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Common upper frequent hypercyclicity

by

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Abstract. Considering a family of upper frequently hypercyclic operators we study the existence of vectors which are upper frequently hypercyclic for any operator of this family. We establish sufficient conditions for an uncountable family of operators to admit such vectors called common upper frequently hypercyclic vectors. Using this result, we then give a construction of such vectors. Finally, we derive some applications to families of weighted shifts.

1. Introduction. In the last decades the notion of hypercyclicity has been the subject of many investigations. An operator T on a Fréchet space X is called *hypercyclic* on X if there exists some vector $x \in X$ such that the set $\{T^n(x) \mid n \geq 0\}$ is dense in X. In this case, x is called *hypercyclic* for T. Birkhoff [13] established an equivalent characterization of hypercyclicity. In addition he proved that the set of hypercyclic vectors for a hypercyclic operator is a dense G_δ-set. In 1969 Rolewicz found one of the first examples of hypercyclic operators: the multiples of the backward shift λB with $\lambda > 1$ on l^p.

Later Salas [21] raised the question of the existence of vectors which are hypercyclic for any multiple of the backward shift λB, $\lambda > 1$. This initiated a new notion: common hypercyclicity. For an arbitrary family $(T(\lambda))_{\lambda\in\Lambda}$ of hypercyclic operators, a vector is called a *common hypercyclic vector* for $(T(\lambda))_{\lambda\in\Lambda}$ if it is hypercyclic for each operator of the family. The Birkhoff theorem combined with the Baire category theorem directly implies that any countable family of hypercyclic operators has a common hypercyclic vector. Actually in this case the set of common hypercyclic vectors is a dense G_δ-set. For uncountable families of operators this argument fails. Indeed, even if Abakumov and Gordon [1] have shown the existence of common hypercyclic

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vectors for $(\lambda B)_{\lambda>1}$, there also exist families of operators without common hypercyclic vectors. For example Bayart and Matheron [9] have proved that the family of hypercyclic weighted shifts on l^2 has no common hypercyclic vector. However, imposing structures on the index set, positive results have been obtained (see [8], [15], [23]).

Most theorems on common hypercyclicity are based on the following generalization of the Birkhoff theorem for families of hypercyclic operators. The first part of this result is due to Saint-Raymond (see [1], [3]). The equivalence has been found by Shkarin [23].

THEOREM (Shkarin [23]). *Let Λ be a σ-compact metric space, X a separable Fréchet space and $(T(\lambda))_{\lambda\in\Lambda}$ a continuous family of operators on X. Then the set of common hypercyclic vectors for $(T(\lambda))_{\lambda\in\Lambda}$ is a G_δ-set. Moreover the following assertions are equivalent:*

(i) *the set of common hypercyclic vectors for $(T(\lambda))_{\lambda\in\Lambda}$ is a dense G_δ-set;*
(ii) *for any compact subset K of Λ and any non-empty open subsets U, V of X, there exists $x \in U$ such that*

$$\forall\lambda \in K,\ \exists n \geq 0, \qquad T^n(\lambda)(x) \in V.$$

The question of the existence of common hypercyclic vectors extends naturally to frequent hypercyclicity, a notion introduced in [6, 7]. Recall that the *lower density* of a subset A of $\mathbb{N}_0$ is given by

$$\underline{\mathrm{dens}}(A) = \liminf_{n\to\infty} \frac{\#(A\cap[0,n])}{n+1}.$$

On the other hand, the *upper density* of a subset A of $\mathbb{N}_0$ is

$$\overline{\mathrm{dens}}(A) = \limsup_{n\to\infty} \frac{\#(A\cap[0,n])}{n+1}.$$

An operator T on X is called *frequently hypercyclic* if there exists a vector $x \in X$ such that for any non-empty open subset U of X, we have

$$\underline{\mathrm{dens}}(\{m \geq 0 \mid T^m(x) \in U\}) > 0.$$

In this case, the vector x is called *frequently hypercyclic* for T. In contrast to hypercyclicity, Moothathu [20], Grivaux and Matheron [16] and Bayart and Ruzsa [10] have proved that the set of frequently hypercyclic vectors for an operator is of first category. Therefore the methods used for common hypercyclicity fail in this case. Nonetheless some positive results have been established by Bayart and Grivaux [5, 7]. Keeping this idea of frequency we then consider a weaker notion introduced by Shkarin [22], the upper frequent hypercyclicity. An operator T on X is called *upper frequently hypercyclic* if there exists a vector $x \in X$ such that for any non-empty open subset U of X,

$$\overline{\mathrm{dens}}(\{m \geq 0 \mid T^m(x) \in U\}) > 0.$$

In this case the vector x is called *upper frequently hypercyclic* for T. The set of upper frequently hypercyclic vectors is denoted by $\mathcal{U}FHC(T)$. The properties of this set are more similar to those of the set of hypercyclic vectors. Indeed, Bayart and Ruzsa [10] have established that, for an upper frequently hypercyclic operator, this set is residual. Furthermore, Bonilla and Grosse-Erdmann [14] have obtained the following analogue to the Birkhoff theorem for upper frequent hypercyclicity.

Theorem (Bonilla and Grosse-Erdmann [14]). *Let X be a separable Fréchet space and T an operator on X. Then the following assertions are equivalent:*

(i) *the set of upper frequently hypercyclic vectors for T is residual in X;*
(ii) *for any non-empty open subset V of X, there exists $\delta > 0$ such that for any non-empty open subset U of X and any $M \in \mathbb{N}$, we have*

$$\exists x \in U,\, \exists n \geq M, \qquad \frac{\#\{0 \leq m \leq n \mid T^m(x) \in V\}}{n+1} > \delta.$$

Using this result we will obtain sufficient conditions for common upper frequent hypercyclicity. This yields the following theorem which serves as a generalization of the Birkhoff theorem.

Theorem 1. *Let Λ be a σ-compact metric space, X a separable Fréchet space and $(T(\lambda))_{\lambda\in\Lambda}$ a continuous family of operators on X. Suppose that for any non-empty open subset V of X and any compact subset K of Λ there exists $\delta > 0$ such that, for any non-empty open subset U of X and any $M \in \mathbb{N}$, we have*

$$\exists x \in U,\, \forall \lambda \in K,\, \exists n \geq M, \qquad \frac{\#\{0 \leq m \leq n \mid T^m(\lambda)(x) \in V\}}{n+1} > \delta.$$

Then the set of common upper frequently hypercyclic vectors for $(T(\lambda))_{\lambda\in\Lambda}$ is residual in X.

