

REMARK ON AN OVERDETERMINED PROBLEM FOR THE
 p -HARMONIC EQUATION IN ANNULAR DOMAINS

BY

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Abstract. We study some overdetermined problems for the p -Laplacian in annular domains and prove the corresponding concentric sphere characterization using a Weinberger-type approach.

1. Introduction. The study of symmetry in overdetermined boundary value problems has become an important field in the theory of partial differential equations. Seminal contributions in this area include the pioneering symmetry results obtained by Serrin [19], which have become classical and remain highly influential. The primary techniques for addressing such problems are the celebrated method of moving planes, originally developed by Alexandrov [3, 4] and further advanced by Serrin [19], as well as Weinberger's approach [21], which relies on maximum principles for P -functions and the Rellich–Pohozaev integral identity.

In the present work, we focus on the radial symmetry of solutions to overdetermined problems in annular domains. Let $\bar{\Omega}_1 \subset \Omega_0$ be two connected bounded C^2 -domains in $\mathbb{R}^n$. In 1990, Philippin [17] considered the following overdetermined problem in the annular domain $\Omega := \Omega_0 \setminus \bar{\Omega}_1$:

$$(1.1) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0, \quad -\frac{\partial u}{\partial \nu} = c_0 & \text{on } \Gamma_0 := \partial\Omega_0, \\ u = 1, \quad -\frac{\partial u}{\partial \nu} = c_1 & \text{on } \Gamma_1 := \partial\Omega_1, \end{cases}$$

where $c_1 > c_0 > 0$ are constants and $\frac{\partial u}{\partial \nu}$ denotes the normal derivative of u oriented away from the origin, which lies inside Γ_1 . Philippin [17] proved that if each domain Ω_i ($i = 0, 1$) is star-shaped, then u is a solution of (1.1) if and only if u is radially symmetric and Ω is a standard annulus (see also [2, 7, 16] for more general results). He obtained this result using Weinberger's

2020 *Mathematics Subject Classification*: Primary 35J25; Secondary 35A23, 31B15.

Key words and phrases: overdetermined problem, p -Laplacian, annular domain.

Received 23 October 2025; revised 10 February 2026 and 21 May 2026.

Published online 28 October 2026.

approach. Later, Reichel [18] and Sirakov [20] obtained the same conclusion for more general cases by means of the moving plane method (see also [8, 9]).

To state our results, we recall the notion of p -capacity of the condenser (Ω_1, Ω_0) . For $p \in (1, n)$, the p -capacity of (Ω_1, Ω_0) is defined as

$$\text{Cap}_p(\Omega_1, \Omega_0) = \inf \left\{ \int_{\Omega_0} |\nabla v|^p dx \mid v \in C_c^\infty(\Omega_0), v \geq 1 \text{ on } \Omega_1 \right\}.$$

The associated p -capacitary potential is the unique weak solution u of the following problem:

$$(1.2) \quad \begin{cases} \Delta_p u = 0 & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \Gamma_0 := \partial\Omega_0, \\ u = 1 & \text{on } \Gamma_1 := \partial\Omega_1. \end{cases}$$

In this paper, we study (1.2) with the overdetermined boundary conditions

$$(1.3) \quad -\frac{\partial u}{\partial \nu} = c_0 \quad \text{on } \Gamma_0,$$

$$(1.4) \quad -\frac{\partial u}{\partial \nu} = c_1 \quad \text{on } \Gamma_1,$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ and $c_1 > c_0 > 0$ are constants. The existence of solutions for (1.2) and (1.4) is related to an interior Bernoulli free boundary problem, while the existence for (1.2) and (1.3) pertains to an exterior Bernoulli free boundary problem (see [1, 5, 11, 10, 12] for details).

Motivated by Philippin's results [17], we use Weinberger's approach to obtain the following theorems.

