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An example concerning Sadullaev's boundary relative extremal functions

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In memory of Józef Siciak

Abstract. We exhibit a smoothly bounded domain Ω with the property that for suitable $K \subset \partial\Omega$ and $z \in \Omega$ the Sadullaev boundary relative extremal functions satisfy $\omega_1(z, K, \Omega) < \omega_2(z, K, \Omega) \leq \omega(z, K, \Omega)$.

1. Introduction. Sadullaev [5] introduced several so-called *boundary relative extremal functions* for compact sets K in the boundary of domains $D \subset \mathbb{C}^n$, and asked whether their regularizations are perhaps always equal. Recently Djire and the author [1, 2] gave a positive answer in certain cases where D and K are particularly nice.

In this note we show that in general equality does not hold. The relevant example is formed by a suitable compact set in the boundary of the domain Ω that was constructed by Fornæss and the author [3] as an example of a domain D where bounded plurisubharmonic functions that are continuous on D cannot be approximated by plurisubharmonic functions that are continuous on $\overline{D}$. We start by briefly recalling the definitions of boundary relative extremal functions and the construction of the domain Ω.

1.1. Boundary relative extremal functions. We follow Sadullaev [5, Section 27]. Let D be a domain with smooth boundary in $\mathbb{C}^n$, $\xi \in \partial D$, and $A_\alpha(\xi) = \{z \in D; |z - \xi| < \alpha\delta_\xi(z)\}$, where $\alpha \geq 1$ and $\delta_\xi(z)$ is the distance from z to the tangent plane at ξ to ∂D. For a function u defined on D, put

$$\tilde{u}(\xi) = \sup_{\alpha>1} \limsup_{\substack{z\to\xi \\ z\in A_\alpha(\xi)}} u(z), \qquad \xi \in \partial D.$$

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Definition 1.1. Let PSH(D) denote the plurisubharmonic functions on D and let $K \subset \partial D$ be compact. We define the following *boundary relative extremal functions*:

$$\omega(z,K,D) = \sup\{u(z) : u \in \mathrm{PSH}(D),\ u \le 0,\ \tilde{u}|_K \le -1\},$$
$$\omega_1(z,K,D) = \sup\{u(z) : u \in \mathrm{PSH}(D) \cap C(\overline{D}),\ u \le 0,\ u|_K \le -1\},$$
$$\omega_2(z,K,D) = \sup\Big\{u(z) : u \in \mathrm{PSH}(D),\ u \le 0,\ \limsup_{\substack{z\to\xi\\ z\in D}} \le -1,\ \text{ for all } \xi \in K\Big\}.$$

The upper *semicontinuous regularization* u^* of a function u on a domain D is defined as

$$u^*(z) = \limsup_{w\to z} u(w).$$

The functions ω^*, ω_1^*, ω_2^* are plurisubharmonic. Observing that $\omega_1(z,K,D) \le \omega_2(z,K,D) \le \omega(z,K,D)$, Sadullaev's question is: *for what j do we have $\omega^*(z,K,D) \equiv \omega_j^*(z,K,D)$?*

1.2. The domain Ω. We briefly recall the construction and properties of the domain Ω from [3]:

$$\Omega = \{(z,w) \in \mathbb{C}^2;\ |w - e^{i\varphi(|z|)}|^2 < r(|z|)\}. \tag{1.1}$$

Here r and φ are in $C^\infty(\mathbb{R})$ with the following properties: $-1 \le r \le 2$; $r(t) \le 0$ for $t \le 1$ and for $t \ge 17$; $r(t) \equiv 1$ for $3 \le t \le 8$ and $10 \le t \le 15$; $r(t)$ takes its maximum value 2 precisely at $t = 2$, 9, and 16. Moreover, $r'(t) > 0$ on $1 \le t < 2$, $8 < t < 9$ and $15 < t < 16$, while $r'(t) < 0$ on $2 < t < 3$, $9 < t < 10$, and $16 < t \le 17$. Next, φ satisfies $\varphi(t) < -\pi/2$ for $t \le 4$ and for $t \ge 14$; $\varphi(t) > \pi/2 + 100$ for $5 \le t \le 6$ and for $12 \le t \le 13$, and $\varphi(t) < -\pi/2 + 100$ for $7 < t < 10$; we demand in addition that $\varphi \le 108$.

From [3] we recall that Ω is a Hartogs domain with smooth boundary, and that the annulus

$$A = \{(z,w);\ w = 0,\ 2 \le |z| \le 15\} \tag{1.2}$$

is contained in $\overline{\Omega}$.

