

# CENTRAL EXTENSIONS OF LIE ALGEBRAS OF SYMPLECTIC AND DIVERGENCE FREE VECTOR FIELDS

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**Abstract.** In this review paper, we present several results on central extensions of the Lie algebra of symplectic (Hamiltonian) vector fields, and compare them with similar results for the Lie algebra of (exact) divergence free vector fields. In particular, we comment on universal central extensions and integrability to the group level.

**1. Perfect commutator ideals.** The Lie algebras of symplectic and divergence free vector fields both have perfect commutator ideals: the Lie algebras of *Hamiltonian* and *exact divergence free* vector fields, respectively.

**1.1. Symplectic vector fields.** Let  $(M, \omega)$  be a compact, connected, symplectic manifold of dimension  $2n$  (the compactness assumption mainly serves to simplify the comparison; most results at the Lie algebra level extend to noncompact manifolds). The Lie algebra of *symplectic vector fields* is denoted by

$$\mathfrak{X}(M, \omega) := \{X \in \mathfrak{X}(M) : L_X \omega = 0\}.$$

The *Hamiltonian vector field*  $X_f$  of  $f \in C^\infty(M)$  is the unique vector field such that  $i_{X_f} \omega = -df$ . Since the Lie bracket of  $X, Y \in \mathfrak{X}(M, \omega)$  is Hamiltonian with

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2010 *Mathematics Subject Classification*: Primary 17B56; Secondary 17B66, 58D05.

*Key words and phrases*: Lie algebra of vector fields, central extension, symplectic manifold, group of diffeomorphisms.

The paper is in final form and no version of it will be published elsewhere.

$i_{[X,Y]}\omega = -d\omega(X, Y)$ , the Lie algebra of Hamiltonian vector fields

$$\mathfrak{X}_{\text{ham}}(M, \omega) := \{X_f : f \in C^\infty(M)\}$$

is an ideal in  $\mathfrak{X}(M, \omega)$ . The map  $X \mapsto [i_X\omega]$  identifies the quotient of  $\mathfrak{X}(M, \omega)$  by  $\mathfrak{X}_{\text{ham}}(M, \omega)$  with the abelian Lie algebra  $H_{\text{dR}}^1(M)$ ,

$$0 \rightarrow \mathfrak{X}_{\text{ham}}(M, \omega) \rightarrow \mathfrak{X}(M, \omega) \rightarrow H_{\text{dR}}^1(M) \rightarrow 0. \tag{1}$$

PROPOSITION 1.1 ([C70]). *The Lie algebra  $\mathfrak{X}_{\text{ham}}(M, \omega)$  of Hamiltonian vector fields is perfect, and coincides with the commutator Lie algebra of the Lie algebra  $\mathfrak{X}(M, \omega)$  of symplectic vector fields.*

We equip the space  $C^\infty(M)$  of  $\mathbb{R}$ -valued, smooth functions on  $M$  with the *Poisson bracket*  $\{f, g\} = \omega(X_f, X_g)$ . The kernel of the Lie algebra homomorphism  $f \mapsto X_f$ , the space of constant functions, is central in  $C^\infty(M)$ . We thus obtain a central extension

$$0 \rightarrow \mathbb{R} \rightarrow C^\infty(M) \rightarrow \mathfrak{X}_{\text{ham}}(M, \omega) \rightarrow 0, \tag{2}$$

which is split for compact  $M$ .

**1.2. Divergence free vector fields.** Let  $(P, \mu)$  be a compact, connected manifold of dimension  $m$ , equipped with a volume form  $\mu$ . Then the Lie algebra of *divergence free vector fields* is denoted by

$$\mathfrak{X}(P, \mu) := \{X \in \mathfrak{X}(P) : L_X \mu = 0\}.$$

The *exact divergence free vector field* of  $\alpha \in \Omega^{m-2}(P)$  is the unique vector field  $X_\alpha$  such that  $d\alpha = i_{X_\alpha} \mu$ . Since the Lie bracket of  $X, Y \in \mathfrak{X}(P, \mu)$  is exact divergence free with  $i_{[X,Y]}\mu = di_X i_Y \mu$ , the Lie algebra of exact divergence free vector fields

$$\mathfrak{X}_{\text{ex}}(P, \mu) := \{X_\alpha : \alpha \in \Omega^{m-2}(P)\}$$

is an ideal in  $\mathfrak{X}(P, \mu)$ . The map  $X \mapsto [i_X \mu]$  identifies the quotient of  $\mathfrak{X}(P, \mu)$  by  $\mathfrak{X}_{\text{ex}}(P, \mu)$  with the abelian Lie algebra  $H_{\text{dR}}^{m-1}(P)$ ,

$$0 \rightarrow \mathfrak{X}_{\text{ex}}(P, \mu) \rightarrow \mathfrak{X}(P, \mu) \rightarrow H_{\text{dR}}^{m-1}(P) \rightarrow 0. \tag{3}$$

PROPOSITION 1.2 ([L74]). *The Lie algebra  $\mathfrak{X}_{\text{ex}}(P, \mu)$  of exact divergence free vector fields is perfect and coincides with the commutator Lie algebra of the Lie algebra  $\mathfrak{X}(P, \mu)$  of divergence free vector fields.*

**2. Lie algebra cocycles from closed forms.** For the symplectic vector fields as well as for the divergence free vector fields, one obtains Lie algebra 2-cocycles from closed forms and also from singular cycles on the manifold.