THEOREM 1.1. *Let $1 < p \leq n$ and let $\Omega = \Omega_0 \setminus \bar{\Omega}_1$ be a bounded, connected annular domain with $C^{2,\alpha}$ boundary. If Γ_i ($i = 0, 1$) are star-shaped, then u is a weak solution of (1.2)–(1.4) if and only if Γ_1 and Γ_0 are concentric spheres whose radii are given by*

$$R_i = \begin{cases} \left(c_i \ln \frac{c_1}{c_0} \right)^{-1}, & p = n, \\ \frac{n-p}{p-1} \left(c_1^{\frac{n-p}{n-1}} - c_0^{\frac{n-p}{n-1}} \right)^{-1} c_i^{-\frac{p-1}{n-1}}, & 1 < p < n. \end{cases}$$

The solution u is radially symmetric and given by

$$u = \begin{cases} \left(\ln \frac{c_0}{c_1} \right)^{-1} \ln \frac{r}{R_0}, & p = n, \\ \left(\frac{n-p}{p-1} \right)^{\frac{n-p}{p-1}} \left(c_1^{\frac{n-p}{n-1}} - c_0^{\frac{n-p}{n-1}} \right)^{-\frac{n-1}{p-1}} \left(r^{-\frac{n-p}{p-1}} - R_0^{-\frac{n-p}{p-1}} \right), & 1 < p < n. \end{cases}$$

THEOREM 1.2. Let $\Omega = B_{R_0} \setminus \bar{\Omega}_1$ (or $\Omega = \Omega_0 \setminus \bar{B}_{R_1}$) be a bounded, connected annular domain with $C^{2,\alpha}$ boundary.

- (1) If $1 < p < n$, then a weak solution u of (1.2)–(1.4) exists if and only if Γ_1 (resp. Γ_0) is a sphere concentric with ∂B_{R_0} (resp. ∂B_{R_1}).
- (2) If $p = n$, then a weak solution u of (1.2) and (1.4) (or (1.2) and (1.3)) exists if and only if Γ_1 (resp. Γ_0) is a sphere concentric with ∂B_{R_0} (resp. ∂B_{R_1}).

REMARK 1.3. (1) A domain is called *star-shaped* with respect to the origin if each component of its boundary can be represented as a graph over a geodesic sphere centered at the origin.

(2) Henrot and Shahgholian [10] proved that if Ω_1 is convex, then the exterior free boundary problem admits a unique convex solution Ω_0 .

2. Proof of Theorem 1.1. We begin with a Rellich–Pohozaev-type identity.

PROPOSITION 2.1. Let $1 < p \leq n$ and let $\Omega_i \subset \mathbb{R}^n$ ($i = 0, 1$) be two bounded star-shaped domains with $C^{2,\alpha}$ boundary such that $\Omega_1 \subset \Omega_0$. If u is a weak solution of (1.2)–(1.4), then

$$(2.1) \quad (n-p)|\Gamma_1|c_1^{p-1} = n(p-1)(|\Omega_1|c_1^p - |\Omega_0|c_0^p).$$

Proof. Let ν be the outer unit normal vector on $\partial\Omega := \Gamma_0 \cup \Gamma_1$. Direct computation yields

$$\begin{aligned} \int_{\partial\Omega} |\nabla u|^p \langle x, \nu \rangle d\sigma &= \int_{\Omega} \operatorname{div}(x|\nabla u|^p) dx \\ &= \int_{\Omega} n|\nabla u|^p dx + p \int_{\Omega} |\nabla u|^{p-2} x_i u_k u_{ki} dx \\ &= \int_{\Omega} n|\nabla u|^p dx + p \int_{\Omega} \frac{\partial}{\partial x_k} (|\nabla u|^{p-2} x_i u_k u_i) dx - p \int_{\Omega} |\nabla u|^p dx \\ &= (n-p) \int_{\Omega} |\nabla u|^p dx + p \int_{\partial\Omega} |\nabla u|^p \langle x, \nu \rangle d\sigma. \end{aligned}$$

Since u is a weak solution of (1.2), we have

$$\int_{\Omega} |\nabla u|^p dx = \int_{\partial\Omega} u |\nabla u|^{p-2} \langle \nabla u, \nu \rangle d\sigma = c_1^{p-1} |\Gamma_1|$$

and

$$\int_{\partial\Omega} |\nabla u|^p \langle x, \nu \rangle d\sigma = c_1^p \int_{\Gamma_1} \langle x, \nu \rangle d\sigma + c_0^p \int_{\Gamma_0} \langle x, \nu \rangle d\sigma = nc_0^p |\Omega_0| - nc_1^p |\Omega_1|.$$

Combining these identities gives (2.1). ■

Next, we introduce the P -function

$$\Phi = \begin{cases} (u + \alpha)^{-\frac{p(n-1)}{n-p}} |\nabla u|^p, & 1 < p < n, \\ e^{-\beta u} |\nabla u|^n, & p = n, \end{cases}$$

where α and β are constants to be chosen later. It follows from [22, Proposition 3.8] that Φ satisfies a strong maximum principle. However, we provide a detailed adaptation here for the reader's convenience, inspired by [15].

