tilde{\kappa}(\theta) = \kappa(\theta) - \kappa(0)$. For the operator $\tau = \Lambda^s$, we note that

$$(3.16) \quad \begin{aligned} [\tau, \tilde{\kappa}(\theta)]F &= \tau(\tilde{\kappa}(\theta)F) - \tilde{\kappa}(\theta)\tau F \\ &= \tau(\kappa(\theta) - \kappa(0))F - (\kappa(\theta) - \kappa(0))\tau F \\ &= \tau(\kappa(\theta)F) - \kappa(\theta)\tau F = [\tau, \kappa(\theta)]F. \end{aligned}$$

Using the Cauchy–Schwarz inequality, we obtain

$$(3.17) \quad |\Pi_i| \leq C\|[\Lambda^\rho, \kappa(\theta)]\partial_i\theta\|_{L^2}\|\Lambda^\rho\nabla\theta\|_{L^2}.$$

For the commutator term on the right-hand side, we successively use (3.16) and (A.1) of Lemma 5.2 to find

$$\begin{aligned} \|[\Lambda^\rho, \kappa(\theta)]\partial_i\theta\|_{L^2} &= \|[\Lambda^\rho, \tilde{\kappa}(\theta)]\partial_i\theta\|_{L^2} \\ &\lesssim \|\nabla\tilde{\kappa}(\theta)\|_{L^\infty}\|\Lambda^\rho\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\Lambda^\rho\tilde{\kappa}(\theta)\|_{L^2}. \end{aligned}$$

We use the identity $\nabla\tilde{\kappa}(\theta) = \tilde{\kappa}'(\theta)\nabla(\theta)$ together with Proposition 3.1(i) and (2.1) to obtain

$$(3.18) \quad \|\nabla\tilde{\kappa}(\theta)\|_{L^\infty} \leq \widetilde{\kappa_1}\|\nabla\theta\|_{L^\infty}.$$

According to Lemma 5.3, and (2.1), we have

$$\begin{aligned} \|\Lambda^\rho\tilde{\kappa}(\theta)\|_{L^2} &= \|\tilde{\kappa}(\theta)\|_{\dot{B}_{2,2}^\rho} \leq C(\widetilde{\kappa_2})(1 + \|\theta\|_{L^\infty})^{1+[\rho]}\|\theta\|_{\dot{B}_{2,2}^\rho} \\ &\leq C(\widetilde{\kappa_2})(1 + \|\theta\|_{L^\infty})^2\|\Lambda^\rho\theta\|_{L^2}, \end{aligned}$$

where we have used the identity $\|\theta\|_{\dot{B}_{2,2}^\rho} = \|\theta\|_{\dot{H}^\rho} = \|\Lambda^\rho\theta\|_{L^2}$. Combining the last three estimates, we get

$$\|[\Lambda^\rho, \kappa(\theta)]\partial_i\theta\|_{L^2} \leq C(\widetilde{\kappa_1}, \widetilde{\kappa_2})(1 + \|\theta\|_{L^\infty})^2\|\nabla\theta\|_{L^\infty}\|\Lambda^\rho\theta\|_{L^2}.$$

Substituting the last estimate into (3.17) and using Young's inequality yields

$$(3.19) \quad |\Pi_i| \leq \frac{\kappa_0}{4}\|\Lambda^\rho\nabla\theta\|_{L^2}^2 + (1 + \|\theta\|_{L^\infty})^4\|\nabla\theta\|_{L^\infty}^2\|\Lambda^\rho\theta\|_{L^2}^2.$$

Now, we are going to bound I. By the Cauchy–Schwarz inequality,

$$(3.20) \quad |\text{I}| \lesssim \|\Lambda^{\rho-1}(v \cdot \nabla\theta)\|_{L^2}\|\Lambda^\rho\nabla\theta\|_{L^2}.$$

First for $\rho \in (1, 2)$, we exploit (A.2) of Lemma 5.2 to get

$$\|\Lambda^{\rho-1}(v \cdot \nabla\theta)\|_{L^2} \lesssim \|v\|_{L^\infty}\|\Lambda^{\rho-1}\nabla\theta\|_{L^2} + \|\nabla\theta\|_{L^{\frac{2}{2-\rho}}}\|\Lambda^{\rho-1}v\|_{L^{\frac{2}{\rho-1}}}.$$

