STUDIA MATHEMATICA
Online First version

Involution similarity preserving linear maps

by

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Abstract. Let X be a Banach space with dimension at least 3. Two operators A and B in $B(X)$ are said to be p-similar if there is a product S of finitely many involutions such that $A = SBS^{-1}$. In this paper, we investigate linear bijections $\Phi : B(X) \to B(X)$ such that $\Phi(A)$ and $\Phi(B)$ are similar whenever A and B are p-similar. We show that such a map is either an isomorphism or an anti-isomorphism plus a p-similarity invariant functional. This result can be used to characterize Lie isomorphisms and Jordan isomorphisms.

1. Introduction and main results. Throughout, all algebras and vector spaces will be over $\mathbb{F}$, where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Given a Banach space X with topological dual X^*, we denote by $B(X)$ and $G(X)$ the Banach algebra of all bounded linear operators on X and the group of all invertible operators in $B(X)$, respectively. Two operators A and B in $B(X)$ are said to be *similar*, denoted $A \sim B$, if $A = S^{-1}BS$ for some $S \in G(X)$. A linear map $\Phi : B(X) \to B(X)$ is said to be *similarity preserving* if $\Phi(A) \sim \Phi(B)$ whenever $A \sim B$. The problem of characterizing similarity preserving linear maps was treated in a series of papers: see, for example, [5, 6, 8, 9, 10, 12]. Among others, Lu and Peng [8] proved that if X is an infinite-dimensional Banach space and $\Phi : B(X) \to B(X)$ is a similarity preserving linear map then Φ must be of the form either $A \mapsto cTAT^{-1} + h(A)I$ or $A \mapsto cTA^*T^{-1} + h(A)I$ for some nonzero scalar c, some invertible operator T and some similarity invariant functional h on $B(X)$. Here, a *similarity invariant functional* h means that $h(A) = h(B)$ whenever $A \sim B$.

In this paper, we will consider maps preserving other types of similarities.

Recall that an operator J in $B(X)$ is called an *involution* if $J^2 = I$, the identity operator. Geometrically, involutions can be viewed as 'reflections' in

2010 *Mathematics Subject Classification*: Primary 47B49.
Key words and phrases: p-similarity, involution, Jordan isomorphism, Lie isomorphism.
Received 11 March 2018; revised 3 June 2018.
Published online 23 April 2019.

DOI: 10.4064/sm180311-5-7

one subspace along another. We denote by $\mathrm{Inv}(X)$ and $\mathrm{P\text{-}Inv}(X)$ the set of all involutions in $B(X)$, and the set of all products of finitely many involutions, respectively. Obviously, $\mathrm{P\text{-}Inv}(X)$ is a subgroup of $G(X)$. What is $\mathrm{P\text{-}Inv}(X)$? Halmos, Custafson and Radjavi [3] showed that if X is finite-dimensional, then $\mathrm{P\text{-}Inv}(X)$ is precisely the set of all operators whose determinant is ± 1; Radjavi [11] proved that if X is an infinite-dimensional Hilbert space, then $\mathrm{P\text{-}Inv}(X) = G(X)$. In the general Banach space case the problem is tricky and the answer is not known (see the discussion in [11]).

We now define similarities via $\mathrm{Inv}(X)$ and $\mathrm{P\text{-}Inv}(X)$. Two operators A and B are said to be *i-similar* (resp. *p-similar*), denoted $A \sim_i B$ (resp. $A \sim_p B$), if $A = S^{-1}BS$ for some $S \in \mathrm{Inv}(X)$ (resp. $S \in \mathrm{P\text{-}Inv}(X)$). A linear map $\Phi : B(X) \to B(X)$ is said to be *i-similarity preserving* (resp. *p-similarity preserving*) if $\Phi(A) \sim \Phi(B)$ whenever $A \sim_i B$ (resp. $A \sim_p B$). A linear functional h on $B(X)$ is said to be *p-similarity invariant* if $h(A) = h(B)$ whenever $A \sim_p B$. Clearly, similarity preserving is stronger than p-similarity preserving.