PROPOSITION 2.2. *Let Ω be a bounded, connected annular domain with $C^{2,\alpha}$ boundary and let u be a weak solution of (1.2)–(1.4). Then, on the set $\{\nabla u \neq 0\}$,*

$$a_{ij}\Phi_{ij} + \mathcal{L}_i\Phi_i \geq \begin{cases} p|\nabla u|^{p-2}(u + \alpha)^{-\frac{p(n-1)}{n-p}} \chi_{ij}\chi_{ij} \geq 0, & 1 < p < n, \\ n|\nabla u|^{n-2}e^{-\beta u} \chi_{ij}\chi_{ij} \geq 0, & p = n, \end{cases}$$

where $\mathcal{L}_i\Phi_i$ denotes terms involving only first derivatives of Φ , and

$$a_{ij} = \delta_{ij} + (p-2) \frac{u_i u_j}{|\nabla u|^2},$$

$$\chi_{ij} = u_{ij} - \frac{u_{ik} u_k u_j}{|\nabla u|^2} + \frac{p-1}{n-1} \frac{u_{ks} u_k u_s}{|\nabla u|^2} \left(\delta_{ij} - \frac{u_i u_j}{|\nabla u|^2} \right).$$

Moreover, Φ cannot attain a local maximum at an interior point of Ω unless Φ is constant.

Proof. We can see that

$$(2.2) \quad \Phi_i = p|\nabla u|^{p-2}(u + \alpha)^{-pp^*} u_k u_{ki} - pp^*(u + \alpha)^{-pp^*-1} |\nabla u|^p u_i,$$

$$(2.3) \quad \Phi_{ij} = p|\nabla u|^{p-2}(u + \alpha)^{-pp^*} \left((p-2) \frac{u_k u_{ki} u_{lj} u_l}{|\nabla u|^2} - pp^* \frac{u_k u_{ki} u_j}{u + \alpha} + u_{kj} u_{ki} \right. \\ \left. + u_k u_{kij} + p^*(pp^* + 1) \frac{u_i u_j |\nabla u|^2}{(u + \alpha)^2} - pp^* \frac{u_k u_{kj} u_i}{u + \alpha} - p^* \frac{u_{ij} |\nabla u|^2}{u + \alpha} \right),$$

where $p^* = \frac{n-1}{n-p}$.

Note that $a_{ij}u_{ij} = 0$. Taking a derivative, we obtain

$$(2.4) \quad a_{ij}u_{ijk} = -a_{ijk}u_{ij} = -(p-2) \left(\frac{(u_{ik}u_j + u_i u_{jk})u_{ij}}{|\nabla u|^2} - 2 \frac{u_{ij}u_i u_j u_l u_{lk}}{|\nabla u|^4} \right).$$

Then, using (2.2)–(2.4), we compute

$$a_{ij}\Phi_{ij} = p|\nabla u|^{p-2}(u + \alpha)^{-pp^*} \left((p-2) \frac{a_{ij}u_k u_{ki} u_{lj} u_l}{|\nabla u|^2} - 2pp^* \frac{a_{ij}u_k u_{ki} u_j}{u + \alpha} \right. \\ \left. + a_{ij}u_{kj} u_{ki} - a_{ijk}u_{ij} u_k + p^*(pp^* + 1) \frac{a_{ij}u_i u_j}{(u + \alpha)^2} |\nabla u|^2 \right)$$