2. Negative answer to Sadullaev's question

Theorem 2.1. *Let $K = \{(z,w) \in \partial\Omega;\ |z| = 2$ or $|z| = 16\}$. Then*

$$\omega_1((z,w),K,\Omega) < \omega_2((z,w),K,\Omega)$$

for (z,w) in an open neighborhood of $\{w = 0,\ |z| = 9\}$.

Proof. Let $u \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$, $u \le 0$, $u|_K \le -1$. Then by the maximum principle, $|u| \le -1$ on the discs $|w - e^{i\varphi(|z|)}| \le 2$, where z is fixed and satisfies $|z| = 2$ or $|z| = 16$, and in particular on the circles $C_1(w) = \{(z,w) : |z| = 2\}$ and $C_2(w) = \{(z,w) : |z| = 16\}$, where $|w| < 1$. Because Ω is a smoothly bounded domain, it follows from [3, Theorem 1] (see also [4] for recent extensions of this theorem) that u can be approximated uniformly on $\overline{\Omega}$ by smooth plurisubharmonic functions v defined on shrinking neighborhoods of $\overline{\Omega}$.

Let $\Omega_\delta = \{\zeta \in \mathbb{C}^2;\, d(\zeta, \overline{\Omega}) < \delta\}$. Then given $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \mathrm{PSH}(\Omega_\delta)$ such that $|u - v| < \varepsilon$ on $\overline{\Omega}$. For $|w| < \delta$ the annulus $A_w = \{(z, w);\, 2 \le |z| \le 16\}$ is contained in Ω_δ. On its boundary, which equals $C_1(w) \cup C_2(w)$, we have $v < -1 + \varepsilon$, hence this also holds on A_w. It follows that $u < -1 + 2\varepsilon$ on $A_w \cap \overline{\Omega}$, in particular $u < -1 + 2\varepsilon$ on the open set $V = \{(z, w);\, 8 < |z| < 10,\, |w| < \delta,\, |w| < r(|z|) - 1\} \subset \Omega$. Consequently, $\omega_1((z, w), K, \Omega) \le -1 + 2\varepsilon$ on V, and therefore also $\omega_1^*((z, w), K, \Omega) \le -1 + 2\varepsilon$ on V.

Next we will construct a plurisubharmonic function in the family that determines ω_2. The construction is as in [3, Section 2]. On $\Omega \cap (\{3 < |z| < 8\} \cup \{10 < |z| < 15\})$ there exists a continuous branch of $\arg w$, denoted by $h(z, w)$, such that

$$\varphi(z) - \pi/2 \le h(z, w) \le \varphi(z) + \pi/2.$$

In [3] we constructed the following plurisubharmonic function:

$$f(z, w) = \begin{cases} 0 & \text{if } |z| < 4 \text{ or } |z| > 14, \\ \max\{0, h(z, w)\} & \text{if } 3 < |z| < 6 \text{ or } 12 < |z| < 14, \\ \max\{100, h(z, w)\} & \text{if } 5 < |z| < 8 \text{ or } 10 < |z| < 13, \\ 100 & \text{if } 7 < |z| < 11. \end{cases} \tag{2.1}$$

It satisfies $f \le 110$ on Ω, $f \equiv 0$ on $\{|z| \le 3\}$ and on $\{|z| \ge 14\}$, hence f extends continuously by 0 to $\overline{\Omega} \cap (\{|z| \le 3\} \cup \{|z| \ge 14\})$, and $f = 100$ on V. The plurisubharmonic function g on Ω defined by

$$g(\zeta) = \frac{f(\zeta) - 110}{110} \qquad (\zeta = (z, w))$$

is negative, identically equal to -1 on $\overline{\Omega} \cap (\{|z| \le 3\} \cup \{|z| \ge 14\})$, and equal to $-10/11$ on V. Hence also $\omega_2^*((z, w), K, \Omega) \ge \omega_2((z, w), K, \Omega) \ge -10/11$ on V. Choosing $\varepsilon < 1/20$ completes the proof. ∎

Remark 2.2. It is perhaps surprising that the inequality

$$\omega_1((z, w), K, \Omega) < \omega_2((z, w), K, \Omega)$$

occurs because the plurisubharmonic functions determining ω_1 are continuous at a part of $\partial\Omega$ that is far away from K. Taking into account that all other boundary relative extremal functions that Sadullaev introduced in [5] are formed by taking the supremum over a set of negative PSH-functions with special boundary behavior at K, it is tempting to introduce

$$\tilde{\omega}_1(z, K, D) = \sup\{u(z) : u \in \mathrm{PSH}(D) \cap C(D \cup K),\, u \le 0,\, u|_K \le -1\}$$

and ask: Do we have

$$\tilde{\omega}_1(z, K, D) = \omega_2(z, K, D) = \omega(z, K, D)?$$

Theorem 2.1 does not give a counterexample.

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