

LOGIC OF DIFFERENTIAL CALCULUS AND THE ZOO OF GEOMETRIC STRUCTURES

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Abstract. Since the discovery of differential calculus by Newton and Leibniz and the subsequent continuous growth of its applications to physics, mechanics, geometry, etc. it was observed that partial derivatives in the study of various natural problems are (self-)organized in certain structures usually called geometric. Tensors, connections, jets, etc. are commonly known examples. This list of classical geometrical structures is sporadically and continuously widening. For instance, Lie algebroids and BV-bracket are popular recent additions to it.

Our goal is to show that the “zoo” of all geometrical structures has a common source in the calculus of functors of differential calculus over commutative algebras, which surprisingly comes from a due mathematical formalization of observability mechanism in classical physics. We also use this occasion for some critical remarks and discussion of some perspectives.

1. Some motivating lessons from the history. The dimension of the subject we shall try to embrace in this short survey forces us to start with some historical and philosophical observations¹ in spite of the expected ironic reaction of the reader.

Zeno of Elea and after him Plato, Aristotle and other Greek philosophers raised the problem of a rational/scientific/mathematical description of motion. Zeno’s famous paradoxes were discussed for two and a half millennia [26, 30, 28, 1] and the discussions are

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¹Philosophy is an attempt to explain in terms of a natural language various aspects of the surrounding reality, tacitly assuming that this is possible. Although this is, as a rule, impossible, a directing philosophy is indispensable in the process of formation of the unique scientific language in which it could be done.

going on as well even now by involving new arguments from quantum mechanics, general relativity ([31]) and such mathematical constructions as nonstandard analysis ([29]). It is very instructive to note that as in the famous competition of Achilles and the tortoise all proposed solutions of Zeno's paradoxes were not fast enough to reach a common consensus before immediate counterarguments forcing them to slide away.

The invention of differential calculus by I. Newton and G. W. Leibniz, on the other hand, made possible an adequate and mathematically exact description of motion, first, in mechanics and later in *classical physics* in general. Highest efficiency and elegance of this approach have led working mathematicians and physicists to a widely shared conviction that all these philosophical discussions are something obsolete and not very relevant. The fact that the long and tortuous path to differential calculus was paved with various paradoxical Zeno-like logical constructions was almost forgotten.

This long history is one of many others that teach us that the information a human can get via his senses cannot be adequately explained in terms of any *natural language*, Greek, Latin, ..., English. Moreover, a more detailed analysis shows that the primary role of a natural language is to transmit information (see Appendix in [25]) but not to explain. Such terms as the famous "infinitesimal" or "wave-particle" come to us, like relic radiation, from the periods of formation of new adequate languages, when their terms were coined as self-contradicting hybrids in the old language by reflecting inadequacy.

A very general lesson to be drawn from this long story is:

Even if a problem/phenomenon is clearly seen, this does not automatically imply that the scientific community is in possession of an adequate language for its exact mathematical description/explanation.

By turning back to the problem of motion or, more generally, to that of "evolution", "continuous change", etc., and recognizing that differential calculus is the native language for these problems, at least, in the context of classical physics the next question to be posed is about the "grammar structure" of this language. It is clear that such concepts as "infinitesimals", "limits", etc., as descriptions of our intuitive ideas in terms of a natural language cannot be used for this purpose. On the other hand, this question, rather natural from philosophical point of view, needs, however, to be put into a more concrete context allowing its scientific analysis. To this end the following observation is of help.

In classical physics the state of a physical system at an instant of time is completely determined by readings of measuring devices of a laboratory. The role of differential calculus is then to elaborate these data in order to predict the further evolution ("motion") of the system or any other information about it. So, it is natural to think that a due mathematical formalization of a physical laboratory should be included into the theory for its completeness.

2. From the observation mechanism in classical physics to differential calculus.

This section is a brief summary of [25] which is our starting point. By a classical physical laboratory \mathcal{L} we mean a set of all relevant measuring devices whose readings completely determine states of the physical system we deal with. With two devices $I_1, I_2 \in \mathcal{L}$ one may associate their sum $I_1 + I_2$. This is a (virtual) device any reading of which is the

sum of corresponding readings of I_1 and I_2 .² Similarly is defined the product $I_1 I_2$ of I_1 and I_2 . For $\lambda \in \mathbb{R}, I \in \mathcal{L}$, λI refers to the device of the same kind as I but with λ -times modified scale. We also need a “stupid” device denoted \mathbb{I} whose reading is constantly 1 independently of the state of the system. The role of this device is that it allows one to shift “zero” on the scale of $I \in \mathcal{L}$ by λ by passing from $I \in \mathcal{L}$ to $I + \lambda \mathbb{I}$. By “constructing” all such virtual devices we obtain a commutative algebra with unit over \mathbb{R} . Real, not virtual devices, measuring devices presented in a concrete laboratory are now interpreted to be generators of this algebra, called the *algebra of observables*.³

Thus we mathematically formalize the concept of a classical physical laboratory as a commutative unitary algebra A over \mathbb{R} . In these terms an observation is the assignment to each “measuring device” $a \in A$ of its “reading” $h(a)$. By definition of A , $h: A \rightarrow \mathbb{R}$ is a homomorphism of unitary algebras. So, the *real spectrum* of A , denoted by

$$\text{Spec}_{\mathbb{R}} A \stackrel{\text{def}}{=} \{\text{all } \mathbb{R}\text{-algebra homomorphisms } h: A \rightarrow \mathbb{R}\},$$

is naturally interpreted as the space of all states of the physical system that we observe.

Similarly is defined the \mathbf{k} -spectrum of a commutative unitary algebra A over a ground field \mathbf{k} , which will be denoted $\text{Spec}_{\mathbf{k}} A$. Supplied with Zariski’s topology, $\text{Spec}_{\mathbf{k}} A$ becomes a topological space. A natural base of this topology consists of the subsets

$$U_a = \{h \in \text{Spec}_{\mathbf{k}} A \mid h(a) \neq 0\} \subset \text{Spec}_{\mathbf{k}} A.$$

A homomorphism $H: A_1 \rightarrow A_2$ of commutative unitary \mathbf{k} -algebras induces a map

$$|H|: \text{Spec}_{\mathbf{k}} A_2 \rightarrow \text{Spec}_{\mathbf{k}} A_1, \quad |H|(h) \stackrel{\text{def}}{=} h \circ H, \quad h \in \text{Spec}_{\mathbf{k}} A_2.$$

$|H|$ is *continuous* in Zariski’s topology. If $A = C^\infty(M)$ with M being a smooth manifold, then there is a natural map $\iota_M: M \rightarrow \text{Spec}_{\mathbb{R}} A$, $M \ni z \mapsto h_z$, where $h_z(f) \stackrel{\text{def}}{=} f(z)$.

The following “spectrum theorem” shows a complete syntony of the above formalization of the observability mechanism in classical physics with well-established facts.

THEOREM 2.1.

1. ι_M is a homeomorphism assuming that M is supplied with the standard topology.
2. Any smooth map $F: M \rightarrow N$ is of the form $F = \iota_N^{-1} \circ |H| \circ \iota_M$ for a homomorphism of unitary algebras $H: C^\infty(N) \rightarrow C^\infty(M)$ and, conversely, any homomorphism of unitary algebras $H: C^\infty(N) \rightarrow C^\infty(M)$ is of the form $H = F^*$ for a smooth map $F: M \rightarrow N$.

Nevertheless, having in mind that differential calculus is the native language of classical physics the most important test of adequacy of the proposed formalization is whether differential calculus is somehow encoded in it. In other words, the question is whether differential calculus is an aspect of commutative algebra. The positive answer comes from the following definition and the subsequent theorem.

²The modern technology allows one to easily construct this and similar devices.

³Taking into consideration all virtual measuring devices we guarantee, among other things, the objectivity of this construction, for instance, independence from national units of physical quantities or various suppliers of laboratory equipment.

Let A be a unitary commutative \mathbf{k} -algebra and P, Q be A -modules. If $a \in A$ and $\nabla : P \rightarrow Q$ is a \mathbf{k} -linear map, then $\delta_a(\nabla) : P \rightarrow Q$ is defined by $\delta_a(\nabla)(p) = \nabla(ap) - a\nabla(p)$, $p \in P$. We also put $\delta_{a_1, \dots, a_r} \stackrel{\text{def}}{=} \delta_{a_1} \circ \dots \circ \delta_{a_r}$ and observe that $\delta_{a_1, \dots, a_r} \stackrel{\text{def}}{=} \delta_{a_1}$ is symmetric with respect to the indices a_i .

DEFINITION 2.1. $\Delta : P \rightarrow Q$ is a linear differential operator of order $\leq m$ if it is \mathbf{k} -linear and $\delta_{a_0, \dots, a_m}(\Delta) = 0$, $\forall a_0, a_1, \dots, a_m \in A$.

So-defined operators preserve all general elementary properties of usual differential operators. For instance, composition of operators of orders $\leq k$ and $\leq l$, respectively, is an operator of order $\leq k + l$, etc.

THEOREM 2.2. Let $A = C^\infty(M)$, $\mathbf{k} = \mathbb{R}$ and $\pi_i : E_i \rightarrow M, i = 1, 2$, be vector bundles over M . Then the notion of a linear differential operator from $P = \Gamma(\pi_1)$ to $Q = \Gamma(\pi_2)$ in the sense of Definition 2.1 coincides with the standard ones.

NOTE. In what follows differential operators (DOs) will be understood in the sense of Definition 2.1. Also, we shall use “commutative algebra” for “commutative associative unitary algebra”.

2.1. Localizability of differential operators. One of the most important properties of DOs, in view of their role in geometry and physics, is their *localizability*. More exactly this means the following.

Recall that a *multiplicative subset* $S \subset A$ of a commutative algebra A is a subset, which is closed with respect to multiplication, contains 1_A and does not contain 0_A . For instance, a multiplicative set $S_U = \{a \in A \mid h(a) \neq 0, \forall h \in U\}$ is naturally associated with an Zariski open subset $U \subset \text{Spec}_{\mathbf{k}} A$. If all elements of S are products of elements $s_1, \dots, s_m \in S$, called *generators* of S , then S is *finitely generated*.

The localization of A over S is the algebra, denoted by $S^{-1}A$, formed by *formal fractions* $a/s, a \in A, s \in S$, i.e., equivalence classes of pairs (a, s) with respect to the equivalence relation:

$$(a_1, s_1) \sim (a_2, s_2) \text{ iff there is an } s \in S \text{ such that } s(a_1s_2 - a_2s_1) = 0.$$

Addition and multiplication of formal fractions are obvious. There is a canonical homomorphism of unitary algebras:

$$\iota = \iota_A : A \rightarrow S^{-1}A, \iota(a) = a/1.$$

The localization $S^{-1}P$ of an A -module P over S is defined similarly just by substituting $p \in P$ for $a \in A$ in the above formulae. Elements of $S^{-1}P$ are formal fractions $p/s, p \in P, s \in S$, and $S^{-1}P$ is an $S^{-1}A$ -module with respect to multiplication $(a/s)(p/s') = ap/ss'$. The canonical map $\iota = \iota_P : P \rightarrow S^{-1}P$ is defined by $\iota(p) = p/1$.

In the sequel $S^{-1}P$ will be considered as an $S^{-1}A$ -module. We shall also write A_S, P_S for $S^{-1}A, S^{-1}P$, respectively, if the context allows.

PROPOSITION 2.1. $|\iota|$ imbeds $\text{Spec}_{\mathbf{k}} A_S$ into $\text{Spec}_{\mathbf{k}} A$ and

$$\text{im}(|\iota|) = \{h \in \text{Spec}_{\mathbf{k}} A \mid h(s) \neq 0, \forall s \in S\} = \bigcap_{s \in S} U_s.$$

If S is finitely generated, then $\text{Spec}_{\mathbf{k}} A_S$ is Zariski open in $\text{Spec}_{\mathbf{k}} A$.

If U is a Zariski open, then $S_U^{-1}A$ (resp., $S_U^{-1}P$) is called the *localization* of A (resp., of the A -module P) with respect to U and will be simply denoted by A_U (resp., P_U).