**2.1. Divergence free vector fields.** Each closed 2-form  $\eta$  on  $P$  defines a continuous 2-cocycle on the Lie algebra of divergence free vector fields, called a *Lichnerowicz cocycle*, by (cf. [L74])

$$\lambda_\eta(X, Y) = \int_P \eta(X, Y) \mu. \tag{4}$$

Cohomologous forms define cohomologous cocycles, so we get a linear map

$$[\eta] \in H_{\text{dR}}^2(P) \mapsto [\lambda_\eta] \in H^2(\mathfrak{X}(P, \mu), \mathbb{R}) \tag{5}$$

of  $H_{\text{dR}}^2(P)$  into the second continuous Lie algebra cohomology of  $\mathfrak{X}(P, \mu)$ . In [R95, §10], Roger outlines a proof of the following statement (we were not able to find a detailed proof in the literature):

**THEOREM 2.1.** *The map (5) is an isomorphism between  $H_{\text{dR}}^2(P)$  and the continuous second Lie algebra cohomology  $H^2(\mathfrak{X}_{\text{ex}}(P, \mu), \mathbb{R})$ .*

If  $m$  is the dimension of  $P$ , then each singular  $(m - 2)$ -cycle  $C$  defines a continuous Lie algebra 2-cocycle on  $\mathfrak{X}(P, \mu)$  by

$$\lambda_C(X, Y) = \int_C i_Y i_X \mu. \tag{6}$$

Again, homologous  $(m - 2)$ -cycles yield cohomologous cocycles, so we obtain a linear map

$$[C] \in H_{m-2}(P, \mathbb{R}) \mapsto [\lambda_C] \in H^2(\mathfrak{X}(P, \mu), \mathbb{R}). \tag{7}$$

**PROPOSITION 2.2** ([V10]). *If the cohomology classes  $[\eta] \in H_{\text{dR}}^2(P)$  and  $[C] \in H_{m-2}(P, \mathbb{R})$  are Poincaré dual to each other, then the cocycles  $\lambda_\eta$  and  $\lambda_C$  lie in the same Lie algebra cohomology class in  $H^2(\mathfrak{X}(P, \mu), \mathbb{R})$ .*

**2.2. Symplectic vector fields.** Each closed 1-form  $\alpha$  on the compact symplectic manifold  $(M, \omega)$  of dimension  $2n$  determines a continuous Lie algebra 2-cocycle on the Lie algebra of Hamiltonian vector fields, the *Roger cocycle*, given by [Ki90, R95]

$$\psi_\alpha(X_f, X_g) := \int_M f \alpha(X_g) \omega^n / n!. \tag{8}$$

This expression does not depend on the choice of the Hamiltonian function  $f$ . More generally, since  $C^\infty(M)$  is a 1-dimensional central extension of the perfect Lie algebra  $\mathfrak{X}_{\text{ham}}(M, \omega)$ , 2-cocycles on  $\mathfrak{X}_{\text{ham}}(M, \omega)$  correspond with 2-cocycles on  $C^\infty(M)$ , and we will not distinguish between the two.

Cohomologous 1-forms define cohomologous 2-cocycles, so we get a linear map

$$[\alpha] \in H_{\text{dR}}^1(M) \mapsto [\psi_\alpha] \in H^2(\mathfrak{X}_{\text{ham}}(M, \omega), \mathbb{R}). \tag{9}$$

The following statement appears first in [R95], and is proven in [JV15a, Thm. 4.17]. In the next section we sketch an outline of this proof.

**THEOREM 2.3.** *The map (9) is an isomorphism between  $H_{\text{dR}}^1(M)$  and the continuous second Lie algebra cohomology  $H^2(\mathfrak{X}_{\text{ham}}(M, \omega), \mathbb{R})$ .*

Each singular  $(2n - 1)$ -cycle  $C \in Z_{2n-1}(M)$  defines a continuous 2-cocycle on the Lie algebra  $\mathfrak{X}_{\text{ham}}(M, \omega)$  by

$$\psi_C(X_f, X_g) := \int_C f dg \wedge \omega^{n-1} / (n - 1)!. \tag{10}$$

**PROPOSITION 2.4** ([JV15a]). *If the cohomology class  $[\alpha] \in H_{\text{dR}}^1(M)$  is Poincaré dual to  $[C] \in H_{2n-1}(M, \mathbb{R})$ , then the cocycle  $\psi_C$  and the Roger cocycle  $\psi_\alpha$  are cohomologous cocycles on  $\mathfrak{X}_{\text{ham}}(M, \omega)$ .*