Thanks to this result, in order to prove the existence of common upper frequently hypercyclic vectors it suffices for each pair U, V of non-empty open subsets of X and for each compact subset K of Λ to find a vector of U that visits V frequently enough under $T(\lambda)$, for any $\lambda \in K$. Adapting methods used for common hypercyclicity we will give a construction of such a vector in Section 2. We will then study applications to families of weighted shifts.

2. A generalization of the Birkhoff theorem for common upper frequent hypercyclicity. Our results involve the same context as common hypercyclicity criteria. In other words, we consider families of operators indexed by a σ-compact metric space. We then introduce an additional notion of continuity for the family of operators. A family $(T(\lambda))_{\lambda\in\Lambda}$ of operators indexed by a metric space is called *continuous* if for any $x \in X$, the map $\Lambda \to X : \lambda \mapsto T(\lambda)(x)$ is continuous on Λ. In this context, we have the following two results from [17, Chapter 11].

PROPOSITION 2. *Let Λ be a metric space, X a Fréchet space and $(T(\lambda))_{\lambda\in\Lambda}$ a continuous family of operators on X. Then the map $\Lambda \times X \to X : (\lambda, x) \mapsto T(\lambda)(x)$ is continuous on $\Lambda \times X$ with respect to the product topology.*

COROLLARY 3. *Let Λ be a metric space, X a Fréchet space and $(T(\lambda))_{\lambda\in\Lambda}$ a continuous family of operators on X. Then for any $n \geq 0$, the family $(T^n(\lambda))_{\lambda\in\Lambda}$ of operators is continuous.*

As announced in the introduction, we start with Theorem 1 which acts as the generalization of the Birkhoff theorem for common hypercyclicity. Before stating it we have to introduce the following notation.

NOTATION. Let X be a Fréchet space, $(T(\lambda))_{\lambda\in\Lambda}$ a family of operators on X, and V an open subset of X. For each $\lambda \in \Lambda$ and each $x \in X$, we set

$$N_\lambda(x, V) := \{n \geq 0 \mid T^n(\lambda)(x) \in V\}.$$

This is the set of visiting times for x in V under the operator $T(\lambda)$.

Proof of Theorem 1. By separability of X there exists a countable base $(V_k)_{k\geq1}$ of non-empty open subsets of X. Moreover the σ-compactness of Λ ensures the existence of a sequence $(K_m)_{m\geq1}$ of compact subsets of Λ such that

$$\Lambda = \bigcup_{m\geq1} K_m. \tag{1}$$

From the hypotheses we deduce that for any $k, m \geq 1$ there exists $\delta_{k,m} > 0$ such that for any $M \geq 1$ and any non-empty open subset U of X,

$$\exists x \in U,\ \forall\lambda \in K_m,\ \exists n \geq M,\qquad \frac{\#(N_\lambda(x, V_k) \cap [0, n])}{n + 1} > \delta_{k,m}.$$

For any $k, m \geq 1$ and any $M \geq 1$, we set

$$E(k, m, M) := \left\{x \in X \;\middle|\; \forall\lambda \in K_m,\ \exists n \geq M,\ \frac{\#(N_\lambda(x, V_k) \cap [0, n])}{n + 1} > \delta_{k,m}\right\}.$$

By the above these sets are clearly dense in X. So to prove the claim it is sufficient by the Baire category theorem to show that

(a) $\bigcap_{k,m\geq1} \bigcap_{M\geq1} E(k, m, M) \subset \bigcap_{\lambda\in\Lambda} \mathcal{U}FHC(T(\lambda))$;

(b) for any $k, m, M \geq 1$, the set $E(k, m, M)$ is open.

To prove (a), let

$$x \in \bigcap_{k,m\geq1} \bigcap_{M\geq1} E(k, m, M).$$

Then

$$\forall k \geq 1,\, \forall m \geq 1,\, \forall \lambda \in K_m, \qquad \overline{\mathrm{dens}}(N_\lambda(x, V_k)) > 0.$$

Since $(K_m)_{m\geq 1}$ covers Λ, we obtain

$$\forall k \geq 1,\, \forall \lambda \in \Lambda, \qquad \overline{\mathrm{dens}}(N_\lambda(x, V_k)) > 0.$$

As $(V_k)_{k\geq 1}$ is a base of non-empty open subsets of X, we conclude that x is a common upper frequently hypercyclic vector for the family $(T(\lambda))_{\lambda\in\Lambda}$, which proves (a).

To show (b), let $k, m, M \geq 1$ and $x \in E(k, m, M)$. By definition there exists a family $(n_\lambda)_{\lambda\in K_m}$ of positive integers such that

$$\forall \lambda \in K_m, \qquad n_\lambda \geq M \quad \text{and} \quad \frac{\#(N_\lambda(x, V_k) \cap [0, n_\lambda])}{n_\lambda + 1} > \delta_{k,m}. \tag{2}$$

We must verify that there exists an open neighbourhood O of x such that $O \subset E(k, m, M)$. First we fix $\lambda \in K_m$. For any $n \in N_\lambda(x, V_k)$, V_k is an open neighbourhood of $T^n(\lambda)(x)$. Moreover from Proposition 2 and Corollary 3, we deduce that for each $n \geq 0$, the map $\Lambda \times X \to X : (\mu, y) \mapsto T^n(\mu)(y)$ is continuous at (λ, x). Hence for any $n \in N_\lambda(x, V_k)$, there exists an open neighbourhood $U_{n,\lambda}$ of λ and an open neighbourhood $O_{n,\lambda}$ of x such that

$$\forall \mu \in U_{n,\lambda},\, \forall y \in O_{n,\lambda}, \qquad T^n(\mu)(y) \in V_k.$$

As the set $N_\lambda(x, V_k) \cap [0, n_\lambda]$ is finite, there exists an open neighbourhood U_λ of λ and an open neighbourhood O_λ of x such that

$$\forall n \in N_\lambda(x, V_k) \cap [0, n_\lambda],\, \forall \mu \in U_\lambda,\, \forall y \in O_\lambda, \qquad T^n(\mu)(y) \in V_k.$$