The central result is as follows.

Theorem 1.1. *Let X be a Banach space with dimension at least* 3. *Then a linear bijection $\Phi : B(X) \to B(X)$ is p-similarity preserving if and only if one of the following holds:*

(1) *There exist a nonzero scalar c, an invertible bounded operator T in $B(X)$ and a p-similarity invariant linear functional h on $B(X)$ with $h(I) \neq -c$ such that $\Phi(A) = cTAT^{-1} + h(A)I$ for all $A \in B(X)$.*
(2) *There exist a nonzero scalar c, an invertible bounded operator $T : X^* \to X$ and a p-similarity invariant linear functional h on $B(X)$ with $h(I) \neq -c$ such that $\Phi(A) = cTA^*T^{-1} + h(A)I$ for all $A \in B(X)$.*

The proof will be given in Section 3. We now give some applications. The most interesting novelty is that we find that Lie (Jordan) isomorphisms are p-similarity preserving (Corollaries 1.4 and 1.5). In the following corollaries, X is a Banach space with dimension at least 3.

First of all, as an immediate consequence, we recapture the main results of [5] and [8].

Corollary 1.2. *A linear bijection $\Phi : B(X) \to B(X)$ is similarity preserving if and only if it is of the form* (1) *or* (2) *in Theorem* 1.1 *with h similarity invariant.*

Corollary 1.3. *A linear bijection $\Phi : B(X) \to B(X)$ is i-similarity preserving if and only if it is of the form* (1) *or* (2) *in Theorem* 1.1.

Proof. The sufficiency is obvious. To show the necessity, it suffices to show that Φ is p-similarity preserving. For this, suppose that $A \sim_p B$

for $A, B \in B(X)$. Then there are involutions $J_1, \ldots, J_n$ such that $A = J_n \cdots J_1 B J_1 \cdots J_n$. Let $B_k = J_k \cdots J_1 B J_1 \cdots J_k$, $k = 1, \ldots, n$. Then

$$B \sim_i B_1 \sim_i \cdots \sim_i B_n = A.$$

It follows that

$$\Phi(B) \sim \Phi(B_1) \sim \Phi(B_2) \sim \cdots \sim \Phi(A).$$

So $\Phi(A) \sim \Phi(B)$. Consequently, Φ is p-similarity preserving, completing the proof. ■

Recall that a linear bijection $\Phi : B(X) \to B(X)$ is a *Jordan isomorphism* if $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for all $A, B \in B(X)$. Since $B(X)$ is prime, it follows from [13] that each Jordan isomorphism of $B(X)$ is either an isomorphism or an anti-isomorphism. Here, as one application of Corollary 1.3, we present a different proof.

COROLLARY 1.4. *Let Φ be a Jordan isomorphism from $B(X)$ onto itself. Then Φ is either an isomorphism or an anti-isomorphism.*

Proof. Note that $\Phi(I) = I$. If J is an involution, then $\Phi(J)^2 = \Phi(J^2) = \Phi(I) = I$ and so $\Phi(J)$ is an involution. If $A = JBJ$, then $\Phi(A) = \Phi(JBJ) = \Phi(J)\Phi(B)\Phi(J)$. So Φ preserves i-similarity. Corollary 1.3 applies. Then $\Phi = c\psi + h$, where c is a non-zero scalar, ψ is an isomorphism or an anti-isomorphism of $B(X)$ and h is a p-similarity invariant linear map from $B(X)$ into $\mathbb{F}I$.