PROPOSITION 2.2. *Let $\pi : E \rightarrow M$ be a vector bundle, $A = C^\infty(M)$ and $P = \Gamma(\pi)$. If U is an open domain in M identified with its image in $\text{Spec}_{\mathbb{R}} A$, then*

$$A_U = C^\infty(U) \quad \text{and} \quad P_U = \Gamma(\pi|_U).$$

Let $\Delta : P \rightarrow Q$ be a DO of order $\leq k$. Its localization $\Delta_S : S^{-1}P \rightarrow S^{-1}Q$ is defined by the formula

$$\Delta_S \left(\frac{p}{s} \right) = \sum_{i=0}^k (-1)^i \binom{k+1}{i+1} \frac{\Delta(s^i p)}{s^{i+1}} \tag{1}$$

If $S = S_U$ we will write Δ_U for Δ_S . Now we have

PROPOSITION 2.3.

1. *Formula (1) correctly defines Δ_S .*
2. *In the conditions of Proposition 2.2, Δ_U is identified with the standard restriction of Δ to U .*

2.2. Geometrization of algebras and modules. Any element $a \in A$ may be “visualized” by the associated \mathbf{k} -valued function f_a on $\text{Spec}_{\mathbf{k}} A$ (“geometrization” of a):

$$f_a(h) \stackrel{\text{def}}{=} h(a), \quad h \in \text{Spec}_{\mathbf{k}} A.$$

These functions form a commutative algebra $A_\Gamma \stackrel{\text{def}}{=} \{f_a | a \in A\}$. A natural surjective homomorphism $\gamma_A : A \rightarrow A_\Gamma$ is not, in general, an isomorphism. Obviously,

$$\ker \gamma_A = \bigcap_{h \in \text{Spec}_{\mathbf{k}} A} \ker h$$

Elements of $\ker \gamma_A$ are in this sense “invisible” and for this reason are called *ghosts*. To stress this fact we put $\text{Ghost}(A) = \ker \gamma_A$. A commutative algebra A without ghosts is called *geometric*. Obviously, $\text{Spec}_{\mathbf{k}} A = \text{Spec}_{\mathbf{k}} A_\Gamma$. Hence A_Γ is geometric and γ_A is the *geometrization* homomorphism. The algebras of smooth functions on smooth manifolds are geometric.

In order to make “visible” elements of an A -module P consider the quotient modules $P_h = P / \ker h \cdot P$, $h \in \text{Spec}_{\mathbf{k}} A$. The A -module P_h may be considered as an A_h -module ($A_h = A / \ker h$). Since h induces an isomorphism of A_h and \mathbf{k} , P_h may be considered as a \mathbf{k} -vector space. This way the family $\{P_h\}_{h \in \text{Spec}_{\mathbf{k}} A}$ of \mathbf{k} -vector spaces is associated with P . If $A = C^\infty(M)$ and $P = \Gamma(\pi)$, then the fiber $\pi^{-1}(x)$, $x \in M$, is naturally identified with P_{h_x} .

For $p \in P$ denote by p_h the coset $[p]_h = p \bmod (\ker h \cdot P) \in P_h$. This way one gets a “section” $\sigma_p : h \mapsto p_h$ of the family $\{P_h\}$. The totality P_Γ of such sections may be viewed either as an A -module or as an A_Γ -module. Indeed, $\sigma_{ap} = f_a \cdot \sigma_p : h \mapsto h(a)p_h = [ap]_h$. By definition, the *support* of P , denoted by $\text{Supp } P$, is

$$\text{Supp } P = \text{Zariski closure of } \{h \in \text{Spec}_{\mathbf{k}} A | P_h \neq 0\}.$$

Denote by γ_P be the canonical projection $P \rightarrow P_\Gamma$. Obviously,

$$\ker \gamma_P = \bigcap_{h \in \text{Spec}_k A} (\ker h \cdot P).$$

For the same reasons as before, elements of $\ker \gamma_P$ are interpreted as “invisible” or *ghosts*, and we put

$$\text{Ghost}(P) = \ker \gamma_P, \quad P_\Gamma = P/\text{Ghost}(P).$$

An A -module without ghosts is called *geometric*. The A -module P_Γ , called the *geometrization* of P , is, obviously, geometric.

Warning: there are non-geometric modules over a geometric algebra and vice versa.

The following theorem by R. Swan completes the spectral theorem.

THEOREM 2.3. *Let $A = C^\infty(M)$, $P = \Gamma(\pi)$ with π being a vector bundle. Then*

1. $P = P_\Gamma$ and P is a finitely generated projective A -module;
2. if Q is a finitely generated projective A -module, then the family $\{Q_{h_x}\}_{x \in M}$ of \mathbb{R} -vector spaces form a vector bundle α over M and Q is canonically isomorphic to $\Gamma(\alpha)$.

If $H : A \rightarrow B$ is a homomorphism of commutative \mathbf{k} -algebras, then any B -module Q acquires an A -module structure with respect to multiplication

$$(a, q) \mapsto a * q \stackrel{\text{def}}{=} H(a)q, \quad a \in A, q \in Q.$$

This structure will be called *H-induced*.

PROPOSITION 2.4. *If Q is a geometric B module, then it is a geometric A -module with respect to the H -induced module structure.*

Finally, we emphasize that importance of *geometricity*, in particular, comes from the fact that the standard differential geometry is a part of differential calculus over commutative algebras in the category of geometric modules over smooth function algebras.

2.3. Basic notation. Here we fix basic notation that will be used in the sequel. Let A be a commutative \mathbf{k} -algebra and P, Q be A -modules. Denote the totality of all DOs $\Delta : P \rightarrow Q$ of order $\leq k$ by $\text{Diff}_k(P, Q)$. $\text{Diff}_k(P, Q)$ has a natural A -bimodule structure. The *left* (resp., *right*) A -module structure of it is defined by

$$(a, \Delta) \mapsto a_Q \circ \Delta \quad (\text{resp.}, (a, \Delta) \mapsto \Delta \circ a_P) \tag{2}$$

where a_R stands for the multiplication by $a \in A$ operator in an A -module R . $\text{Diff}_k^<(P, Q)$ and $\text{Diff}_k^>(P, Q)$ denote the corresponding left and right A -modules, respectively. The totality $\text{Diff}(P, Q)$ of all DOs from P to Q is a filtered A -bimodule

$$\text{Hom}_A(P, Q) = \text{Diff}_0(P, Q) \subset \dots \subset \text{Diff}_k(P, Q) \subset \text{Diff}_{k+1}(P, Q) \subset \dots \text{Diff}(P, Q)$$

$\text{Diff}(P, P)$ is a filtered \mathbf{k} -algebra with respect to composition of DOs, and $\text{Diff}(P, Q)$ is a left (resp., right) filtered $\text{Diff}(Q, Q)$ -module (resp., $\text{Diff}(P, P)$ -module). Also, we use the short notation $\text{Diff } P$ for $\text{Diff}(A, P)$. Similarly, we define $\text{Diff}_k^< P$, etc.

3. Back to Zeno and tangent vectors. The key notion of classical differential calculus, namely, that of derivative, opened the way to constructively manipulate with previously intuitive ideas of velocity, acceleration, etc. in mechanics and of tangency, curvature, etc. in geometry. It is remarkable that the mechanism of observability allows one to transform the “antique Greek intuition” into a rigorous definition in a very simple and direct way. Indeed, if h_t is the state of the mechanical system at time t , then its motion is described by the curve $\gamma: t \mapsto h_t$ on $\text{Spec}_{\mathbb{R}} A$ with A being the algebra of observables. Intuitively, the velocity of this motion at time t is the tangent vector to γ and, therefore, to $\text{Spec}_{\mathbb{R}} A$ at the point h_t . In old-fashioned terms this vector should be

$$\xi = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (h_{t+\Delta t} - h_t).$$

This expression, as an \mathbb{R} -linear map from A to \mathbb{R} , is well-defined assuming that the limit exists. It is easily deduced that the relation ξ with the multiplicative structure of A is described by the “Leibniz rule”

$$\xi(ab) = \xi(a)b + a\xi(b), \quad \forall a, b \in A.$$

In the general algebraic context these heuristic considerations motivate the following definition.

DEFINITION 3.1. Let A be a commutative algebra over \mathbf{k} . A linear map $\xi: A \rightarrow \mathbf{k}$ is a tangent vector to $\text{Spec}_{\mathbf{k}} A$ at its point h if $\xi(ab) = \xi(a)h(b) + h(a)\xi(b)$, $\forall a, b \in A$.

In the case $A = C^\infty(M)$, $\mathbf{k} = \mathbb{R}$ this definition gives standard tangent vectors to M if M is identified with $\text{Spec}_{\mathbb{R}} A$ via the spectrum theorem. Similarly, derivations of a commutative \mathbf{k} -algebra A are naturally interpreted as vector fields on $\text{Spec}_{\mathbf{k}} A$, etc.

Definition 3.1 has the following useful interpretation. With a point $h \in \text{Spec}_{\mathbf{k}} A$ one can associate an A -module structure in the field \mathbf{k} by defining the module product to be $a \cdot \lambda \stackrel{\text{def}}{=} h(a)\lambda$, $a \in A$, $\lambda \in \mathbf{k}$. Denote this A -module by \mathbf{k}_h . Then a tangent vector to $\text{Spec}_{\mathbf{k}} A$ at the point h may be viewed as a first order DO $\xi: A \rightarrow \mathbf{k}_h$ such that $\xi(1_A) = 0$. This illustrates universality of Definition 2.1 and the way to transform spectra of commutative algebras into objects of a new differential geometry.

EXAMPLE 3.1. Let N be a closed subset of a smooth manifold M . Define the smooth function algebra on N by putting $C^\infty(N) \stackrel{\text{def}}{=} C^\infty(M)|_N$. The subset N supplied with this algebra will be called a smooth set. The spectrum of the algebra $C^\infty(N)$ is naturally identified with N and, therefore, differential geometry of N can be developed along the lines as above. It is nontrivial even for rather exotic smooth subsets. For instance, tangent spaces to the Cantor set are 1-dimensional and they are 2-dimensional for the Peano curve. Also, vector fields on the Cantor set are nontrivial and trivial on the Peano curve. The algebra of differential forms (see below) is nontrivial on these smooth sets.

EXAMPLE 3.2. All DOs of order greater than zero over the algebra of continuous functions on a smooth manifold M are trivial, while the \mathbb{R} -spectrum of this algebra is naturally identified with M .

The above general algebraic formalization of the intuitive idea of velocity makes it possible to “prove” impossibility to adequately describe the phenomenon of motion in

a natural language. To this end one has to analyze a system of statements (propositions) pretending to such a description, which formally takes part of a *propositional* or Boolean algebra. Recall that the basic operations of a Boolean algebra are conjunction (\wedge), disjunction (\vee) and negation (\neg). However, for computations of the truth value of propositions the operations of addition ($x \oplus y \stackrel{\text{def}}{=} (x \wedge \neg y) \vee (\neg x \wedge y)$) and multiplication ($x \cdot y \stackrel{\text{def}}{=} x \wedge y$) are more convenient. This leads to the equivalent notion of a Boolean ring, i.e., a commutative unitary algebra (A, \oplus, \cdot) over the field \mathbb{F}_2 of integers modulo 2 with the property $a \cdot a = a, \forall a \in A$ (*idempotence*). In this setting truth values are interpreted to be algebra homomorphisms $A \rightarrow \mathbb{F}_2$, i.e., elements of the spectrum $\text{Spec}_{\mathbb{F}_2} A$. So, informally, a Boolean ring may be viewed as an \mathbb{F}_2 -algebra of observables, i.e., a “laboratory” whose “measuring devices” are supplies with the {false, true}-scale. A simple computation based on idempotence property of A proves

PROPOSITION 3.1. *All DOs of order greater than zero over a Boolean ring A are trivial. In particular, all tangent vectors to $\text{Spec}_{\mathbb{F}_2} A$ are trivial.*

Therefore, by adopting the observability principle we see that the phenomenon of motion is inexpressible in terms of a natural language. In particular, Zeno’s paradoxes are not, in fact, paradoxical from this point of view, since the truth value of any reasoning depends on truth values assigned to single propositions forming this reasoning. This, however, is too personal as one can see from texts dedicated to Zeno’s paradoxes and hence cannot be objectively resolved in terms of this language.