$$\begin{aligned}
 &= p|\nabla u|^{p-2}(u+\alpha)^{-pp^*} \left(u_{ki}u_{ki} + (p^*)^2 \left(\left(\frac{1}{p^*} - 2 \right) (p-1) + (p-2) \right) \frac{|\nabla u|^4}{(u+\alpha)^2} \right) \\
 &\quad + \text{lower-order terms in } \Phi_i \\
 &= p|\nabla u|^{p-2}(u+\alpha)^{-pp^*} \left(u_{ki}u_{ki} - \frac{u_k u_{ki} u_l u_{li}}{|\nabla u|^2} + \frac{2(p-1)}{n-1} \frac{u_i u_j u_{ij} \Delta u}{|\nabla u|^2} \right. \\
 &\quad \left. + \frac{(p-1)(p-3)}{n-1} \left(\frac{u_i u_j u_{ij}}{|\nabla u|^2} \right)^2 \right) + \text{lower-order terms in } \Phi_i \\
 &= p|\nabla u|^{p-2}(u+\alpha)^{-pp^*} \chi_{ij} \chi_{ij} + \text{lower-order terms in } \Phi_i,
 \end{aligned}$$

where “lower-order terms in Φ_i ” means terms that involve only first derivatives of Φ . In the third equality we have used

$$\begin{aligned}
 (p-2) \left(\frac{u_i u_j u_{ij}}{|\nabla u|^2} \right)^2 &= - \frac{u_i u_j u_{ij} \Delta u}{|\nabla u|^2} \\
 &= (p-2)(p^*)^2 \frac{|\nabla u|^4}{(u+\alpha)^2} + \text{lower-order terms in } \Phi_i,
 \end{aligned}$$

and

$$\frac{u_k u_{ki} u_l u_{li}}{|\nabla u|^2} = (p^*)^2 \frac{|\nabla u|^4}{(u+\alpha)^2} + \text{lower-order terms in } \Phi_i.$$

Similarly, for $p = n$ we obtain

$$a_{ij} \Phi_{ij} + \mathcal{L}_i \Phi_i \geq n |\nabla u|^{n-2} e^{-\beta u} \chi_{ij} \chi_{ij} \geq 0.$$

This completes the proof of Proposition 2.2. ■

Now we show that Φ is constant in Ω . Choose α such that

$$\left(1 + \frac{1}{\alpha} \right)^{-\frac{n-1}{n-p}} = \frac{c_0}{c_1}.$$

It is easy to see that $\Phi|_{\Gamma_1} = \Phi|_{\Gamma_0}$ is constant.

From Proposition 2.2, either Φ is constant or Φ attains its maximum on $\partial\Omega$. In either case we have $\Phi_\nu \geq 0$ on Γ_0 and $\Phi_\nu \leq 0$ on Γ_1 . Using the p -Laplace equation

$$0 = \Delta_p u = |\nabla u|^{p-2} ((p-1)u_{\nu\nu} + (n-1)Hu_\nu),$$

where H denotes the mean curvature, we obtain

(2.5)

$$\begin{aligned}
 \Phi_\nu &= \frac{p}{p-1} |\nabla u|^{p-2} (u+\alpha)^{-\frac{p(n-1)}{n-p}} \left((p-1)u_k u_{k\nu} - \frac{(n-1)(p-1)}{n-p} \frac{u_\nu |\nabla u|^2}{u+\alpha} \right) \\
 &= \frac{p}{p-1} |\nabla u|^{p-2} (u+\alpha)^{-\frac{p(n-1)}{n-p}} \left(-(n-1)H_0 u_\nu - \frac{(n-1)(p-1)}{n-p} \frac{|\nabla u|^2}{u+\alpha} \right) u_\nu \\
 &\geq 0 \quad \text{on } \Gamma_0,
 \end{aligned}$$

and similarly

(2.6)

$$\begin{aligned} \Phi_\nu &= \frac{p}{p-1} |\nabla u|^{p-2} (u+\alpha)^{-\frac{p(n-1)}{n-p}} \left(-(n-1)H_1 u_\nu - \frac{(n-1)(p-1)}{n-p} \frac{|\nabla u|^2}{u+\alpha} \right) u_\nu \\ &\leq 0 \quad \text{on } \Gamma_1, \end{aligned}$$

where H_i is the mean curvature of Γ_i .

