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TULCZYJEW MECHANICS FOR PARTICLES IN GAUGE FIELDS

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Abstract. It is briefly reported here that, via an idea due to S. Sternberg and a few other mathematicians and physicists, W. M. Tulczyjew's ingenious approach to the dynamics of a particle works equally well for the dynamics of an electrically charged particle in gauge fields.

1. Introduction. Tulczyjew's unified approach to Lagrangian and Hamiltonian descriptions of particle dynamics is quite appealing to geometry-oriented minds. An advantage of this geometric approach is its flexibility in the sense that it can be easily adapted to different settings, especially to systems with singular Lagrangians or subject to constraints. As one more demonstration of its flexibility, in this article it is reported that Tulczyjew's approach also works for particle dynamics in which an electrically charged particle moves in the presence of an external gauge field, either abelian or non-abelian.

This brief report is primarily devoted to the main ideas and motivations. Readers who wish to know more details should consult [M1].

The work presented here is a natural consequence of the interaction of three ingredients: (i) a canonical symplectomorphism [D], (ii) Tulczyjew's original idea [T], (iii) an idea which was to a certain extent already known to a few people such as S. K. Wong [Wo], made more explicit by S. Sternberg [S], and elaborated further by A. Weinstein [We] and R. Montgomery [M2]. Note that ingredient (i) is in principle known to Tulczyjew in its general form, and in any case its special form has been used by Tulczyjew in [T].

For the convenience of readers, let us list some standard mathematical symbols that are going to be used here: if X is a smooth manifold, then $\tau_X\colon TX \to X$ denotes the tangent bundle of X, $\pi_X\colon T^*X \to X$ denotes the cotangent bundle of X, ϑ_X denotes

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the Liouville form on T^*X and $\omega_X := \mathrm{d}\vartheta_X$ denotes the tautological symplectic form on T^*X. Also, for notational sanity in this report, we shall use the same notation for both a differential form (or a map) and its pullback under a fiber bundle projection map. For example, for a differential form Ω on manifold F, its pullback under projection $X\times F \to F$ is also denoted by Ω.

We shall use the following general fact: if X is a smooth manifold and f is a smooth real function on X, then the total differential $\mathrm{d}f$ of f is a section of the cotangent bundle $T^*X \to X$, moreover, the image of $\mathrm{d}f$, denoted by D_f, is a Lagrangian submanifold of T^*X. More generally, if Y is a submanifold of X and f is a smooth real function on Y,

$$p:\ T^*X|_Y \to T^*Y$$

is the canonical surjective vector bundle homomorphism, then

$$D_f := p^{-1}(\mathrm{Im}\,\mathrm{d}f) \tag{1}$$

is a Lagrangian submanifold of T^*X, and will be referred to as the *Lagrangian submanifold generated by f*. Here $T^*X|_Y = \bigcup_{y\in Y} T^*_yX$.

2. A canonical symplectomorphism. Let V be a real vector space, V^* be its dual space. It is well-known that $V\oplus V^*$ is a canonical symplectic vector space. Indeed, by convention, the canonical symplectic structure on $V\oplus V^*$ is the following skew-symmetric bilinear form

$$\eta\big((x_1,\zeta_1),(x_2,\zeta_2)\big) = \langle\zeta_1, x_2\rangle - \langle\zeta_2, x_1\rangle\,.$$

Let X be a smooth manifold, $\alpha \in T^*X$, and $x = \pi_X(\alpha)$. Then the tangent space $T_\alpha(T^*X)$, being equal to $T_xX \oplus T^*_xX$, is a canonical symplectic space; moreover, if q^i is a system of local coordinates around the point x, and q^i, p_j is the resulting system of local coordinates on T^*X, then $(T_\alpha(T^*X), \mathrm{d}p_i\wedge \mathrm{d}q^i|_\alpha)$ is the canonical symplectic space mentioned above. Since the differential 2-form $\mathrm{d}p_i \wedge \mathrm{d}q^i$ is closed, we conclude that T^*X is a symplectic manifold whose symplectic form is denoted by ω_X, as mentioned early. So, locally $\omega_X = \mathrm{d}p_i \wedge \mathrm{d}q^i$.

Let V^{**} be the double dual of V and ι: $V \to V^{**}$ be the natural identification: $\iota(u)(\alpha) = \alpha(u)$ for any $u \in V$ and any $\alpha \in V^*$. One can see that the map

$$V\oplus V^* \to V^*\oplus V^{**}, \qquad (u,\alpha)\mapsto(\alpha, -\iota(u)), \tag{2}$$

is a symplectic isomorphism. Now, if we view V and V^* as affine spaces, then we have symplectic manifolds $T^*V = V\times V^*$ and $T^*V^* = V^*\times V^{**}$, each of which is the Cartesian product of two affine spaces. In view of symplectic isomorphism (2), the diffeomorphism

$$T^*V \to T^*V^*, \qquad (x,\xi)\mapsto(\xi, -\iota(x)), \tag{3}$$

is a symplectomorphism. Indeed, the linearization of smooth map (3) is (naturally identified with) symplectic isomorphism (2).

