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Reducts of Hrushovski's constructions of a higher geometrical arity

by

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Abstract. Let $\mathbb{M}_n$ denote the structure obtained from Hrushovski's (non-collapsed) construction with an n-ary relation and $\mathrm{PG}(\mathbb{M}_n)$ its associated pregeometry. It was shown by Evans and Ferreira (2011) that $\mathrm{PG}(\mathbb{M}_3) \ncong \mathrm{PG}(\mathbb{M}_4)$. We show that $\mathbb{M}_3$ has a reduct $\mathbb{M}^{\mathrm{clq}}$ such that $\mathrm{PG}(\mathbb{M}_4) \cong \mathrm{PG}(\mathbb{M}^{\mathrm{clq}})$. To achieve this we show that $\mathbb{M}^{\mathrm{clq}}$ is a slightly generalised Fraïssé–Hrushovski limit incorporating non-eliminable imaginary sorts in $\mathbb{M}^{\mathrm{clq}}$.

1. Introduction. The class of combinatorial pregeometries associated with a structure (or a theory) is an important invariant in geometric stability theory and its bifurcations, going back to Baldwin–Lachlan [BL71], Zilber's trichotomy conjecture [Zil84] and its many applications, e.g. [HZ96], Shelah's analysis of super-stable theories [She90, Chapters V, IX, X], [Hru87] and more.

In the late 1970s Zilber suggested a classification of the geometries associated with strongly minimal theories based on the algebraic structures interpreted by those theories: trivial if no group is interpretable, projective if the theory is 1-based but non-trivial, or the geometry associated with an algebraically closed field otherwise.

It follows from the fundamental theorem of projective geometry (e.g. [Art57, §II]) that (infinite) projective geometries are in one-to-one correspondence with projective spaces, and are classified by their (possibly non-commutative) fields of scalars. Though there is no simple characterisation of the geometries associated with algebraically closed fields, those are characterised by the characteristic of the field [EH95].

Combined with [Hru87] and [Rab93] (and more generally, [HS17]), these observations imply that if T is strongly minimal and its geometry is either

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that of an algebraically closed field or locally modular, then the geometries of reducts of T are partially ordered. Namely, if T' is a reduct of T as above and $\mathrm{PG}(T) \not\cong \mathrm{PG}(T')$ then the geometry of T' is strictly coarser than the geometry of T. Thus, if T is the theory of an algebraic curve over an algebraically closed field and T' is a reduct of T, then either T' interprets an algebraically closed field, in which case T and T' have isomorphic geometries, or T' is locally modular. In the latter case a reduct T'' of T' is either trivial, or T' and T'' have the geometries of projective spaces over fields $F' \geq F''$ (respectively) [Pil96, Corollary 4.5.9].

Similar results can be obtained for reducts of o-minimal structures (for o-minimal structures this is an immediate consequence of the o-minimal Trichotomy Theorem, [PS98], for the more general claim, this follows from [HOP10]).

In the present note we show that the above is not true in Hrushovski's (non-collapsed) construction. Let $\mathbb{M}_n$ denote the generic structure associated with Hrushovski's construction in the language consisting of a single n-ary relation (see Section 3 for precise definitions) and $\mathrm{PG}(\mathbb{M}_n)$ the pregeometry associated with its unique regular type of rank ω. Ferreira and Evans [EF11] show that $\mathrm{PG}(\mathbb{M}_n) \cong \mathrm{PG}(\mathbb{M}_m)$ if and only if $n = m$, and the same remains true even locally. It is an easy exercise to verify that if $n < m$ we can, possibly naming finitely many parameters, identify $\mathbb{M}_n$ with a (proper) reduct of $\mathbb{M}_m$. E.g., to find $\mathbb{M}_3$ in $\mathbb{M}_4$ we can consider the reduct to the relation $S(x, y, z) := R(x, x, y, z)$ (assuming R allows repetitions). In case R does not allow repetitions we can, fixing a generic a, consider the reduct $S(x, y, z) := R(a, x, y, z)$.

Keeping the above notation, our main result is:

THEOREM. *Let $n > 2$ be a natural number, $0 < r < n$ and $s = n - r + 1$. Then there exists a reduct $\mathbb{M}^{\mathrm{clq}}$ of $\mathbb{M}_n$ such that:*

(1) $\mathbb{M}^{\mathrm{clq}}$ *is a Fraïssé–Hrushovski limit with respect to a predimension function allowing a polynomial number of relations.*
(2) $\mathrm{PG}(\mathbb{M}^{\mathrm{clq}}) \cong \mathrm{PG}(\mathbb{M}_{rs})$.