To treat the two terms $\|\nabla\theta\|_{L^{\frac{2}{2-\rho}}}$ and $\|\Lambda^{\rho-1}v\|_{L^{\frac{2}{\rho-1}}}$ we successively use the Sobolev embeddings $\dot{H}^{\rho-1} \hookrightarrow L^{\frac{2}{2-\rho}}$ and $\dot{H}^{2-\rho} \hookrightarrow L^{\frac{2}{\rho-1}}$ to find

$$\|\Lambda^{\rho-1}(v \cdot \nabla\theta)\|_{L^2} \lesssim (\|v\|_{L^\infty} + \|\nabla v\|_{L^2})\|\Lambda^\rho\theta\|_{L^2}.$$

Substituting the last estimate in (3.20), by Young's inequality we deduce

$$|\text{I}| \lesssim \frac{\kappa_0}{4}\|\Lambda^\rho\nabla\theta\|_{L^2}^2 + (\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^2}^2)\|\Lambda^\rho\theta\|_{L^2}^2.$$

In the case $\rho = 2$, we successively use integration by parts, the Cauchy–Schwarz inequality, Lemma 5.2, and Young’s inequality to obtain

$$\begin{aligned}
 (3.21) \quad |I| &\leq C \| \Lambda(v \cdot \nabla \theta) \|_{L^2} \| \Lambda^\rho \theta \|_{L^2} \\
 &\leq (\|v\|_{L^\infty} \| \Lambda \nabla \theta \|_{L^2} + \| \nabla \theta \|_{L^\infty} \| \Lambda v \|_{L^2}) \| \Lambda^2 \nabla \theta \|_{L^2} \\
 &\leq (\|v\|_{L^\infty} \| \Lambda^2 \theta \|_{L^2} + \| \nabla \theta \|_{L^\infty} \| \nabla v \|_{L^2}) \| \Lambda^2 \nabla \theta \|_{L^2} \\
 &\leq \frac{\kappa_0}{4} \| \Lambda^2 \nabla \theta \|_{L^2}^2 + \|v\|_{L^\infty}^2 \| \Lambda^2 \theta \|_{L^2}^2 + \| \nabla v \|_{L^2}^2 \| \nabla \theta \|_{L^\infty}^2.
 \end{aligned}$$

By inserting (3.17) and (3.20) into (3.15), we deduce that

$$\begin{aligned}
 (3.22) \quad \frac{d}{dt} \| \Lambda^\rho \theta(t) \|_{L^2}^2 + \kappa_0 \| \Lambda^\rho \nabla \theta \|_{L^2}^2 \\
 \lesssim (\|v\|_{L^\infty}^2 + \| \nabla v \|_{L^2}^2 + (1 + \|\theta\|_{L^\infty})^4 \| \nabla \theta \|_{L^\infty}^2) \| \Lambda^\rho \theta \|_{L^2}^2 \\
 + \| \nabla v \|_{L^2}^2 \| \nabla \theta \|_{L^\infty}^2.
 \end{aligned}$$

The interpolation inequality together with the Calderón–Zygmund estimate (2.4) yields

$$\begin{aligned}
 (3.23) \quad \frac{d}{dt} \| \Lambda^\rho \theta(t) \|_{L^2}^2 + \kappa_0 \| \Lambda^\rho \nabla \theta \|_{L^2}^2 \\
 \lesssim (\|v\|_{L^2}^{\frac{q-2}{q-1}} \| \omega \|_{L^q}^{\frac{q}{q-1}} + \| \nabla v \|_{L^2}^2 + (1 + \|\theta\|_{L^\infty})^4 \| \nabla \theta \|_{L^\infty}^2) \| \Lambda^\rho \theta \|_{L^2}^2 \\
 + \| \nabla v \|_{L^2}^2 \| \nabla \theta \|_{L^\infty}^2.
 \end{aligned}$$

In order to handle the term $\| \nabla \theta \|_{L^\infty}$, we invoke the logarithmic interpolation inequality

$$(3.24) \quad \| \nabla \theta \|_{L^\infty} \leq C(1 + \| \nabla \theta \|_{L^2} + \| \Delta \theta \|_{L^2} \sqrt{\ln(e + \| \Lambda^\rho \nabla \theta \|_{L^2})}), \quad \rho > 1.$$