It follows that

$$(2.7) \quad \begin{cases} H_0 \leq \frac{p-1}{n-p} \frac{c_0}{\alpha} & \text{on } \Gamma_0, \\ H_1 \geq \frac{p-1}{n-p} \frac{c_1}{1+\alpha} & \text{on } \Gamma_1. \end{cases}$$

The Minkowski integral formula (see [17, (2.21)]) states that

$$\int_{\Gamma} \langle x, \nu \rangle H \, d\sigma = |\Gamma|.$$

Since Γ_0 and Γ_1 are star-shaped, we obtain

$$(2.8) \quad \begin{cases} |\Gamma_0| = \int_{\Gamma_0} \langle x, \nu \rangle H_0 \, d\sigma \leq \frac{n(p-1)}{n-p} \frac{c_0 |\Omega_0|}{\alpha}, \\ |\Gamma_1| = \int_{\Gamma_1} \langle x, \nu \rangle H_1 \, d\sigma \geq \frac{n(p-1)}{n-p} \frac{c_1 |\Omega_1|}{1+\alpha}. \end{cases}$$

From the above we deduce

$$\begin{aligned} (2.9) \quad |\Gamma_0| c_0^{p-1} &\leq \frac{n(p-1)}{n-p} \frac{c_0^p |\Omega_0|}{\alpha} \\ &= \frac{n(p-1) c_1^p |\Omega_1| - (n-p) |\Gamma_1| c_1^{p-1}}{(n-p)\alpha} \\ &\leq \frac{n(p-1) c_1^p |\Omega_1|}{(n-p)(1+\alpha)} \leq c_1^{p-1} |\Gamma_1|. \end{aligned}$$

Note that

$$(2.10) \quad 0 = \int_{\Omega} \Delta_p u \, dx = c_1^{p-1} |\Gamma_1| - c_0^{p-1} |\Gamma_0|.$$

Thus all inequalities in (2.5)–(2.9) become equalities. Consequently, Ω_1 and Ω_0 are balls, and Φ is constant on $\bar{\Omega}$.

Finally, we prove that the balls are concentric. Since Φ is constant in Ω , we have

$$(2.11) \quad 0 = u_{ij} - \frac{u_{ik} u_k u_j}{|\nabla u|^2} + \frac{p-1}{n-1} \frac{u_{ks} u_k u_s}{|\nabla u|^2} \left(\delta_{ij} - \frac{u_i u_j}{|\nabla u|^2} \right)$$

and

$$(2.12) \quad u_{ik}u_iu_j(u+\alpha) = \frac{n-1}{n-p}u_ku_j|\nabla u|^2.$$

Substituting (2.11) into (2.12) yields

$$(2.13) \quad |\nabla u|^{\frac{n+p-2}{n-1}}(|\nabla u|^{-\frac{n+p-2}{n-1}}u_i)_j + \frac{p-1}{n-p}(u+\alpha)^{-1}|\nabla u|^2\delta_{ij} = 0.$$

On the other hand, from $\Phi = c_0^p\alpha^{-\frac{p(n-1)}{n-p}}$ we deduce

$$(2.14) \quad (u+\alpha)^{-1} = |\nabla u|^{-\frac{n-p}{n-1}}\left(c_1^{\frac{n-p}{n-1}} - c_0^{\frac{n-p}{n-1}}\right).$$

Combining (2.13) and (2.14) gives

$$(|\nabla u|^{-\frac{n+p-2}{n-1}}u_i)_j = -\frac{p-1}{n-p}\left(c_1^{\frac{n-p}{n-1}} - c_0^{\frac{n-p}{n-1}}\right)\delta_{ij}.$$

Integrating, we obtain

$$|\nabla u|^{-\frac{n+p-2}{n-1}}u_i = -\frac{p-1}{n-p}\left(c_1^{\frac{n-p}{n-1}} - c_0^{\frac{n-p}{n-1}}\right)x_i.$$

Thus,

$$u_i = \left(\frac{n-p}{p-1}\right)^{\frac{n-p}{p-1}}\left(c_1^{\frac{n-p}{n-1}} - c_0^{\frac{n-p}{n-1}}\right)^{-\frac{n-1}{p-1}}\left(r^{-\frac{n-p}{p-1}}\right)_i,$$

which implies

$$u = \left(\frac{n-p}{p-1}\right)^{\frac{n-p}{p-1}}\left(c_1^{\frac{n-p}{n-1}} - c_0^{\frac{n-p}{n-1}}\right)^{-\frac{n-1}{p-1}}\left(r^{-\frac{n-p}{p-1}} - R_0^{-\frac{n-p}{p-1}}\right).$$

From the boundary conditions, the radii of the spheres Γ_i are

$$R_i = \frac{n-p}{p-1}\left(c_1^{\frac{n-p}{n-1}} - c_0^{\frac{n-p}{n-1}}\right)^{-1}c_i^{-\frac{p-1}{n-1}}, \quad i = 0, 1.$$

Similarly, for $p = n$ we obtain

$$u = \left(\ln \frac{c_0}{c_1}\right)^{-1} \ln \frac{r}{R_0}, \quad R_i = \left(c_i \ln \frac{c_1}{c_0}\right)^{-1}.$$

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2. We treat the two cases separately.