It is a fact that essentially all results for vector spaces can be extended to vector bundles. So it is not a surprise that the above canonical symplectomorphism extends as well.

Theorem 2.1 (J. P. Dufour, 1990). *Let $E \to X$ be a real vector bundle and $E^* \to X$ be its dual vector bundle. Then there is a canonical symplectomorphism*

$$\kappa : T^*E^* \overset{\cong}{\to} T^*E \tag{4}$$

that generalizes the inverse of symplectomorphism (3).

This theorem was essentially known to Tulczyjew long time ago. Indeed, the case that the real vector bundle in the theorem is $TX \to X$ was used by him in [T].

3. Tulczyjew's original idea. In 1974 Tulczyjew introduced a geometric approach to classical mechanics which brings the Hamiltonian and Lagrangian formalisms under a common geometric roof. In this approach the dynamics of a particle with configuration space X is determined by a Lagrangian submanifold D of TT^*X (the total tangent space of T^*X), and the description of D by its Hamiltonian $H\colon T^*X \to \mathbb{R}$ (resp. its Lagrangian $L\colon TX \to \mathbb{R}$) yields the Hamilton (resp. Euler–Lagrange) equation. In fact, this approach works in a much more general setting, as one can see from a plethora of papers authored by members of the Polish school [TU1, U, GGU1, GGU2, TU2, TU3, PT, ST].

3.1. Tulczyjew triple. Let us now consider the dynamics of a mass point with configuration space X, a smooth manifold. In its Hamiltonian formulation, the dynamic equation can be derived from its Hamiltonian $H\colon T^*X \to \mathbb{R}$. In its Lagrangian formulation, the dynamic equation can be derived from its Lagrangian $L\colon TX \to \mathbb{R}$. Therefore, we have a Lagrangian submanifold D_{-H} of T^*T^*X and a Lagrangian submanifold D_L of T^*TX, in the sense of Eq. (1). It is an observation of Tulczyjew that, under the canonical symplectomorphism

$$\kappa : T^*T^*X \to T^*TX,$$

the two Lagrangian submanifolds D_{-H} and D_L get identified.

Since T^*X is a symplectic manifold, the total cotangent space T^*T^*X can be identified with its total tangent space TT^*X via diffeomorphism

$$TT^*X \overset{\beta}{\to} T^*T^*X, \qquad v \mapsto v \lrcorner (\omega_X|_\xi) \tag{5}$$

where $\xi = \tau_{T^*X}(v)$ and $\lrcorner$ is the interior product. Consequently, if we let $\alpha = \kappa \circ \beta$, then we arrive at the following commutative triangle of diffeomorphisms:

$$\begin{array}{ccc} T^*T^*X & \xleftarrow{\quad\beta\quad} & TT^*X \\ & {\scriptstyle\kappa}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & T^*TX & \end{array}$$

Since κ is a symplectomorphism, there is a unique symplectic structure on TT^*X such that the above triangle is a commutative triangle of symplectomorphisms. Indeed, this unique symplectic structure on TT^*X is the tangent lift of the canonical symplectic structure on T^*X. On the other hand, κ does not preserve the underlying Liouville structure, i.e., $\kappa^*\vartheta_{TX} \neq \vartheta_{T^*X}$, so we have two underlying Liouville structures on TT^*X (i.e., $\alpha^*\vartheta_{TX}$ and $\beta^*\vartheta_{T^*X}$) for the aforementioned unique symplectic structure on TT^*X. Finally, since $\kappa^{-1}(D_L) = D_{-H}$, we have $\beta^{-1}(D_{-H}) = \alpha^{-1}(D_L)$, a Lagrangian submanifold of TT^*X.

In summary, we have the following facts:

(i) On TT^*X, there are two canonical Liouville structures which gives rise to the same canonical symplectic structure.
(ii) For the dynamics of a mass point with configuration space X, Hamiltonian H, and Lagrangian L, there is a unique Lagrangian submanifold of TT^*X, i.e., $D := \beta^{-1}(D_{-H})$.