With (3.24) substituted into (3.22), we arrive at

$$\begin{aligned}
 \frac{d}{dt} \| \Lambda^\rho \theta(t) \|_{L^2}^2 + \kappa_0 \| \Lambda^\rho \nabla \theta \|_{L^2}^2 \\
 \leq C(1 + \|v\|_{L^2}^{\frac{q-2}{q-1}} \| \omega \|_{L^q}^{\frac{q}{q-1}} + \| \nabla v \|_{L^2}^2 + (1 + \|\theta\|_{L^\infty})^4 \| \nabla \theta \|_{L^2}^2) \| \Lambda^\rho \theta \|_{L^2}^2 \\
 + C \| \Delta \theta \|_{L^2}^2 \ln(e + \| \Lambda^\rho \nabla \theta \|_{L^2}^2) \| \Lambda^\rho \theta \|_{L^2}^2 \\
 + C \| \nabla v \|_{L^2}^2 \| \Delta \theta \|_{L^2}^2 \ln(e + \| \Lambda \nabla \theta \|_{L^2}^2).
 \end{aligned}$$

From Proposition 3.1(i), we infer

$$\begin{aligned}
 (3.25) \quad \frac{d}{dt} (1 + \| \Lambda^\rho \theta(t) \|_{L^2}^2) + \kappa_0 \| \Lambda^\rho \nabla \theta \|_{L^2}^2 \\
 \leq C(1 + \|\theta_0\|_{L^2})^4 ((1 + \|v\|_{L^2}^{\frac{q-2}{q-1}} \| \omega \|_{L^q}^{\frac{q}{q-1}} + \| \nabla v \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2) (1 + \| \Lambda^\rho \theta \|_{L^2}^2) \\
 + C(1 + \| \nabla v \|_{L^2}^2) \| \Delta \theta \|_{L^2}^2 \ln(e + \| \Lambda^\rho \nabla \theta \|_{L^2}^2) (1 + \| \Lambda^\rho \theta \|_{L^2}^2)).
 \end{aligned}$$

In light of the estimate

$$\|\Delta\theta\|_{L^2}^2 \leq C\|\theta\|_{L^2}^{\frac{2\rho-2}{\rho+1}} \|A^\rho \nabla \theta\|_{L^2}^{\frac{4}{\rho+1}} \leq \|A^\rho \nabla \theta\|_{L^2}^{\frac{4}{\rho+1}},$$

where $\frac{4}{\rho+1} < 2$ since $\rho > 1$, and combining this with Lemma 5.4, Propositions 3.1 and 3.2, and Corollary 3.3, we conclude that

$$\|A^\rho \theta(t)\|_{L^2}^2 + \int_0^t \|A^\rho \nabla \theta(\tau)\|_{L^2}^2 d\tau \leq C_0 \exp(\exp(C_0 t^2)).$$

The proof of Proposition 3.4 is complete. ■

3.4. The L^∞ bound of the vorticity. The following important result is a direct consequence of the above proposition.

COROLLARY 3.5. *Let v be a smooth divergence-free vector field. Then for every smooth solution (ω, θ) of the system (VD_μ) we have, for every $t \geq 0$,*

$$\|\omega(t)\|_{L^\infty} \leq C_0 \exp(\exp(C_0 t^2)).$$

In addition,

$$\|v\|_{L_t^\infty L^\infty} \leq C_0 \exp(\exp(C_0 t^2)).$$

Proof. The maximum principle for the ω -equation gives

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \int_0^t \|\nabla \theta(\tau)\|_{L^\infty} d\tau.$$

Since $H^{\rho+1} \hookrightarrow \text{Lip}$ for $\rho > 1$, we have

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \|\theta\|_{L_t^1 H^{\rho+1}}.$$

By the Cauchy–Schwarz inequality with respect to time and Proposition 3.4, we conclude that

$$(3.26) \quad \|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \|\theta\|_{L_t^2 H^{\rho+1}} t^{1/2} \leq C_0 \exp(\exp(C_0 t^2)).$$

From the Biot–Savart law, $v = \nabla^\perp \Delta^{-1} \omega$, and applying Bernstein’s inequality, we obtain the equivalence

$$\forall q \in \mathbb{N}, \quad \|\Delta_q v\|_{L^\infty} \approx 2^{-q} \|\Delta_q \omega\|_{L^\infty}.$$

Hence,

$$(3.27) \quad \|v\|_{L^\infty} \lesssim \|v\|_{L^2} + \|\omega\|_{L^\infty}.$$

Inserting (3.26) into (3.27) and using Proposition 3.1(ii), we conclude that

$$(3.28) \quad \|v(t)\|_{L^\infty} \leq C_0 \exp(\exp(C_0 t^2)),$$

which is the desired estimate. ■

4. Inviscid limit. This section concerns the inviscid limit for system (B_μ) towards (B_0) as $\mu \rightarrow 0$, and establishes the rate of convergence of the velocities and densities.