It should be stressed that triviality of DOs over a Boolean ring is not due to discreteness of the ground field \mathbb{F}_2 . For instance, the spectrum of the algebra $\mathbb{F}_2[x]$ of polynomials with coefficients in \mathbb{F}_2 consists of two points $h_0 : p(x) \mapsto p(0)$ and $h_1 : p(x) \mapsto p(1), p(x) \in \mathbb{F}_2[x]$. Then it directly follows from the definition that there is exactly one nontrivial tangent vector ξ_ϵ at $h_\epsilon, \epsilon = 0, 1$, namely, $\xi : p(x) \mapsto p'(\epsilon)$ where $p'(x)$ stands for the formal derivative of $p(x)$. Moreover, the vector field $X : h_\epsilon \mapsto \xi_\epsilon$ on $\text{Spec}_{\mathbb{F}_2}(\mathbb{F}_2[x])$ generates the flow $A_t, t \in \mathbb{F}_2$, defined by

$$A_t^* \stackrel{\text{def}}{=} e^{tX} : (\mathbb{F}_2[x])_\Gamma \rightarrow (\mathbb{F}_2[x])_\Gamma, t \in \mathbb{F}_2,$$

which is well-defined since $X^2 = 0$. It is easy to see that $A_0 = \text{id}$ and A_1 interchanges h_0 and h_1 . (Note that $\mathbb{F}_2[x]$ is not geometric.)

REMARK 3.1. This simple example gives a counterexample to our intuition that “discreteness” and “differential calculus” are incompatible matters.

4. The shortest way from observability to Hamiltonian mechanics. Here we shall show how the mathematical framework for Hamiltonian mechanics can be rediscovered by answering a natural question from the “observability philosophy”:

*What is the algebra of observables for T^*M assuming that $C^\infty(M)$ is the algebra of observables for M ?*

First, we associate with filtered modules and algebras $\text{Diff}(P, Q), \text{Diff } P$, etc. the corresponding graded objects called (main) *symbols*:

$$\text{Smb}l_k(P, Q) = \frac{\text{Diff}_k(P, Q)}{\text{Diff}_{k-1}(P, Q)}, \quad k \geq 0,$$

assuming that $\text{Diff}_{-1}(P, Q) = 0$, and

$$\text{Smb}l(P, Q) = \bigoplus_{k \geq 0} \text{Smb}l_k(P, Q).$$

If $\Delta \in \text{Diff}_k(P, Q)$, then $\text{smb}l_k \Delta \stackrel{\text{def}}{=} (\Delta \bmod \text{Diff}_{k-1}) \in \text{Smb}l_k(P, Q)$ is called the (main) symbol of Δ . The composition of DOs induces the *composition of symbols*

$$\text{Smb}l_k(P, Q) \times \text{Smb}l_l(Q, R) \rightarrow \text{Smb}l_{k+l}(P, R). \tag{3}$$

The left and right A -module structures on $\text{Smb}l_k(P, Q)$ induced from $\text{Diff}_k(P, Q)$ coincide, since $\delta_a(\text{Diff}_k(P, Q)) \subset \text{Diff}_{k-1}(P, Q)$. So, the composition of symbols (3) is A -bilinear. In particular, $\text{Smb}l(R, R)$ is an associative graded A -algebra, and $\text{Smb}l(P, Q)$ is a right graded $\text{Smb}l(P, P)$ -module and a left graded $\text{Smb}l(Q, Q)$ -module. As in the case of differential operators we shall use $\text{Smb}l P$ for $\text{Smb}l(A, P)$. In particular, $\text{Smb}l A$ is a graded A -algebra and $\text{Smb}l P$ is a graded $\text{Smb}l A$ -module.

The natural Hamiltonian formalism is based on the following elementary lemma.

LEMMA 4.1. *If an A -module P is 1-dimensional, then the algebra $\text{Smb}l(P, P)$ is commutative and hence $[\Delta, \nabla] \in \text{Diff}_{k+l-1}(P, P)$ if $\Delta \in \text{Diff}_k(P, P)$ and $\nabla \in \text{Diff}_l(P, P)$.*

This allows one to define the *Poisson bracket* in $\text{Smb}l(P, P)$ for a 1-dimensional A -module P by putting

$$\{\text{smb}l_k \Delta, \text{smb}l_l \nabla\} \stackrel{\text{def}}{=} \text{smb}l_{k+l-1}[\Delta, \nabla]. \tag{4}$$

This bracket inherits skew-commutativity and the Jacobi identity from the commutator $[\cdot, \cdot]$. So, $(\text{Smb}l(P, P), \{\cdot, \cdot\})$ is a Lie algebra over \mathbf{k} . Moreover, $\{\cdot, \cdot\}$ is a *bi-derivation*, as follows from the elementary property $[\Delta \circ \nabla, \square] = \Delta \circ [\nabla, \square] + [\Delta, \square] \circ \nabla$ of commutators. So, $X_s \stackrel{\text{def}}{=} \{s, \cdot\}$, $s \in \text{Smb}l(P, P)$, is a derivation of $\text{Smb}l(P, P)$ and hence may be viewed as a vector field on $\text{Spec}_{\mathbf{k}}(\text{Smb}l(P, P))$, which will be called *Hamiltonian*.

THEOREM 4.1.

1. *The algebra $\text{smb}l C^\infty(M)$ is geometric.*
2. *$\text{Spec}_{\mathbb{R}}(\text{Smb}l C^\infty(M))$ is canonically identified with the total space of the cotangent bundle $T^*(M)$ of M so that the elements of $\text{smb}l C^\infty(M)_\Gamma$ are identified with smooth functions on $T^*(M)$, which are polynomial along fibers of $T^*(M) \rightarrow M$.*
3. *The Poisson bracket in $\text{Smb}l C^\infty(M)$ is identified with the standard Poisson bracket on $T^*(M)$.*
4. *If $f \in C^\infty(M)$, then for the section $\sigma_{df}: M \rightarrow T^*(M)$, $x \mapsto d_x f$ of the cotangent bundle we have*

$$\sigma_{df}^*(\text{smb}l_k \Delta) = \frac{1}{k!} \delta_f^k(\Delta)$$

where $\text{smb}l_k \Delta$ is interpreted as a function on $T^*(M)$.

See [25] for a proof.

Construction of symbols is naturally localizable. Indeed, there is a natural homomorphism

$$\iota_{\text{smb}} : \text{Smb}_k(P, Q)_S \rightarrow \text{Smb}_k(P_S, Q_S), \iota(\text{smb}_k \Delta) \mapsto \text{smb}_k \Delta_S, \Delta \in \text{Diff}_k(P, Q),$$

of A -modules, which respects the product of symbols and hence all module structures involved and, as a consequence, the Poisson bracket.

Theorem 4.1 reveals the true nature of the standard Poisson and, therefore, of symplectic structure on $T^*(M)$ and as such has important consequences. In particular, it directly leads to a conceptual definition, namely, as $\text{Spec}_k(\text{Smb } A)$, of the cotangent bundle to the spectrum $\text{Spec}_k A$ of a commutative algebra A . Its mechanical interpretation is: if $\text{Spec}_k A$ is the configuration space of a “mechanical system”, then $\text{Spec}_k(\text{Smb } A)$ is its “phase space”. Probably, the most important consequence of this interpretation is that it offers a direct algorithm for developing Hamiltonian mechanics/formalism over graded commutative algebras, for instance, over super-manifolds. This is a powerful instrument of constructing appropriate mathematical models in physics, whose potential is far from being exploited.

The somehow mysterious role of symbols in this construction is, in fact, absolutely natural in view of the fact that *propagation of fold-type singularities* of solutions of (non-linear) PDE \mathcal{E} is described by the Hamiltonian vector field with the main symbol of \mathcal{E} as its Hamiltonian. The corresponding Hamilton-Jacobi equation (= the “eikonal equation” in geometrical optics) is only one in the system that completely describes the behavior of fold-type singularities [18, 20, 24, 32, 41, 21, 19]. Unfortunately, this fact seems not to be sufficiently known. Additionally, it sheds a new light on the quantization problem [41, 50].

EXAMPLE 4.1. Fold-type singularities of solutions of $u_{xx} - \frac{1}{c^2}u_{tt} - mu^2 = 0$ are described by the following equations assuming that the wave front is of the form $x = \varphi(t)$ and $y \stackrel{\text{def}}{=} u|_{\text{wavefront}}, h \stackrel{\text{def}}{=} u_x|_{\text{wave front}}$:

$$\begin{cases} \ddot{y} + (cm)^2 g = \pm 2c\dot{h} \\ 1 - \frac{1}{c^2}\dot{\varphi}^2 = 0 \Leftrightarrow \dot{\varphi} = \pm c \end{cases} \Leftarrow \begin{cases} \text{Equations describing} \\ \text{behaviour of fold-type} \\ \text{singularities} \end{cases}$$

In particular, if $\dot{h} = 0$ (“resting particle”), then $\ddot{y} + (cm)^2 g = 0 \Rightarrow \boxed{v = mc}$.

The following toy example illustrates another aspect of the above approach, namely, the possibility to develop “discrete” Hamiltonian formalism.

EXAMPLE 4.2. Let $A = \mathbb{F}_2[x]$. Then $\text{Spec}_{\mathbb{F}_2}(\text{Smb } A)$ consists of 4 points and Hamiltonian vector fields on it form a 3-dimensional Lie algebra $\{e_1, e_2, e_3\}$ over \mathbb{F}_2 with $[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_2$ (see the end of Section 3).

Finally, we note that by adding to the above mathematical scheme the physical principles prescribing how to associate the energy function with a given mechanical system one gets a complete physical theory of mechanics in the Hamiltonian form. It is worth stressing that in this approach, fundamental concepts of traditional mechanics like force etc. are consequences of the above general principles. Unfortunately, a textbook based on this approach is still waiting to be written. Finally, by remembering the long and not

very straight-line history of Hamiltonian mechanics one may note that the conceptual algebraic approach to differential calculus is a good shortcut.

5. Basic functors of differential calculus. In this section we give some basic examples of *functors of differential calculus* (FDC) and of natural transformations relating them. These two ingredients form what can be informally called the “logic of differential calculus”.

From now on A stands for a commutative \mathbf{k} -algebra and P, Q, \dots for A -modules.

5.1. A play between the left and the right. The most obvious FDCs are $\text{Diff}_k^< : P \mapsto \text{Diff}_k^< P$, $k \geq 0$, and similarly for $\text{Diff}_k^>$ and Diff_k . They will be used to construct further FDCs and some natural transformations relating them.

We start by noticing that the operators

$$\text{id}_k^< : \text{Diff}_k^<(P, Q) \rightarrow \text{Diff}_k^>(P, Q), \quad \text{id}_k^> : \text{Diff}_k^>(P, Q) \rightarrow \text{Diff}_k^<(P, Q),$$

which are identity maps as maps of the set $\text{Diff}_k(P, Q)$, are, generally, k -th order DOs. Indeed, it directly follows from $\delta_a(\text{id}_k^<) = (a_Q - a_P)|_{\text{Diff}_k(P, Q)}$ and similarly for $\text{id}_k^>$ (see (2)).

The operator $\mathcal{A}_k^< : \text{Diff}_k^< P \rightarrow P$, $\Delta \mapsto \Delta(1_A)$ is, obviously, a homomorphism of A -modules. i.e. a 0-th order DO, while the operator $\mathcal{A}_k^> = \mathcal{A}_k^< \circ \text{id}_k^<$, which coincides with $\mathcal{A}_k^<$ as a map of sets, is, generally, of order k . The operator $\mathcal{A}_k^>$ is *co-universal* in the following sense.

