

THE WORKS OF WILLIAM ROWAN HAMILTON IN GEOMETRICAL OPTICS AND THE MALUS–DUPIN THEOREM

CHARLES-MICHEL MARLE

Université Pierre et Marie Curie, Paris, France

E-mail: charles-michel.marle@math.cnrs.fr, cmm1934@orange.fr

Abstract. The works of William Rowan Hamilton in geometrical optics are presented, with emphasis on the Malus–Dupin theorem. According to that theorem, a family of light rays depending on two parameters can be focused to a single point by an optical instrument made of reflecting or refracting surfaces if and only if, before entering the optical instrument, the family of rays is rectangular (i.e., admits orthogonal surfaces). Moreover, the theorem states that a rectangular system of rays remains rectangular after an arbitrary number of reflections through, or refractions across, smooth surfaces of arbitrary shape. The original proof of that theorem due to Hamilton is presented, along with another proof founded in symplectic geometry. It was the proof of that theorem that led Hamilton to introduce his *characteristic function* in optics, then in dynamics under the name of *action integral*.

1. Introduction. It was a pleasure and a honour to present this work at the international meeting “Geometry of Jets and Fields” in honour of Professor Janusz Grabowski.

The works of Joseph Louis Lagrange (1736–1813) and Siméon Denis Poisson (1781–1840) during the years 1808–1810 on the slow variations of the orbital elements of planets in the solar system, are one of the main sources of contemporary symplectic geometry. Another source of symplectic geometry, equally important but maybe not so well known, are the works on optics due to Pierre de Fermat (1601–1665), Christian Huygens (1629–1695), Étienne-Louis Malus (1775–1812), Charles François Dupin (1784–1873) and William Rowan Hamilton (1805–1865). I will deal mainly in what follows with the Malus–Dupin theorem in geometrical optics, according to which a family of light rays smoothly depending on two parameters which, when entering an optical system, has a property

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called *rectangularity*¹, keeps that property in each transparent medium in which it propagates. The optical system may be made of any number of homogeneous and isotropic transparent media of various refractive indices separated by smooth surfaces and of reflecting smooth surfaces of arbitrary shapes.

In view of explaining what is *rectangularity* of a family of light rays, let me recall a few basic concepts of geometrical optics.

In geometrical optics, luminous phenomena are described in terms of *light rays*. In classical (non-relativistic) physics, the physical space in which we live and in which light propagates is mathematically described as an affine three-dimensional space \mathcal{E} endowed, once a unit of length has been chosen, with a Euclidean structure. The set of all oriented straight lines in \mathcal{E} will be denoted by \mathcal{L} . In a homogeneous and isotropic transparent medium, a *light ray* is a connected component of the part of an oriented straight line contained in that medium, hence a segment of an element of \mathcal{L} . For dealing with reflections and refractions, it will be convenient to consider the full oriented straight line which bears that segment. A reflection in a mirror mathematically described as a smooth surface, or a refraction across a smooth surface which separates two transparent media with different refractive indices, are therefore described as *transformations* of the space \mathcal{L} , i.e. as maps defined on an open subset of \mathcal{L} , which associates to each light ray which hits the reflecting or refracting surface the corresponding reflected or refracted light ray.

It will be proven below that \mathcal{L} can be very naturally endowed with a four-dimensional smooth manifold structure.

DEFINITION 1.1. A *family of light rays smoothly depending on n parameters* ($1 \leq n \leq 4$) is an immersed (not necessarily embedded) submanifold \mathcal{F} of dimension n of \mathcal{L} . In short, it will be called *an n -parameters family of rays*, the smoothness being tacitly assumed.

DEFINITION 1.2. A *regular point* of a light ray L_0 in a two-parameter family \mathcal{F} of rays is a point $m_0 \in L_0$ with the following property: for any smooth surface $S \subset \mathcal{E}$ containing m_0 and transverse at that point to the ray R_0 , there exists an open neighbourhood U of L_0 in \mathcal{F} and an open neighbourhood V of m_0 in S , such that each ray $L \in U$ meets V at a unique point m and that the map $L \mapsto m$ is a diffeomorphism of U onto V .

DEFINITION 1.3. A two-parameter family \mathcal{F} of rays is said to be *rectangular* when for each regular point of each ray $L \in \mathcal{F}$, there exists a smooth surface orthogonally crossed by L and by all the rays in a neighbourhood of L in \mathcal{F} .

I can now indicate a mathematically precise statement of the Malus–Dupin theorem.

MAIN THEOREM 1.4 (Malus–Dupin theorem). *A rectangular family of light rays which enters into an optical system with any number of smooth reflecting or refracting surfaces remains rectangular in each homogeneous and isotropic transparent medium in which it propagates.*

COMMENTS 1.5. Regularity of a point on a ray in a two-parameter family, and rectangularity of a two-parameter family of rays, are local properties. When a point m_0 on a ray

¹It is the terminology used by Hamilton.

R_0 in a two-parameter family of rays \mathcal{F} is regular, there exists an open neighbourhood U of R_0 in \mathcal{F} and an open neighbourhood W of m_0 in \mathcal{E} such that each ray in U meets W and that each point in W belongs to a unique ray in U and is regular on that ray. By taking, for each point $m \in W$, the plane through m orthogonal to the ray $R \in U$ which bears that point, one obtains a rank 2 distribution on W . The rectangularity of the family of rays \mathcal{F} means that all the distributions so obtained are integrable in the sense of [19] definition 5.2 p. 130.

Very often, a two-parameter family of rays is such that in some parts of the physical space \mathcal{E} several sheets of that family of rays are superposed. The set of non-regular points on rays of such a family constitute the *caustic surfaces* of the family. These surfaces were studied by Hamilton as soon as 1824, when he was 19 years old [6]².

EXAMPLES 1.6. The family of rays emitted by a luminous point in a homogeneous and isotropic transparent medium is rectangular, since all the spheres centred on the luminous point are orthogonal to all rays. On each ray, any point other than the luminous point is regular.

Similarly, the family of rays emitted in a homogeneous and isotropic transparent medium by a smooth luminous surface, when each point of that surface emits only one ray in a direction orthogonal to the surface, is rectangular: it is indeed a well known geometric property of the family of straight lines orthogonal to a smooth surface.

In the three-dimensional affine Euclidean space \mathcal{E} , let D_1 and D_2 be two straight lines, orthogonal to each other, which have no common point. Let \mathcal{F} be the two-parameter family of straight lines which meet both D_1 and D_2 , oriented from D_1 to D_2 . Frobenius' theorem ([19] theorem 5.2 p. 134) proves that \mathcal{F} is not rectangular.