It is now sufficient to show that $c = 1$ and $h = 0$. We thank the reviewer for showing us the following shorter proof. Since Jordan isomorphism preserves Jordan triple product, it preserves rank one operators by applying characterization of rank-one operators via triple product: "A is of rank one if and only if for every $B \in B(X)$ there exists a scalar λ such that $ABA = \lambda A$." It follows that $h(F) = 0$ for every rank-one operator F. As Jordan isomorphism preserves idempotents, in particular rank-one idempotents, we have $c = 1$. Then we can further deduce $h(A) = 0$ for all $A \in B(X)$, completing the proof. ■

Recall that a linear bijection $\Phi : B(X) \to B(X)$ is called a *Lie isomorphism* if $\Phi([A, B]) = [\Phi(A), \Phi(B)]$ for all $A, B \in B(X)$, where $[A, B] = AB - BA$. It is well-known that such a Φ is *standard*, that is, $\Phi = \psi + \tau$, where ψ is an isomorphism or the negative of an anti-isomorphism and τ is a linear map from $B(X)$ into to $\mathbb{F}I$ vanishing at each commutator. Here, as another application of Corollary 1.3, we present a different proof.

COROLLARY 1.5. *Let Φ be a Lie isomorphism from $B(X)$ onto itself. Then Φ is standard.*

Proof. We first show that Φ is i-similarity preserving. Let $A, B \in B(X)$ and J be a nontrivial involution such that $B = JAJ$. Let $P = (I - J)/2$. Then P is idempotent and $J = I - 2P$. By [14, Lemma 2.2], there is a scalar λ and an idempotent Q such that $\Phi(P) = \lambda I + Q$. Let $E = I - 2Q$. Then E is an involution. Since $\Phi(I) \in \mathbb{F}I$, there is a scalar μ such that $\Phi(J) = \mu I + E$. Thus, from $2(A - B) = [J, [J, A]]$, we have

$$2(\Phi(A) - \Phi(B)) = \big[\Phi(J), [\Phi(J), \Phi(A)]\big] = 2\big(\Phi(A) - E\Phi(A)E\big).$$

So $\Phi(B) = E\Phi(A)E$, which implies $\Phi(A) \sim \Phi(B)$.

Corollary 1.3 applies. Assume $\Phi = c\psi + \tau$, where c is a nonzero scalar, ψ is an isomorphism or an anti-isomorphism and τ is a linear map from $B(X)$ into $\mathbb{F}I$. Thus, for $A, B \in B(X)$, from $\Phi([A, B]) = [\Phi(A), \Phi(B)]$ we see that if ψ is an isomorphism then

$$(c^2 - c)[\psi(A), \psi(B)] = \tau([A, B]);$$

if ψ is an anti-isomorphism then

$$(c^2 + c)[\psi(A), \psi(B)] = \tau([A, B]).$$

It follows from the Kleinecke–Shirokov theorem (see, for example, [4, Question 232]) that $\tau([A, B]) = 0$ for all $A, B \in B(X)$. Hence by putting non-commuting A, B in the last two displayed equations, we see that if ψ is an isomorphism then $c^2 - c = 0$ and hence $c = 1$; if ψ is an anti-isomorphism then $c^2 + c = 0$ and hence $c = -1$. ∎

2. Preliminary results. Throughout this section, X is a Banach space with dimension ≥ 3. For nonzero vectors $x \in X$ and $f \in X^*$, the rank one operator $x \otimes f$ is defined as the map $y \mapsto f(y)x$. We denote by $F(X)$ the set of all finite rank operators in $B(X)$. Also, $F_0(X)$ denotes the subspace of all trace zero finite rank operators.

PROPOSITION 2.1. *Suppose that N is in $F(X)$ with $N^2 = 0$. Then $I + N$ is a product of two involutions.*

Proof. We can write $N = \sum_{i=1}^{n} x_i \otimes f_i$, where $x_1, \ldots, x_n \in X$ are linearly independent, $f_1, \ldots, f_n \in X^*$ are linearly independent, and $f_i(x_j) = 0$, $i, j = 1, \ldots, n$. Take $y_1, \ldots, y_n \in X$ such that $f_i(y_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. Let $P = \sum_{i=1}^{n} y_i \otimes f_i$. Then $P^2 = P$, $NP = N$ and $PN = 0$. Now it is easy to see that $(I - 2P)^2 = I$, $(I + N - 2P)^2 = I$ and $(I - 2P)(I + N - 2P) = I + N$, completing the proof. ∎

Observe that p-similarity satisfies reflexivity (i.e. $A \sim_p A$), symmetry (i.e., if $A \sim_p B$ then $B \sim_p A$) and transitivity (i.e. if $A \sim_p B$ and $B \sim_p C$ then $A \sim_p C$).