In contrast to the Lichnerowicz cocycles  $\lambda_\eta$  that are defined directly on  $\mathfrak{X}(P, \mu)$ , not every Roger cocycle  $\psi_\alpha$  can be extended from  $\mathfrak{X}_{\text{ham}}(M, \omega)$  to  $\mathfrak{X}(M, \omega)$ : the obstruction

depends on the cohomology class  $[\alpha] \in H_{\text{dR}}^1(M)$ . To write it down, we introduce a skew symmetric 2-form and an alternating 4-form on  $H_{\text{dR}}^1(M)$  defined by

$$([\alpha], [\beta]) := \int_M \alpha \wedge \beta \wedge \omega^{n-1} / (n-1)!,$$

$$([\alpha], [\beta_1], [\beta_2], [\beta_3]) = \int_M \alpha \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \omega^{n-2} / (n-2)!.$$

**THEOREM 2.5** ([V05]). *The Roger cocycle  $\psi_\alpha$  on  $\mathfrak{X}_{\text{ham}}(M, \omega)$  is extendible to a cocycle on  $\mathfrak{X}(M, \omega)$  if and only if the class  $[\alpha] \in H_{\text{dR}}^1(M)$  satisfies*

$$([\alpha], b_1, b_2, b_3) = \frac{1}{\text{vol}_\omega(M)} \sum_{\text{cycl}} ([\alpha], b_1)(b_2, b_3)$$

for all  $b_1, b_2, b_3 \in H_{\text{dR}}^1(M)$ , where the symplectic volume is  $\text{vol}_\omega(M) = \int_M \omega^n / n!$ .

The proof uses the Hochschild–Serre spectral sequence associated to (1). For the  $2n$ -torus and for Thurston’s symplectic manifold, all Roger cocycles extend to the Lie algebra of symplectic vector fields, while for a surface of genus  $g \geq 2$  no Roger cocycle can be extended.

**3. Universal central extension.** A continuous, linearly split central extension  $\mathfrak{z} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g}$  of the locally convex Lie algebra  $\mathfrak{g}$  is called *universal* if for every continuous, linearly split central extension  $\mathfrak{a} \rightarrow \mathfrak{g}^\sharp \rightarrow \mathfrak{g}$ , there exists a unique continuous Lie algebra homomorphism  $\phi: \widehat{\mathfrak{g}} \rightarrow \mathfrak{g}^\sharp$  such that the following diagram is commutative:

$$\begin{array}{ccccc} \mathfrak{z} & \longrightarrow & \widehat{\mathfrak{g}} & \xrightarrow{q} & \mathfrak{g} \\ \downarrow \phi|_{\mathfrak{z}} & & \downarrow \phi & & \downarrow \text{Id} \\ \mathfrak{a} & \longrightarrow & \mathfrak{g}^\sharp & \longrightarrow & \mathfrak{g}. \end{array}$$

If a universal central extension exists, then it is unique up to continuous isomorphisms. If  $\mathfrak{z}$  is finite-dimensional, then the continuous second Lie algebra cohomology satisfies  $H^2(\mathfrak{g}, \mathbb{R}) \simeq \mathfrak{z}^*$ .

**3.1. Divergence free vector fields.** The Lie algebra  $\mathfrak{X}_{\text{ex}}(P, \mu)$  of exact divergence free vector fields admits a central extension

$$0 \rightarrow Z^{m-2}(P) \rightarrow \Omega^{m-2}(P) \xrightarrow{q} \mathfrak{X}_{\text{ex}}(P, \mu) \rightarrow 0$$

on the level of *Leibniz algebras* (this observation is due to Brylinski, cf. [R95]). The projection  $q$  is given by  $q(\alpha) = X_\alpha$ , and the (left) Leibniz bracket on  $\Omega^{m-2}(P)$  is given by

$$[\alpha, \beta] = L_{X_\alpha} \beta = i_{X_\alpha} i_{X_\beta} \mu + di_{X_\alpha} \beta. \tag{11}$$

Indeed, the left Leibniz identity  $[\alpha, [\beta, \gamma]] = [[\alpha, \beta], \gamma] + [\beta, [\alpha, \gamma]]$  follows from  $[X_\alpha, X_\beta] = X_{[\alpha, \beta]}$ . Since the Leibniz bracket is skew-symmetric modulo exact terms, the Fréchet space  $\Omega^{m-2}(P)/d\Omega^{m-3}(P)$  inherits the Lie bracket

$$[[\alpha], [\beta]] = [i_{X_\alpha} i_{X_\beta} \mu].$$

The map  $q : [\alpha] \mapsto X_\alpha$  then becomes a Lie algebra homomorphism, making

$$H_{\text{dR}}^{m-2}(P) \rightarrow \Omega^{m-2}(P)/d\Omega^{m-3}(P) \xrightarrow{q} \mathfrak{X}_{\text{ex}}(P, \mu) \tag{12}$$

a central extension of Lie algebras, called the *Lichnerowicz central extension*. A consequence of Theorem 2.1 is:

**THEOREM 3.1.** *The universal central extension of  $\mathfrak{X}_{\text{ex}}(P, \mu)$  is the Lichnerowicz central extension (12).*