After a short presentation of the historical background of the Malus–Dupin theorem, I will explain its original proof due to Hamilton. Then I will present another proof founded in symplectic geometry.

2. Historical background. Étienne Louis Malus de Mitry (1775–1812) was a soldier in the French army, a mathematician and a physicist. He studied the properties of families of oriented straight lines in view of applications in optics. Moreover, he developed the undulatory theory of light due to Christian Huygens (1629–1695), discovered and studied the phenomena of *light polarization* and of *birefringence* which occurs when light propagates in some crystals. He was engaged in the disastrous military campaign launched by Napoléon in Egypt (1798–1801). In Egypt he fell ill of a terrible disease, the plague, and was miraculously cured. In 1811 he became Director of Studies at the French École Poly-

²Hamilton assumes implicitly that on each ray of a two-parameter family there exist regular points. I will not give a proof of this property, nor will I study the caustic surfaces. Readers interested in these advanced topics related to the theory of singularities are referred to [1], chapter 9, section 46, pages 248–258 and Appendix 11, pages 438–439. They will find in that book and in [5] (Introduction, pp. 1–150) applications of symplectic geometry in geometrical optics much more advanced than those discussed here.

technique. Weakened by the diseases caught in Egypt, he died of tuberculosis in 1812. During the Egypt campaign he kept a journal which was published eighty years after his death [17].

Malus proved [15] that the family of rays emitted by a luminous point (which, as seen above, is rectangular) is still rectangular after *one reflection* in a smooth mirror or *one refraction* across a smooth surface. But he was in doubt whether this property is still satisfied for *several* successive reflections or refractions [16]³. Malus' works on families of oriented straight lines were later used and much extended by Hamilton [6, 7, 8, 9, 12].

Charles François Dupin (1784–1873) was a French naval engineer and mathematician. His name is linked to several mathematical objects: *Dupin's cyclids*, remarkable surfaces he discovered when he was a young student of Gaspard Monge (1746–1818) at the French École Polytechnique; *Dupin's indicatrix* which describes the shape of a smooth surface near one of its points. I think that Arnold's theorem stated in the footnote below does not apply to these objects, which are indeed due to Dupin. He spent several years in Corfu (Greece) where he renovated the naval dockyard, while participating in the creation, then in the works of the Ionian Academy. He became Professor at the French *Conservatoire des Arts et Métiers*, where he lectured and wrote several books for the education of working classes. He had very modern ideas: he thought that young girls should receive as good an education as young boys and believed that a general increase of the education level would have beneficial effects on the whole society. Unfortunately these generous ideas are still not everywhere in application in today's world. He was exceptionally shrewd: according to Wikipedia [20], he inspired the poet and novelist *Edgar Allan Poe* (1809–1849), the character of *Auguste Dupin* appearing in the three detective stories *The Murders in the Rue Morgue*, *The Mystery of Marie Roget* and *The Purloined Letter*. He found a very neat geometric proof of the Malus–Dupin theorem for reflections [4] and he knew that the same result was true for refractions but did not publish his proof.

According to [2], Adolphe Quetelet (1796–1874) and Joseph Diaz Gergonne (1771–1859) obtained in 1825 a proof of the Malus–Dupin theorem both for reflections and for refractions. A little later, the great Irish mathematician William Rowan Hamilton (1805–1865) independently obtained a complete proof of that theorem [7]. He knew the previous works of Malus on the subject and quoted them in his own works, but it seems that he did not know the works of Dupin, Quetelet and Gergonne. Maybe this explains why that theorem, called in French textbooks on optics the *Malus–Dupin theorem* [3], is generally called *Malus' theorem* in other countries.

3. Hamilton's proof of the Malus–Dupin theorem. I present in this section Hamilton's proof of the Malus–Dupin theorem ([7]), first for a reflection, then for a refraction. While scrupulously following Hamilton's ideas, I use today's vector notations in use in

³This is a very nice example of application of the famous Arnold theorem: when a theorem or a mathematical object is named by a person's name, that person *is not* the person who proved that theorem or who created that mathematical object. V. Arnold used to add: *of course, my theorem applies to itself!*

mathematics and physics. Moreover I use figures to illustrate Hamilton’s reasoning, although there are none in his publications⁴.

3.1. A reflection. Hamilton considers a two-parameter family \mathcal{F} of rays reflected by a smooth surface M . He assumes that each ray in \mathcal{F} meets M transversally. His arguments are local since they apply to a small neighbourhood of each ray in \mathcal{F} . He proves successively three results. First, he proves that if the reflected family of rays is focused to a single point, the incident family \mathcal{F} is rectangular. Then he proves that if the incident family \mathcal{F} is rectangular, for each ray in \mathcal{F} one can choose the position and the shape of a smooth mirror in such a way that by reflection that ray and all rays in a small neighbourhood of it are focused to a single point. Moreover the point at which the reflected rays are focused can be almost freely chosen, with very few restrictions. These two results are in a way more precise than the Malus–Dupin theorem, since they precisely indicate the nature of the reflected family. Finally, Hamilton proves the Malus–Dupin theorem itself.

3.1.1. Mathematical formulae for reflections. Let me first prove some formulae which follow from the laws of reflection. Let $k = (k_1, k_2)$ be local coordinates in the family of rays \mathcal{F} defined on an open neighbourhood V of $(0, 0)$ in \mathbb{R}^2 . For each $k \in V$, the ray in \mathcal{F} with coordinates k will be denoted by $L_1(k)$. Let $\overrightarrow{u_1(k)}$ be its unitary directing vector, $P(k)$ be the point at which this ray hits the mirror M , $\overrightarrow{n(k)}$ be the unitary vector orthogonal to M at $P(k)$ directed towards the reflecting side of that mirror, $L_2(k)$ be the corresponding reflected ray and $\overrightarrow{u_2(k)}$ its unitary directing vector. The maps which associate $P(k)$, $\overrightarrow{u_1(k)}$ and $\overrightarrow{u_2(k)}$ to each $k \in V$ of course are smooth. Let $M_1(k)$ be a point on L_1 and $M_2(k)$ be a point on $L_2(k)$ (Figure 1). Their

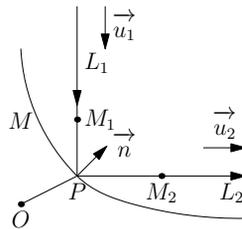


Fig. 1. Reflection

choice is up to now relatively free, it is only assumed that the maps $k \mapsto M_1(k)$ and $k \mapsto M_2(k)$ are smooth. Let O be a fixed point in the physical space \mathcal{E} taken as origin.