This result implies, in particular, that the geometries associated with Hrushovski constructions are not linearly ordered by reducts. Indeed, consider $\mathbb{M}_4$ and identify $\mathbb{M}_3$ as a (proper) reduct thereof. By our theorem (with $n = 3$, $r = s = 2$) there is a (proper) reduct $\mathbb{M}^{\mathrm{clq}}$ of $\mathbb{M}_3$ such that $\mathrm{PG}(\mathbb{M}^{\mathrm{clq}}) \cong \mathrm{PG}(\mathbb{M}_4)$. Finally, $\mathrm{PG}(\mathbb{M}_4) \not\cong \mathrm{PG}(\mathbb{M}_3)$ by [EF11].

In [Eva05] Evans gives an example of a theory T which is, in the terminology of [Goo91], trivial for freedom (namely, if a, b, c are pairwise independent over a parameter set A then a and b are independent over Ac) and 1-based with a reduct isomorphic to $\mathbb{M}_n$ (so neither trivial nor 1-based). The example of [Eva05] is, however, strictly stable and therefore not of a geometric nature. Our example takes place in an ω-stable theory, and the geometry we study is precisely that of the unique regular type of rank ω. In view of the results of [EF12] there is good reason to believe that the results of the present paper can be reproduced in the context of strongly minimal theories. Although we see no obvious obstacles, the technicalities to sort out seem to go beyond the scope of the present note.

Finally, we remark that though, in the notation of the main theorem, $\mathrm{PG}(\mathbb{M}^{\mathrm{clq}}) \cong \mathrm{PG}(\mathbb{M}_{rs})$, the two structures are not isomorphic (or even elementarily equivalent). Thus, it does not follow from our result that $\mathbb{M}^{\mathrm{clq}}$ itself has a reduct whose geometry is isomorphic to that of $\mathbb{M}_k$ for some $k > rs$. The following remains open:

Question. Let $\mathbb{M}_n$ be the generic of Hrushovski's (non-collapsed) ab initio construction with a single n-ary relation. Is there, for all $k > n$, a reduct $\mathcal{M}(k)$ of $\mathbb{M}_n$ such that $\mathrm{PG}(\mathcal{M}(k)) \cong \mathrm{PG}(\mathbb{M}_k)$?

1.1. Preliminaries. A category whose objects form a class of finite (relational) structures $\mathbb{C}$, closed under isomorphisms and substructures, and whose morphisms, $\leqslant$, are (not necessarily all) embeddings, is an *amalgamation class* (or has the *Amalgamation property* and *Joint Embedding property*) if:

(AP) If $A, B_1, B_2 \in \mathbb{C}$ are such that $A \leqslant B_1, B_2$, then there exists some $D \in \mathbb{C}$ and embeddings $f_i : B_i \to D$ such that $f_i[B_i] \leqslant D$, $f_1 \restriction A = f_2 \restriction A$, and $f_1[A] \leqslant D$.

(JEP) If $A_1, A_2 \in \mathbb{C}$, then there exists some $B \in \mathbb{C}$ and embeddings $f_i : A_i \to B$ such that $f_i[A_i] \leqslant B$ for $i = 1, 2$.

By Fraïssé's Theorem, to every amalgamation class with countably many isomorphism types is associated a unique (up to isomorphism) countable structure $\mathbb{M}$ satisfying:

(1) Every finite substructure of $\mathbb{M}$ is an element of $\mathbb{C}$.
(2) Whenever $A \leqslant \mathbb{M}$ and $A \leqslant D \in \mathbb{C}$, there is an embedding $f : D \to \mathbb{M}$ fixing A pointwise such that $f[D] \leqslant \mathbb{M}$.

We call $\mathbb{M}$ a *generic structure* for $\mathbb{C}$.

Hrushovski showed that if $\mathcal{L}$ is a countable finite relational language ([1]), and for a finite $\mathcal{L}$-structure A we let $\delta(A) := |A| - k(A)$, where $k(A)$ is the number of $\mathcal{L}$-relations in (powers of) A, then the class $\mathbb{C}$ of all finite $\mathcal{L}$-structures A such that $\delta(B) \geq 0$ for all $B \subseteq A$ is an amalgamation class with respect to a class of so called self-sufficient (or strong) embeddings. Provided the language contains at least one n-ary relation for $n \geq 3$ or two binary

([1]) This generalises easily to infinite languages, provided every finite structure supports only finitely many non-empty relations.

relations, the $\mathbb{C}$-generic structure $\mathbb{M}$ is ω-stable with a unique non-trivial regular type, $p_{\mathbb{C}}$. We call (the pregeometry of) $p_{\mathbb{C}}$ the pregeometry of $\mathbb{M}$.