**3.2. Symplectic vector fields.** In order to describe the universal central extension of  $\mathfrak{X}_{\text{ham}}(M, \omega)$ , we introduce the *canonical homology*  $H_{\bullet}^{\text{can}}(M, \pi)$  of a Poisson manifold  $(M, \pi)$ . It is defined as the homology of the complex  $\Omega^\bullet(M)$ , equipped with the degree decreasing differential ([K84])

$$\delta = i_\pi d - di_\pi .$$

For the special case of a Poisson structure  $\pi = \omega^\sharp$  derived from a symplectic form, the following result of Brylinski expresses  $\delta$  in terms of the *symplectic Hodge star operator*  $*$ :  $\Omega^k(M) \rightarrow \Omega^{2n-k}(M)$ . This is the natural analogue of the ordinary (Riemannian) Hodge star operator in the symplectic context, defined uniquely by the requirement that  $\alpha \wedge * \beta = \wedge^k \omega(\alpha, \beta) \omega^n / n!$  for all  $\alpha, \beta \in \Omega^k(M)$ .

**THEOREM 3.2** ([Br88]). *If  $\pi = \omega^\sharp$  for a symplectic manifold  $(M, \omega)$ , then the differential  $\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$  is*

$$\delta = (-1)^{k+1} * d * .$$

In particular, this shows that the  $*$ -operator yields an isomorphism between the canonical complex

$$\dots \rightarrow \Omega^2(M) \xrightarrow{\delta} \Omega^1(M) \xrightarrow{\delta} \Omega^0(M) \rightarrow 0 \tag{13}$$

and the de Rham complex

$$\dots \rightarrow \Omega^{2n-2}(M) \xrightarrow{d} \Omega^{2n-1}(M) \xrightarrow{d} \Omega^{2n}(M) \rightarrow 0. \tag{14}$$

We thus obtain an isomorphism  $H_k^{\text{can}}(M, \omega^\sharp) \simeq H_{\text{dR}}^{2n-k}(M)$ .

**3.2.1. Description of the universal central extension.** The universal central extension of  $\mathfrak{X}_{\text{ham}}(M, \omega)$  is now obtained from (13) by quotienting out  $\delta\Omega^2(M)$ . The quotient space  $\Omega^1(M)/\delta\Omega^2(M)$  becomes a Lie algebra with the bracket ([JV15a])

$$[[\alpha], [\beta]] = [\delta\alpha d\delta\beta], \tag{15}$$

and the Lie algebra homomorphism  $[\alpha] \mapsto X_{\delta\alpha}$  yields the central extension

$$H_1^{\text{can}}(M, \omega^\sharp) \rightarrow \Omega^1(M)/\delta\Omega^2(M) \xrightarrow{q} \mathfrak{X}_{\text{ham}}(M, \omega). \tag{16}$$

A different description of this extension is obtained if we identify the Lie algebra  $\Omega^1(M)/\delta\Omega^2(M)$  with  $\Omega^{2n-1}(M)/d\Omega^{2n-2}(M)$  using the symplectic Hodge star operator. The Lie bracket (15) becomes

$$[[\gamma_1], [\gamma_2]] = [f_1 df_2 \wedge \omega^{n-1} / (n-1)!], \tag{17}$$

where the function  $f \in C^\infty(M)$  is uniquely defined by the form  $\gamma \in \Omega^{2n-1}(M)$  via  $f\omega^n/n! = d\gamma$ . The central extension (16) thus becomes

$$H_{\text{dR}}^{2n-1}(M) \rightarrow \Omega^{2n-1}(M)/d\Omega^{2n-2}(M) \rightarrow \mathfrak{X}_{\text{ham}}(M, \omega). \tag{18}$$

**THEOREM 3.3** ([JV15a]). *The central extension (16) of  $\mathfrak{X}_{\text{ham}}(M, \omega)$ , and hence also the isomorphic extension (18), is universal.*

*Proof.* We give only an outline of the proof, and refer to [JV15a] for details. The main point is to classify continuous  $\mathbb{R}$ -values 2-cocycles on  $C^\infty(M)$ . Every such cocycle gives rise to an antisymmetric derivation  $D : C^\infty(M) \rightarrow C^\infty(M)'$  by  $D(f)g = \psi(f, g)$ . One first shows that  $\psi$  must be diagonal, in the sense that  $\psi(f, g) = 0$  whenever the supports of  $f$  and  $g$  are disjoint. This implies that  $D$  is support decreasing, hence a differential operator of locally finite order by Peetre's theorem. Using the fact that  $D$  is a derivation, an explicit calculation in local coordinates shows that this differential operator is of order 1. In fact,  $\psi$  is of the form  $\psi(f, g) = S(f dg)$  for some current  $S \in \Omega^1(M)'$ . The closedness of  $\psi$  then implies that  $S$  vanishes on  $\delta\Omega^2(M)$ , so that it factors through a functional  $\bar{S} : \Omega^1(M)/\delta\Omega^2(M) \rightarrow \mathbb{R}$ .