⁴According to the editors of Hamilton’s mathematical works, the total lack of figures in his published works could be due to Lagrange’s influence. Indeed Lagrange proudly writes in the preface of his famous book [13]: “On ne trouvera point de figure dans cet Ouvrage. Les méthodes que j’y expose ne demandent ni constructions, ni raisonnements géométriques ou mécaniques, mais seulement des opérations algébriques, assujetties à une marche régulière et uniforme. Ceux qui aiment l’Analyse, verront avec plaisir la Mécanique en devenir une nouvelle branche, et me sauront gré d’en avoir étendu ainsi le domaine”.

For short, $\overrightarrow{M_1(k)}$, $\overrightarrow{M_2(k)}$ and $\overrightarrow{P(k)}$ stand for the vectors $\overrightarrow{OM_1(k)}$, $\overrightarrow{OM_2(k)}$ and $\overrightarrow{OP(k)}$. We have

$$\overrightarrow{M_1(k)P(k)} = \overrightarrow{M_1(k)P(k)u_1(k)}, \quad \overrightarrow{P(k)M_2(k)} = \overrightarrow{P(k)M_2(k)u_2(k)},$$

where $\overrightarrow{M_1(k)P(k)}$ and $\overrightarrow{P(k)M_2(k)}$ are the algebraic values of the vectors $\overrightarrow{M_1(k)P(k)}$ and $\overrightarrow{P(k)M_2(k)}$, i.e. the lengths of the straight line segments $(M_1(k), P(k))$ and $(P(k), M_2(k))$ with the sign + if the light propagates from $M_1(k)$ towards $P(k)$, or from $P(k)$ towards $M_2(k)$ (as in Figure 1) and – otherwise. The points $M_1(k)$ and $M_2(k)$ can indeed be chosen behind the mirror M on the straight lines which support the light rays $L_1(k)$ and $L_2(k)$. By differentiating these equalities, we obtain

$$\begin{aligned} d\overrightarrow{P(k)} - d\overrightarrow{M_1(k)} &= d\overrightarrow{M_1(k)P(k)u_1(k)} + \overrightarrow{M_1(k)P(k)} d\overrightarrow{u_1(k)}, \\ d\overrightarrow{M_2(k)} - d\overrightarrow{P(k)} &= d\overrightarrow{P(k)M_2(k)u_2(k)} + \overrightarrow{P(k)M_2(k)} d\overrightarrow{u_2(k)}. \end{aligned}$$

Let us take the scalar product with $\overrightarrow{u_1(k)}$ (respectively, with $\overrightarrow{u_2(k)}$) of both sides of the first (respectively second) equality, and let us add the two equalities so obtained. Since the vectors $\overrightarrow{u_1(k)}$ and $\overrightarrow{u_2(k)}$ are unitary, $\overrightarrow{u_1(k)} \cdot d\overrightarrow{u_1(k)} = 0$ and $\overrightarrow{u_2(k)} \cdot d\overrightarrow{u_2(k)} = 0$ Therefore

$$\begin{aligned} (\overrightarrow{u_1(k)} - \overrightarrow{u_2(k)}) \cdot d\overrightarrow{P(k)} &= \overrightarrow{u_1(k)} \cdot d\overrightarrow{M_1(k)} - \overrightarrow{u_2(k)} \cdot d\overrightarrow{M_2(k)} \\ &+ d(\overrightarrow{M_1(k)P(k)} + \overrightarrow{P(k)M_2(k)}), \end{aligned} \tag{*}$$

where the dot \cdot stands for the scalar product of vectors. The laws of reflection show that the vectors $\overrightarrow{u_1(k)} - \overrightarrow{u_2(k)}$ and $\overrightarrow{n(k)}$ are parallel. Moreover any infinitesimal variation $d\overrightarrow{P(k)}$ of the vector $\overrightarrow{P(k)}$ is tangent to the surface M at $P(k)$, therefore is orthogonal to $\overrightarrow{n(k)}$. So

$$(\overrightarrow{u_1(k)} - \overrightarrow{u_2(k)}) \cdot d\overrightarrow{P(k)} = 0. \tag{**}$$

It follows from the above equalities (*), (**) that

$$\overrightarrow{u_2(k)} \cdot d\overrightarrow{M_2(k)} - \overrightarrow{u_1(k)} \cdot d\overrightarrow{M_1(k)} = d(\overrightarrow{M_1(k)P(k)} + \overrightarrow{P(k)M_2(k)}). \tag{***}$$

3.1.2. First result proven by Hamilton. Hamilton assumes that the reflection in the mirror M concentrates all the reflected rays to a single point M_2 , which can be real (before the mirror) or virtual (behind the mirror). He chooses all the points $M_2(k)$ coincident with M_2 . Therefore $d\overrightarrow{M_2(k)} = 0$, since $M_2(k) = M_2$ does not depend on k , so equality (***) above becomes

$$-\overrightarrow{u_1(k)} \cdot d\overrightarrow{M_1(k)} = d(\overrightarrow{M_1(k)P(k)} + \overrightarrow{P(k)M_2}).$$

Hamilton chooses the point $M_1(k)$ on each incoming light ray $L_1(k)$ in such a way that

$$\overrightarrow{M_1(k)P(k)} + \overrightarrow{P(k)M_2} = \text{Constant}.$$

Then

$$\overrightarrow{u_1(k)} \cdot d\overrightarrow{M_1(k)} = 0,$$

which proves that any infinitesimal variation of $M_1(k)$ is orthogonal to $\overrightarrow{u_1(k)}$. If the point $M_1(0, 0)$ is regular on the ray $L_1(0, 0)$, the points $M_1(k)$ draw a small smooth surface when k varies around $(0, 0)$, which is orthogonally crossed by the rays $L_1(k)$. This proves that the family of rays \mathcal{F} is rectangular in a neighbourhood of $L_1(0, 0)$.

3.1.3. Second result proven by Hamilton. Hamilton now assumes that the family of rays \mathcal{F} is rectangular. He tacitly assumes that there exists a regular point on the ray $L_1(0, 0)$, and he takes that point for $M_1(0, 0)$. The definition of rectangularity 1.3 shows that there exists a small smooth open surface Σ containing $M_1(0, 0)$ orthogonally crossed by the rays $L_1(k)$ for all k in some open neighbourhood V' of $(0, 0)$, $V' \subset V$. For each $k \in V'$ Hamilton chooses for $M_1(k)$ the point at which $L_1(k)$ crosses Σ .