Proof of Theorem 1.4. Taking the difference between (B_μ) and (B_0) , by setting $\bar{v}_\mu = v_\mu - v$, $\bar{\theta}_\mu = \theta_\mu - \theta$ and $\bar{p}_\mu = p_\mu - p$ we find that the triplet $(\bar{v}_\mu, \bar{\theta}_\mu, \bar{p}_\mu)$ is governed by the following evolution system:

$$(D_\mu) \quad \begin{cases} \partial_t \bar{v}_\mu + v_\mu \cdot \nabla \bar{v}_\mu - \mu \Delta \bar{v}_\mu = \mu \Delta v - \nabla \bar{p}_\mu + \bar{\theta} \vec{e}_2 - \bar{v}_\mu \cdot \nabla v, \\ \partial_t \bar{\theta}_\mu + v_\mu \cdot \nabla \bar{\theta}_\mu - \nabla \cdot (\kappa(\theta_\mu) \nabla \bar{\theta}_\mu) \\ \quad \quad \quad = -\bar{v}_\mu \cdot \nabla \theta + \nabla \cdot ((\kappa(\theta_\mu) - \kappa(\theta)) \nabla \theta), \\ \nabla \cdot \bar{v}_\mu = 0, \\ (\bar{v}_\mu, \bar{\theta}_\mu)|_{t=0} = (\bar{v}_0^\mu, \bar{\theta}_0^\mu). \end{cases}$$

Taking the L^2 inner product of the first equation in (D_μ) with $\bar{v}_\mu$, and integrating by parts over $\mathbb{R}^2$ bearing in mind that $\nabla \cdot v_\mu = \nabla \cdot v = 0$, shows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{v}_\mu(t)\|_{L^2}^2 + \mu \int_{\mathbb{R}^2} |\nabla \bar{v}_\mu|^2 dx &= \mu \int_{\mathbb{R}^2} \Delta v \bar{v}_\mu dx + \int_{\mathbb{R}^2} \bar{v}_\mu \bar{\theta}_\mu \vec{e}_2 dx \\ &\quad - \int_{\mathbb{R}^2} \bar{v}_\mu \cdot \nabla v \bar{v}_\mu dx. \end{aligned}$$

Then, by integration by parts and Hölder's inequality we obtain

$$(4.1) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{v}_\mu(t)\|_{L^2}^2 + \mu \|\nabla \bar{v}_\mu\|_{L^2}^2 \\ \lesssim \mu \|\nabla v\|_{L^2} \|\nabla \bar{v}_\mu\|_{L^2} + \|\bar{v}_\mu(t)\|_{L^2} \|\bar{\theta}_\mu(t)\|_{L^2} + \|\nabla v\|_{L^q} \|\bar{v}_\mu\|_{L^{2q'}}^2 \end{aligned}$$

for $q \in (2, \infty)$, where q' denotes the conjugate exponent of q . We next successively apply the Biot–Savart law, Young's inequality, and interpolation to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{v}_\mu(t)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla \bar{v}_\mu\|_{L^2}^2 &\lesssim \frac{\mu}{2} \|\omega\|_{L^2}^2 + \frac{1}{2} (\|\bar{v}_\mu(t)\|_{L^2}^2 + \|\bar{\theta}_\mu(t)\|_{L^2}^2) \\ &\quad + q \|\omega\|_{L^q} \|\bar{v}_\mu\|_{L^2}^{2/q'} \|\bar{v}_\mu\|_{L^\infty}^{2/q}. \end{aligned}$$

Thus, we have

$$(4.2) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{v}_\mu(t)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla \bar{v}_\mu\|_{L^2}^2 \\ \lesssim \frac{\mu}{2} \|\omega\|_{L^2}^2 + \frac{1}{2} (\|\bar{v}_\mu(t)\|_{L^2}^2 + \|\bar{\theta}_\mu(t)\|_{L^2}^2) \\ + q \|\omega\|_{L^q} \|\bar{v}_\mu\|_{L^2}^{2(1-1/q)} (\|v_\mu\|_{L^\infty} + \|v\|_{L^\infty})^{2/q}. \end{aligned}$$