Altogether, for any $\lambda \in K_m$, there exists an open neighbourhood U_λ of λ and an open neighbourhood O_λ of x such that

$$\forall \mu \in U_\lambda,\, \forall y \in O_\lambda, \qquad N_\lambda(x, V_k) \cap [0, n_\lambda] \subset N_\mu(y, V_k). \tag{3}$$

In particular $(U_\lambda)_{\lambda\in K_m}$ is a family of open subsets of Λ whose union contains K_m. By compactness of K_m, there exist $I \geq 1$ and $\lambda_1, \ldots, \lambda_I \in K_m$ such that

$$K_m \subset \bigcup_{1\leq i\leq I} U_{\lambda_i}.$$

We finally set

$$O := \bigcap_{1\leq i\leq I} O_{\lambda_i}.$$

By (3) we then obtain

$$\forall y \in O,\, \forall \mu \in K_m,\, \exists 1 \leq i \leq I, \qquad N_{\lambda_i}(x, V_k) \cap [0, n_{\lambda_i}] \subset N_\mu(y, V_k).$$

On the other hand, we deduce from (2) that

$$\forall 1 \leq i \leq I, \qquad n_{\lambda_i} \geq M \quad \text{and} \quad \frac{\#(N_{\lambda_i}(x, V_k) \cap [0, n_{\lambda_i}])}{n_{\lambda_i} + 1} > \delta_{k,m}.$$

Combining these facts we conclude that

$$\forall y \in O,\, \forall \mu \in K_m,\, \exists n > M, \qquad \frac{\#(N_\mu(y, V_k) \cap [0,n])}{n+1} > \delta_{k,m}.$$

Therefore (b) is satisfied. Applying the Baire category theorem we end the proof. ∎

3. Common upper frequent hypercyclicity criterion. The previous theorem gives sufficient conditions to have common upper frequently hypercyclic vectors. In this section we will use this result to establish a theorem in the same vein as the common hypercyclicity theorem from [17, Chapter 11, Theorem 11.9]. Before stating this theorem we have to recall the definition of unconditional and uniform convergence of a family of series. To this end we will work with an F-norm on X. This notion can be found in [18].

Definition 4. Let X be a vector space. An *F-norm* on X is a functional $\|\_\| : X \to [0, +\infty[$ such that

(i) for any $x, y \in X$, $\|x+y\| \le \|x\| + \|y\|$;
(ii) for any scalar λ and any $x \in X$, if $|\lambda| \le 1$ then $\|\lambda x\| \le \|x\|$;
(iii) for any $x \in X$, $\lim_{\lambda \to 0} \|\lambda x\| = 0$;
(iv) for any $x \in X$, if $\|x\| = 0$ then $x = 0$.

Considering a Fréchet space X endowed with a separating increasing sequence $(p_k)_{k\ge1}$ of seminorms, we denote by $\|\_\|$ the F-norm on X defined by

$$\|x\| := \sum_{k\ge1} \frac{1}{2^k} \min(1, p_k(x)) \qquad \text{for } x \in X.$$

This F-norm induces the topology of X and allows us to work with a norm-like functional.

Definition 5. Let X be a Fréchet space and $(x_{\lambda,n})_{(\lambda,n)\in\Lambda\times\mathbb{N}}$ a family of vectors of X. We say that the series $\sum_{n\ge1} x_{\lambda,n}$ converges *unconditionally and uniformly* for $\lambda \in \Lambda$ if for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\Big\| \sum_{n\in F} x_{\lambda,n} \Big\| < \varepsilon \qquad \text{for any } \lambda \in \Lambda \text{ and any finite } F \subset \{N_0, N_0+1, \ldots\}.$$

For the application to families of weighted shifts we will use the following sufficient conditions for uniform unconditional convergence from [17, Chapter 11].

Remark 6. Let X be a Fréchet space, $(e_n)_{n\ge1}$ a sequence of vectors of X and $(a_{\lambda,n})_{(\lambda,n)\in\Lambda\times\mathbb{N}}$ a family of scalars. Suppose that there exists a sequence $(c_n)_{n\in\mathbb{N}}$ of positive numbers such that

- for any $\lambda \in \Lambda$ and any $n \geq 1$, $|a_{\lambda,n}| \leq c_n$;
- $\sum_{n\geq 1} c_n e_n$ converges unconditionally.

Then $\sum_{n\geq 1} a_{\lambda,n} e_n$ converges unconditionally and uniformly for $\lambda \in \Lambda$.

Theorem 7 (Common upper frequent hypercyclicity criterion). *Let Λ be a real interval, X a separable Fréchet space and $(T(\lambda))_{\lambda\in\Lambda}$ a continuous family of operators on X. Suppose that for any compact subinterval K of Λ, there exists a dense subset X_0 of X and maps $S_n(\lambda) : X_0 \to X$ for $n \geq 0$ and $\lambda \in K$ such that for any $x \in X_0$,*

(i) *the series $\sum_{n=0}^{m} T^m(\lambda)(S_{m-n}(\mu_n)(x))$ converges unconditionally and uniformly for $m \geq 0$, $\mu_0 \geq \cdots \geq \mu_m \in K$ and $\lambda \in K$;*
(ii) *the series $\sum_{n\geq 0} T^m(\lambda)(S_{m+n}(\mu_n)(x))$ converges unconditionally and uniformly for $m \geq 0, (\mu_n)_{n\geq 0}$ any non-decreasing sequence from K and $\lambda \leq \mu_0 \in K$;*
(iii) *for any $\varepsilon > 0$, there exists a decreasing sequence $(d_n)_{n\geq 1}$ of positive numbers such that*
   (a) *for any $n \geq 1$ and any $\lambda, \mu \in K$,*
   $$\text{if} \quad 0 \leq \mu - \lambda \leq d_n \quad \text{then} \quad \|T^n(\lambda)(S_n(\mu)(x)) - x\| \leq \varepsilon;$$
   (b) *for any $c \in \mathbb{N}$, the series $\sum_{t\geq 1} d_{c^t}$ diverges;*
(iv) *$(T^n(\lambda)(x))_{n\geq 0}$ converges uniformly to 0 for $\lambda \in K$.*

Then the set of common upper frequently hypercyclic vectors for $(T(\lambda))_{\lambda\in\Lambda}$ is residual in X, and in particular non-empty.