We see that every 2-cocycle  $\psi$  on  $\mathfrak{X}_{\text{ham}}(M, \omega)$  is of the form

$$(q^*\psi)([\alpha], [\beta]) = \psi(X_{\delta\alpha}, X_{\delta\beta}) = \bar{S}([\delta\alpha d\delta\beta]) \stackrel{(15)}{=} \bar{S}([\alpha], [\beta]).$$

Thus  $q^*H^2(\mathfrak{X}_{\text{ham}}(M, \omega), \mathbb{R}) = 0$ , which shows the universality of the central extension (16). ■

Since  $H^2(\mathfrak{g}, \mathbb{R})$  is isomorphic to the dual of the centre of (16), we find in particular

$$H^2(\mathfrak{X}_{\text{ham}}(M, \omega), \mathbb{R}) \simeq H_1^{\text{can}}(M, \omega^\sharp)^*. \quad (19)$$

Identifying  $H_1^{\text{can}}(M, \omega^\sharp)^*$  with the singular homology  $H_{2n-1}(M, \mathbb{R})$ , we find that the map  $[C] \mapsto [\psi_C]$  of (10) is an isomorphism

$$H_{2n-1}(M, \mathbb{R}) \simeq H^2(\mathfrak{X}_{\text{ham}}(M, \omega), \mathbb{R}). \quad (20)$$

Similarly, we can identify  $H_1^{\text{can}}(M, \omega^\sharp)^*$  with  $H_{\text{dR}}^1(M)$  using Poincaré duality. This shows that the map  $[\alpha] \mapsto [\psi_\alpha]$  of (8) is an isomorphism

$$H_{\text{dR}}^1(M) \simeq H^2(\mathfrak{X}_{\text{ham}}(M, \omega), \mathbb{R}), \quad (21)$$

as conjectured by Roger in [R95].

**4. Integrability of Lie algebra extensions.** Not every Lie algebra 2-cocycle is integrable to a central Lie group extension: the main obstruction is the period homomorphism [N02a]. In this section we give conditions for integrability of the restriction of Lichnerowicz cocycles to  $\mathfrak{X}_{\text{ex}}(P, \mu)$  as well as integrability of Roger cocycles both on  $C^\infty(M)$  and  $\mathfrak{X}_{\text{ham}}(M, \omega)$ .

**4.1. Non-linear Grassmannians.** The *non-linear Grassmannian*  $\text{Gr}_k(M)$  is the space of  $k$ -dimensional, compact, oriented, embedded, submanifolds  $N \subseteq M$ . It is a Fréchet manifold in a natural way, see [KM97, §44]. The tangent space of  $\text{Gr}_k(M)$  at  $N$  can be naturally identified with the space of smooth sections of the normal bundle  $TN^\perp := (TM|_N)/TN$ .

We have a transgression map  $\tau : \Omega^{k+r}(M) \rightarrow \Omega^r(\text{Gr}_k(M))$  defined by

$$(\tau\alpha)([Y_1], \dots, [Y_r]) := \int_N i_{Y_r} \dots i_{Y_1}(\alpha|_N).$$

Here all  $[Y_j]$  are tangent vectors at  $N \in \text{Gr}_k(M)$ , i.e. sections of the normal bundle  $TN^\perp$ . The expression above is independent of the vector fields  $Y_j$  on  $M$  along  $N$  chosen to represent  $[Y_j]$ .

The Lie group  $\text{Diff}(M)$  acts smoothly on  $\text{Gr}_k(M)$  by  $(\varphi, N) \mapsto \varphi(N)$ . For this action, the fundamental vector field  $\tau_X \in \mathfrak{X}(\text{Gr}_k(M))$  of  $X \in \mathfrak{X}(M)$  is given by  $\tau_X(N) = [X|_N]$ . The transgression enjoys the following functorial properties:

$$\begin{aligned} d \circ \tau &= \tau \circ d, & \varphi^* \circ \tau &= \tau \circ \varphi^*, \\ i_{\tau_X} \circ \tau &= \tau \circ i_X, & L_{\tau_X} \circ \tau &= \tau \circ L_X. \end{aligned} \tag{22}$$

In particular, a closed differential form  $\alpha \in \Omega^{k+2}(M)$  transgresses to a closed 2-form  $\Omega = \tau\alpha$  on  $\text{Gr}_k(M)$ . Just as in the finite-dimensional case, a (possibly degenerate) symplectic manifold  $(\mathcal{M}, \Omega)$  is called *prequantizable* if it admits a *prequantum bundle*  $\mathcal{P} \rightarrow \mathcal{M}$ , i.e. a principal  $S^1$ -bundle with connection form  $\Theta \in \Omega^1(\mathcal{P})$  whose curvature is  $\Omega$ .

**THEOREM 4.1** ([I96], [HV04]). *Let  $\alpha \in \Omega^{k+2}(M)$  be a closed differential form with integral cohomology class. Then the non-linear Grassmannian  $\text{Gr}_k(M)$  endowed with the closed 2-form  $\Omega = \tau\alpha$  is prequantizable.*