On the other hand, proceeding in the same way for the $\bar{\theta}_\mu$ -equation, we find

$$\begin{aligned}
 (4.3) \quad & \frac{1}{2} \frac{d}{dt} \|\bar{\theta}_\mu(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} \kappa(\theta_\mu) \nabla \bar{\theta}_\mu \cdot \nabla \bar{\theta}_\mu \, dx \\
 &= - \int_{\mathbb{R}^2} \bar{v}_\mu \cdot \nabla \theta \bar{\theta}_\mu \, dx - \int_{\mathbb{R}^2} (\kappa(\theta_\mu) - \kappa(\theta)) \nabla \theta \cdot \nabla \bar{\theta}_\mu \, dx =: \text{I} + \text{II}.
 \end{aligned}$$

To estimate the first term, we successively use the incompressibility condition $\nabla \cdot \bar{v}_\mu = 0$, integration by parts, Hölder's inequality, and Young's inequality:

$$\begin{aligned}
 (4.4) \quad \text{I} &= - \int_{\mathbb{R}^2} \operatorname{div}(\bar{v}_\mu \theta) \bar{\theta}_\mu \, dx = \int_{\mathbb{R}^2} \bar{v}_\mu \theta \cdot \nabla \bar{\theta}_\mu \, dx \\
 &\lesssim \|\theta\|_{L^\infty} \|\bar{v}_\mu\|_{L^2} \|\nabla \bar{\theta}_\mu\|_{L^2} \leq C(\varepsilon) \|\theta\|_{L^\infty}^2 \|\bar{v}_\mu\|_{L^2}^2 + \frac{1}{2\varepsilon} \|\nabla \bar{\theta}_\mu\|_{L^2}^2.
 \end{aligned}$$

For the second term II, we first exploit Taylor's formula to write

$$\begin{aligned}
 \kappa(\theta_\mu) - \kappa(\theta) &= (\theta_\mu - \theta) \int_0^1 \kappa'(\theta_\mu + \vartheta(\theta - \theta_\mu)) \, d\vartheta \\
 &= \bar{\theta}_\mu \int_0^1 \kappa'(\theta_\mu + \vartheta(\theta - \theta_\mu)) \, d\vartheta.
 \end{aligned}$$

Thus, from (1.1), we have

$$|\kappa(\theta_\mu) - \kappa(\theta)| \leq \kappa_0 |\bar{\theta}_\mu|.$$

Hence, by Hölder's inequality, Ladyzhenskaya's inequality, (3.11), and Young's inequality we get

$$\begin{aligned}
 |\text{II}| &\lesssim \int_{\mathbb{R}^2} |\bar{\theta}_\mu \nabla \theta \cdot \nabla \bar{\theta}_\mu| \, dx \lesssim \|\bar{\theta}_\mu\|_{L^4} \|\nabla \theta\|_{L^4} \|\nabla \bar{\theta}_\mu\|_{L^2} \\
 &\lesssim \|\bar{\theta}_\mu\|_{L^2}^{1/2} \|\nabla \bar{\theta}_\mu\|_{L^2}^{1/2} \|\theta\|_{L^\infty}^{1/2} \|\Delta \theta\|_{L^2}^{1/2} \|\nabla \bar{\theta}_\mu\|_{L^2} \\
 &\leq C(\varepsilon) \|\bar{\theta}_\mu\|_{L^2} \|\nabla \bar{\theta}_\mu\|_{L^2} \|\theta\|_{L^\infty} \|\Delta \theta\|_{L^2} + \frac{1}{2\varepsilon} \|\nabla \bar{\theta}_\mu\|_{L^2}^2 \\
 &\leq C(\varepsilon) \|\theta\|_{L^\infty}^2 \|\Delta \theta\|_{L^2}^2 \|\bar{\theta}_\mu\|_{L^2}^2 + \frac{1}{4\varepsilon} \|\nabla \bar{\theta}_\mu\|_{L^2}^2.
 \end{aligned}$$