Let $\Delta \in \text{Diff}_k(P, Q)$. The homomorphism

$$h^\Delta : P \ni p \mapsto \Delta_p \in \text{Diff}_k^> Q, \quad \Delta_p(a) \stackrel{\text{def}}{=} \Delta(ap)$$

makes the diagram

$$\begin{array}{ccc} P & \xrightarrow{h^\Delta} & \text{Diff}_k^> Q \\ & \searrow \Delta & \downarrow \mathcal{A}_k^> \\ & & Q \end{array} \tag{5}$$

commutative. Then it is easy to see that the map

$$\text{Diff}_k^>(P, Q) \rightarrow \text{Hom}_A(P, \text{Diff}_k^> Q), \quad \Delta \mapsto h^\Delta \tag{6}$$

is an isomorphism of A -modules, which *co-represents* the functor $P \mapsto \text{Diff}_k(P, Q)$ (Q is fixed) in the category of A -modules. The mapping (6) considered in the left A -module structures is an isomorphism of A -modules as well:

$$\text{Diff}_k^<(P, Q) \rightarrow \text{Hom}_A^<(P, \text{Diff}_k^> Q). \tag{7}$$

The upper index “ $<$ ” in $\text{Hom}_A^<(P, \text{Diff}_k^> Q)$ tells us that the set of A -homomorphisms from P to $\text{Diff}_k^> Q$ is supplied with the A -module structure induced by the left A -module structure in $\text{Diff}_k Q$. By specifying (5) for $\Delta = \text{id}_k^<$ we have

$$\text{Diff}_k^<(P, Q) \xrightarrow{h^{\text{id}_k^<}} \text{Diff}_k^>(\text{Diff}_k^>(P, Q)) \text{ and if } P = A, \text{ Diff}_k^< Q \rightarrow \text{Diff}_k^>(\text{Diff}_k^> Q). \tag{8}$$

The l -th *prolongation* h_l^Δ of h^Δ , $l \geq 0$, is defined to be $h_l^\Delta \stackrel{\text{def}}{=} h^{h^\Delta \circ \mathcal{A}_l^\Delta}$. It makes the following diagram commutative:

$$\begin{array}{ccc}
 \text{Diff}_l^\Delta P & \xrightarrow{\mathcal{A}_l^\Delta} & P \\
 h_l^\Delta \downarrow & & \downarrow h^\Delta \quad \Delta \\
 \text{Diff}_l^\Delta (\text{Diff}_k^\Delta Q) & \xrightarrow{\mathcal{A}_l^\Delta} & \text{Diff}_k^\Delta Q \xrightarrow{\mathcal{A}_k^\Delta} Q
 \end{array} \tag{9}$$

Since the order of the DO $\Delta_{(l)} = \Delta \circ \mathcal{A}_l^\Delta$ is $\leq k + l$, the diagram

$$\begin{array}{ccc}
 \text{Diff}_l^\Delta P & \xrightarrow{h^{\Delta_{(l)}}} & \text{Diff}_{k+l}^\Delta P \\
 \mathcal{A}_l^\Delta \downarrow & \searrow \Delta_{(l)} & \downarrow \mathcal{A}_{k+l}^\Delta \\
 P & \xrightarrow{\Delta} & Q
 \end{array} \tag{10}$$

whose upper triangle is (5) for $\Delta_{(l)}$, is commutative. The diagram

$$\begin{array}{ccc}
 \text{Diff}_l^\Delta (\text{Diff}_k^\Delta Q) & \xrightarrow{c_{l,k}} & \text{Diff}_{k+l}^\Delta Q \\
 \mathcal{A}_l^\Delta \downarrow & & \downarrow \mathcal{A}_{k+l}^\Delta \\
 \text{Diff}_k^\Delta Q & \xrightarrow{\mathcal{A}_k^\Delta} & Q
 \end{array} \tag{11}$$

is the particular case of (10) for $\Delta = \mathcal{A}_k^\Delta$. By definition, $c_{l,k} = h^\square$ with $\square = \mathcal{A}_k^\Delta \circ \mathcal{A}_l^\Delta : \text{Diff}_l^\Delta (\text{Diff}_k^\Delta Q) \rightarrow Q$.

Now, by combining diagrams (9)-(11), we get the commutative diagram

$$\begin{array}{ccc}
 \text{Diff}_l^\Delta P & \xrightarrow{\mathcal{A}_l^\Delta} & P \\
 h^{\Delta_{(l)}} \downarrow & \searrow \Delta_{(l)} & \downarrow \Delta \\
 \text{Diff}_{k+l}^\Delta Q & \xrightarrow{\mathcal{A}_{k+l}^\Delta} & Q
 \end{array} \tag{12}$$

and the homomorphism of filtered A -modules

$$h_*^\Delta : \text{Diff}^\Delta P \rightarrow \text{Diff}^\Delta Q, \quad h_*^\Delta|_{\text{Diff}_l^\Delta P} = h^{\Delta_{(l)}}.$$

Obviously, $h_*^\Delta(\square) = \Delta \circ \square$, $\square \in \text{Diff}^\Delta P$.

Objects representing the functors Diff_k are *jets* (see Section 7) and some constructions with them come from diagrams (9)-(12).

EXAMPLE 5.1. The symbol $\langle \text{Diff}_1^\Delta A \otimes_A P \rangle$ refers to the A -module that as a \mathbf{k} -vector space coincides with $\text{Diff}_1^\Delta A \otimes_A P$, while its A -module structure is induced by the left multiplication in $\text{Diff}_1 A$. One of possible definitions of a connection in an A -module P is an A -homomorphism $\kappa : \langle \text{Diff}_1^\Delta A \otimes_A P \rangle \rightarrow P$ such that $\kappa(\text{id}_A \otimes p) = p$. In this terms the covariant derivative $\nabla_X : P \rightarrow P$, $X \in D(A)$ is defined by $\nabla_X(p) \stackrel{\text{def}}{=} \kappa(X \otimes p)$. The “right” analogue of it, a *right connection* in P , is defined as an A -homomorphism $\xi : \rangle \text{Diff}_1^\Delta A \otimes_A P \rightarrow P$. The *right* covariant derivative ${}_X \nabla$ corresponding to $X \in D(A)$ is defined by ${}_X \nabla(p) \stackrel{\text{def}}{=} -\kappa(X \otimes p)$. Such a connection is *flat* if $[_X \nabla, {}_Y \nabla] = [_X, {}_Y \nabla] \nabla$.

Right connections are used in the construction of integral forms (see Subsection 7.7 and [23, 54]).

5.2. Multi-derivation functors. Now we shall introduce non-obvious functors, which correspond to multi-vector fields on manifolds for smooth function algebras. We start from the *derivation* functor D :

$$\begin{aligned}
 D: P \mapsto D(P) &= \{\Delta \in \text{Diff}_1^<(P) \mid \Delta(1) = 0\} \\
 &\equiv \{\Delta : A \longrightarrow P \mid \underbrace{\Delta(ab) = a\Delta(b) + b\Delta(a)}_{\text{derivations}}\}.
 \end{aligned}$$

If the A -module $D(A)$ is geometric, then its elements are vector fields on $\text{Spec } A$. The functor D is followed by the functors

$$D_2: P \mapsto D_2(P) \stackrel{\text{def}}{=} D(D(P) \subset \text{Diff}_1^> P), \tag{13}$$

$$\mathfrak{D}_2: P \mapsto \mathfrak{D}_2^{(>)}(P) \stackrel{\text{def}}{=} \text{Diff}_1^{(>)}(D(P) \subset \text{Diff}_1^> P). \tag{14}$$

Here the symbol $D(D(P) \subset \text{Diff}_1^> P)$ stands for the totality of all derivations $\Delta: A \rightarrow \text{Diff}_1^> P$ such that $\Delta(A) \subset D(A)$ supplied with the A -module structure $(a, \Delta) \mapsto a_P \circ \Delta$. The symbol $\text{Diff}_1(D(P) \subset \text{Diff}_1^> P)$ has a similar meaning, while $\text{Diff}_1^{(>)}(D(P) \subset \text{Diff}_1^> P)$ refers to the same \mathbf{k} -vector space but supplied with the A -module structure $(a, \Delta) \mapsto \Delta \circ a_A$.

Now it is important to observe that the imbedding $D_2(P) \subset \mathfrak{D}_2^{(>)}(P)$ of \mathbf{k} -vector spaces is a 1-st order DO over A . This makes the following inductive definition meaningful:

$$D_m(P) \stackrel{\text{def}}{=} D(D_{m-1}(P) \subset \mathfrak{D}_{m-1}^{>}(P)), \tag{15}$$

$$\mathfrak{D}_m^{(>)}(P) = \text{Diff}_1^{(>)}(D_{m-1}(P) \subset \mathfrak{D}_{m-1}^{>}(P)). \tag{16}$$

If $A = C^\infty(M)$, then $D_m(A)$ consists of m -vector fields on M .

It is not difficult to deduce from the above definitions natural embeddings of A -modules

$$D_{m+n}(P) \subset D_m(D_n(P)), \quad D_m(P) \subset D_{m-1}^{<}(\text{Diff}_1^>(P)) \tag{17}$$

and the splitting $\mathfrak{D}_m(P) = D_{m-1}(P) \oplus D_m(P)$. In other words, there are natural transformations of FDOs

$$D_{m+n} \rightarrow D_m \circ D_n, \quad D_m \rightarrow D_{m-1}^{<}(\text{Diff}_1^>), \quad D_{m-1} \leftarrow \mathfrak{D}_m \rightarrow D_m. \tag{18}$$

If $A = C^\infty(M)$, then the module $\Lambda^m(M)$ of m -th order differential forms on M is the representing object for the FDC D_m in the category of geometrical A -modules. Below we shall see that transformations (18) are “responsible” for well-known properties of differential forms.

5.3. The meaning of multi-derivation functors and Diff-Spencer complexes.

Definitions (13)–(16), which at first glance look enigmatic, in fact come naturally from

the following commutative diagram:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & D_2(P) & \longrightarrow & D^<(\text{Diff}_1^> P) & \xrightarrow{c_{1,1}} & \text{Diff}_2(P) & \xrightarrow{\mathcal{H}_2} & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \downarrow = & & \\
 0 & \longrightarrow & \mathfrak{D}_2(P) & \longrightarrow & \text{Diff}_1^<(\text{Diff}_1^>(P)) & \xrightarrow{c_{1,1}} & \text{Diff}_2(P) & \xrightarrow{\mathcal{H}_2} & P & \longrightarrow & 0
 \end{array} \tag{19}$$

where all non-labeled maps are natural embeddings. It directly follows from the definitions that $\mathfrak{D}_2(P)$ is exactly the description of the kernel of $c_{1,1}$, and $D_2(P)$ of the kernel of its restriction to $D^<(\text{Diff}_1^> P)$. Moreover, both lines in (19) are complexes, and the top line is called the *2-nd order Diff-Spencer complex*. Similarly, a more complicated diagram chasing shows that $D_m(P)$ is the kernel of a natural map $D_{m-1}^<(\text{Diff}_1^> P) \rightarrow D_{m-2}^<(\text{Diff}_2^> P)$. The imbedding

$$D_m(P) \hookrightarrow D_{m-1}^<(\text{Diff}_1^> P) \tag{20}$$

may be called the *(m - 1)-th exterior co-differential*, since it is represented by the exterior differential $d: \Lambda^{m-1}(M) \rightarrow \Lambda^m(M)$ if $A = C^\infty(M)$ (see below). Also, by applying (20) to $\text{Diff}_k^>(P)$ instead of P we obtain

$$D_m^<(\text{Diff}_k^>(P)) \rightarrow D_{m-1}^<(\text{Diff}_1^>(\text{Diff}_k^>(P))) \rightarrow D_{m-1}^<(\text{Diff}_{k+1}^>(P))$$

where the right map is induced by $c_{1,k}: \text{Diff}_1^>(\text{Diff}_k^>(P)) \rightarrow \text{Diff}_{k+1}^>(P)$. This composition is the differential in the $(m + k)$ -th *Diff-Spencer complex* $\text{Sp}_{m+k}(P)$:

$$0 \longrightarrow D_{m+k}(P) \longrightarrow \dots \longrightarrow D_m^<(\text{Diff}_k^>(P)) \longrightarrow D_{m-1}^<(\text{Diff}_{k+1}^>(P)) \longrightarrow \dots \tag{21}$$

Differentials of this complex are A -homomorphisms and hence its homology, called *Diff-Spencer homology*, are A -modules. They naturally describe singularities of both A (or $\text{Spec}_k A$) and P (see, for instance, [3]).

The *infinite order Diff-Spencer complex* $\text{Sp}_*(P)$

$$\dots \longrightarrow D_s^<(\text{Diff}^> P) \longrightarrow D_{s-1}^<(\text{Diff}^>(P)) \longrightarrow \dots \longrightarrow \text{Diff}^>(P) \longrightarrow 0 \tag{22}$$

is the direct limit of natural embeddings $\dots \subset \text{Sp}_m(P) \subset \text{Sp}_{m+1}(P) \subset \dots$. The homomorphism h_*^Δ induces a chain map $\mathfrak{h}^\Delta: \text{Sp}_*(P) \rightarrow \text{Sp}_*(Q)$. The homology related to \mathfrak{h}^Δ plays an important role in the geometric theory of PDEs (see [14, 40, 45, 17]). In particular, in Secondary Calculus computation of this homology for the universal linearization operators of (nonlinear) PDEs is the most powerful method for finding higher symmetries, conservation laws, field Poisson structures, etc. (see [42, 14, 13, 16]).