PROPOSITION 2.2. *All nilpotent operators of rank one in $B(X)$ are p-similar to each other.*

Proof. We first prove two claims.

CLAIM 1. *If* $0 \neq f \in X^*$ *and* $0 \neq x, y \in \ker(f)$, *then* $x \otimes f \sim_p y \otimes f$.

First suppose that x and y are linearly independent. Choose $g, h \in X^*$ such that $g(x) = h(y) = 1$ and $g(y) = h(x) = 0$. Let $P = x \otimes g + y \otimes h$ and $J = x \otimes h + y \otimes g + I - P$. Then it is not difficult to verify that $J^2 = I$ and $Jx \otimes fJ = y \otimes f$. So, $x \otimes f \sim_p y \otimes f$.

Next suppose that x and y are linearly dependent. Since $\dim X \geq 3$, we can choose $z \in \ker(f)$ such that z and x as well as z and y are linearly independent. Then by the previous case, we have $x \otimes f \sim_p z \otimes f$ and $y \otimes f \sim_p z \otimes f$. It follows that $x \otimes f \sim_p y \otimes f$.

CLAIM 2. *If* $0 \neq f, g \in X^*$ *and* $0 \neq x \in \ker(f) \cap \ker(g)$, *then* $x \otimes f \sim_p x \otimes g$.

If f and g are linearly dependent, then we are done by Claim 1. Now suppose that f and g are linearly independent. Choose $y, z \in X$ such that $f(y) = g(z) = 0$ and $f(z) = g(y) = 1$. Let $P = z \otimes f + y \otimes g$ and $J = y \otimes f + z \otimes g + I - P$. Then $J^2 = I$ and $Jx \otimes fJ = y \otimes g$. So $x \otimes f \sim_p y \otimes f$.

Now, let $x \otimes f$ and $y \otimes g$ be nilpotent operators of rank one. Since $\dim X \geq 3$, we can choose a nonzero $z \in \ker(f) \cap \ker(g)$ as follows: take linearly independent vectors $x_1, x_2 \in \ker(f)$, and then let $z = g(x_2)x_1 - g(x_1)x_2$ if $g(x_1) \neq 0 \neq g(x_2)$ or let $z = x_i$ if $g(x_i) = 0$. Now by Claim 1,

$$x \otimes f \sim_p z \otimes f.$$

By Claim 2,

$$z \otimes f \sim_p z \otimes g.$$

By Claim 1 again,

$$z \otimes g \sim_p y \otimes g.$$

It follows that $x \otimes f \sim_p y \otimes g$. ∎

LEMMA 2.3 ([7, Lemma 2.4]). *Suppose that* X *is infinite-dimensional. Let* A *and* B *be in* $B(X)$. *Assume that for every* $x \in X$ *the vector* Ax *belongs to the linear span of* x *and* Bx. *Then* A, B *and* I *are linearly dependent.*

LEMMA 2.4 ([2, Lemma 2.1]). *Let* $A \in B(X)$, *not of the form scalar plus finite rank. Then for every positive integer* n *there exist* $x_1, \ldots, x_n \in X$ *such that* $x_1, \ldots, x_n, Ax_1, \ldots, Ax_n$ *are linearly independent.*

LEMMA 2.5. *Suppose that* X *is infinite-dimensional. Let* $A \notin \mathbb{F}I$. *Then there exist* $B_1, B_2 \in F_0(X)$ *such that:*

(i) B_1 *and* B_2 *are linearly independent, and*
(ii) $A + B_i \sim_p A$, $i = 1, 2$.