3.2. Tulczyjew mechanics. It is observed by Tulczyjew that the equation of motion is completely determined by the Lagrangian submanifold D of TT^*X. To see this, we let $c\colon I \to T^*X$ be a smooth map from an open interval I to T^*X and

$$\dot{c} : I \to TT^*X$$

be its derivative. Then, the equation of motion in Tulczyjew's approach is just the statement that the image of $\dot{c}$, $\operatorname{Im}\dot{c}$, lies inside D. Indeed, if we denote the local coordinates on X by q, the resulting local coordinates on T^*X by (q,p), and the resulting local coordinates on TT^*X by $(q,p,\dot{q},\dot{p})$. Then $c(t) = (q(t),p(t))$, $\dot{c} = (q,p,\frac{\mathrm{d}q}{\mathrm{d}t},\frac{\mathrm{d}p}{\mathrm{d}t})$, and

$$D = \left\{(q,p,\dot{q},\dot{p}) \mid \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}\right\}.$$

Therefore, the condition $\operatorname{Im}\dot{c} \subset D$ is nothing but the equations

$$\frac{\mathrm{d}q}{\mathrm{d}t}(t) = \frac{\partial H}{\partial p}(q(t),p(t)), \quad \frac{\mathrm{d}p}{\mathrm{d}t}(t) = -\frac{\partial H}{\partial q}(q(t),p(t)),$$

i.e., the Hamilton equations written in local coordinates.

In general, a dynamics in the sense of Tulczyjew is just a Lagrangian submanifold of TT^*X, which may or may not have a Lagrangian or a Hamiltonian. Formally, we have

DEFINITION 3.1 (Tulczyjew Dynamics). Let X be a smooth manifold.

(1) A Lagrangian submanifold D of TT^*X is called a *Tulczyjew dynamics* with preconfiguration space X.
(2) A smooth real function H on a submanifold of T^*X is called a *Hamiltonian* for the Tulczyjew dynamics D if $\beta(D) = D_{-H}$.
(3) A smooth real function L on a submanifold of TX is called a *Lagrangian* for the Tulczyjew dynamics D if $\alpha(D) = D_L$.

Note that a constrained dynamics is just a Tulczyjew dynamics for which $\tau_{T^*X}(D)$ is a proper subset of T^*X.

4. Sternberg phase space. The third (also the last) ingredient we need is an idea that was first made explicit by S. Sternberg [S], though it is really hidden in the work by S. K. Wong [Wo] and is likely known to a few other mathematicians and physicists.

Let us start with a technical setup:

G	a compact connected Lie group
$\mathfrak{g}$, $\mathfrak{g}^*$	the Lie algebra of G and its dual
$\langle\, ,\, \rangle$	the pairing of $\mathfrak{g}$ with $\mathfrak{g}^*$
$P \to X$	a principal G-bundle over X
Θ	a fixed principal connection form on $P \to X$
F	a Hamiltonian G-space
Ω	the symplectic form on F
$\Phi : F \to \mathfrak{g}^*$	the G-equivariant moment map
$\mathcal{F} \to X$	the associated fiber bundle with fiber F
$\mathcal{F}^\sharp$	the limit of diagram $T^*X \to X \leftarrow \mathcal{F}$
$\mathcal{F}_\sharp$	the limit of diagram $TX \to X \leftarrow \mathcal{F}$

In this setup, the curvature of Θ will serve as the background magnetic field, and moment map Φ will serve as the electric charge carried by the particle. Recall that, for notational sanity here, we shall use the same notation for both a differential form (or a map) and its pullback under a fiber bundle projection map.

THEOREM 4.1 (Sternberg, 1977). *The following statements are valid.*

(i) *There is a closed real differential two-form Ω_Θ on $\mathcal{F}$ which is of the form $\Omega - \mathrm{d}\langle A, \Phi\rangle$ under a local trivialization of $P \to X$ in which the connection form Θ is represented by the $\mathfrak{g}$-valued differential one-form A on X.*

(ii) *The differential two-form $\omega_\Theta := \omega_X + \Omega_\Theta$ is a symplectic form on $\mathcal{F}^\sharp$, where ω_X is the canonical symplectic form on T^*X, pulled back under $\mathcal{F}^\sharp \to T^*X$, and Ω_Θ is the pullback of Ω_Θ under $\mathcal{F}^\sharp \to \mathcal{F}$.*

We would like to remark that Ω_Θ is the right substitute for Ω when we go from a product bundle with the product connection to a generic bundle; also, if $G = \mathrm{U}(1)$, then $(\mathcal{F}^\sharp, \omega_\Theta) = (T^*X, \omega_X - q_e\, \mathrm{d}A)$ where q_e is the electric charge of the particle; finally, in the Hamiltonian formalism, as shown by Sternberg and others, the Sternberg phase space $(\mathcal{F}^\sharp, \omega_\Theta)$ is the right substitute for (T^*X, ω_X) when particles move in a background gauge field.