5.4. Higher analogues of multi-derivation functors and Diff-Spencer complexes.

Now we can discover many other FDCs just by substituting

$$D_{(k)}(P) \stackrel{\text{def}}{=} \{ \Delta \in \text{Diff}_k^<(P) \mid \Delta(1) = 0 \}$$

for $D(P)$ in the definition of multi-derivation functors. Namely, the generalization of (13) is as follows:

$$D_{(k,l)}(P) \stackrel{\text{def}}{=} D_{(l)}^<(D_{(k)}(P) \subset \text{Diff}_k^>(P)), \tag{23}$$

$$\mathfrak{D}_{(k,l)}(P) \stackrel{\text{def}}{=} \text{Diff}_l^<(D_{(k)}(P) \subset \text{Diff}_k^>(P)), \tag{24}$$

and, in general,

$$D_{(k_1, \dots, k_m)}(P) \stackrel{\text{def}}{=} D_{(k_m)}(D_{(k_1, \dots, k_{m-1})}(P) \subset \mathfrak{D}_{k_1, \dots, k_{m-1}}^>(P)), \tag{25}$$

$$\mathfrak{D}_{(k_1, \dots, k_m)}^{\langle \rangle}(P) \stackrel{\text{def}}{=} \text{Diff}_{(k_m)}^{\langle \rangle}(D_{(k_1, \dots, k_{m-1})}(P) \subset \mathfrak{D}_{k_1, \dots, k_{m-1}}^>(P)). \tag{26}$$

A higher analogue of the Diff-Spencer complex is associated with a sequence of positive integers $\sigma = (s_1, \dots, s_m)$, $s_i \in \mathbb{Z}_+$. It is denoted by $\text{Sp}_\sigma(P)$ and looks like this:

$$\begin{aligned} 0 \rightarrow D_{\sigma_m}(P) \rightarrow \dots \rightarrow D_{\sigma_i}^>(\text{Diff}_{m_i}^<(P)) \rightarrow D_{\sigma_{i-1}}^<(\text{Diff}_{m_{i+1}}^>(P)) \rightarrow \dots \\ \dots \rightarrow \text{Diff}_{m_0}(P) \xrightarrow{d_{k_0}} P \rightarrow 0 \end{aligned} \tag{27}$$

with $\sigma_i = (s_1, \dots, s_i)$ and $m_i = s_{i+1} + \dots + s_m$.

If $\sigma \leq \tau$, then $\text{Sp}_\sigma(P) \subset \text{Sp}_\tau(P)$. Diff-Spencer complex can be also defined for σ infinite from the right:

$$\dots \rightarrow D_{\sigma_i}^<(\text{Diff}^>(P)) \rightarrow D_{\sigma_{i-1}}^<(\text{Diff}^>(P)) \rightarrow \dots \rightarrow \text{Diff}(P) \xrightarrow{d_\infty} P \rightarrow 0. \tag{28}$$

For other related FDCs see [37].

5.5. Absolute and relative functors. The previously discussed FDCs may be treated symbolically without reference to concrete algebras and modules. For instance, by using such symbol as $D^<(D \subset \text{Diff}_1^>)$ instead of $D^<(D(P) \subset \text{Diff}_1^>P)$ we stress that this is an *absolute functor*, i.e., that its construction does not depend on the ground algebra A . From now on we shall use these symbols. A natural transformation of one absolute FDC Φ to another Ψ will be denoted by $\Phi \rightarrow \Psi$.

Natural transformations of the form $\Phi \rightarrow \Psi^<(\text{Diff}_k^>)$ are to be distinguished, since they are the source of *natural differential operators* (see [10]) between objects representing Φ, Ψ . For example, the naturality property of the exterior differential d , namely, that $d \circ F^* = F^* \circ d$ for any smooth map F , reflects existence of $D_k \rightarrow D_{k-1}^<(\text{Diff}_1^>)$ (see Section 7.3).

An example of a *relative functor* of differential calculus is the functor $Q \mapsto \text{Diff}_k(P, Q)$ for a fixed A -module P . Relative functors depend on concrete algebras and modules. In particular, they naturally appear when the ground algebra A or some special modules over it have some relevant peculiarities such as Poincaré duality, etc. An instance of that we shall see below in connection with *integral forms*.

It is important to stress that the notion of a FDC includes multifunctors. For example, $\text{Diff}_k(\cdot, \cdot)$ is an absolute bifunctor. Multi-functors of the form

$$\text{Diff}_{k_1}^{\varepsilon_1}(P_1, \text{Diff}_{k_2}^{\varepsilon_2}(P_2, \dots, \text{Diff}_{k_m}^{\varepsilon_m}(P_m, Q) \dots))$$

with $\varepsilon_i = “<”$ or “ $>$ ” generalize this simple example. Because of space limitations, in this paper we cannot give a due attention to this topic.

6. The graded generalization. Naturality of the above general approach also appears in the fact that it automatically generalizes to graded commutative algebras and, in particular, to supermanifolds. Indeed, all that is needed to this end is the graded version of δ_a 's. In particular, this makes algorithmic finding analogues of the “usual” geometrical

structures in the graded context. Moreover, it allows one to discover the *conceptual meaning* of various well-known quantities, for instance, tensors, that are traditionally defined by a description. These points are illustrated below.

6.1. Differential operators over graded commutative algebras. Recall that a *graded associative algebra* over a field \mathbf{k} is a pair (A, G) where

1. A is an associative \mathbf{k} -algebra;
2. G is a commutative monoid written additively;
3. $A = \bigoplus_{g \in G} A_g$ with A_g 's being \mathbf{k} -vector spaces and $A_g \cdot A_h \subset A_{g+h}$;

If $\beta(\cdot, \cdot)$ is an \mathbb{F}_2 -valued bi-additive form on G , then A is β -commutative if we have $ba = (-1)^{\beta(g,h)}ab$ for $a \in A_g, b \in A_h$.⁴ Similarly, an A -module P is *graded* if $P = \bigoplus_{g \in G} P_g$ and $A_g \cdot P_h \subset P_{g+h}$. In the sequel it will be tacitly assumed that all constructions and operations with graded objects respect gradings. For instance, a homomorphism $F: P \rightarrow Q$ of graded A -modules is *graded* of degree h if $F(P_g) \subset Q_{g+h}, \forall g \in G$. Accordingly, the notation $\text{Hom}_A^h(P, Q)$ will refer to the \mathbf{k} -vector space of all A -homomorphisms from P to Q of degree h and $\text{Hom}_A(P, Q)$ to the module of all graded A -homomorphisms, i.e., $\text{Hom}_A(P, Q) = \bigoplus_{h \in G} \text{Hom}_A^h(P, Q)$. An element $p \in P_g$ of a graded module P is called *homogeneous of degree g* . We shall also adopt the simplifying notation $(-1)^{ST}$ for $(-1)^{\beta(\deg S, \deg T)}$ for homogeneous elements S and T of graded A -modules.

If $\Delta \in \text{Hom } \mathbf{k}(P, Q)$ and $a \in A$ are homogeneous, then we put

$$\delta_a(\Delta) = \Delta \circ a_P - (-1)^{a\Delta} a_Q \circ \Delta \quad \text{and} \quad \delta_{a_1, \dots, a_m} = \delta_{a_1} \circ \dots \circ \delta_{a_m}$$

DEFINITION 6.1. Let P and Q be graded modules over a graded commutative algebra A . Then $\Delta \in \text{Hom}_{\mathbf{k}}(P, Q)$ is a (graded) differential operator of order $\leq k$ if

$$\delta_{a_0, a_1, \dots, a_k}(\Delta) = 0 \quad \text{for all homogeneous } a_0, a_1, \dots, a_k \in A.$$

With this definition all above constructions of FDOs, Hamiltonian formalism, etc. automatically generalize to the graded case just by literally repeating the corresponding definitions.

6.2. Example: Lie algebroids and dioles. Recall that a *Lie algebroid* over a non-graded commutative \mathbf{k} -algebra A is an A -module P supplied with a Lie algebra structure $[\cdot, \cdot]_P$ and a homomorphism $\alpha: P \rightarrow D(A)$, called the *anchor*, such that

1. $[p, aq]_P = a[p, q]_P + \alpha(p)q, \forall a \in A, p, q \in P$;
2. α is a Lie algebra homomorphism from $(P, [\cdot, \cdot]_P)$ to $(D(A), [\cdot, \cdot])$.

This is a general algebraic version of the standard definition when $A = C^\infty(M)$ and $P = \Gamma(\pi)$ with π being a vector bundle over M .

This geometrical object, at first glance not very usual, is in fact a graded analogue of a Poisson manifold. More exactly, the corresponding \mathbb{Z} -graded algebra \mathcal{A} , called the algebra of *dioles* or *diole algebra*, is defined to be

1. $\mathcal{A}_0 = A, \mathcal{A}_1 = P$ and $\mathcal{A}_i = \{0\}, i \neq 0, 1$;

⁴For simplicity we do not consider here a more general notion of commutativity (see [52, 33]).

- 2. the product in $\mathcal{A}_0 \subset \mathcal{A}$ is that in A and the product $\mathcal{A}_0 \cdot \mathcal{A}_1 \subset \mathcal{A}_1$ is the A -module product $(A, P) \rightarrow P$.

Note that $\mathcal{A}_1 \cdot \mathcal{A}_1 \subset \mathcal{A}_2 = \{0\}$ and that \mathcal{A} is graded commutative with respect to the trivial sign form β . See [51] for further details.

Let $\{\cdot, \cdot\}$ be a Poisson structure in \mathcal{A} of degree -1 , i.e., a graded Lie algebra structure in \mathcal{A} such that $\{P, P\} \subset P$, $\{P, A\} \subset A$, which is additionally a biderivation of \mathcal{A} . The ‘‘Hamiltonian’’ field $\alpha(p) \stackrel{\text{def}}{=} \{p, \cdot\}|_A$, $p \in P$, is, obviously, a derivation of A . From the biderivation property $\{ap, b\} = \{a, b\}p + a\{p, b\}$, $a, b \in A$, $p \in P$, and $\{A, A\} \subset \mathcal{A}_{-1} = \{0\}$ we see that $\alpha(ap) = a\alpha(p)$, i.e., that $\alpha: P \rightarrow A$ is a homomorphism of A -modules. Then the Jacobi identity

$$\{\{p, q\}, a\} + \{\{a, p\}, q\} + \{\{q, a\}, p\} = 0 \iff \alpha(\{p, q\})(a) = [\alpha(a), \alpha(q)](a)$$

tells us that α is a homomorphism of Lie algebras. Similarly, by putting $[\cdot, \cdot]_P \stackrel{\text{def}}{=} \{\cdot, \cdot\}|_P$ we see that condition (1) in the definition of algebroid is exactly the Leibniz rule $\{p, aq\} = \{p, a\}q + a\{p, q\}$. So, Lie algebroids are nothing but Poisson structures on algebras of diodes of degree -1 or *fat Poisson manifolds* in the sense of [5]. An advantage of this interpretation is that it puts algebroids into the rich context of differential calculus over diode algebras and, in particular, makes obvious the analogy with the standard Poisson geometry.

In this connection one may be curious about Poisson structures of different degrees over A . Among these only structures of degrees from -2 to 1 may be nontrivial. Their ‘‘non-graded’’ description is as follows.

A Poisson structure of degree -2 is just an A -bilinear and A -valued skew-symmetric form on P . Poisson structures of degree 1 are elements of $D_2(P)$. Structures of degree 0 are more complicated. Each of them consists of a Poisson structure $\{\cdot, \cdot\}_A$ on A and a flat *Hamiltonian connection*, which lifts the ‘‘Hamiltonian vector field’’ $X_a \stackrel{\text{def}}{=} \{a, \cdot\}_A$, $a \in A$, to the derivation $\nabla_a: P \rightarrow P$ of P over X_a . This means that $\nabla_a(bp) = X_a(b)p + b\nabla_a(p)$, $b \in A$, $p \in P$. Additionally, it is required that $\nabla_{ab} = a\nabla_b + b\nabla_a$.