Plugging (4.4) and the last estimate into (4.3), and using (1.1), we obtain

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\bar{\theta}_\mu(t)\|_{L^2}^2 + \kappa_0^{-1} \|\nabla \bar{\theta}_\mu\|_{L^2}^2 &\lesssim \|\theta\|_{L^\infty}^2 \|\bar{v}_\mu\|_{L^2}^2 + \|\theta\|_{L^\infty}^2 \|\Delta \theta\|_{L^2}^2 \|\bar{\theta}_\mu\|_{L^2}^2 \\
 &\quad + \frac{3}{4\varepsilon} \|\nabla \bar{\theta}_\mu\|_{L^2}^2,
 \end{aligned}$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|\bar{\theta}_\mu(t)\|_{L^2}^2 + \kappa_0^{-1} \|\nabla \bar{\theta}_\mu\|_{L^2}^2 \lesssim \|\theta\|_{L^\infty}^2 \|\bar{v}_\mu\|_{L^2}^2 + \|\theta\|_{L^\infty}^2 \|\Delta \theta\|_{L^2}^2 \|\bar{\theta}_\mu\|_{L^2}^2.$$

Combining the last estimate with (4.2), we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \Pi_\mu(t) &\lesssim \frac{\mu}{2} \|\omega\|_{L^2}^2 + \Pi_\mu(t) + (\|\theta\|_{L^\infty}^2 + \|\theta\|_{L^\infty}^2 \|\Delta\theta\|_{L^2}^2) \Pi_\mu(t) \\ &\quad + q \|\omega\|_{L^q} (\|v_\mu\|_{L^\infty} + \|v\|_{L^\infty})^{2/q} \Pi_\mu(t)^{1-1/q}. \end{aligned}$$

Consequently,

$$(4.5) \quad \frac{1}{2} \frac{d}{dt} \Pi_\mu(t) \lesssim \frac{\mu}{2} \|\omega\|_{L^2}^2 + (\alpha(t) + \beta(t) + 1) (\Pi_\mu(t) + q \Pi_\mu(t)^{1-1/q})$$

with $\alpha = \|\theta\|_{L^\infty}^2 + \|\theta\|_{L^\infty}^2 \|\Delta\theta\|_{L^2}^2$ and $\beta = \|\omega\|_{L^2 \cap L^\infty} (\|v_\mu\|_{L^\infty} + \|v\|_{L^\infty})^{2/q}$. Choosing $q = 2 - 2 \log \Pi$, we get

$$(4.6) \quad q \Pi_\mu^{1-1/q} \leq 2e^{1/2} \Pi_\mu \log(e^2 / \Pi_\mu).$$

By substituting the last estimate into (4.5) and integrating in time, we find that

$$\begin{aligned} \Pi_\mu(t) &\lesssim \Pi_\mu(0) + \mu t \|\omega\|_{L_t^\infty L^2}^2 \\ &\quad + \int_0^t (\alpha(\tau) + \beta(\tau) + 1) \Pi_\mu(\tau) \log\left(\frac{e^2}{\Pi_\mu(\tau)}\right) d\tau \\ &\lesssim \Pi_\mu(0) + \mu t \|\omega\|_{L_t^\infty L^2}^2 \\ &\quad + \sup_{\tau \in [0, t]} (\alpha(\tau) + \beta(\tau) + 1) \int_0^t \Pi_\mu(\tau) \log\left(\frac{e^2}{\Pi_\mu(\tau)}\right) d\tau. \end{aligned}$$

Thanks to Proposition 3.1 Proposition 3.2 and (3.28), we deduce that

$$\Pi_\mu(t) \lesssim \Pi_\mu(0) + \mu t \exp(C_0 t) + \exp(\exp(C_0 t^2)) \int_0^t \Pi_\mu(\tau) \log\left(\frac{e^2}{\Pi_\mu(\tau)}\right) d\tau.$$

However, $\Pi_\mu(0) = 0$. Thus, applying Osgood's Lemma 5.1 gives

$$\begin{aligned} \Pi_\mu(t) &\lesssim (\mu t \|\omega\|_{L_t^\infty L^2}^2)^{\exp(-\sup_{\tau \in [0, t]} (\alpha(\tau) + \beta(\tau) + 1)t)} \\ &\quad \times e^{1 - \exp(-\sup_{\tau \in [0, t]} (\alpha(\tau) + \beta(\tau) + 1)t)}. \end{aligned}$$

Thanks to Propositions 3.1 and 3.2 and 3.28, we deduce that

$$\begin{aligned} \|\bar{v}_\mu(t)\|_{L^2}^2 + \|\bar{\theta}_\mu(t)\|_{L^2}^2 &\leq (C_0(1+t)\mu t)^{\exp(-\exp(\exp(C_0 t^2))t)} \\ &\quad \times e^{1 - \exp(-\exp(\exp(C_0 t^2))t)}. \end{aligned}$$