In the present note (Section 2) we show that Hrushovski's construction can be carried out in a similar way if, e.g., in the context of a unique s-ary relation (possibly on r-tuples, rather than singletons), instead of defining $\delta(A) := |A| - k(A)$ we let $\lambda(A) := |A| - \sum_K (|K| - s + 1)$ where the sum ranges over all maximal (large enough) *cliques* in A, allowing A^r to support a uniformly bounded polynomial (rather than linear) number of relations. A clique K is determined by any $(s-1)$-tuple of its members, and thus can be viewed as an equivalence relation on $(s-1)$-tuples. In model-theoretic terms cliques can be viewed as an imaginary sort that need not be eliminable in the generic structure. From that perspective, the above predimension function is merely the straightforward adaptation of Hrufshovski's original predimension function to this two-sorted structure, and the non-linear number of relations is an artifact of counting (wrongly) the number of relations in definable congruence classes (of r-tuples)—namely, cliques.

In Section 3.1 we show that the generic structure associated with the clique construction is isomorphic to a (proper) reduct of Hrushovski's original construction. From the point of view described above, this reduct, unlike the original construction, does not admit geometric elimination of imaginaries, and our construction can be viewed as a first step towards generalising the construction to incorporate imaginary elements. It is, apparently, a necessary step in classifying all reducts of Hrushovski's ab initio construction. We do not know, at this stage, how to identify all imaginaries associated with an arbitrary such reduct.

As a test case we suggest the following: Let

$$T(x_1, x_2, x_3, x_4) := \exists y_1, y_2, y_3\, R(y_1, y_2, y_3) \wedge R(y_1, x_1, x_2) \wedge R(y_2, x_2, x_3) \wedge R(y_3, x_3, x_4)$$

and let M_T be the reduct of $\mathbb{M}_3$ whose unique atomic relation is T. Is M_T an ab initio structure? With respect to what predimension function? What is its geometry?

Finally, in Section 3.2 we conclude the proof of our main theorem.

2. The construction. Fix some natural $n \geq 2$ and $0 < r < n$, and denote $s = n - r + 1$. Let $\mathcal{L} = \{T_k : k \geq s\}$ be the relational language where T_k is of arity $r \cdot k$. Throughout, we think of T_k as a relation on k distinct r-tuples (condition (1) below), and additionally, we assume (2) and (3):

(1) $T_k(\bar{x}_1, \ldots, \bar{x}_k) \Rightarrow \bigwedge_{i<j\leq k} \bar{x}_i \neq \bar{x}_j$,
(2) $T_k(\bar{x}_1, \ldots, \bar{x}_k) \Rightarrow \bigwedge_{\sigma \in S_k} T_k(\bar{x}_{\sigma(1)}, \ldots, \bar{x}_{\sigma(k)})$,
(3) $T_k(\bar{x}_1, \ldots, \bar{x}_k) \Rightarrow \bigwedge_{s \leq i < k} T_i(\bar{x}_1, \ldots, \bar{x}_i)$.

DEFINITION 2.1. (1) For an $\mathcal{L}$-structure A, we say that $K \subseteq A^r$ with $|K| \geq s$ is a *clique* if for all $k \geq s$, whenever $\bar{x}_1, \dots, \bar{x}_k \in K$ are distinct, then $(\bar{x}_1, \dots, \bar{x}_k) \in T_k^A$.

We say that K is a *maximal* clique if there is no clique $K' \subseteq A^r$ such that $K' \supset K$. For an $\mathcal{L}$-structure A, define $\mathrm{M}(A)$ to be the set of maximal cliques of A.

(2) Define $\mathcal{C}_0^{\mathrm{clq}}$ to be the class of finite $\mathcal{L}$-structures A such that whenever $K_1, K_2 \in \mathrm{M}(A)$ are distinct, then $|K_1 \cap K_2| < s$.

REMARK 2.2. The language $\mathcal{L}$ has relations of infinitely many arities, so that for any collection M of sets each of cardinality at least s and pairwise incomparable with respect to inclusion there exists a unique $\mathcal{L}$-structure A with $\mathrm{M}(A) = M$. Specifically, A can be interpreted in the obvious way in the two-sorted structure $(\bar{A}, M, I)$ where $\bar{A} = \bigcup M$ is the universe of A and $I \subseteq \bar{A}^r \times \mathrm{M}(A)$ is interpreted as inclusion.

For structures $A \in \mathcal{C}_0^{\mathrm{clq}}$, due to the restriction on intersections of cliques, the language $\mathcal{L} = \{T_s, T_{s+1}\}$ suffices for this purpose (T_s itself would not suffice: take $s+1$ pairwise distinct cliques, any two intersecting in a set of size $s-1$, such that their union, A, has size $s+1$. Then A is not contained in any maximal clique, but this cannot be discerned using T_s alone). The requirement $|K_1 \cap K_2| < s$ is put in place in order to simplify the construction to come, e.g., it implies Observation 2.6. For the purpose of getting the guarantee of the previous paragraph from a finite part of $\mathcal{L}$, any uniform bound on $|K_1 \cap K_2|$ will have sufficed.