7. Representing objects. In our approach covariant tensors, jets and other covariant objects of the standard differential geometry appear as elements of objects representing FDCs in suitable subcategories of the category $A\text{Mod}$ of A -modules. Here are the main details of this construction.

7.1. Representing objects: generalities. For simplicity we shall consider only the non-graded case. In the first approximation an object representing a FDC Φ in a category \mathcal{K} of A modules is an A -module $\mathcal{O}_{\mathcal{K}}(\Phi)$ such that Φ is equivalent to the functor $P \mapsto \text{Hom}_A(\mathcal{O}_{\mathcal{K}}(\Phi), P)$, $P \in \text{Ob } \mathcal{K}$. Objects representing the same functor are naturally isomorphic. The homomorphism $\mathcal{O}_{\mathcal{K}}(\Phi) \rightarrow P$ representing $\theta \in \Phi(P)$ will be denoted by h_θ . A natural transformation $\Phi \rightarrow \Psi$ of FDCs is then represented by a homomorphism $\mathcal{O}_{\mathcal{K}}(\Psi) \xrightarrow{\Upsilon} \mathcal{O}_{\mathcal{K}}(\Phi)$ of A -modules. Namely, by identifying $\Phi(P)$ and $\text{Hom}_A(\mathcal{O}_{\mathcal{K}}(\Phi), P)$, etc. we have:

$$\text{Hom}_A(\mathcal{O}_{\mathcal{K}}(\Phi), P) \longrightarrow \text{Hom}_A(\mathcal{O}_{\mathcal{K}}(\Psi), P), \quad h_\theta \mapsto h_\theta \circ \Upsilon.$$

Let A and B be commutative algebras. For an A -module P , an B -module R and an (A, B) -bimodule Q we have the canonical isomorphism

$$\text{Hom}_B(P \otimes_A Q, R) = \text{Hom}_A(P, \text{Hom}_B(Q, R)). \tag{29}$$

An obvious consequence is that $\mathcal{O}_{\mathcal{K}}(\Phi) \otimes_A \mathcal{O}_{\mathcal{K}}(\Psi)$ represents the composition $\Phi \circ \Psi$, i.e., the functor $P \mapsto \Phi(\Psi(P))$.

Assume now that Ψ is A -bimodule-valued. Then by labeling the corresponding two A -module structures by “ $<$ ” and “ $>$ ” we obtain two A -module-valued functors, $\Psi^<$ and $\Psi^>$. Each multiplication by $a \in A$ is a natural transformation of Ψ and hence induces an endomorphism of $\mathcal{O}_{\mathcal{K}}(\Psi)$. This way $\mathcal{O}_{\mathcal{K}}(\Psi)$ acquires an A -bimodule structure. Accordingly, the corresponding A -module structures will be also denoted by “ $<$ ” and “ $>$ ”. The A -module $\Phi^<(\Psi^>(P))$ is defined as the \mathbf{k} -vector space coinciding with $\Phi(\Psi^>(P))$, in which the A -module structure is induced by the $<$ -module structure in $\Psi(P)$. This way we get the functor $\Phi^<(\Psi^>)$. Once again it follows from isomorphism (29) that the representing object for $\Phi^<(\Psi^>)$ is ${}^<\mathcal{O}_{\mathcal{K}}(\Psi)> \otimes_A \mathcal{O}_{\mathcal{K}}(\Phi)$. This symbol tells us that the tensoring is taken with respect to the $>$ -structure of $\mathcal{O}_{\mathcal{K}}(\Psi)$, while the A -module structure of the resulting tensor product is induced by the $<$ -structure in $\mathcal{O}_{\mathcal{K}}(\Psi)$.

7.2. Existence of representing objects. Representing objects in the category $A\text{Mod}$ of all A -modules exist for all FDCs. The techniques sketched in the previous subsection reduce, basically, their construction to that for functors $\text{Diff}_k(P, \cdot)$ (see [14, 35]). These objects are called k -th order jets of P and are denoted by $\mathcal{J}^k(P)$. The construction of these A -modules is rather elementary. Indeed, consider the A -module $A \otimes_{\mathbf{k}} P$ and associate with an $a \in A$ the homomorphism

$$\delta^a : A \otimes_{\mathbf{k}} P \rightarrow A \otimes_{\mathbf{k}} P, \quad \delta^a(a' \otimes_{\mathbf{k}} p) = a' \otimes_{\mathbf{k}} ap - aa' \otimes_{\mathbf{k}} p.$$

Next, denote by μ_{k+1} the submodule of $A \otimes_{\mathbf{k}} P$ generated by all elements of the form $(\delta^{a_0} \circ \delta^{a_1} \circ \dots \circ \delta^{a_k})(a \otimes_{\mathbf{k}} p)$ and put

$$\mathcal{J}^k(P) \stackrel{\text{def}}{=} \frac{A \otimes_{\mathbf{k}} P}{\mu_{k+1}} \quad \text{and} \quad j_k : P \rightarrow \mathcal{J}^k(P), \quad j_k(p) = 1 \otimes_{\mathbf{k}} p \pmod{\mu_{k+1}}.$$

PROPOSITION 7.1. *For any $\Delta \in \text{Diff}_k(P, Q)$ there is a unique A -homomorphism h_Δ that makes the following diagram commutative:*

$$\begin{array}{ccc} P & \xrightarrow{j_k} & \mathcal{J}^k(P) \\ & \searrow \Delta & \downarrow h_\Delta \\ & & Q \end{array} \tag{30}$$

The correspondence $\Delta \mapsto h_\Delta$ establishes an isomorphism of the A -modules $\text{Diff}_k^<(P, Q)$ and $\text{Hom}_A(\mathcal{J}^k(P), Q)$. Moreover, $\mathcal{J}^k(P) = {}^<\mathcal{J}^k(A)> \otimes_A P$.

It is easy to see that j_k is a k -th order DO and Proposition 7.1 tells us that it is *universal*. According to Subsection 7.1, the embedding of functors $\text{Diff}_l(P, \cdot) \rightarrow \text{Diff}_k(P, \cdot)$, $l \leq k$, induces a natural projection $\pi_{k,l} : \mathcal{J}^k(P) \rightarrow \mathcal{J}^l(P)$.

Conceptually, *k-th order differential forms* are defined to be elements of the A -module representing functor D_k in $AMod$, which is denoted by $\Lambda^k(A)$. This module may be constructed by the methods above. A natural splitting $\text{Diff}_1^< = \text{id} \oplus D$ where $\text{id} = \text{Diff}_0^<$ is the identity functor suggests to define $\Lambda^1(A)$ as the kernel of $\pi_{1,0}: \mathcal{J}^1(A) \rightarrow \mathcal{J}^0(A) = A$. In this approach, the differential $d: A \rightarrow \Lambda^1(A)$ is defined by the formula $da \stackrel{\text{def}}{=} j_1(a) - aj_1(1_A)$.

It follows from the last construction in Subsection 7.1 and Proposition 7.1 that the A -module representing the functor $D_{k-1}^<(\text{Diff}_1^>)$ is $\mathcal{J}^1(\Lambda^{k-1}(A))$. The diagram analogous to (30)

$$\begin{array}{ccc} \Lambda^{k-1}(A) & \xrightarrow{j_k} & \mathcal{J}^k(\Lambda^{k-1}(A)) \\ & \searrow d & \downarrow h \\ & & \Lambda^k(A), \end{array}$$

in which h is the homomorphism representing the transformation of functors $D_k \rightarrow D_{k-1}^<(\text{Diff}_1^>)$, is the definition of the exterior differential d . Now it is not difficult to see that the *jet-Spencer complex*

$$0 \leftarrow \Lambda^n(A) \leftarrow \dots \mathcal{J}^k(\Lambda^{n-k}(A)) \leftarrow \mathcal{J}^{k+1}(\Lambda^{n-k-1}(A)) \leftarrow \dots \tag{31}$$

represents the functor $P \mapsto \text{Sp}_n(P)$ (see 21). For more examples of this kind see [14] (chapter I) and [37, 35, 45].

It is possible to construct an A -module representing a single FDC Φ in a category \mathcal{K} as the quotient module $\mathcal{O}_{AMod}(\Phi)/K$ where K is the intersection of the kernels of all homomorphisms $\mathcal{O}_{AMod}(\Phi) \rightarrow Q, Q \in \text{Ob } \mathcal{K}$. However, this module does not, generally, belong to \mathcal{K} . This makes it impossible to represent in \mathcal{K} all FDCs and natural DOs connecting them. Categories of A -modules that contain so-defined single representing A -modules are called *differentially closed* (see [35] for more details).

An important example of differentially closed categories is the category $\Gamma AMod$ of geometric A -modules over a geometrical algebra A . Representing A -modules in this category are geometrizations of representing A -modules in $AMod$ (see Subsection 2.2). Importance of the category of geometric modules is that it is in full compliance with the observability principle. Indeed, we have

PROPOSITION 7.2. *Representing objects in the category of geometric A -modules over $A = C^\infty(M)$ are identical to the corresponding objects in standard differential geometry.*

This means that differential forms, jets, etc. in the ordinary sense of these terms are nothing else than elements of $C^\infty(M)$ -modules that represent the corresponding FDC in the category of geometric modules. The following example illustrate the drastic difference between the categories of all $C^\infty(M)$ -modules and of geometric $C^\infty(M)$ -modules.

EXAMPLE 7.1. Let d_{alg} denote the exterior differential in $AMod$. If $A = C^\infty(\mathbb{R})$, then $d_{alg}(e^x) \neq e^x d_{alg}x$. In other words, $(d_{alg}(e^x) - e^x d_{alg}x) \in \text{Ghost}(C^\infty(\mathbb{R}))$.

The reader will find more about it in [53].

7.3. Naturality of d and related topics. A remarkable property of ordinary differential forms, jets, etc. is their *naturality*. This means that any smooth map $F: M \rightarrow N$ is accompanied by a map $F^*: \Lambda^i(N) \rightarrow \Lambda^i(M)$. We denote by $\Lambda^i(L)$ the $C^\infty(L)$ -module of (ordinary) i -th order differential forms on the manifold L . This “experimental” fact has the following explanation.

Let $H: A_1 \rightarrow A_2$ be a homomorphism of commutative algebras and Φ an absolute FDC. In this situation any A_2 -module Q acquires an A_1 -module structure with the A_1 -module multiplication $(a_1, q) \mapsto H(a_1)q, a_1 \in A_1, q \in Q$.

Now assume that $\mathcal{K}_i, i = 1, 2$, is a differentially closed category of A_i -modules and that any $Q \in \text{Ob } \mathcal{K}_2$ belongs to \mathcal{K}_1 as A_1 -module. Also change the notation by putting $\Lambda_{\mathcal{K}}^i(A) = \mathcal{O}_{\mathcal{K}}(D_i), \mathcal{J}_{\mathcal{K}}^k(A) = \mathcal{O}_{\mathcal{K}}(\text{Diff}_k^<)$ and denote by $d_{\mathcal{K}}: \Lambda_{\mathcal{K}}^i(A) \rightarrow \Lambda_{\mathcal{K}}^{i+1}(A)$ the exterior differential in \mathcal{K} . The composition $X = d_{\mathcal{K}_2} \circ H$ is a derivation of A_1 with values in $\Lambda_{\mathcal{K}_2}^1(A_2)$ considered as an A_1 -module in \mathcal{K}_1 . By universality of $d_{\mathcal{K}_1}, X = h_X \circ d_{\mathcal{K}_1}$. So, by putting $H_{\Lambda^1} \stackrel{\text{def}}{=} h_X$ we get a commutative diagram on the left:

$$\begin{array}{ccc} \Lambda_{\mathcal{K}_1}^1(A_1) & \xrightarrow{H_{\Lambda^1}} & \Lambda_{\mathcal{K}_2}^1(A_2) \\ d_{\mathcal{K}_1} \uparrow & & \uparrow d_{\mathcal{K}_2} \\ A_1 & \xrightarrow{H} & A_2 \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \Lambda^1(N) & \xrightarrow{F^*} & \Lambda^1(M) \\ d \uparrow & & \uparrow d \\ C^\infty(N) & \xrightarrow{F^*} & C^\infty(M) \end{array}$$

The diagram on the right expressing naturality of 1st order differential forms and the exterior differential d is the specialization of the left diagram to $H = F^*$ and categories of geometrical modules \mathcal{K}_i 's in view of Theorem 2.1 and Proposition 7.2.