Consequently,

$$\|v_\mu - v\|_{L_t^\infty L^2} + \|\theta_\mu - \theta\|_{L_t^\infty L^2} \leq (\mu t)^{\frac{1}{2} \exp(-\exp(\exp(C_0 t^2))t)} \Phi(t),$$

where

$$\Phi(t) = C_0(1+t)^{\frac{1}{2} \exp(-\exp(\exp(C_0 t^2))t)} e^{\frac{1}{2} [1 - \exp(-\exp(\exp(C_0 t^2))t)]}.$$

This completes the proof. ■

5. Appendix. In this appendix, we present some technical lemmas used in this article. We begin by recalling the following version of Osgood's lemma, which is a particular case of [2, Lemma 3.4].

LEMMA 5.1. *Let f be a measurable function from $[t_0, T]$ to $[0, a]$, and γ a locally integrable function from $[t_0, T]$ to $\mathbb{R}^+$. Assume that, for some nonnegative real number c , the function f satisfies*

$$f(t) \leq c + \int_{t_0}^t \gamma(\tau) f(\tau) \log\left(\frac{a}{f(\tau)}\right) d\tau \quad \text{for } a, t \in [t_0, T].$$

Then

$$f(t) \leq c^{\exp(-\int_{t_0}^t \gamma(\tau) d\tau)} a^{1-\exp(-\int_{t_0}^t \gamma(\tau) d\tau)}.$$

The following two estimates, related to fractional differential operators, can be found in various classical works; see, for example, [2, 34].

LEMMA 5.2. *Let $p, p_1, p_3 \in (1, \infty)$ and $p_2, p_4 \in [1, \infty]$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then for $s > 0$, there exists a positive constant C such that

$$(A.1) \quad \|[A^s, f]g\|_{L^p} \leq C(\|A^{s-1}g\|_{L^{p_1}} \|\nabla f\|_{L^{p_2}} + \|A^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}).$$

In particular,

$$(A.2) \quad \|A^s(fg)\|_{L^p} \leq C(\|A^s g\|_{L^{p_1}} \|f\|_{L^{p_2}} + \|A^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}).$$

Next, we state the boundedness result concerning the effect of left composition with a smooth function on Besov spaces [2, Theorem 2.61].

LEMMA 5.3. *Let $s > 0$, $p, r \in [1, \infty] \times [1, \infty]$ and $f \in \dot{B}_{p,r}^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Assume that $u(\cdot)$ is a smooth function on $\mathbb{R}$ with $u(0) = 0$. Then*

$$\|u(f)\|_{\dot{B}_{p,r}^s} \leq C(1 + \|f\|_{L^\infty})^{[s]+1} \|f\|_{\dot{B}_{p,r}^s},$$

where the constant C depends on

$$\sup_{k \leq [s]+2, |a| \leq \|f\|_{L^\infty}} \|u^{(k)}(a)\|_{L^\infty}.$$

We conclude this section with a refined logarithmic Gronwall inequality (see, for example, [38, Lemma A.4]).

LEMMA 5.4. *Let $T > 0$, and let A and B be absolutely continuous and nonnegative functions on $(0, T)$ satisfying*

$$A'(t) + B(t) \leq [l(t) + m(t) \ln(A + e) + n(t) \ln(A + B + e)](A + e) + f(t)$$

for any $t \in (0, T)$, where $l(t)$, $m(t)$, $n(t)$ and $f(t)$ are all nonnegative and integrable functions on $(0, T)$. Assume further that there are constants $K, \alpha \in$

$[0, \infty)$ and $\beta \in [0, 1)$ such that for any $t \in (0, T)$,

$$n(t) \leq K(A(t) + e)^\alpha (A(t) + B(t) + e)^\beta.$$

Then, for any $t \in (0, T)$, we have

$$A(t) + \int_0^t B(\tau) d\tau \leq e^\Omega,$$

where

$$\begin{aligned} \Omega = & \left(X(0) + \int_0^t (C(\alpha, \beta, K) + f(\tau) + l(\tau)) d\tau \right) \\ & \times \exp \left(\int_0^t (m(\tau) + C(\alpha, \beta)n(\tau)) d\tau \right). \end{aligned}$$

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