5. Tulczyjew's approach for particles in gauge fields. Since both Sternberg's work and Tulczyjew's work are quite natural, there should be a very natural setting to combine them. Indeed, there is one, as we shall see in this section. As a result, we obtain Tulczyjew's approach for particles in gauge fields as well as the Lagrangian side of Sternberg's work.

To start, recall that $\mathcal{F}_\sharp$ is the limit of diagram $TX \to X \leftarrow \mathcal{F}$. Note that $\mathcal{F}_\sharp \to \mathcal{F}$ is a real vector bundle and its dual is vector bundle $\mathcal{F}^\sharp \to \mathcal{F}$. So $T^*\mathcal{F}^\sharp \cong T^*\mathcal{F}_\sharp$ by Dufour's theorem. Let us denote the diffeomorphism

$$T\mathcal{F}^\sharp \to T^*\mathcal{F}^\sharp, \qquad v \mapsto v \lrcorner\, (\omega_\Theta|_\xi), \tag{6}$$

by β_M (here $\xi = \tau_{\mathcal{F}^\sharp}(v)$), and the diffeomorphism $\kappa \circ \beta_M$ by α_M, then we arrive at the magnetized version of the Tulczyjew triple:

$$\begin{array}{ccc} T^*\mathcal{F}^\sharp & \xleftarrow{\ \beta_M\ } & T\mathcal{F}^\sharp \\ & \searrow^{\kappa} \quad \swarrow_{\alpha_M} & \\ & T^*\mathcal{F}_\sharp & \end{array}$$

Since κ is a symplectomorphism, there is a unique symplectic structure on $T\mathcal{F}^\sharp$ such that the above triangle is a commutative triangle of symplectomorphisms. On the other hand, κ does not preserve the underlying Liouville structure, so we have two underlying Liouville structures on $T\mathcal{F}^\sharp$ (i.e., $\alpha_M^*\vartheta_{\mathcal{F}_\sharp}$ and $\beta_M^*\vartheta_{\mathcal{F}^\sharp}$) for the aforementioned unique symplectic structure on $T\mathcal{F}^\sharp$.

Definition 5.1 (Magnetized Tulczyjew Dynamics). Assume the setup in Section 4.

(1) A Lagrangian submanifold D of $T\mathcal{F}^\sharp$ is called a *magnetized Tulczyjew dynamics* with pre-configuration space $\mathcal{F}$. This dynamics is called a *constrained system* if $\tau_{\mathcal{F}^\sharp}(D)$ is a proper subset of $\mathcal{F}^\sharp$.
(2) A smooth real function H on a submanifold of $\mathcal{F}^\sharp$ is called a *Hamiltonian* for the magnetized Tulczyjew dynamics D if $\beta_M(D) = D_{-H}$.
(3) A smooth real function L on a submanifold of $\mathcal{F}_\sharp$ is called a *Lagrangian* for the magnetized Tulczyjew dynamics D if $\alpha_M(D) = D_L$.

Note that a Lagrangian or a Hamiltonian for a magnetized Tulczyjew dynamics may not exist. Also, one can verify that Sternberg's work in [S] is really the Hamiltonian side for unconstrained systems. However, the Lagrangian side seems to be new.

Theorem 5.2. *Assume the setup in Section 4. Suppose that F is a homogeneous Hamiltonian G-space and the Lagrangian L exists and is defined on the whole $\mathcal{F}_\sharp$. Then locally we have the following equation of motion:*

$$\boxed{\begin{array}{l} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot q^i}\right) = \dfrac{\partial L}{\partial q^i} + \dfrac{dq^j}{dt}\langle F_{ji}, \Phi\rangle + \{L, \langle A_i, \Phi\rangle\}_F \\[2ex] \qquad \dfrac{Dz}{dt}\lrcorner\,\Omega = \partial_F L \end{array}} \tag{7}$$

Here q is a local system of coordinates on X, z is a local system of coordinates on F, $(q, \dot q, z)$ is the resulting local system of coordinates on $\mathcal{F}_\sharp$ upon a choice of a local trivialization of the principal bundle $P \to X$ over the coordinate chart on X, $\frac{1}{2}F_{ij}\,\mathrm{d}q^i \wedge \mathrm{d}q^j$ is the gauge field strength, $\frac{Dz}{dt}$ is the covariant derivative of z, and

$$\{f, g\}_F := \Omega^{\alpha\beta}\frac{\partial f}{\partial z^\alpha}\frac{\partial g}{\partial z^\beta}, \qquad \partial_F L := \frac{\partial L}{\partial z^\alpha}\,\mathrm{d}z^\alpha.$$

After communicating with J. Grabowski and P. Urbański, the author learned that there had been an approach to electrically charged particles based on affine geometries [TU1, U, GGU1]; this is associated with $\mathbb{R}$-principal bundles, but a reasonable extension to non-abelian gauge fields may also be possible.

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