A *quantomorphism* of a prequantum bundle  $\mathcal{P}$  is a connection-preserving automorphism. We denote the quantomorphism group of  $\mathcal{P}$  by

$$\text{Diff}(\mathcal{P}, \Theta) = \{\varphi \in \text{Diff}(\mathcal{P}) : \varphi^*\Theta = \Theta\},$$

and its identity component by  $\text{Diff}(\mathcal{P}, \Theta)_0$ . We denote by  $\text{Diff}_{\text{ham}}(\mathcal{M}, \Omega)$  the group of Hamiltonian diffeomorphisms of the underlying symplectic manifold  $(\mathcal{M}, \Omega)$ . The *prequantization central extension* of  $(\mathcal{M}, \Omega)$  is ([Ko70, S70])

$$S^1 \rightarrow \text{Diff}(\mathcal{P}, \Theta)_0 \rightarrow \text{Diff}_{\text{ham}}(\mathcal{M}, \Omega). \tag{23}$$

If  $\mathcal{M}$  is finite-dimensional, then this is a central extension of Fréchet Lie groups [RS81]. If  $\mathcal{M}$  is infinite-dimensional, then (23) is merely a central extension of topological groups. Nonetheless, we have the following theorem:

**THEOREM 4.2** ([NV03]). *Let  $\mathcal{P} \rightarrow \mathcal{M}$  be a prequantum bundle over a connected, symplectic, Fréchet manifold  $(\mathcal{M}, \Omega)$ , and let  $G \rightarrow \text{Diff}_{\text{ham}}(\mathcal{M}, \Omega)$  be a Hamiltonian action on  $\mathcal{M}$  by a connected, Fréchet Lie group  $G$ . Then the pullback of the prequantization central extension (23) by a Hamiltonian Lie group action  $G \rightarrow \text{Diff}_{\text{ham}}(\mathcal{M}, \Omega)$  provides a central Lie group extension of  $G$ .*

The corresponding Lie algebra extension of  $\mathfrak{g} = \text{Lie}(G)$  is defined by the Lie algebra 2-cocycle  $(X, Y) \mapsto \Omega(X_{\mathcal{M}}, Y_{\mathcal{M}})(x_0)$ , where  $x_0 \in \mathcal{M}$  and  $X_{\mathcal{M}}, Y_{\mathcal{M}}$  are the fundamental vector fields on  $\mathcal{M}$  with generators  $X, Y \in \mathfrak{g}$ .

**4.2. Divergence free vector fields.** The group  $\text{Diff}_{\text{ex}}(P, \mu)$  of exact volume preserving diffeomorphisms, defined as the kernel of Thurston’s flux homomorphism [B97, §3], is a Fréchet Lie group with Lie algebra  $\mathfrak{X}_{\text{ex}}(P, \mu)$ . We show that under certain integrality conditions, the restriction of the Lichnerowicz cocycles to the commutator ideal  $\mathfrak{X}_{\text{ex}}(P, \mu) \subseteq \mathfrak{X}(P, \mu)$  are integrable to Lie group extensions of  $\text{Diff}_{\text{ex}}(P, \mu)$ .

Since the cohomology class  $\frac{k}{\text{vol}_\mu(P)}[\mu]$  is integral for every  $k \in \mathbb{Z}$ , the non-linear Grassmannian  $\text{Gr}_{m-2}(P)$  with symplectic form  $\Omega = \frac{k}{\text{vol}_\mu(P)}\tau\mu$  is prequantizable by Theorem 4.1. Since the natural action of  $\text{Diff}_{\text{ex}}(P, \mu)$  on  $\text{Gr}_{m-2}(P)$  is Hamiltonian, we can apply Theorem 4.2 to the connected component  $\mathcal{M}$  of  $Q \in \text{Gr}_{m-2}(P)$ . This yields the central Lie group extension

$$S^1 \rightarrow \widehat{\text{Diff}}_{\text{ex}}(P, \mu) \rightarrow \text{Diff}_{\text{ex}}(P, \mu). \tag{24}$$

The corresponding Lie algebra extension of  $\mathfrak{X}_{\text{ex}}(P, \mu)$  is described by a multiple of the cocycle  $\lambda_Q$  of equation (6), namely

$$\left(\frac{k}{\text{vol}_\mu(P)}\tau\mu\right)_Q(\tau_X, \tau_Y) = \frac{k}{\text{vol}_\mu(P)} \int_Q i_Y i_X \mu. \tag{25}$$

**THEOREM 4.3** ([I96], [HV04]). *Let  $Q \subset P$  be a codimension two embedded submanifold. Then the Lie algebra extensions defined by integral multiples of the cocycle  $\frac{1}{\text{vol}_\mu(P)}\lambda_Q$  integrate to central Lie group extensions of the group of exact volume preserving diffeomorphisms  $\text{Diff}_{\text{ex}}(P, \mu)$ .*

A class  $[\eta] \in H_{\text{dR}}^2(P)$  is integral if and only if it is Poincaré dual to the singular homology class  $[Q] \in H_{m-2}(P, \mathbb{R})$  of a closed, codimension 2 submanifold of  $P$  (cf. [BT82, §12]). We infer that if  $[\eta]$  is integral, then the integral multiples  $\frac{k}{\text{vol}_\mu(P)}\lambda_\eta$  of the Lichnerowicz cocycle give rise to integrable Lie algebra extensions of  $\mathfrak{X}_{\text{ex}}(P, \mu)$ .