The finite sums in the hypothesis (i) are regarded as infinite series by adding 0 terms.

Let us explain the main idea of this result before proving it.

Main idea. In view of Theorem 1, in order to obtain common upper frequent hypercyclicity it is sufficient to find a vector z from an open subset U which visits V frequently enough under $T(\lambda_t)$, for some $\lambda_0 < \cdots < \lambda_\tau \in \Lambda$. We construct this vector by blocks. The first block will ensure that $z \in U$. Then in order to visit V we will add perturbations of a vector y in V. More specifically, we take

$$z := x + \sum_{t=1}^{\tau} \underbrace{(S_{l_t}(\lambda_t)(y) + S_{l_t+s_0}(\lambda_t)(y) + \cdots + S_{l_t+L_t s_0}(\lambda_t)(y))}_{\text{to visit } V \text{ under } T(\lambda_t)\ L_t+1 \text{ times}},$$

where $x \in U$, and for any $t \in \{1, \ldots, \tau\}$, we denote by

- s_0 the *gap* between two successive approximations in each block;

- l_t the *beginning* of the block for λ_t;
- $L_t + 1$ the *number of visits* in the block for λ_t.

Notice that we only consider the visiting times with a constant gap. This is a technical necessity. In the proof we will choose $(L_t)_{t\geq 1} := (l_t)_{t\geq 1}$.

As announced, this result is similar to the common hypercyclicity theorem from [17, Chapter 11, Theorem 11.9]. The main difference is the third condition. For common hypercyclicity we only require the divergence of $\sum_{n\geq 1} d_n$. Here we also must have that divergence but in a specific way. We select a particular subsequence of $(d_n)_{n\geq 1}$. This provides enough time between approximations for two successive parameters λ.

Proof of Theorem 7. We will apply Theorem 1. We have to show that $(T(\lambda))_{\lambda\in\Lambda}$ satisfies the hypotheses of that result. Let K be a compact subset of Λ. We can assume without loss of generality that $K := [a, b]$ is a subinterval of Λ. By the hypotheses there exists a dense subset X_0 of X and maps $S_n(\lambda) : X_0 \to X$, $n \geq 0$, $\lambda \in K$, such that (i)–(iv) are satisfied. Let V be a non-empty open subset of X. Since X_0 is dense in X we can suppose that $V := B(y, \varepsilon)$ with some $y \in X_0$ and $\varepsilon > 0$. By (iii) there exists a decreasing sequence $(d_n)_{n\geq 1}$ of positive numbers such that

(a) for any $n \geq 1$ and any $\lambda, \mu \in K$,
$$\text{if} \quad 0 \leq \mu - \lambda \leq d_n \quad \text{then} \quad \|T^n(\lambda)(S_n(\mu)(y)) - y\| \leq \varepsilon/4;$$

(b) for any $c \in \mathbb{N}$, the series $\sum_{t\geq 1} d_{c^t}$ diverges.

On the other hand, by (i) and (ii) there exists a positive integer s_0 such that for any finite $F \subset \{s_0, s_0 + 1, \ldots\}$ and any $m \geq 0$ we have

(I) for any $\mu_0 \geq \cdots \geq \mu_m \in [a, b]$ and any $\lambda \in [a, b]$,
$$\Big\| \sum_{n\in F\cap[0,m]} T^m(\lambda)(S_{m-n}(\mu_n)(y)) \Big\| < \frac{\varepsilon}{4};$$

(II) for any non-decreasing sequence $(\mu_n)_{n\geq 0} \subset [a,b]$ and any $\lambda \leq \mu_0 \in [a,b]$,
$$\Big\| \sum_{n\in F} T^m(\lambda)(S_{m+n}(\mu_n)(y)) \Big\| < \frac{\varepsilon}{4}.$$

Set $\delta := 1/(2 + s_0) > 0$. Let U be a non-empty open subset of X and $M \geq 1$. Once again, as X_0 is dense in X we can assume that $U := B(x, r)$ with some $x \in X_0$ and $r > 0$. From (ii) with $m = 0$ and (iv) there exists a positive integer N_0 such that for any finite $F \subset \{N_0, N_0 + 1, \ldots\}$,

(III) for any non-decreasing sequence $(\mu_n)_{n\geq 0} \subset [a, b]$,
$$\Big\| \sum_{n\in F} S_n(\mu_n)(y) \Big\| < r;$$

(IV) for any $n \geq N_0$ and any $\lambda \in [a, b]$, $\|T^n(\lambda)(x)\| < \varepsilon/4$.

Furthermore by taking $c := \max(N_0, 2+s_0, M)$, the hypothesis (b) ensures that the series $\sum_{t\geq 1} d_{c^{t+1}}$ diverges. Therefore there exists an integer $\tau \geq 1$ such that

$$a + \sum_{t=1}^{\tau-1} d_{c^{t+1}} \leq b < a + \sum_{t=1}^{\tau} d_{c^{t+1}}.$$

So we obtain the subdivision $(\lambda_t)_{0\leq t\leq \tau}$ of $[a,b]$ defined by

$$\lambda_\tau := b, \qquad \lambda_t := a + \sum_{s=1}^{t} d_{c^{s+1}} \quad \text{for any } 0 \leq t \leq \tau - 1.$$

We finally consider the sequence $(l_t)_{t\geq 1} := (c^t)_{t\geq 1}$ and the vector z given by

$$z := x + \sum_{t=1}^{\tau} \sum_{l=0}^{l_t} S_{l_t + l s_0}(\lambda_t)(y).$$

We observe that $l_t + l_t s_0 < l_{t+1}$ for any $1 \leq t \leq \tau$, which implies that the integers $l_t + l s_0$ with $0 \leq l \leq l_t$ and $t \geq 1$ are pairwise distinct. Combining this with (III) entails that

$$\|z - x\| = \Big\| \sum_{t=1}^{\tau} \sum_{l=0}^{l_t} S_{l_t + l s_0}(\lambda_t)(y) \Big\| < r.$$