Proof. This follows from Proposition 2.1 and [8, proof of Lemma 2.3]. ∎

Lemma 2.6. *Suppose X is infinite-dimensional. Let $\Phi : B(X) \to B(X)$ be a bijective p-similarity preserving linear map. Assume that there exists $A \in B(X)$ such that $A \notin \mathbb{F}I + F(X)$ and $\Phi(A) = \lambda I + F$ for some $\lambda \in \mathbb{F}$ and some finite rank operator F. Denote $r = \operatorname{rank}(F)$. Then for any positive integer k, there exists a trace-zero operator $C \in B(X)$ with* $\operatorname{rank}(C) = 2k$ *such that* $\operatorname{rank}(\Phi(C)) \leq 2r$.

Proof. Let k be any positive integer. By Lemma 2.4, there exist $2k$ vectors $x_1, \ldots, x_{2k} \in X$ such that $x_1, \ldots, x_{2k}, Ax_1, \ldots, Ax_{2k}$ are linearly independent. Choose $f_1, \ldots, f_k \in X^*$ such that

$$\begin{aligned} f_i(x_j) &= 0, \qquad i = 1, 2, \ldots, k;\ j = 1, \ldots, 2k, \\ f_i(Ax_j) &= 0, \qquad i, j = 1, \ldots, k, \\ f_i(Ax_{k+j}) &= \delta_{ij}, \quad i, j = 1, \ldots, k. \end{aligned}$$

We claim that $f_1, \ldots, f_k, A^*f_1, \ldots, A^*f_k$ are linearly independent. Indeed, suppose $\sum_{i=1}^n \alpha_i f_i + \sum_{i=1}^n \beta_i A^* f_i = 0$ for some $\alpha_i, \beta_i \in \mathbb{F}$, $i = 1, \ldots, k$. Then for each $1 \leq j \leq k$, we have $\sum_{i=1}^n \alpha_i f_i(x_{k+j}) + \sum_{i=1}^n \beta_i A^* f_i(x_{k+j}) = 0$ and hence $\beta_j = 0$. Further, for each $1 \leq j \leq k$, we have $\sum_{i=1}^n \alpha_i f_i(Ax_{k+j}) = 0$ and hence $\alpha_j = 0$.

Let $N = \sum_{i=1}^n x_i \otimes f_i$. Then we have $N^2 = 0$ and $NAN = 0$. Let $B = (I + N)A(I - N)$ and $C = B - A$. It is not difficult to see that C has trace zero and rank $2k$. By Proposition 2.1, $A \sim_p B$ and so $\Phi(A) \sim \Phi(B)$. Suppose $\Phi(B) = T\Phi(A)T^{-1}$ for some invertible $T \in B(X)$. Then from

$$\Phi(C) = \Phi(B) - \Phi(A) = T\Phi(A)T^{-1} - \Phi(A) = TFT^{-1} - F$$

it follows that $\operatorname{rank}(\Phi(C)) \leq 2r$. ∎

Proposition 2.7 ([8, Lemma 2.5]). *Let $x, y \in X$ and $f, g \in X^*$. Then $I - (x \otimes f + y \otimes g)$ is invertible if and only if $(f(x) - 1)(g(y) - 1) \neq f(y)g(x)$.*

Proposition 2.8 ([8, Proposition 2.6]). *Let X be an infinite-dimensional Banach space. Let $\Phi : B(X) \to B(X)$ be an injective linear map such that $\Phi(N)$ is nilpotent of rank one for every rank one nilpotent operator $N \in B(X)$. Then one of the following holds:*