The same arguments explain naturality of modules $\mathcal{J}_{\mathcal{K}}^k(A)$ and the DO $j_k: A \rightarrow \mathcal{J}_{\mathcal{K}}^k(A)$. Also, together with inductive arguments used in the definition of functors $D_{\mathcal{K}}^k$ s they explain naturality of higher order differential forms and exterior differentials, jet-Spencer complexes and so on. Higher analogues of de Rham and Spencer complexes are examples of natural differential operators, which can be hardly discovered by traditional methods (compare with [10]).

7.4. Multiplicative structure in \mathcal{J}^k . Representing modules may have various additional structures coming from specific natural transformations of FDCs. This point is illustrated below. Assuming that a differentially closed category \mathcal{K} is fixed we shall omit the subscript \mathcal{K} in the notation of representing modules in this category.

From the defining formula $D_{k+1} = D^<(D_k \subset \text{Diff}_1^>)$ follows the inclusion transformation $D_{k+1} \rightarrow D \circ D_k$. By iterating this procedure we can construct an inclusion $D_{k+l} \rightarrow D_l \circ D_k$. Then, according to the general principles of Subsection 7.1, the last inclusion is represented by a homomorphism $\wedge_{k,l}: \Lambda^k(A) \otimes_A \Lambda^l(A) \rightarrow \Lambda^{k+l}(A)$ of representing modules. If \mathcal{K} is the category of geometric modules over $C^\infty(M)$, then $\wedge_{k,l}$ is the standard wedge product of differential forms on M . As before, the naturality of the so-defined wedge product straightforwardly follows from its definition.

Another example concerns modules of jets. The *diagonal transformation* of functors $\text{Diff}_k^< \xrightarrow{\mathbf{m}_k} \text{Diff}_k^< \circ \text{Diff}_k^<:$

$$\text{Diff}_k^< P \ni \Delta \mapsto \mathbf{m}_k(\Delta) \in \text{Diff}_k^<(\text{Diff}_k^< P), \quad \mathbf{m}_k(\Delta)(a) = \Delta \circ a_P \tag{32}$$

is represented by the homomorphism $\mathcal{J}^k(A) \otimes_A \mathcal{J}^k(P) \xrightarrow{\mathbf{m}^k} \mathcal{J}^k(P)$ of the representing modules. This supplies $\mathcal{J}^k(P)$ with a natural $\mathcal{J}^k(A)$ -module structure. It is easily deduced from this definition that

$$j_k(a) \cdot j_k(p) = j_k(ap), \quad a \in A, p \in P \quad \text{where} \quad j_k(a) \cdot j_k(p) \stackrel{\text{def}}{=} \mathbf{m}^k(j_k(a) \otimes j_k(p))$$

In particular, if $P = A$, then \mathbf{m}^k supplies $\mathcal{J}^k(A)$ with a commutative algebra structure.

The standard operator of insertion of a vector field X to differential forms is due to a natural transformation of *relative* functors $i^X: D_k \rightarrow D_{k+1}$, $X \in D(A)$. For instance, the insertion operator $i_X: \Lambda^2 \rightarrow \Lambda^1$ represents the transformation of functors

$$i^X: D(P) \ni Y \mapsto \{a \mapsto X(a)Y - Y(a)X\} \in D_2(P).$$

Similarly can be defined the Lie derivative of differential forms as well as higher order analogues of it and of the insertion operation (see [34, 35]).

7.5. Tensors conceptually. According to the standard coordinate-free definition, covariant tensors are $C^\infty(M)$ -multilinear functions on vector fields on M . This definition, being descriptive, does not tell us anything about the role of these objects in the structure of differential calculus. For instance, it does not explain why skew-symmetric tensors, i.e., differential forms, are related by a natural DO, the exterior differential, while symmetric tensors do not. Another similar question is why a natural connection, namely, that of Levi-Civita, is associated only with non-degenerate symmetric tensors. In this regard it is instructive to notice that various attempts to construct an analogue of the Levi-Civita connection for symplectic manifolds have been made recently without any positive result.

Below we shall sketch the conceptual approach to tensors and, as a byproduct, shall answer the above two questions. The decisive idea is to interpret covariant tensors as special first order differential forms over the algebra of *iterated differential forms* (see [47]). For simplicity we shall discuss only the non-graded case.

We begin from the algebra $\Lambda_1 \stackrel{\text{def}}{=} \Lambda_{\mathcal{K}}^*(A)$ of differential forms in a suitable differentially closed category of A -modules, for instance, the category of geometrical modules over $C^\infty(M)$. In order to save “space-time” we shall omit all references to \mathcal{K} including the notation. Λ_1 is a \mathbb{Z} -graded commutative algebra with the ordinary multiplication $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ for the sign form. Therefore, the algebra of differential forms over Λ_1 denoted $\Lambda_2 \stackrel{\text{def}}{=} \Lambda^*(\Lambda_1)$ is well-defined. The exterior differential $d = d_1$ is a derivation or a “vector field” over Λ_1 . It naturally extends as the Lie derivative L_{d_1} to Λ_2 . Hence L_{d_1} commutes with the exterior differential d_2 in Λ_2 . It is convenient to abuse the notation by using d_1 for L_{d_1} . Note also that $\Lambda_1 = \Lambda^0(\Lambda_1) \subset \Lambda_2$. By continuing this process we inductively construct the algebra $\Lambda_k \stackrel{\text{def}}{=} \Lambda^*(\Lambda_{k-1})$ of *k-times iterated differential forms* with commuting differentials d_1, \dots, d_k . The natural inclusions $\Lambda_{k-1} = \Lambda^0(\Lambda_{k-1}) \subset \Lambda_k$ allow one to define the *algebra of iterated differential forms*

$$\Lambda_\infty \stackrel{\text{def}}{=} \bigcup_{0 \leq k} \Lambda_k \quad \text{with} \quad \Lambda_0 = A.$$

This algebra is *conceptually closed*, i.e., $\Lambda^*(\Lambda_\infty)$ is naturally isomorphic to Λ_∞ .

In the category of geometric modules over $A = C^\infty(M)$ the above-mentioned interpretation of covariant tensors as iterated differential forms looks very simple and is as

follows:

$$i_p: T^k(M) \ni df_1 \otimes_A df_2 \otimes_A \cdots \otimes_A df_k \mapsto d_1 f_1 \wedge d_2 f_2 \wedge \cdots \wedge d_k f_k \in \Lambda_k$$

In particular, if $\tau' = \tau_{ij} dx^i \otimes dx^j \in T^2(M)$, then $\tau \stackrel{\text{def}}{=} i_2(\tau') = \tau_{ij} d_1 x^i d_2 x^j \in \Lambda_2^1$ and

$$d_2(d_1 \tau) = \tau_{i,j,k} d_1 x^i d_1 x^k d_2 x^j d_2 x^l - \gamma_{ijk} d_1 x^i d_2 x^j d_2(d_1 x^k) - g_{ij} d_2(d_1 x^i) d_2(d_1 x^j)$$

where $\tau_{ij,m} = \partial \tau_{ij} / \partial x^m$, etc. $\gamma_{ijk} = \tau_{ik,j} + \tau_{kj,i} - \gamma_{ij,k}$, $g_{ij} = 1/2(\tau_{ij} + \tau_{ji})$ and we omit symbols of wedge products. Note that γ_{ijk} is the double first kind Christoffel symbol of the Levi-Civita connection if the tensor τ' is symmetric, i.e., $\tau' = g \stackrel{\text{def}}{=} g_{ij} dx^i \otimes dx^j$.

Assume now that g is non-degenerate. Then the map $\text{grad}_g: f \mapsto \text{grad}_g f$ is well-defined. It is a $D(A)$ -valued derivation of A and hence $\text{grad}_g = h_g \circ d$ with $h_g = h^{\text{grad}_g}: \Lambda^1(A) \rightarrow D(A)$. The isomorphism h_g naturally extends to an A -linear derivation of $\Lambda_1 = \Lambda^*(A)$ with values in the $\Lambda^*(A)$ -module $\Lambda^*(A) \otimes_A D(A)$ denoted h_g^* . Similarly, h_g^* , being a derivation of Λ_1 induces a Λ_1 -linear derivation

$$h_g^{(2)}: \Lambda_2 \rightarrow \Lambda_2 \otimes_{\Lambda_1} (\Lambda_1 \otimes_A D(A)) = \Lambda_2 \otimes_A D(A).$$

In coordinates $h_g^{(2)} = g^{l\alpha} i_{\partial_\alpha} \otimes_A i_{\partial_\alpha}^{(2)}$ with $i_{\partial_\alpha}^{(2)} = i_{i\partial_\alpha}$. Then

$$\Gamma(\tau) \stackrel{\text{def}}{=} \frac{1}{2} h_g^{(2)}(d_2(d_1 \tau)) = (d_2(d_1 x^\alpha) + \Gamma_{ij}^\alpha d_1 x^i d_2 x^j) \otimes_A i_{\partial_\alpha} \tag{33}$$

where $\Gamma_{ij}^\alpha = 1/2 g^{\alpha k} \gamma_{kji}$. If $\tau' = g$, then Γ_{ij}^α 's are the Christoffel symbols of the pseudo-metric g . Call $\Gamma(\tau)$ the *Levi-Civita form of τ* . Since $\Gamma(\tau)$ is a vector-valued graded form, its graded Frolicher-Nijenhuis square $[\cdot, \cdot]^{FN}$ is well-defined and gives the *curvature form*:

$$[\Gamma(\tau), \Gamma(\tau)]^{FN} = R_{ijk}^\alpha d_1 x^k d_2 x^j d_2 x^i \otimes_A i_{\rho_\alpha} \tag{34}$$

with $R_{ijk}^\alpha = \partial_i \Gamma_{jk}^\alpha - \partial_j \Gamma_{ik}^\alpha + \Gamma_{i\beta}^\alpha \Gamma_{jk}^\beta - \Gamma_{j\beta}^\alpha \Gamma_{ik}^\beta$. Similarly, all other standard quantities related with the Levi-Civita connection can be obtained by applying to $\Gamma(\tau)$ natural operators of differential calculus over Λ_1 (see [48]). So, all these facts lead to recognize that, conceptually, $\Gamma(\tau)$ is what should be called the Levi-Civita connection associated with a second order covariant tensor with non-degenerate symmetric part. This interpretation inserts Riemannian geometry into the rich machinery of differential calculus over iterated differential forms and, in particular, allows one to avoid seemingly natural questions related with tensors, which are, in fact, conceptually ill-posed as, for example, the question about an analogue of the Levi-Civita connection for symplectic manifolds.

7.6. An example of application: natural equations in general relativity. A richer and conceptually certain mathematical language offers more possibilities to mathematically formalize various situations in physics. This is especially important when the subject is of a non-intuitive character. We shall illustrate this common place with an application of iterated forms to general relativity (see [48]).

Let $\tau = g + \omega$ be a covariant second order tensor field on a 4-fold M with g and ω being its symmetric and skew-symmetric parts, respectively. We shall interpret τ, g and ω as iterated $(1, 1)$ -forms. In the context of general relativity it is natural to interpret g as the metric in the space shaped by the matter field ω , i.e., by the “fermionic” part of τ . By analogy with the classical vacuum Einstein equation $\text{Ric}(g) = 0$ we assume that

g and ω are connected by the equation $\text{Ric}(\tau) = 0$ where $\text{Ric}(\tau)$ is the Ricci tensor of the connection $\Gamma(\tau)$. We describe it in coordinates.

If R_{ijk}^α 's are as in (34), then $\text{Ric}(\tau) = R_{ik}d_1x^i d_2x^k$ with $R_{ik} = R_{ijk}^j$. Denote by ∇_g the covariant differential of the connection $\Gamma(g)$ and put $\text{Ric}(g) = R_{ik}^{(g)}d_1x^i d_2x^k$. Then $\text{Ric}(\tau) = 0$ reads as follows:

$$R_{ij}^{(g)} + \frac{9}{16}g^{kl}g^{mn}\partial_{[m}\omega_{i]l}\partial_{[k}\omega_{j]n} = 0, \tag{35}$$

$$\nabla_g(\partial_{[i}\omega_{jk]}) = 0.$$

The second of these equations describes a perfect irrotational fluid. A remarkable feature of known exact solutions of this equation is that they describe an expanding universe. At the moment physical interpretation of this fluid is not very clear. Speculatively, one may think that ‘‘molecules’’ forming it are galaxies. Concluding we stress that ω is a rather simplified model of matter. But, on the other hand, various richer models can be proposed by varying the algebra of observables.