**4.3. Symplectic vector fields.** We study the integrability of Roger cocycles in the special case of a compact, prequantizable, symplectic manifold  $(M, \omega)$  of dimension  $2n$  with prequantum bundle  $\pi: P \rightarrow M$ . The connection form  $\theta \in \Omega^1(P)$  is then a contact form on the  $(2n + 1)$ -dimensional manifold  $P$ , because

$$\mu = \frac{1}{(n + 1)!} \theta \wedge (d\theta)^n \tag{26}$$

is a volume form. The contact manifold  $(P, \theta)$  is called the *regular* or *Boothby–Wang* contact manifold associated to  $(M, \omega)$  [BW58].

The group of quantomorphisms  $\text{Diff}(P, \theta)$  is a Fréchet Lie group. Its Lie algebra  $\mathfrak{X}(P, \theta) = \{X \in \mathfrak{X}(P) : L_X\theta = 0\}$  can be identified with the Poisson Lie algebra  $(C^\infty(M), \{\cdot, \cdot\})$ , by means of the Lie algebra isomorphism

$$\mathfrak{X}(P, \theta) \rightarrow C^\infty(M) : X \mapsto \theta(X).$$

Denote its inverse  $C^\infty(M) \rightarrow \mathfrak{X}(P, \theta)$  by  $f \mapsto \zeta_f$ , uniquely determined by two conditions:  $i_{\zeta_f}\theta = \pi^*f$  and  $i_{\zeta_f}d\theta = -\pi^*df$ . Note that  $\zeta_f \in \mathfrak{X}(P, \theta)$  is exact divergence free with respect to the volume form (26).

**PROPOSITION 4.4.** *The group of quantomorphisms  $\text{Diff}(P, \theta)$  is a subgroup of  $\text{Diff}(P, \mu)$ , while its identity component  $\text{Diff}(P, \theta)_0$  is a subgroup of  $\text{Diff}_{\text{ex}}(P, \mu)$ .*

Let  $Q \in \text{Gr}_{2n-1}(P)$  be a codimension two submanifold of  $P$ . By Theorem 4.3, the Lichnerowicz cocycle  $\frac{k}{\text{vol}_\mu(P)}\lambda_Q$  on  $\mathfrak{X}_{\text{ex}}(P, \mu)$ , with  $k \in \mathbb{Z}$ , can be integrated to a central extension of  $\text{Diff}_{\text{ex}}(P, \mu)$ . Its pullback along the inclusion  $\iota: \text{Diff}(P, \theta)_0 \hookrightarrow \text{Diff}_{\text{ex}}(P, \mu)$



yields a central Lie group extension

$$S^1 \rightarrow \widehat{\text{Diff}}(P, \theta)_0 \rightarrow \text{Diff}(P, \theta)_0.$$

The corresponding Lie algebra extension is described by the pullback of the Lichnerowicz cocycle  $\frac{k}{\text{vol}_\mu(P)} \lambda_Q$  along  $\iota_* \circ \zeta: C^\infty(M) \rightarrow \mathfrak{X}_{\text{ex}}(P, \mu)$ . One checks ([JV15b, Prop. 4.8]) that the resulting cocycle is  $-\frac{k}{\text{vol}_\mu(P)} \psi_C$ , with  $\psi_C$  the 2-cocycle on  $C^\infty(M)$  of equation (10), associated to the singular  $(2n - 1)$ -cycle  $C = \pi_* Q$  on  $M$ . Using the relation  $\text{vol}_\mu(P) = \frac{2\pi}{n+1} \text{vol}_\omega(M)$  with  $\text{vol}_\omega(M) = \int_M \omega^n/n!$  the Liouville volume, we get the following integrability theorem:

**THEOREM 4.5** ([JV15b]). *Let  $\pi : (P, \theta) \rightarrow (M, \omega)$  be a prequantum line bundle, and  $Q \subseteq P$  a codimension two embedded submanifold. Then the Lie algebra extensions defined by integral multiples of the cocycle*

$$\frac{n + 1}{2\pi \text{vol}_\omega(M)} \psi_{\pi_* Q}$$

*integrate to Lie group extensions of the quantomorphism group  $\text{Diff}(M, \theta)_0$ .*

In other words, the classes  $[\psi_C] \in H^2(C^\infty(M), \mathbb{R})$  corresponding to the lattice

$$\frac{n + 1}{2\pi \text{vol}_\omega(M)} \pi_* H_{n-1}(P, \mathbb{Z}) \subseteq H_{n-1}(M, \mathbb{R})$$

give rise to integrable cocycles of  $C^\infty(M)$ . Combining Theorem 4.5 with Poincaré duality (Proposition 2.4), one sees that the same holds for Roger cocycles  $[\psi_\alpha]$  with  $[\alpha]$  in the lattice

$$\frac{n + 1}{2\pi \text{vol}_\omega(M)} \pi_! H_{\text{dR}}^2(P)_{\mathbb{Z}} \subseteq H_{\text{dR}}^{n-1}(M),$$

where the map  $\pi_!: H_{\text{dR}}^k(P) \rightarrow H_{\text{dR}}^{k-1}(M)$  is given by fibre integration and  $H_{\text{dR}}^2(P)_{\mathbb{Z}}$  denotes the space of integral cohomology classes.