Moreover, for $1 \leq t \leq \tau$, $\lambda \in [\lambda_{t-1}, \lambda_t]$ and $0 \leq l \leq l_t$, we have

$$\begin{aligned} T^{l_t + l s_0}(\lambda)(z) = T^{l_t+ls_0}(\lambda)(x) &+ \sum_{s=1}^{t-1} \sum_{k=0}^{l_s} T^{l_t+ls_0}(\lambda)(S_{l_s + k s_0}(\lambda_s)(y)) \\ &+ \sum_{k=0}^{l-1} T^{l_t+ls_0}(\lambda)(S_{l_t+ks_0}(\lambda_t)(y)) + T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0}(\lambda_t)(y)) \\ &+ \sum_{k=l+1}^{l_t} T^{l_t+ls_0}(\lambda)(S_{l_t+ks_0}(\lambda_t)(y)) \\ &+ \sum_{s=t+1}^{\tau} \sum_{k=0}^{l_s} T^{l_t+ls_0}(\lambda)(S_{l_s+ks_0}(\lambda_s)(y)), \end{aligned}$$

which may be written equivalently as

$$\begin{aligned} T^{l_t+ls_0}(\lambda)(z) = T^{l_t+ls_0}(\lambda)(x) &+ T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0}(\lambda_t)(y)) \\ &+ \Big(\sum_{s=1}^{t-1} \sum_{k=0}^{l_s} T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0-(l_t+ls_0-l_s-ks_0)}(\lambda_s)(y)) \\ &\qquad + \sum_{k=0}^{l-1} T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0-(l_t+ls_0-l_t-ks_0)}(\lambda_t)(y)) \Big) \end{aligned}$$

$$+\Big(\sum_{k=l+1}^{l_t} T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0+(l_t+ks_0-l_t-ls_0)}(\lambda_t)(y)) + \sum_{s=t+1}^{\tau}\sum_{k=0}^{l_s} T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0+(l_s+ks_0-l_t-ls_0)}(\lambda_s)(y))\Big).$$

As observed previously, the integers $l_s + ks_0$ with $0 \le k \le l_s$ and $s \ge 1$ are pairwise distinct. Since $\lambda \in [\lambda_{t-1}, \lambda_t]$, it follows from (I) and (II) that

$$\|T^{l_t+ls_0}(\lambda)(z) - y\| < \|T^{l_t+ls_0}(\lambda)(x)\| + \|T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0}(\lambda_t)(y)) - y\| + \varepsilon/4 + \varepsilon/4.$$

Moreover by definition, $l_t + ls_0 \ge c = \max(N_0, 2 + s_0, M)$, so $l_t + ls_0 \ge N_0$. From (IV) we then deduce that

$$\|T^{l_t+ls_0}(\lambda)(z) - y\| < \tfrac{3}{4}\varepsilon + \|T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0}(\lambda_t)(y)) - y\|. \tag{4}$$

On the other hand, since $\lambda \in [\lambda_{t-1}, \lambda_t]$, we have

$$0 \le \lambda_t - \lambda \le \lambda_t - \lambda_{t-1} \le d_{c^{t+1}} = d_{l_{t+1}}.$$

But $(d_n)_{n\ge 1}$ is decreasing and $l_t + ls_0 < l_{t+1}$. This entails that

$$0 \le \lambda_t - \lambda \le d_{l_t+ls_0}.$$

So by (a) we have

$$\|T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0}(\lambda_t)(y)) - y\| \le \varepsilon/4.$$

Together with (4) we conclude that

$$\|T^{l_t+ls_0}(\lambda)(z) - y\| < \varepsilon.$$

Altogether we have shown that $z \in U$ and

$$\forall 1 \le t \le \tau,\ \forall \lambda \in [\lambda_{t-1}, \lambda_t],\ \forall 0 \le l \le l_t, \quad T^{l_t+ls_0}(\lambda)(z) \in V.$$

In particular this implies that

$$\forall 1 \le t \le \tau,\ \forall \lambda \in [\lambda_{t-1}, \lambda_t], \quad \frac{\#(N_\lambda(z,V) \cap [0, l_t + l_t s_0])}{l_t + l_t s_0 + 1} > \frac{1}{2 + s_0}.$$

As $l_1 \ge M$, we thus have $z \in U$ and

$$\forall \lambda \in [a,b],\ \exists n \ge M, \quad \frac{\#(N_\lambda(z,V) \cap [0,n])}{n+1} > \frac{1}{2+s_0},$$

which ends the proof. ∎

4. Application to weighted shifts. In this section we give examples of families that have common upper frequently hypercyclic vectors. To this end we will consider families of weighted shifts. These operators form a rich source of examples in linear dynamics.

DEFINITION 8. Let $p \geq 1$ and $w = (w_n)_{n\geq 1}$ be a sequence of non-zero scalars. The *weighted shift* B_w is the map on l^p defined by

$$B_w(x) := (w_{n+1}x_{n+1})_{n\geq 0} \quad \text{for } x = (x_n)_{n\geq 0} \in l^p,$$

where l^p is the Banach space of sequences $(x_n)_{n\geq 0}$ such that the series $\sum_{n\geq 0} |x_n|^p$ converges. Weighted shifts on c_0, the Banach space of sequences converging to 0, are defined similarly.

We mention that a weighted shift B_w is an operator on l^p or on c_0 if and only if the sequence w is bounded. Furthermore, a weighted shift is upper frequently hypercyclic on l^p if and only if

$$\sum_{\nu\geq 1} \frac{1}{|w_1 \dots w_\nu|^p} < \infty.$$

A characterization of upper frequent hypercyclicity for c_0 is also known but more complicated. These characterizations are due to Bayart and Ruzsa [10].

More generally we may define these operators on Fréchet spaces with an unconditional basis.

DEFINITION 9. Let X be a Fréchet space, $(e_n)_{n\geq 0}$ an unconditional basis of X and $w = (w_n)_{n\geq 1}$ a sequence of non-zero scalars. The *weighted shift* B_w is the map on X defined by

$$B_w(x) := \sum_{n\geq 0} w_{n+1}x_{n+1}e_n \quad \text{for } x = \sum_{n\geq 0} x_n e_n \in X,$$

where we suppose that this series converges in X.

We derive from Theorem 7 the following result for families of weighted shifts. This result is based on a special case of a common hypercyclicity theorem for families of weighted shifts due to Bayart and Matheron [8].