Let $P(0, 0)$ be any regular point on the ray $L_1(0, 0)$. Of course it is possible to take $P(0, 0) = M_1(0, 0)$ but any other regular point can be chosen. That point will be the point at which $L_1(0, 0)$ hits the mirror which will be constructed. The comments 1.5 prove that there exists an open neighbourhood W of $P(0, 0)$ in \mathcal{E} whose all points are regular on the rays in V' which cross them. By restricting eventually V' and W it is possible to arrange things so that each ray in V' meets W and that each point in W is crossed by a unique ray in V' . The map $X \mapsto k(X)$, which associates to each point $X \in W$ the coordinates $k = (k_1, k_2)$ of the unique ray in V' which crosses X , is smooth. Hamilton builds in W the reflecting surface which concentrates the reflected rays to a single point.

Let M_2 be a point in the physical space \mathcal{E} which will be the point on which the reflected rays are focused. The only restriction on the choice of that point is that it must be other than $P(0, 0)$. The two functions defined on W

$$X \mapsto F_\varepsilon(X) = \overline{M_1(k(X))} X + \varepsilon \|\overrightarrow{X M_2}\|, \quad X \in W, \quad \text{with } \varepsilon = \pm 1$$

are smooth and their first differentials at $P(0, 0)$ vanish. Consider the two subsets

$$M_\varepsilon = \{X \in W; F_\varepsilon(X) = F_\varepsilon(P(0, 0))\}, \quad \varepsilon = \text{either } + \text{ or } -.$$

By eventually restricting again W and V' , things can be arranged so that the two subsets M_ε are two small open surfaces containing the point $P(0, 0)$, transversally met by all rays in V' . If M_2 is situated on the ray $L_1(0, 0)$ *before* $P(0, 0)$ (with respect to the orientation of the oriented straight line $L_1(0, 0)$), it necessarily will be a *real* convergence point of the reflected rays and one has to choose $\varepsilon = +$. If M_2 is on the ray $L_1(0, 0)$ *after* $P(0, 0)$, it necessarily will be a *virtual* convergence point of the reflected rays and one has to choose $\varepsilon = -$. For all other possible choices of M_2 , the following calculations prove that both $\varepsilon = +$ and $\varepsilon = -$ can be chosen. By taking the surface M_ε as a mirror, for any point $P \in M_\varepsilon$,

$$\varepsilon \|\overrightarrow{P M_2}\| = \overline{P M_2}$$

and for any infinitesimal variation of P on M_ε ,

$$d\left(\overline{M_1(k(P))} P + \overline{P M_2}\right) = 0, \quad \overline{u_1(k(P))} \cdot d\overline{M_1(k(P))} = 0.$$

Since $M_2(k(P)) = M_2$ does not depend on P , the equality (*) above shows that

$$\left(\overrightarrow{u_1(k(P))} - \overrightarrow{u_2(k(P))}\right) \cdot d\overrightarrow{P} = 0.$$

This proves that for each $P \in M_\varepsilon$, reflection in the mirror M_ε transforms the oriented straight $L_1 \in \mathcal{F}$ which meets M_ε at P into the oriented straight line L_2 through the point P with $\overrightarrow{u_2(k(P))}$ as unitary directing vector.

3.1.4. Proof of the Malus–Dupin theorem for a reflection. Hamilton finally proves the Malus–Dupin theorem for a reflection: if a rectangular two-parameter family of rays hits transversally a smooth reflecting surface of any shape, the corresponding family of reflected rays is rectangular. For proving this result, he chooses a regular point $M_1(0, 0)$ on the ray $L_1(0, 0)$ and a smooth surface containing this point crossed orthogonally by the rays $L_1(k)$ for all k near enough to $(0, 0)$. As point $M_1(k)$ on the incoming ray $L_1(k)$ he chooses the point at which that ray crosses that surface orthogonally, and on the corresponding reflected ray $L_2(k)$ he chooses $M_2(k)$ so that

$$\overline{M_1(k) P(k)} + \overline{P(k) M_2(k)} = \text{Constant}.$$

The above equality (***) proves that

$$\overrightarrow{u_2(k)} \cdot d\overline{M_2(k)} = 0.$$

If $M_2(0, 0)$ is a regular point on the ray $L_2(0, 0)$, the small displacements of $M_2(k)$ when k varies in a neighbourhood of $(0, 0)$ draw a smooth surface orthogonally crossed by the rays $L_2(k)$. The family of reflected rays therefore is rectangular.

3.2. A refraction. Hamilton considers a two-parameter family of rays \mathcal{F} refracted across a smooth surface R . He assumes that at the point at which an incoming ray reaches the refracting surface R , that ray and the corresponding refracted ray are transverse to that surface. He proves successively three results which correspond to those previously proven for a reflection, his proofs resting on the following equalities.

3.2.1. Mathematical formulae for a refraction. The notations are the same as those in Section 3.1.1: \mathcal{F} is a two-parameter family of rays refracted across a smooth refracting surface R . Local coordinates on \mathcal{F} , denoted by $k = (k_1, k_2)$, take their values in a neighbourhood V of $(0, 0)$ in \mathbb{R}^2 . For each $k \in V$, $L_1(k) \in \mathcal{F}$ is the ray of coordinates k , $L_2(k)$ the corresponding refracted ray, $\overrightarrow{u_1(k)}$ and $\overrightarrow{u_2(k)}$ their respective unitary directing vectors, $P(k)$ the point at which $L_1(k)$ meets the surface R , $\overrightarrow{n(k)}$ a unitary vector orthogonal to R at $P(k)$, directed for example towards the side containing the ray $L_1(k)$. Let $M_1(k)$ be a point on $L_1(k)$ and $M_2(k)$ be a point on $L_2(k)$ chosen so that the maps $k \mapsto M_1(k)$ and $k \mapsto M_2(k)$ are smooth (Figure 2).