CASE 1: $1 < p < n$. Assume first that $\Omega_0 = B_{R_0}$. Then $\Gamma_0 = \partial B_{R_0}$ has mean curvature $H_0 = 1/R_0$. From (2.7) we have

$$(3.1) \quad \frac{1}{R_0} \leq \frac{p-1}{n-p} \frac{c_0}{\alpha} \quad \text{on } \Gamma_0.$$

Note that $n|B_{R_0}| = |\partial B_{R_0}|R_0$. Using (2.1), (2.10) and (3.1) we obtain

$$\begin{aligned} (n-p)c_1^{p-1}|\Gamma_1| &= n(p-1)c_1^p|\Omega_1| - (p-1)R_0c_0^{p-1}|\partial B_{R_0}|c_0 \\ &\leq n(p-1)c_1^p|\Omega_1| - (n-p)\alpha|\Gamma_1|c_1^{p-1}, \end{aligned}$$

which simplifies to

$$(3.2) \quad c_1 \geq \frac{n-p}{p-1} \frac{(1+\alpha)|\Gamma_1|}{n|\Omega_1|}.$$

Substituting (3.2) into (2.7) yields

$$H_1 \geq \frac{|\Gamma_1|}{n|\Omega_1|}.$$

Hence $H_1 > 0$ and

$$\int_{\Gamma_1} \frac{1}{H_1} d\sigma \leq n|\Omega_1|.$$

The Heintze–Karcher inequality (see [13, 23]) gives

$$\int_{\Gamma_1} \frac{1}{H_1} d\sigma \geq n|\Omega_1|,$$

with equality only when Ω_1 is a ball. Hence Ω_1 is a ball. By Theorem 1.1, Ω is a concentric annulus.

If instead $\Omega = \Omega_0 \setminus \bar{B}_{R_1}$, then by Remark 1.3(2), Ω_0 is convex. A similar argument shows that Ω_0 is a ball, and consequently Ω is a concentric annulus.

CASE 2: $p = n$. Suppose $\Omega_1 = B_{R_1}$. From the proofs of (2.1) and (2.10) we have

$$(3.3) \quad R_1 \int_{\partial B_{R_1}} |\nabla u|^n d\sigma = nc_0^n|\Omega_0|, \quad \int_{\partial B_{R_1}} |\nabla u|^{n-1} d\sigma = c_0^{n-1}|\Gamma_0|.$$

By Hölder's inequality,

$$(3.4) \quad \int_{\partial B_{R_1}} |\nabla u|^{n-1} d\sigma \leq \left(\int_{\partial B_{R_1}} |\nabla u|^n d\sigma \right)^{\frac{n-1}{n}} |\partial B_{R_1}|^{1/n}.$$

Combining (3.3) and (3.4) gives

$$|\Gamma_0|^{\frac{n}{n-1}} \leq n|\Omega_0| |\mathbb{S}^{n-1}|^{\frac{1}{n-1}},$$

where $\mathbb{S}^{n-1}$ is the unit sphere. The isoperimetric inequality provides the opposite bound:

$$|\Gamma_0|^{\frac{n}{n-1}} \geq n|\Omega_0| |\mathbb{S}^{n-1}|^{\frac{1}{n-1}}.$$

Thus equality holds, which implies that Ω_0 is a ball.

Similarly, if $\Omega_0 = B_{R_0}$, we obtain

$$|\Gamma_1|^{\frac{n}{n-1}} = n|\Omega_1| |\mathbb{S}^{n-1}|^{\frac{1}{n-1}},$$

so Ω_1 is a ball.

This completes the proof of Theorem 1.2.

Funding. This work was supported by the National Natural Science Foundation of China (Grant No. 12201138), the Mathematics Tianyuan Fund (Grant No. 12226350) and NSF of Henan Province (Grant No. 262300421869).

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