THEOREM 10. *Let Λ be a real interval, X a Fréchet space, $(e_n)_{n\geq 0}$ an unconditional basis of X and $(w_n(\lambda))_{(n,\lambda)\in\mathbb{N}\times\Lambda}$ a family of positive numbers such that for any $\lambda \in \Lambda$, $B_{w(\lambda)}$ is an operator on X. Suppose that*

(i) *for any $n \geq 1$, the function $w_n : \Lambda \to \mathbb{R} : \lambda \mapsto w_n(\lambda)$ is increasing;*
(ii) *for any $\lambda \in \Lambda$, the series*

$$\sum_{\nu\geq 1} \frac{1}{w_1(\lambda)\dots w_\nu(\lambda)} e_\nu \text{ converges in } X;$$

(iii) *for any compact subinterval K of Λ and any $n \in \mathbb{N}$, there exists a positive number $L_n(K)$ such that*

$$\forall \lambda, \mu \in K, \qquad |\log(w_n(\lambda)) - \log(w_n(\mu))| \leq L_n(K)|\lambda - \mu|;$$

(iv) *for any compact subinterval K of Λ and any $s \in \mathbb{N}$, the series*

$$\sum_{t\geq 1}\Big(\sum_{i=1}^{s^t} L_i(K)\Big)^{-1} \quad \textit{diverges.}$$

Then the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda\in\Lambda}$ is residual in X, and in particular non-empty.

Proof. To apply Theorem 7, we must prove that $(B_{w(\lambda)})_{\lambda\in\Lambda}$ satisfies the hypotheses of that result. We begin by proving that $(B_{w(\lambda)})_{\lambda\in\Lambda}$ is a continuous family of operators on X. Let $x \in X$ and $K = [a, b]$ be a compact subinterval of Λ. As $B_{w(b)}$ is an operator on X and $(e_n)_{n\geq 0}$ is an unconditional basis of X, the series

$$\sum_{n\geq 0} w_{n+1}(b)x_{n+1}e_n$$

converges unconditionally. Together with (i) this entails by Remark 6 that

$$\sum_{n\geq 0} w_{n+1}(\lambda)x_{n+1}e_n$$

converges unconditionally and uniformly for $\lambda \in K$. Using the continuity of $\Lambda \to \mathbb{C} : \lambda \mapsto w_n(\lambda)$, $n \geq 0$, we then easily deduce the continuity of $\Lambda \to X : \lambda \mapsto B_{w(\lambda)}(x)$ on K. Thus $(B_{w(\lambda)})_{\lambda\in\Lambda}$ is continuous.

Now we will show that the main conditions of Theorem 7 hold. We fix $K = [a, b] \subset \Lambda$. The set $X_0 := \operatorname{span}\{e_\nu \mid \nu \geq 0\}$ is dense in X. For any $\lambda \in K$, we define the map $F(\lambda)$ on X_0 by

$$\forall x = \sum_{\nu=0}^{J} x_\nu e_\nu \in X_0, \qquad F(\lambda)(x) = \sum_{\nu=0}^{J} \frac{1}{w_{\nu+1}(\lambda)} x_\nu e_{\nu+1}.$$

We then set $S_n(\lambda) := F^n(\lambda)$ for any $\lambda \in K$ and $n \geq 0$. We have to show that for any $x \in X_0$, the main conditions of Theorem 7 hold. We observe that the conditions (i), (ii) and (iv) of Theorem 7 need only be checked for e_ν, $\nu \geq 0$. Firstly we remark that for any non-negative integer ν,

$$\text{(5)} \quad \forall\lambda \in K, \forall n \geq 0, \qquad B^n_{w(\lambda)}(e_\nu) = \begin{cases} w_{\nu-n+1}(\lambda)\dots w_\nu(\lambda)e_{\nu-n} & \text{if } \nu \geq n,\\ 0 & \text{else.}\end{cases}$$

By definition of $(F(\lambda))_{\lambda\in K}$, for any non-negative integer ν,

$$\text{(6)} \qquad \forall\lambda \in K, \forall n \geq 0, \qquad F^n(\lambda)(e_\nu) = \frac{1}{w_{\nu+1}(\lambda)\dots w_{\nu+n}(\lambda)} e_{\nu+n}.$$

We immediately deduce from (5) that the condition (iv) of Theorem 7 is satisfied for each e_ν, $\nu \geq 0$.

For $\nu \geq 0$, it follows from (5) and (6) that

$$\forall\lambda, \mu \in K, \forall m \geq n > \nu, \qquad B^m_{w(\lambda)}(F^{m-n}(\mu)(e_\nu)) = 0,$$

which gives the condition (i) of Theorem 7. Let $m \geq 0$, $(\mu_n)_{n\geq 0}$ be a non-decreasing sequence from K and $\lambda \leq \mu_0 \in K$. By (5) and (6) we have

$$\forall n \geq 0, \qquad B^m_{w(\lambda)}(F^{m+n}(\mu_n)(e_\nu)) = \frac{w_{\nu+n+1}(\lambda)\dots w_{\nu+m+n}(\lambda)}{w_{\nu+1}(\mu_n)\dots w_{\nu+m+n}(\mu_n)} e_{\nu+n}.$$

Thus

$$(7) \qquad \sum_{n\geq 0} B^m_{w(\lambda)}(F^{m+n}(\mu_n)(e_\nu)) = \sum_{n\geq 0} \frac{w_{\nu+n+1}(\lambda)\dots w_{\nu+m+n}(\lambda)}{w_{\nu+1}(\mu_n)\dots w_{\nu+m+n}(\mu_n)} e_{\nu+n}.$$

Furthermore, since $K = [a, b]$, by the hypothesis (i) we have

$$(8) \qquad \forall n \geq 0, \qquad \left| \frac{w_{\nu+n+1}(\lambda)\dots w_{\nu+m+n}(\lambda)}{w_{\nu+1}(\mu_n)\dots w_{\nu+m+n}(\mu_n)} \right| \leq \frac{1}{w_{\nu+1}(a)\dots w_{\nu+n}(a)}.$$

However, by the hypothesis (ii) the series

$$\sum_{n\geq 0} \frac{1}{w_{\nu+1}(a)\dots w_{\nu+n}(a)} e_{\nu+n} \quad \text{converges in } X.$$

Combining this with (7) and (8) we deduce from Remark 6 that the assumption (ii) of Theorem 7 is satisfied.