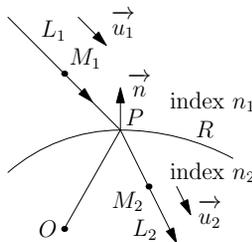


Fig. 2. Refraction

Let $O \in \mathcal{E}$ be a fixed point taken as origin. As before $\overrightarrow{M_1(k)}$, $\overrightarrow{M_2(k)}$ and $\overrightarrow{P(k)}$ stand for the vectors $\overrightarrow{OM_1(k)}$, $\overrightarrow{OM_2(k)}$ and $\overrightarrow{OP(k)}$. Let n_1 and n_2 be the refractive indices of the two transparent media separated by the surface R . Arguments similar to those used in section 3.1.1 for a reflection easily lead to the following equality, which corresponds to equality (*) of that section.

$$\begin{aligned} (n_1 \overrightarrow{u_1(k)} - n_2 \overrightarrow{u_2(k)}) \cdot d\overrightarrow{P(k)} &= n_1 \overrightarrow{u_1(k)} \cdot d\overrightarrow{M_1(k)} - n_2 \overrightarrow{u_2(k)} \cdot d\overrightarrow{M_2(k)} \\ &\quad + d(n_1 \overrightarrow{M_1(k)P(k)} + n_2 \overrightarrow{P(k)M_2(k)}). \end{aligned}$$

Equality (**) obtained for a reflection must be replaced by

$$(n_1 \overrightarrow{u_1(k)} - n_2 \overrightarrow{u_2(k)}) \cdot d\overrightarrow{P(k)} = 0,$$

which expresses Snell–Descartes’ laws of refraction.

Using these two equalities, Hamilton briefly sketches the proof of the three following results, which correspond to those he already obtained for a reflection.

3.2.2. First result proven by Hamilton. If a two-parameter family of rays is concentrated to a single point by refraction across a smooth surface which separates two transparent media with different refractive indices, before reaching that surface the family of ongoing rays is rectangular.

3.2.3. Second result proven by Hamilton. Conversely, if a two-parameter family of rays contained in a transparent medium with refractive index n_1 is rectangular, for any ray in this family one can choose the position and the shape of a smooth refracting surface separating that transparent medium from another transparent medium with refractive index $n_2 \neq n_1$ so that after refraction all the rays contained in some neighbourhood of that ray are concentrated to a single point.

3.2.4. Proof of the Malus–Dupin theorem for refraction. When a rectangular family of rays contained in a transparent medium with refractive index n_1 is refracted, through a smooth surface, into another transparent medium of refractive index $n_2 \neq n_1$, the family of refracted rays is rectangular.

3.3. The characteristic function. After proving the Malus–Dupin theorem for reflection and for refraction, Hamilton observes that this theorem remains valid for an optical device with any number of smooth reflecting or refracting surfaces: when a rectangular family of rays enters that optical device, the outgoing family of rays is also rectangular.

Hamilton introduces the notion of *characteristic function* of an optical device. He successively provides several, more and more general definitions of that notion [7, 8, 9, 12]. In its most general definition, it is a function, $(M_1, M_2) \mapsto V(M_1, M_2)$, which depends on two points M_1 and M_2 taken in the part of physical space occupied by the device, equal to the *optical length* of a light ray joining these two points, travelling into the optical device and obeying the laws of reflection and of refraction at each encounter of a reflecting or refracting smooth surface. For a device containing only reflecting surfaces, it is simply the sum of the lengths of all straight line segments constituting the light ray joining M_1 and M_2 . When the device contains refractive surfaces, it is the sum of the products of the length of each straight line segment by the refractive index of the transparent medium

which contains that segment. In [12] Hamilton even considers optical devices made by a continuous transparent media whose refractive index may depend on the considered point in space, on the direction of the light ray and on a *chromatic index* to allow the treatment of non-monochromatic light. The value $V(M_1, M_2)$ is then expressed by an *action integral* taken on the path of the light ray joining M_1 and M_2 . Applying this method to the propagation of light in birefringent crystals, Hamilton discovers the very remarkable phenomenon of *conical refraction*, whose existence was confirmed in 1833 by experiments, using an aragonite crystal, he suggested to his colleague Humphrey Lloyd of the *Trinity College* in Dublin.

Hamilton proves that the value of the action integral expressing $V(M_1, M_2)$ is *stationary* with respect to infinitesimal variations of the path on which that integral is calculated, the end points M_1 and M_2 remaining fixed. This very important result relates optics to the calculus of variations, in agreement with the principle stated in 1657 by Pierre de Fermat (1601–1665). As in optics, the characteristic function is used by Hamilton as a key concept in his famous works on dynamics [10, 11].

4. A symplectic proof of the Malus–Dupin theorem. It is proven in this section that the set \mathcal{L} of oriented straight lines in an affine Euclidean three-dimensional space \mathcal{E} is endowed with a very naturally defined symplectic form $\omega_{\mathcal{L}}$ (4.1) and that rectangular families of oriented straight lines are immersed Lagrangian submanifolds of $(\mathcal{L}, \omega_{\mathcal{L}})$ (4.3). Then it is proven that any reflection on a smooth surface is a symplectomorphism of an open subset of $(\mathcal{L}, \omega_{\mathcal{L}})$ onto another open subset of that symplectic manifold (4.4). Similarly it is proven that any refraction through a smooth surface which separates two transparent media with refractive indices n_1 and n_2 is a symplectomorphism of an open subset of $(\mathcal{L}, n_1\omega_{\mathcal{L}})$ onto an open subset of $(\mathcal{L}, n_2\omega_{\mathcal{L}})$. The Malus–Dupin theorem is an easy consequence of these results.

PROPOSITION 4.1. *Let \mathcal{E} be an affine Euclidean space of dimension 3 and \mathcal{L} be the set of oriented straight lines in \mathcal{E} . The choice of a point $O \in \mathcal{E}$ determines a one-to-one map of \mathcal{L} onto the cotangent bundle $T^*\Sigma$ to a sphere Σ of dimension 2. The pull-backs by this map of the topology, the differential manifold structure and the affine bundle structure of $T^*\Sigma$ endow \mathcal{L} with a topology, a differential manifold structure and an affine bundle structure which do not depend on the choice of O . Moreover the pull-back of the exterior differential $d\theta_{\Sigma}$ of the Liouville form θ_{Σ} on $T^*\Sigma$ is a symplectic form $\omega_{\mathcal{L}}$ on \mathcal{L} which does not depend on the choice of O .*