- (i) *There exists a nonzero $x \in X$ and an injective linear map τ from $F_0(X)$ to $\{f \in X^* : f(x) = 0\}$ such that $\Phi(A) = x \otimes \tau(A)$ for every $A \in F_0(X)$.*
- (ii) *There exists a nonzero $x \in X$ and an injective linear map δ from $F_0(X)$ to $\{x \in X : f(x) = 0\}$ such that $\Phi(A) = \delta(A) \otimes f$ for every $A \in F_0(X)$.*
- (iii) *There exist injective linear maps $T : X \to X$ and $S : X^* \to X^*$ such that $\Phi(x \otimes f) = Tx \otimes Sf$ for every rank one nilpotent operator $x \otimes f$.*
- (iv) *There exist injective linear maps $T : X^* \to X$ and $S : X \to X^*$ such that $\Phi(x \otimes f) = Tf \otimes Sx$ for every rank one nilpotent operator $x \otimes f$.*

REMARK 2.9 ([8, Remark 2.7]). Suppose that $\Phi(F_0(X)) = F_0(X)$. Then we have only case (iii) or (iv) with T and S bijective and continuous. Furthermore, if (iii) occurs, then there is a nonzero scalar c such that $\Phi(A) = cTAT^{-1}$ for every $A \in F_0(X)$; if (iv) occurs, then there is a nonzero scalar c such that $\Phi(A) = cTA^*T^{-1}$ for every $A \in F_0(X)$.

3. The proof of Theorem 1.1. In this section, we will complete the proof of Theorem 1.1. The sufficiency is a straightforward verification (cf. [8, proof of Theorem 3.1]). So we just need to verify the necessity. For clarity, we will organize the proof into a series of lemmas. In what follows, X is a Banach space of dimension at least 3, and Φ is a bijective linear map of $B(X)$ satisfying $\Phi(A) \sim \Phi(B)$ whenever $A \sim_p B$.

We begin by showing that Φ sends scalar operators to scalar ones.

LEMMA 3.1. *$\Phi(\mathbb{F}I) = \mathbb{F}I$.*

Proof. By surjectivity, there exists $A \in B(X)$ such that $\Phi(A) = I$. If $A \notin \mathbb{F}I$, by Lemma 2.5 there exists a nonzero operator $B \in B(X)$ such that $A + B \sim_p A$. Then we have $I + \Phi(B) \sim I$ and hence $\Phi(B) = 0$, conflicting with the injectivity of Φ. ∎

LEMMA 3.2. *$\Phi(\mathcal{N}_1(X)) \subseteq \mathcal{N}_1(X)$. Moreover, $\Phi(F_0(X)) \subseteq F_0(X)$.*

Proof. Take a rank one operator $B \in B(X)$ and choose $A \in B(X)$ such that $\Phi(A) = B$. Clearly, A is not a scalar operator by Lemma 3.1. Thus, we can find a vector $x \in X$ such that x and Ax are linearly independent. Then there exists $f \in X^*$ such that $f(x) = 0$ and $f(Ax) = 1$. Set $N = x \otimes f$. Then $N^2 = 0$ and $NAN = N$. For every $\lambda \in \mathbb{F}$, by Proposition 2.1, we have $(I + \lambda N)A(I - \lambda N) \sim_p A$. It follows that $\Phi((I + \lambda N)A(I - \lambda N)) \sim B$. Thus,

$$\Phi((I + \lambda N)A(I - \lambda N)) = -\lambda^2 \Phi(N) + \lambda \Phi(AN - NA) + B$$

is of rank one for every scalar λ. Dividing by λ^2, sending λ to infinity, and applying the fact that the set of all operators of rank at most one is closed, we arrive at $\operatorname{rank}(\Phi(N)) = 1$. Moreover, by Proposition 2.2, $N \sim_p 2N$, and therefore $\Phi(N) \sim 2\Phi(N)$. As $\Phi(N)$ is of rank one, it has to be nilpotent. By Proposition 2.2 again, we conclude that Φ maps nilpotents of rank one into nilpotents of rank one. ∎