Finally, it remains to show that the condition (iii) of Theorem 7 holds for each $x \in X_0$. Let $x \in X_0$. By definition, there exist $J \geq 0$ and scalars $x_0, \dots, x_J$ such that

$$x := \sum_{\nu=0}^{J} x_\nu e_\nu.$$

Let $\varepsilon > 0$. By the hypothesis (iii), there exists a sequence $(L_n)_{n\geq 1}$ of positive numbers such that

$$(9) \qquad \forall n \geq 1, \forall \lambda, \mu \in K, \qquad |\log(w_n(\lambda)) - \log(w_n(\mu))| \leq L_n |\lambda - \mu|.$$

On the other hand, by properties of F-norms, there exists $\eta > 0$ such that

$$(10) \qquad \forall \xi \in \mathbb{C}, \qquad \left[|\xi| \leq \eta \Rightarrow \left(\forall 0 \leq \nu \leq J, \ \|\xi x_\nu e_\nu\| \leq \frac{\varepsilon}{J+1} \right) \right].$$

We define a sequence $(d_n)_{n\geq 1}$ of positive numbers by

$$\forall n \geq 1, \qquad d_n := \eta \Big(\sum_{i=1}^{n+J} L_i \Big)^{-1}.$$

The hypothesis (iv) directly gives the condition (iii)(b) of Theorem 7. Let $n \geq 1$ and $\lambda, \mu \in K$ be such that $0 \leq \mu - \lambda \leq d_n$. By (5) and (6) we obtain

$$(11) \qquad \|B^n_{w(\lambda)}(F^n(\mu)(x)) - x\| \leq \sum_{\nu=0}^{J} \left\| \left(\frac{w_{\nu+1}(\lambda)\dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu)\dots w_{\nu+n}(\mu)} - 1 \right) x_\nu e_\nu \right\|.$$

Moreover as $\lambda \le \mu$, the hypothesis (i) implies that

$$\forall 0 \le \nu \le J, \qquad \left|\frac{w_{\nu+1}(\lambda)\dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu)\dots w_{\nu+n}(\mu)} - 1\right| = 1 - \frac{w_{\nu+1}(\lambda)\dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu)\dots w_{\nu+n}(\mu)},$$

which may be written equivalently as

$$\forall 0 \le \nu \le J, \qquad \left|\frac{w_{\nu+1}(\lambda)\dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu)\dots w_{\nu+n}(\mu)} - 1\right| = 1 - \mathrm{e}^{-\sum_{i=1}^{n}(\log(w_{\nu+i}(\mu)) - \log(w_{\nu+i}(\lambda)))}.$$

In other words,

$$\forall 0 \le \nu \le J, \qquad \left|\frac{w_{\nu+1}(\lambda)\dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu)\dots w_{\nu+n}(\mu)} - 1\right| = \int\limits_{-\sum_{i=1}^{n}(\log(w_{\nu+i}(\mu)) - \log(w_{\nu+i}(\lambda)))}^{0} \mathrm{e}^{\xi}\, d\xi.$$

Hence

$$\forall 0 \le \nu \le J, \qquad \left|\frac{w_{\nu+1}(\lambda)\dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu)\dots w_{\nu+n}(\mu)} - 1\right| \le \sum_{i=1}^{n}\big(\log(w_{\nu+i}(\mu)) - \log(w_{\nu+i}(\lambda))\big).$$

Thus by (9),

$$\forall 0 \le \nu \le J, \qquad \left|\frac{w_{\nu+1}(\lambda)\dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu)\dots w_{\nu+n}(\mu)} - 1\right| \le \Big(\sum_{i=1}^{n} L_{\nu+i}\Big)(\mu - \lambda).$$

Since $0 \le \mu - \lambda \le d_n$, it follows from the definition of d_n that

$$\forall 0 \le \nu \le J, \qquad \left|\frac{w_{\nu+1}(\lambda)\dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu)\dots w_{\nu+n}(\mu)} - 1\right| \le \eta.$$

By (10) and (11) we conclude that

$$\|B^n_{w(\lambda)}(F^n(\mu)(x)) - x\| \le \varepsilon.$$

Therefore the condition (iii)(a) of Theorem 7 is also satisfied. Altogether we have shown that Theorem 7 can be applied to $(B_{w(\lambda)})_{\lambda\in\Lambda}$, as desired. ∎

COROLLARY 11. *Let Λ be a real interval, X a Fréchet space, $(e_n)_{n\ge 0}$ an unconditional basis of X and $(w_n(\lambda))_{(n,\lambda)\in\mathbb{N}\times\Lambda}$ a family of positive numbers such that for any $\lambda \in \Lambda$, $B_{w(\lambda)}$ is an operator on X. Suppose that*

(i) *for any $n \ge 1$, the function $w_n : \Lambda \to \mathbb{R} : \lambda \mapsto w_n(\lambda)$ is increasing;*
(ii) *for any $\lambda \in \Lambda$, the series*

$$\sum_{\nu\ge 1} \frac{1}{w_1(\lambda)\dots w_\nu(\lambda)} e_\nu \text{ converges in } X;$$

(iii) *for any compact subinterval K of Λ, there exists a sequence $(L_n(K))_{n\ge 1}$ of positive numbers such that*
   (a) *the series $\sum_{n\ge 1} L_n(K)$ converges;*
   (b) *for any $n \ge 1$ and any $\lambda, \mu \in K$,*

$$|\log(w_n(\lambda)) - \log(w_n(\mu))| \le L_n(K)|\lambda - \mu|.$$

Then the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda\in\Lambda}$ is residual in X, and in particular non-empty.

Proof. This follows directly from Theorem 10. ■

As an application of this corollary we obtain the propositions below which give us a general form of some families with common upper frequently hypercyclic vectors on l^p and on c_0.

PROPOSITION 12. *Let $p \geq 1$, and let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of nonnegative numbers. Define*

$$\forall \lambda \in \mathbb{R}, \qquad w(\lambda) := (a_n \, \mathrm{e}^{\lambda b_n})_{n\geq 1}.$$

Suppose that

(I) *$(a_n)_{n\geq 1}$ is a bounded sequence of positive numbers;*
(II) *the series $\sum_{n\geq 1} b_n$ and $\sum_{n\geq 1}(\prod_{i=1}^{n} a_i)^{-p}$ converge.*

Then the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda\in\Lambda}$ is residual in l^p, and in particular non-empty.