Proof. Let Σ be a sphere of any radius R , for example $R = 1$, centred on a point $C \in \mathcal{E}$, and let O be another point in \mathcal{E} . For each oriented straight line $L \in \mathcal{L}$, let $m_L \in \Sigma$ be the point such that $\vec{u}_L = \overrightarrow{Cm_L}$ is a directing vector of L , and η_L be the linear form on the tangent space $T_{m_L}\Sigma$ given by

$$\eta_L(\vec{w}) = \overrightarrow{OP} \cdot \vec{w}, \quad \vec{w} \in T_m\Sigma,$$

where P is any point on L and where the dot \cdot stands for the scalar product of vectors. (Figure 3). Clearly the map $\Phi_O : L \mapsto \eta_L$ so defined is one-to-one from \mathcal{L} onto $T^*\Sigma$. That map does not depend on the choice of the centre C of Σ (if two spheres of the same radius

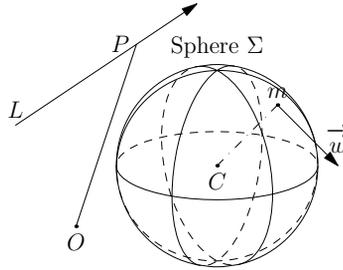


Fig. 3. The space of oriented straight lines and the cotangent bundle to a sphere

are identified by means of the translation which transports the centre of one sphere onto the centre of the other sphere). When O is replaced by another point $O' \in \mathcal{E}$, the covector $\eta_L \in T_{m_L}^* \Sigma$ is replaced by $\eta'_L \in T_{m_L}^* \Sigma$, given by $\eta'_L(\vec{w}) = \eta_L(\vec{w}) + \overrightarrow{O'O} \cdot \vec{w}$, where \vec{w} is any vector in $T_{m_L} \Sigma$. The covector η' can be expressed as $\eta'_L = \eta_L + df_{O'O}(m_L)$, where $f_{O'O} : \Sigma \rightarrow \mathbb{R}$ is the smooth function $f_{O'O}(n) = \overrightarrow{O'O} \cdot \overrightarrow{Cn}$, with $n \in \Sigma$. The one-to-one maps $\Phi_{O'}$ and Φ_O therefore are related by $\Phi_{O'} = \Psi_{O'O} \circ \Phi_O$, where $\Psi_{O'O} : T^* \Sigma \rightarrow T^* \Sigma$ is the diffeomorphism $\eta \mapsto \eta + df_{O'O}(\pi_\Sigma(\eta))$, $\pi_\Sigma : T^* \Sigma \rightarrow \Sigma$ being the canonical projection.

The identification of \mathcal{L} with $T^* \Sigma$ by means of the map Φ_O allows the transfer on \mathcal{L} of the mathematical structure of $T^* \Sigma$. So the set \mathcal{L} becomes endowed with a topology, a differentiable manifold structure, a vector bundle structure and a 1-form $\Phi_O^* \theta_\Sigma$, pull-back of the Liouville form θ_Σ on $T^* \Sigma$. When O is replaced by O' the topology, the differentiable manifold structure and the affine bundle structure of \mathcal{L} remain unchanged, while its vector bundle structure is modified; the 1-form $\Phi_O^* \theta_\Sigma$ is replaced by

$$\Phi_{O'}^* \theta_\Sigma = \Phi_O^*(\Psi_{O'O}^* \theta_\Sigma) = \Phi_O^*(\theta_\Sigma + \pi_\Sigma^*(df_{O'O})).$$

Therefore $\Phi_{O'}^*(d\theta_\Sigma) = \Phi_O^*(d\theta_\Sigma)$ is a symplectic form $\omega_\mathcal{L}$ on \mathcal{L} which does not depend on the choice of O . ■

REMARK 4.2. Let $\widehat{\mathcal{L}}$ be the set of *pointed oriented straight lines* in \mathcal{E} , i.e. the set of pairs (L, P) where $L \in \mathcal{L}$ is an oriented straight line and P is a point of L . This set is a smooth manifold of dimension 5 which projects onto the manifold \mathcal{L} of dimension 4, the projection $(L, P) \mapsto L$ amounting to “forgetting” the point P . Let O be any fixed point in \mathcal{E} . An element (L, P) in $\widehat{\mathcal{L}}$ can be represented by the pair of vectors $(\overrightarrow{OP}, \vec{u})$ made by \overrightarrow{OP} and the unitary directing vector \vec{u} of the oriented line L . There exists⁵ on $\widehat{\mathcal{L}}$ an exact differential 2-form $\omega_{\widehat{\mathcal{L}}}$ given by

$$\omega_{\widehat{\mathcal{L}}}(L, P) = \sum_{i=1}^3 dp_i \wedge du_i = d\overrightarrow{OP} \wedge d\vec{u},$$

where (p_1, p_2, p_3) and (u_1, u_2, u_3) are the components of \overrightarrow{OP} and \vec{u} in an orthonormal

⁵The choice of a point $O \in \mathcal{E}$ allows the identification of $\widehat{\mathcal{L}}$ with the restriction $T_\Sigma^* \mathcal{E}$ to the sphere Σ of the cotangent bundle $T^* \mathcal{E}$. With this identification $\omega_{\widehat{\mathcal{L}}}$ is the form induced on $T_\Sigma^* \mathcal{E}$ by the canonical symplectic form $d\lambda_\mathcal{E}$ of the cotangent bundle $T^* \mathcal{E}$.

basis. The symbol \wedge on the right hand side is a combination of scalar and exterior products. The form $\omega_{\widehat{\mathcal{L}}}$ projects onto \mathcal{L} and its projection is the symplectic form $\omega_{\mathcal{L}}$, which therefore can be expressed as

$$\omega_{\mathcal{L}}(L) = d\overrightarrow{OP} \wedge d\overrightarrow{u}, \quad L \in \mathcal{L},$$

since the right hand side does not depend on the choice of the point P on the oriented straight line L , nor on that of the point O in \mathcal{E} . This very convenient expression of $\omega_{\mathcal{L}}$ is particularly well suited when used in conjunction with the usual vector calculus in a three-dimensional Euclidean vector space.