LEMMA 3.3. *Assume that X is infinite-dimensional. Let A be in $B(X)$ and suppose that $\Phi(A) \in \mathbb{F}I + F(X)$. Then $A \in \mathbb{F}I + F(X)$.*

Proof. Assume on the contrary that $A \notin \mathbb{F}I + F(X)$ and $\Phi(A) = \lambda I + F$ for some $\lambda \in \mathbb{F}$ and some finite rank operator F. Denote $\operatorname{rank}(F) = r$. Then, by Lemma 2.6, there exists a trace-zero operator $C = \sum_{i=1}^{2k} x_i \otimes f_i$ with $\operatorname{rank}(C) = 2k$ $(k > r)$ such that $\operatorname{rank}(\Phi(C)) \le 2r$. Since Φ maps the set of nilpotents of rank one into itself, we can apply Proposition 2.8. Assume that case (iii) holds. Since T and S are injective, $\Phi(C) = \sum_{i=1}^{2k} Tx_i \otimes Sf_i$ is of rank $2k > 2r$, a contradiction. In a similar way we can prove that case (iv) cannot occur either. So, we have either (i) or (ii). We will just consider case (i) since the proof for (ii) goes in almost the same way. Thus, there exists a nonzero $x \in X$ and a linear map $\tau : F_0(X) \to \{f \in X^* : f(x) = 0\}$ such that $\Phi(A) = x \otimes \tau(A)$ for every $A \in F_0(X)$. Let $y \in X$ be linearly independent of x. Choose a nonzero $g \in X^*$. By surjectivity, there exists $D \in B(X)$ such that $\Phi(D) = y \otimes g$. Obviously, D is not a scalar operator. By Lemma 2.5 we find linearly independent N_1, $N_2 \in F_0(X)$ such that

$$D + N_1 \sim_p D \quad \text{and} \quad D + N_2 \sim_p D.$$

It follows that

$$y \otimes g + x \otimes \tau(N_1) \sim y \otimes g \quad \text{and} \quad y \otimes g + x \otimes \tau(N_2) \sim y \otimes g.$$

Since y and x are linearly independent, we infer that g, $\tau(N_1)$ are linearly dependent and g, $\tau(N_2)$ are linearly dependent. Hence, $\Phi(N_1)$ and $\Phi(N_2)$ are linearly dependent, contradicting the bijectivity of Φ. ∎

LEMMA 3.4. *One of the following holds:*

(1) *There is an invertible operator T in $B(X)$ and a nonzero scalar c such that $\Phi(A) = cTAT^{-1}$ for all $A \in F_0(X)$.*
(2) *There is an invertible operator $T : X^* \to X$ and a nonzero scalar c such that $\Phi(A) = cTA^*T^{-1}$ for all $A \in F_0(X)$.*

Proof. If X is finite-dimensional, this follows from Lemma 3.2 and [1, Theorem].

In what follows, we suppose that X is infinite-dimensional. Then from Lemmas 3.2 and 3.3 we have

$$\Phi(F_0(X)) \subseteq F_0(X) \subseteq F(X) \subseteq F(X) + \mathbb{F}I \subseteq \Phi(F(X) + \mathbb{F}I).$$

Note that $F(X) = F_0(X) \oplus \mathbb{F}P$ for any idempotent P of rank one. Therefore $F_0(X)$ is a subspace of codimension 1 in $F(X)$. Also, $F(X)$ is of codimension 1 in $F(X) + \mathbb{F}I$. Consequently, $\Phi(F_0(X))$ is of codimension 2 in $\Phi(F(X) + \mathbb{F}I)$. It follows that

$$\Phi(F_0(X)) = F_0(X) \quad \text{and} \quad F(X) + \mathbb{F}I = \Phi(F(X) + \mathbb{F}I).$$

Now Remark 2.9 applies, completing the proof. ∎

LEMMA 3.5. *Suppose that Lemma* 3.4(1) *holds. Then Theorem* 1.1(1) *holds.*

Proof. Composing Φ with a similarity transformation, and then multiplying it by c^{-1}, we may assume that $\Phi(A) = A$ for every $A \in F_0(X)$.