PROPOSITION 13. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of nonnegative numbers. Define*

$$\forall \lambda \in \mathbb{R}, \qquad w(\lambda) := (a_n \, \mathrm{e}^{\lambda b_n})_{n\geq 1}.$$

Suppose that

(I) *$(a_n)_{n\geq 1}$ is a bounded sequence of positive numbers;*
(II) *$\sum_{n\geq 1} b_n$ converges;*
(III) *$(\prod_{i=1}^{n} a_i)_{n\geq 1}$ converges to ∞.*

Then the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda\in\Lambda}$ is residual in c_0, and in particular non-empty.

We also give an example which is not covered by Proposition 12. This actually illustrates the fact that the hypotheses of Corollary 11 are stronger than those of Theorem 10.

EXAMPLE 14. Let $p \geq 1$. Define

$$\forall \lambda > 1/p, \qquad w(\lambda) := \left(\left(\frac{n+1}{n}\right)^{\lambda}\right)_{n\geq 1}.$$

We observe that for any $n \geq 1$ and any $\lambda, \mu > 1/p$,

$$|\log(w_n(\lambda)) - \log(w_n(\mu))| = \log\left(\frac{n+1}{n}\right)|\lambda - \mu|.$$

By taking $(L_n(K))_{n\geq 1} := \big(\log\big(\frac{n+1}{n}\big)\big)_{n\geq 1}$, we deduce from Theorem 10 that the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda>1/p}$ is residual in l^p.

Finally, we consider the historical example of families with common hypercyclic vectors: the multiples of the backward shift, $(\lambda B)_{\lambda>1}$. We easily observe that this family does not satisfy the conditions of our results. Actually we have shown in [19] that the family of all multiples of an operator does not admit any common upper frequently hypercyclic vector. This will be the subject of another paper.

5. Common $\mathcal{A}$-hypercyclicity. We end this paper with a generalization to $\mathcal{A}$-hypercyclicity. This notion, created by Bès, Menet, Peris and Puig [12], generalizes hypercyclicity and upper frequent hypercyclicity. To define it, we consider Furstenberg families.

DEFINITION 15. Let $\mathcal{A}$ be a non-empty family of subsets of $\mathbb{N}_0$. We say that $\mathcal{A}$ is a *Furstenberg family* if it is *hereditary upward*, that is,

$$\forall A \in \mathcal{A}, \forall B \subset \mathbb{N}_0, \quad (A \subset B \Rightarrow B \in \mathcal{A}).$$

Moreover a Furstenberg family $\mathcal{A}$ is called *upper* if $\mathcal{A}$ does not contain the empty set and there exists a set D, a countable set M and a family $(\mathcal{A}_{\delta,\mu})_{(\delta,\mu)\in D\times M}$ of subsets of $\mathbb{N}_0$ such that $\mathcal{A}$ can be written as

$$\mathcal{A} = \bigcup_{\delta\in D} \mathcal{A}_\delta \quad \text{with} \quad \mathcal{A}_\delta := \bigcap_{\mu\in M} \mathcal{A}_{\delta,\mu}$$

and has the following properties:

(i) $\mathcal{A}$ is *uniformly left-invariant*, that is,

$$\forall A \in \mathcal{A}, \exists \delta \in D, \forall k \in \mathbb{N}_0, \quad (A - k) \cap \mathbb{N}_0 \in \mathcal{A}_\delta;$$

(ii) for any $\delta \in D$ and any $\mu \in M$, the family $\mathcal{A}_{\delta,\mu}$ is *finitely hereditary upward*, that is,

$$\forall A \in \mathcal{A}_{\delta,\mu}, \exists F \subset \mathbb{N}_0 \text{ finite}, \forall B \subset \mathbb{N}_0, \quad (A \cap F \subset B \Rightarrow B \in \mathcal{A}_{\delta,\mu}).$$

DEFINITION 16. Let X be a Fréchet space, T an operator on X, and $\mathcal{A}$ a Furstenberg family. We say that T is *$\mathcal{A}$-hypercyclic* on X if there exists $x \in X$ such that for any non-empty open subset V of X,

$$\{n \geq 0 \mid T^n(x) \in V\} \in \mathcal{A}.$$

In this case, x is called *$\mathcal{A}$-hypercyclic* for T.

Actually Bonilla and Grosse-Erdmann [14] have obtained an analogue to the Birkhoff theorem for $\mathcal{A}$-hypercyclicity with $\mathcal{A}$ an upper Furstenberg family. Using similar arguments to those in the proof of Theorem 1, we establish the following generalization of Theorem 1 for common $\mathcal{A}$-hypercyclicity.

THEOREM 17. *Let $\mathcal{A}$ be an upper Furstenberg family written as*

$$\mathcal{A} := \bigcup_{\delta\in D} \bigcap_{\mu\in M} \mathcal{A}_{\delta,\mu}.$$

Let Λ be a σ-compact metric space, X a separable Fréchet space and $(T(\lambda))_{\lambda\in\Lambda}$ a continuous family of operators on X. Suppose that for any non-empty open subset V of X and for any compact subset K of Λ there exists $\delta \in D$ such that, for any non-empty open subset U of X and any $\mu \in M$, we have

$$\exists x \in U, \forall\lambda \in K, \quad N_\lambda(x, V) \in \mathcal{A}_{\delta,\mu}.$$

Then the set of common $\mathcal{A}$-hypercyclic vectors for $(T(\lambda))_{\lambda\in\Lambda}$ is residual in X.

This theorem allows us to adapt the results of Sections 2 and 3 to common $\mathcal{A}$-hypercyclicity. So we can obtain common $\mathcal{A}$-hypercyclicity criteria for some families $\mathcal{A}$ (see [19]).

In this paper we only study the existence of common upper frequently hypercyclic vectors by adapting results for common hypercyclicity. It will be interesting to investigate the existence of common upper frequently hypercyclic subspaces. Indeed, there are some results on common hypercyclic subspaces (see [2, 4]). Moreover Bès and Menet [11] have recently obtained results on upper frequently hypercyclic subspaces.

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