**REMARK 4.6.** By Lie’s Second Theorem for regular Lie groups [KM97, §40], the Lie algebra homomorphism  $\kappa : \mathfrak{X}_{\text{ham}}(M) \rightarrow C^\infty(M)$  that splits (2),

$$\kappa(X_f) := f - \frac{1}{\text{vol}_\omega(M)} \int_M f \omega^n/n!,$$

integrates to a group homomorphism on the universal covering group of the group of Hamiltonian diffeomorphisms:

$$K : \widetilde{\text{Diff}}_{\text{ham}}(M, \omega) \rightarrow \text{Diff}(P, \theta)_0.$$

The above extensions of the quantomorphism group can be pulled back by  $K$  to obtain extensions of the universal covering group  $\widehat{\text{Diff}}_{\text{ham}}(M, \omega)$  that integrate the cocycles  $\frac{n+1}{2\pi \text{vol}_\omega(M)} \psi_\alpha$  and  $\frac{n+1}{2\pi \text{vol}_\omega(M)} \psi_C$ , subject to the integrality conditions  $[C] \in \pi_* H_{2n-1}(P, \mathbb{Z})$  and  $[\alpha] \in \pi_! H_{\text{dR}}^2(P)_{\mathbb{Z}}$ .

**Acknowledgments.** C. Vizman was supported by the grant PN-II-ID-PCE-2011-3-0921 of the Romanian National Authority for Scientific Research. B. Janssens was supported by the NWO grant 613.001.214 “Generalised Lie algebra sheaves”.

## References

- [B97] A. Banyaga, *The Structure of Classical Diffeomorphism Groups*, Math. Appl. 400, Kluwer Academic Publ., Dordrecht 1997.
- [BW58] W. M. Boothby, H. C. Wang, *On contact manifolds*, Ann. of Math. (2) 68 (1958), 721–734.
- [BT82] R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, Grad. Texts in Math. 82, Springer, New York, 1982.
- [Br88] J.-L. Brylinski, *A differential complex for Poisson manifolds*, J. Differential Geom. 28 (1988), 93–114.
- [C70] E. Calabi, *On the group of automorphisms of a symplectic manifold*, in: Problems in Analysis (the Symposium in honor of S. Bochner), Princeton Univ. Press, Princeton, 1970, 1–26.
- [HV04] S. Haller, C. Vizman, *Non-linear Grassmannians as coadjoint orbits*, Math. Ann. 329 (2004), 771–785.
- [I96] R. S. Ismagilov, *Representations of infinite-dimensional groups*, Transl. Math. Monogr. 152, Amer. Math. Soc., Providence, RI, 1996.
- [JV15a] B. Janssens, C. Vizman, *Universal central extension of the Lie algebra of Hamiltonian vector fields*, Int. Math. Res. Not. IMRN 2016 (2016), 4996–5047.
- [JV15b] B. Janssens, C. Vizman, *Integrability of central extensions of the Poisson Lie algebra via prequantization*, arXiv:1508.00841.
- [Ki90] A. A. Kirillov, *La géométrie des moments pour les groupes de difféomorphismes*, in: Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory (Paris, 1989), Progr. Math. 92, Birkhäuser, Boston, 1990, 73–83.
- [Ko70] B. Kostant, *Quantization and Unitary Representations*, in: Lectures in Modern Analysis and Applications III, Lecture Notes in Math. 170, Springer, Berlin, 1970, 87–208.
- [K84] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, in: The Mathematical Heritage of Élie Cartan (Lyon, 1984), Astérisque, 1985, Numéro Hors Série, 257–271.
- [KM97] A. Kriegl, P. W. Michor, *The Convenient Setting of Global Analysis*, Math. Surveys Monogr. 53, Amer. Math. Soc., Providence, RI, 1997.
- [L74] A. Lichnerowicz, *Algèbre de Lie des automorphismes infinitésimaux d'une structure unimodulaire*, Ann. Inst. Fourier (Grenoble) 24 (1974), 219–266.
- [N02a] K.-H. Neeb, *Central extensions of infinite-dimensional Lie groups*, Ann. Inst. Fourier (Grenoble) 52 (2002), 1365–1442.
- [NV03] K.-H. Neeb, C. Vizman, *Flux homomorphisms and principal bundles over infinite dimensional manifolds*, Monatsh. Math. 139 (2003), 309–333.
- [RS81] T. Raţiu, R. Schmid, *The differentiable structure of three remarkable diffeomorphism groups*, Math. Z. 177 (1981), 81–100.
- [R95] C. Roger, *Extensions centrales d'algèbres et de groupes de Lie de dimension infinie, algèbre de Virasoro et généralisations*, Rep. Math. Phys. 35 (1995), 225–266.
- [S70] J.-M. Souriau, *Structure des systèmes dynamiques*, Maîtrises de Mathématiques, Dunod, Paris 1970.
- [V05] C. Vizman, *Central extensions of the Lie algebra of symplectic vector fields*, J. Lie Theory 16 (2005), 297–309.
- [V10] C. Vizman, *Lichnerowicz cocycles and central Lie group extensions*, An. Univ. Vest. Timiş. Ser. Mat.-Inform. 48 (2010), 285–297.