If $A \in \mathbb{F}I$, then by Lemma 3.1, $\Phi(A) = A + h(A)I$ for some scalar $h(A)$.

We now suppose $A \notin \mathbb{F}I$. Let $B = \Phi(A)$. We can assume that B is invertible. Actually, if B is not invertible, we may replace A by $A + \lambda I$ for an appropriate scalar λ, since $\Phi(\mathbb{F}I) = \mathbb{F}I$. Let $x \in X$ and $f \in X^*$ be such that $f(x) = f(B^{-1}x) = 0$. For all $\alpha \in \mathbb{F}$, we have

$$(I + \alpha x \otimes f)A(I - \alpha x \otimes f) = A + F_\alpha,$$

where $F_\alpha = \alpha x \otimes A^* f - \alpha Ax \otimes f - \alpha^2 f(Ax)x \otimes f$. Obviously, $F_\alpha \in F_0(X)$ and $\operatorname{rank}(F_\alpha) \leq 2$. By Proposition 2.1, we have $A \sim_p A + F_\alpha$. It follows that $B \sim B + F_\alpha$. Hence $I + B^{-1}F_\alpha = B^{-1}(B + F_\alpha)$ is invertible. Applying Proposition 2.7 and the fact that $f(B^{-1}x) = 0$, we get

$$f(AB^{-1}x)f(B^{-1}Ax)\alpha^2 - (f(AB^{-1}x) - f(B^{-1}Ax))\alpha - 1 \neq 0$$

for every $\alpha \in \mathbb{F}$. Thus, $f(AB^{-1}x)f(B^{-1}Ax) = 0$ and $f(AB^{-1}x) - f(B^{-1}Ax) = 0$. It follows that $f(AB^{-1}x) = 0$. Since x, f are arbitrary vectors satisfying $f(x) = f(B^{-1}x) = 0$, it follows from Lemma 2.3 that I, B^{-1}, AB^{-1} are linearly dependent, i.e., there exist scalars c_1, c_2, c_3, not all zero, such that $c_1 I + c_2 B^{-1} + c_3 AB^{-1} = 0$. Because $A \notin \mathbb{F}I$, we then have $B = g(A)A + h(A)I$ for some scalars $g(A)$, $h(A)$.

We now show that $g(A) = 1$ for all $A \in B(X)$. Indeed, choose $C \in F_0(X)$ such that A, C and I are linearly independent. Then $\Phi(A + C) = g(A + C)(A + C) + h(A + C)I$ and $\Phi(A + C) = \Phi(A) + \Phi(C) = g(A)A + h(A)I + C$. Comparing those two equations and using the fact A, C, I are linearly independent, we get $g(A) = g(A + C) = 1$.

Finally, we show that h is p-similarity invariant. Suppose that $A \sim_p B$ for $A, B \in B(X)$. Then $A = S_1 B S_1^{-1}$ and $\Phi(A) = S_2 \Phi(B) S_2^{-1}$ for some $S_1 \in$ P-Inv(X) and $S_2 \in G(X)$. Thus, $A + h(A)I = S_2(B + h(B)I)S_2^{-1}$ and hence $S_1 S_2^{-1} A - A S_1 S_2^{-1} = (h(A) - h(B)) S_1 S_2^{-1}$. It follows from the Kleinecke–Shirokov theorem that $h(A) - h(B) = 0$, completing the proof. ∎

Similarly, we have

LEMMA 3.6. *Suppose that Lemma* 3.4(2) *holds. Then Theorem* 1.1(2) *holds.*

Thus we have completed the proof of Theorem 1.1.

Acknowledgements. The authors would like to thank the referee for a very thorough reading of the paper and many helpful comments.

The research was supported by the National Natural Science Foundation of China (Grant No. 11571247).

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