PROPOSITION 4.3. *A two-parameter family of rays is rectangular in the sense of 1.3 if and only if it is an immersed Lagrangian submanifold of the symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$ of oriented straight lines in the affine Euclidean space \mathcal{E} .*

Proof. Each element L_0 in a two-parameter family \mathcal{F} of oriented straight lines has an open neighbourhood in \mathcal{F} which is the image of an injective smooth map $L : (k_1, k_2) \mapsto L(k_1, k_2)$, defined on an open subset of \mathbb{R}^2 containing $(0, 0)$, with values in \mathcal{L} , such that $L(0, 0) = L_0$. For each $k = (k_1, k_2)$ in the open subset of \mathbb{R}^2 on which the map L is defined, let $\overrightarrow{u}(k)$ be the unitary directing vector of $L(k)$ and $P(k)$ a point of $L(k)$. The points $P(k)$ are not uniquely determined but it is always possible to choose them to make smooth the map $k \mapsto (P(k), \overrightarrow{u}(k))$. The Remark 4.2 allows us to write

$$\begin{aligned} L^*\omega_{\mathcal{L}} &= d(\overrightarrow{P}(k) \cdot d\overrightarrow{u}(k)) = d(d(\overrightarrow{P}(k) \cdot \overrightarrow{u}(k)) - \overrightarrow{u}(k) \cdot d\overrightarrow{P}(k)) \\ &= -d(\overrightarrow{u}(k) \cdot d\overrightarrow{P}(k)), \end{aligned}$$

where $\overrightarrow{P}(k)$ stands for the vector $\overrightarrow{OP}(k)$, O being any fixed point in \mathcal{E} . The immersed submanifold \mathcal{F} is Lagrangian in a neighbourhood of L_0 if and only if $L^*\omega_{\mathcal{L}} = 0$ ([14] p. 92, or [18] p. 123), in other words if and only if the differential one-form $\overrightarrow{u}(k) \cdot d\overrightarrow{P}(k)$ is closed. Poincaré’s lemma ([19], theorem 4.1 page 121) asserts that a one-form is closed if and only if it is locally the differential of a smooth function. The immersed submanifold \mathcal{F} therefore is Lagrangian near L_0 if and only if there exists a smooth function $k \mapsto F(k)$, defined on a neighbourhood of $(0, 0)$, such that

$$\overrightarrow{u}(k) \cdot d\overrightarrow{P}(k) = dF(k). \tag{*}$$

The vector $\overrightarrow{u}(k)$ being unitary, for any constant $c \in \mathbb{R}$ we have

$$dF(k) = \overrightarrow{u}(k) \cdot d((F(k) + c)\overrightarrow{u}(k)).$$

If a smooth function F satisfying (*) exists, it also satisfies, for any constant $c \in \mathbb{R}$,

$$\overrightarrow{u}(k) \cdot d(\overrightarrow{P}(k) - (F(k) + c)\overrightarrow{u}(k)) = 0. \tag{**}$$

Let us assume that \mathcal{F} is Lagrangian in a neighbourhood of L_0 , and let F be a smooth function, defined in a neighbourhood of $(0, 0)$, which satisfies (*). Let Q_0 be a regular point on L_0 . There exists a constant $c \in \mathbb{R}$ such that $\overrightarrow{P}(0, 0) - (F(0, 0) + c)\overrightarrow{u}(0, 0) = \overrightarrow{Q}_0$, where \overrightarrow{Q}_0 stands for the vector \overrightarrow{OQ}_0 . Since the points near enough to Q_0 are regular on the rays near enough to L_0 which cross them, the variations of $\overrightarrow{P}(k) - (F(k) + c)\overrightarrow{u}(k)$ when k varies around $(0, 0)$ generate a smooth surface containing Q_0 which, as shown by

the equality (**), is orthogonally crossed by the oriented straight lines $L(k)$ for all k near enough to $(0, 0)$. The family \mathcal{F} therefore is rectangular near L_0 .

Conversely, let us assume that \mathcal{F} is rectangular near L_0 . As was tacitly done by Hamilton, I assume that there exists a regular point on L_0 . There exists a smooth surface containing this point crossed orthogonally by L_0 and by the oriented straight lines $L(k)$ for all k near enough to $(0, 0)$. This surface is made by the points $P(k) - F(k)\vec{u}(k)$, when k varies around $(0, 0)$, F being a smooth function. The function F satisfies equality (*), therefore \mathcal{F} is Lagrangian near L_0 . ■

PROPOSITION 4.4. *Let $M \subset \mathcal{E}$ be a smooth open reflecting surface in the Euclidean three-dimensional affine space \mathcal{E} . The set U of oriented straight lines which meet M transversally on its reflecting side is an open subset of the symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$ of all oriented straight lines. The map which associates to each element in U which bears a light ray the oriented straight line which bears the corresponding reflected ray is a symplectomorphism of U onto the open subset of \mathcal{L} made by the straight lines in U with the opposite orientation.*

Proof. Being determined by strict inequalities, U is an open subset of \mathcal{L} . Let L_1 be a variable element in U , P the point at which L_1 meets the reflecting surface M , L_2 the oriented straight line which bears the corresponding reflected ray, \vec{u}_1 and \vec{u}_2 the unitary directing vectors of L_1 and L_2 , respectively (Figure 1). As in Section 3.1.1, \vec{P} stands for the vector \vec{OP} , $O \in \mathcal{E}$ being a fixed point taken as origin. The expression of the symplectic form $\omega_{\mathcal{L}}$ given in Remark 4.2 shows that it is enough to prove the equality $d\vec{P} \wedge d\vec{u}_2 = d\vec{P} \wedge d\vec{u}_1$. The laws of reflection shows that $\vec{u}_2 - \vec{u}_1 = 2(\vec{u}_1 \cdot \vec{n})\vec{n}$. Therefore

$$d\vec{P} \wedge d(\vec{u}_2 - \vec{u}_1) = 2d\vec{P} \wedge d((\vec{u}_1 \cdot \vec{n})\vec{n}) = -2d((\vec{u}_1 \cdot \vec{n})(\vec{n} \cdot d\vec{P})) = 0,$$

since $\vec{n} \cdot d\vec{P} = 0$, the vectors $d\vec{P}$ and \vec{n} being, respectively, tangent to and orthogonal to the surface M at P . ■

PROPOSITION 4.5. *Let $R \subset \mathcal{E}$ be a smooth open refracting surface which separates two transparent media with refractive indices n_1 and $n_2 \neq n_1$, respectively. The set U of oriented straight lines which meet transversally the surface R on the side of the medium with refractive index n_1 under an angle such that the laws of refraction allow the existence, in the medium with refractive index n_2 , of a refracted ray transverse to R , is an open subset U of the symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$ of all oriented straight lines. The map which associates to each element in U which bears a light ray the oriented straight line which bears the corresponding refracted ray is a symplectomorphism of U endowed with the symplectic form $n_1\omega_{\mathcal{L}}$ onto another open subset of \mathcal{L} endowed with the symplectic form $n_2\omega_{\mathcal{L}}$.*