7.7. Integration. In the category of smooth orientable manifolds *integrands*, i.e., the quantities to be integrated, are top differential forms. In the case of supermanifolds differential forms can be of any positive degree and, therefore, none of them can be considered as integrand. By introducing *integral forms* F. Berezin had overcome this difficulty (see [2]). Further development of the original Berezin approach has led to a general construction of integral forms over general graded commutative algebras (see [33]). The idea of this construction is to pass from the complex of differential forms

$$0 \rightarrow A = \Lambda^0(A) \xrightarrow{d_0} \Lambda^1(A) \xrightarrow{d_1} \dots \xrightarrow{d_i} \Lambda^i(A) \xrightarrow{d_{i+1}} \dots \tag{36}$$

to the complex of adjoint operators

$$0 \leftarrow \Sigma^0(A) \xleftarrow{\hat{d}_0} \Sigma^1(A) \xleftarrow{\hat{d}_1} \dots \xleftarrow{\hat{d}_{i-1}} \Sigma^i(A) \xleftarrow{\hat{d}_i} \dots \tag{37}$$

If $A = C^\infty(M)$, $\dim M = n$ and $\hat{P} \stackrel{\text{def}}{=} \text{Hom}_A(P, \Lambda^n(A))$, then $\Sigma^i(A) = \hat{\Lambda}^i(A) = \Lambda^{n-i}(A)$ and $\hat{d}_i = \pm d_{n-i-1}$. In particular, $\Sigma^0(A) = \Lambda^n(A)$ (see [38, 40]) and elements of $\Sigma^0(A)$ may be interpreted as integrands. This is crucial, since these elements belong to the initial part of a complex and not to its top term, which does not, generally, exist. A point that is not very satisfactory here is that the definition of \hat{P} is based on existence of the top term $\Lambda^n(A)$. Fortunately, this inconvenience can be resolved by observing that \hat{P} is the cohomology of the complex $\text{Diff}(P, \Lambda(A))$ of A -homomorphosms

$$0 \rightarrow \text{Diff}^>(P, A) \xrightarrow{w_P} \text{Diff}^>(P, \Lambda^1(A)) \xrightarrow{w_P} \dots \xrightarrow{w_P} \text{Diff}^>(P, \Lambda^i(A)) \xrightarrow{w_P} \dots, \tag{38}$$

$$w_P(\Delta) \stackrel{\text{def}}{=} d \circ \Delta, \quad \Delta \in \text{Diff}(P, \Lambda(A))$$

denoted by $\hat{P} = H(w_P)$, $\hat{P} = \oplus_i \hat{P}^i$ with $\hat{P}^i = H^i(w_P)$. Note that complex (38) is well-defined for any G -graded algebra and that \hat{P} is $(\mathbb{Z} \times G)$ -graded. If $A = C^\infty(M)$, then the only nontrivial component of \hat{P} is \hat{P}^n . The module \hat{P} is called the *adjoint* to P .

A DO $\square \in \text{Diff}(P, Q)$ induces a cochain map $\text{Diff}(Q, \Lambda(A)) \rightarrow \text{Diff}(P, \Lambda(A))$ and the induced map in cohomology $\hat{\square}: \hat{Q} \rightarrow \hat{P}$ is called the *adjoint* to \square operator. With these definitions the adjoint to the de Rham complex (37) is well-defined with $\Sigma^i(A) \stackrel{\text{def}}{=} \hat{\Lambda}^i(A)$.

Elements of $\Sigma^s(A)$ are called *s-order integral forms*. The A -module $\mathcal{B}(A) \stackrel{\text{def}}{=} \Sigma^0(A)$ is called the *Berezinian* (in the category \mathcal{K} of A -modules) (see [33]). Originally, integral forms and Berezinians appeared in the context of supermanifolds (see [2, 55, 27]) and this is the generalisation of these notions to arbitrary graded commutative algebras.

REMARK 7.1. For some “good” algebras A the Berezinian can be thought of as a 1-dimensional projective A -module supplied with a flat right connection (see [23, 54]). This simple interpretation could be convenient in practical computations but is not satisfactory as a concept.

Computations of Berezinians for many algebras of interest essentially proceed along the same lines as in [38, 40]. For example, with this method can be described the Berezinian of the algebra of iterated forms $\Lambda_k(A)$ for $A = C^\infty(M)$. It turns out that the only nontrivial homogeneous component of $\mathcal{B}(\Lambda_k)$ is of order $2^{k-1}n$, which is canonically isomorphic to Λ_k (see [49]).

Finally, in this approach integration is the map that associates with an integral form $\omega \in \mathcal{B}(A)$ its homology class in complex (37).

8. Conclusions. In this quick trip through differential calculus over commutative algebras we have tried to show expressive capacity and universality of this new language. The standpoint based on it forces some different views on both traditional and modern parts of mathematics and theoretical physics. We shall outline some of them by starting from the following historical parallel with the purpose to avoid formal theorizing.

8.1. Sheaves as “broken lines”. Since there are no means to study arbitrary (smooth) curves and other “curvilinear objects” within the Euclidean geometry framework, geometers in antiquity reached a psychological comfort with the idea that a curve is the limit of inscribed broken lines. Even if the intuitive term “limit” was well defined this would not be a self-consistent definition. Nevertheless, this intuition helped to compute the length or other geometrical characteristics of some particular curves before the invention of differential calculus. Moreover, broken-lines-like considerations were among decisive factors that had led to discovery of differential calculus.

This is one of many situations one may meet either in the history or in contemporary mathematics when mathematical objects are constructed from pieces still treatable in the old language while the new adequate language is not yet in hand. Sheaves are well-known examples of this kind. For instance, the necessity of sheaves in modern complex geometry is due to the fact that a complex manifold is defined as something sewn from open pieces of \mathbb{C}^n . As a consequence all other relevant geometrical quantities in complex geometry are defined as cohomology of suitable sheaves. The passage to the limit in the definition of sheaf cohomology is pretty parallel to the curves understood as limits of broken lines. Implicitly, this reflects the fact that a complex manifold is not, generally, the spectrum of its algebra of holomorphic functions. In this sense complex manifolds are not “observable”. But the observability can be immediately reached if a complex manifold is defined to be a smooth manifold supplied with an integrable Nijenhuis tensor. In this approach all necessary holomorphic objects, for instance, tensors,

are defined as compatible with the Nijenhuis tensor. This way sheaves can be eliminated from complex geometry together with the corresponding heavy technical instruments like derived functors, etc. This not only simplifies and enriches the theory but also leads to new important generalizations. As an example we mention the theory of singularities of solutions of PDEs where analogues of complex geometry are of fundamental importance (see [46]).

This parallel between sheaves and broken lines allows one to foresee that the language of differential calculus over commutative algebras will inevitably substitute the language of sheaves in the future. At the same time the fundamental role of “sheaf technologies” in the past should be highly recognised. In particular, they implicitly contributed to preparing the land for differential calculus over commutative algebras.

8.2. Some general expectations. Essentially the same arguments as in the preceding subsection are applied as well to other areas of contemporary mathematics and theoretical physics. We shall briefly indicate those of them where all-round implementation of the DCCA-based methods looks most promising.

Algebraic geometry. Algebraic geometry is an area, which suggests itself introduction of DCCA. For that it is sufficient to change the object of study by passing, according to the “philosophy of observability”, from affine or projective varieties to the corresponding algebras. This allows direct application of methods of differential geometry. For instance, the Spencer cohomology of algebraic varieties, defined as in Subsections 5.3 and 7.2, are fine invariants of their singularities (see [3]). As such it suggests an alternative approach to the problem of resolution of singularities. Also, existence of singularities does not prevent direct definition of the de Rham and other basic cohomology of algebraic varieties. For some other examples see [15]. This is why it is natural to think that a systematic review of foundations of algebraic geometry on the basis of DCCA would give a strong new impetus to this classical area.

Geometry of PDEs and secondary calculus. Originally, one of the main stimuli to develop DCCA sprang out of the necessity to construct the complete analogue of differential geometry on the “space of all solutions of a given PDE” (see [14]). The intuitive idea of such a “space” is formalised with the concept of a *diffiety*. Diffieties form a special class of, generally, infinite-dimensional manifolds where traditional methods and means fail to work. On the contrary, DCCA makes this problem naturally solvable (see [40, 43, 17, 45, 50]). Moreover, computations of basic quantities related with nonlinear PDEs (symmetries, conservation laws, hamiltonian structures, Bäcklund transformations, etc.) are essentially based on DCCA (see [42, 16, 13]). Secondary calculus, a natural language for the modern geometrical theory of nonlinear PDEs, is the specialization of DCCA to diffieties. This explains the key importance of the DCCA-based methods, especially, cohomological ones for the theory of nonlinear PDEs. For further details see surveys [44] and [50].

Graded differential geometry. As we have already pointed out the definition of graded analogues of all objects of the standard differential geometry in terms of DCCA is identical to the non-graded ones assuming that the latter are defined *conceptually*. However, the

search for conceptual definition of concrete quantities, the *conceptualization problem*, could be a nontrivial task as one can see from the previous discussion of tensors and integral forms. On the other hand, the recent history of formation of super-geometry when this opportunity was not taken into consideration illustrates the complications that could be otherwise avoided. The necessity to systematically develop fundamentals of graded differential geometry on the basis of DCCA is now urged by perspectives of important applications to both geometry and physics. The advantages that can be gained by putting problems in ordinary differential geometry into the graded framework are illustrated by the above “conceptualization” of tensors in terms of iterated differential forms. Also, it is rather plausible that the problem of discretization of differential geometry for computer implementations is one of those that are naturally inscribed in this context.

Bohr correspondence principle and observability in quantum physics. Natural relations of quantum physics with the classical one that are partially expressed by the famous Bohr correspondence principle must be duly reflected in the mathematics formalising them. The commonly adopted von Neumann’s proposal to appoint self-adjoint operators in Hilbert spaces as observables in quantum mechanics manifestly violates the so generalised Bohr’s principle. Indeed, the very rich language of differential calculus that includes all the necessary for classical physics differential geometry has nothing in common with the poor and even rude language of Hilbert spaces. For instance, Hilbert spaces of functions defined on domains of different shapes and dimensions are isomorphic. Moreover, this language is not localisable in the space-time and also fails to work in QFT.

So, the problem of adequate mathematical formalisation of the observability mechanism in quantum mechanics is still open as well as in QFT. The principal difficulty of this problem is that the physical factors “responsible” for observability should be duly formalised and incorporated into this mechanism. The arguments of a smooth and natural correspondence with the language of classical physics suggest looking for the details of this mechanism in graded differential geometry. The reader will find some more concrete ideas in [44] and [9].

Of special interest in this list of problems is the long-standing problem of mathematically rigorous theory of integration by paths, which is crucial for modern QFT. It is plausible that the solution would be an analogue in secondary calculus of the cohomological theory discussed in Subsection 7.7.

8.3. The language barrier. One of the goals of this brief survey was to show with sufficiently simple examples that the language and methods of DCCA are both simplifying and unifying. Not less important is that it reveals natural relations, which not infrequently remain hidden in the standard descriptive approach and, as a consequence, allows one to better foresee the lines of the future developments by avoiding eventual misleading ideas and intuition that may come from the “descriptiveness”. And, finally, DCCA makes possible a large expansion of methods of traditional “differential mathematics” to new emerging areas of mathematics and theoretical physics and also to not yet “differentialised” domains of mathematics itself.

This is why we think that a systematic introduction of the language and methods of DCCA into the above indicated areas could have a notable positive effect. Unfortunately, the language barriers slow down realisation in full of these promising and intriguing possibilities. Other barriers are due to elevated diversity, complexity and dimension of the problems arising here, which require organisation of large scale research programs, as is common in experimental physics. In this connection a systematic introduction of fundamentals of DCCA into university courses could turn the situation to the best.

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