Proof. The notations being the same as those in the proof of 4.4, let \vec{v}_1 be the orthogonal projection of \vec{u}_1 onto the tangent plane to the surface R at P . The Snell–Descartes’ laws of refraction indicate that if a refracted ray L_2 corresponding to L_1 exists, the orthogonal projection \vec{v}_2 of its unitary directing vector \vec{u}_2 onto the tangent plane to the surface R

at P satisfies the equality

$$n_2 \vec{v}_2 = n_1 \vec{v}_1, \quad \text{or} \quad n_2(\vec{u}_2 - (\vec{u}_2 \cdot \vec{n})\vec{n}) = n_1(\vec{u}_1 - (\vec{u}_1 \cdot \vec{n})\vec{n}).$$

When it can be satisfied that equality determines \vec{u}_2 , therefore determines the oriented straight line L_2 which bears the refracted ray. When $n_1 \leq n_2$ that equality can always be satisfied, but when $n_1 > n_2$ it can be satisfied by a straight line L_2 transverse to R if and only if $(n_1/n_2)\|\vec{v}_1\| < 1$, i.e. if and only if the angle α_1 made by L_1 with the vector orthogonal to R at P satisfies $\sin \alpha_1 < n_2/n_1$ ⁶. That inequality being strict, U is an open subset of \mathcal{L} .

As in 4.4 it is enough to prove that $n_2 d\vec{P} \wedge d\vec{u}_2 = n_1 d\vec{P} \wedge d\vec{u}_1$. We have

$$\begin{aligned} d\vec{P} \wedge (n_2 d\vec{u}_2 - n_1 d\vec{u}_1) &= d\vec{P} \wedge d(n_2(\vec{u}_2 \cdot \vec{n})\vec{n} - n_1(\vec{u}_1 \cdot \vec{n})\vec{n}) \\ &= -d((n_2(\vec{u}_2 \cdot \vec{n}) - n_1(\vec{u}_1 \cdot \vec{n}))(\vec{n} \cdot d\vec{P})) \\ &= 0, \end{aligned}$$

since $\vec{n} \cdot d\vec{P} = 0$, the vectors $d\vec{P}$ and \vec{n} being, respectively, tangent to and orthogonal to the surface R at P . ■

5. Conclusion. Reflections on and refractions across a smooth surface being symplectomorphisms, the propagation of light through an optical device made by several reflecting and refracting smooth surfaces is a symplectomorphism (since by composition of several symplectomorphisms one gets a symplectomorphism). The image by a symplectomorphism of an immersed Lagrangian submanifold is another immersed Lagrangian submanifold, which proves the Malus–Dupin theorem 1.4.

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References

- [1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. Springer, New York, 1978.
- [2] A. W. Conway and J. L. Synge, Appendix to *Sir William Rowan Hamilton Mathematical Works, Vol. I*, Cambridge Univ. Press, London, 1931, 463–464.
- [3] J.-M. Courty, *Notes de cours de l'Université Pierre et Marie Curie, L3 Physique, PGA Chapitre VI page 48*, http://www.edu.upmc.fr/physique/phys325/Documents/Poly_II_Chap_6.pdf.
- [4] C. Dupin, *Applications de la géométrie*, Mémoire présenté à l'Académie des Sciences en 1816, publié à Paris en 1822.
- [5] V. Guillemin, S. Sternberg, *Symplectic Techniques in Physics*, Cambridge Univ. Press, Cambridge, 1984.

⁶ If that inequality is not satisfied the light ray supported by L_1 is totally reflected.

- [6] W. R. Hamilton, *On caustics, part first*, manuscript, 1824; in: Sir William Rowan Hamilton Mathematical Works, Vol. I, Chapter XV, Cambridge Univ. Press, London, 1931.
- [7] W. R. Hamilton, *Theory of systems of rays, part first and part second*, part first: Trans. Roy. Irish Acad. 15 (1828), 69–174; part second: manuscript; in: Sir William Rowan Hamilton Mathematical Works, Vol. I, Chapter I, Cambridge Univ. Press, London, 1931.
- [8] W. R. Hamilton, *Supplement to an essay on the theory of systems of rays*, Trans. Roy. Irish Acad. 16 (1830), 1–61; in: Sir William Rowan Hamilton Mathematical Works, Vol. I, Chapter II, Cambridge Univ. Press, London, 1931.
- [9] W. R. Hamilton, *Second supplement to an essay on the theory of systems of rays*, Trans. Roy. Irish Acad. 16 (1831), 93–125; in: Sir William Rowan Hamilton Mathematical Works, Vol. I, Chapter III, Cambridge Univ. Press, London, 1931.
- [10] W. R. Hamilton, *On a general method in Dynamics*, Philos. Trans. Roy. Soc. 1834, 247–308; in: Sir William Rowan Hamilton Mathematical Works, Vol. IV, Cambridge Univ. Press, London, 1931.
- [11] W. R. Hamilton, *Second essay on a general method in dynamics*, Philos. Trans. Roy. Soc. 1835, 95–144; in: Sir William Rowan Hamilton Mathematical Works, Vol. IV, Cambridge Univ. Press, London, 1931.
- [12] W. R. Hamilton, *Third supplement to an essay on the theory of systems of rays*, Trans. Roy. Irish Acad. 17 (1837), 1–144; in: Sir William Rowan Hamilton Mathematical Works, Vol. I, Chapter IV, Cambridge Univ. Press, London, 1931.
- [13] J.-L. Lagrange, *Mécanique analytique*, Première édition chez la veuve Desaint, Paris, 1808. Quatrième édition, dans *Œuvres de Lagrange*, volumes XI et XII, Gauthier-Villars, Paris, 1888.
- [14] P. Libermann, C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht, 1987.
- [15] É. L. Malus, *Journal de l'École Polytechnique* 7 (1808), 1–44 et 84–129.
- [16] É. L. Malus, *Traité d'optique*, Mémoires présentés à l'Institut par divers savants, 2 (1811), 214–302.
- [17] É. L. Malus, *L'agenda de Malus, Souvenirs de l'expédition d'Égypte 1798–1801*, Honoré Champion, Paris, 1892.
- [18] J.-P. Ortega, T. S. Ratiu, *Momentum Maps and Hamiltonian Reduction*, Birkhäuser, Boston, 2004.
- [19] S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, NJ, 1964.
- [20] Wikipedia, *Charles Dupin*, http://fr.wikipedia.org/wiki/Charles_Dupin.