OBSERVATION 2.3. For an $\mathcal{L}$-structure A and a substructure $B \subseteq A$,

$$\mathrm{M}(B) = \{K \cap B^r : K \in \mathrm{M}(A),\ |K \cap B^r| \geq s\}.$$

NOTATION 2.4. For a finite set X denote $|X|_* = \max\{0, |X| - (s-1)\}$.

DEFINITION 2.5. For every finite $\mathcal{L}$-structure A define

$$t(A) = \sum_{K \in \mathrm{M}(A)} |K|_* \quad \text{and} \quad \lambda(A) = |A| - t(A).$$

OBSERVATION 2.6. For $A \in \mathcal{C}_0^{\mathrm{clq}}$, whenever $B \subseteq A$ and $K \in \mathrm{M}(B)$, there is a unique extension of K to a maximal clique of A. In particular,

$$t(B) = \sum_{K \in \mathrm{M}(A)} |K \cap B|_*.$$

LEMMA 2.7. *The function $\lambda : \mathcal{C}_0^{\mathrm{clq}} \to \mathbb{Z}$ is submodular. That is,*

$$\lambda(A \cup B) + \lambda(A \cap B) \leq \lambda(A) + \lambda(B)$$

whenever $D \in \mathcal{C}_0^{\mathrm{clq}}$ and $A, B, A \cup B, A \cap B \subseteq D$ are induced substructures.

Proof. For each $K \in \mathrm{M}(A \cup B)$ let K_A, K_B, K_{AB} denote the sets $K \cap A^r$, $K \cap B^r$, $K \cap (A \cap B)^r$, respectively. Observe that

$$|K|_* + |K_{AB}|_* \geq |K_A|_* + |K_B|_*$$

for each $K \in \mathrm{M}(A \cup B)$. Thus, by Observation 2.6,

$$\begin{aligned} t(A \cup B) + t(A \cap B) &= \sum_{K \in \mathrm{M}(A \cup B)} |K|_* + \sum_{K \in \mathrm{M}(A \cup B)} |K_{AB}|_* \\ &\geq \sum_{K \in \mathrm{M}(A \cup B)} |K_A|_* + \sum_{K \in \mathrm{M}(A \cup B)} |K_B|_* \\ &= t(A) + t(B), \end{aligned}$$

proving the statement. ∎

For a finite $\mathcal{L}$-structure A and a substructure $B \subseteq A$ define $\lambda(A/B) = \lambda(A \cup B) - \lambda(B)$. Extend this definition to an infinite $\mathcal{L}$-structure A and a cofinite substructure B by defining $\lambda(A/B) = \inf\{\lambda(X/X \cap B) : X \supseteq A \setminus B, |X| < \infty\}$. The definitions coincide on finite structures, by submodularity. Write $B \leqslant A$ if $\lambda(X/B) \geq 0$ for every $B \subseteq X \subseteq A$. By submodularity again, the relation $\leqslant$ is transitive.

Definition 2.8. Define $\mathcal{C}^{\mathrm{clq}}$ to be the class of $\mathcal{L}$-structures $A \in \mathcal{C}_0^{\mathrm{clq}}$ such that $\emptyset \leqslant A$.

Remark 2.9. One can define an analogue of $\mathcal{C}^{\mathrm{clq}}$, where distinct cliques are allowed to intersect in arbitrarily large sets, and the only requirement on an $\mathcal{L}$-structure A is that $\emptyset \leqslant A$. In that case, the guarantee of a sufficient finite language of the second paragraph of Remark 2.2 may not hold, seemingly. However, it can be shown that there is some finite $k = k(r, s)$ such that $\{T_s, \ldots, T_k\}$ does suffice. The proof is not entirely trivial, but we omit it.

Definition 2.10. Let $A_1, A_2 \in \mathcal{C}_0^{\mathrm{clq}}$ and let $B = A_1 \cap A_2$ be a common induced substructure. Define the *standard amalgam* of A_1 and A_2 over B to be the unique $\mathcal{L}$-structure D whose universe is $A_1 \cup A_2$ such that $\mathrm{M}(D) = M \cup M'$ where

$$\begin{aligned} M &= \{K \in \mathrm{M}(A_1) \cup \mathrm{M}(A_2) : |K \cap B^r| < s\}, \\ M' &= \{K_1 \cup K_2 : K_1 \in \mathrm{M}(A_1),\ K_2 \in \mathrm{M}(A_2),\ |K_1 \cap K_2| \geq s\}. \end{aligned}$$

Observation 2.11. Let $A_1, A_2 \in \mathcal{C}^{\mathrm{clq}}$ be such that $B = A_1 \cap A_2$ is a common substructure. Let D be the standard amalgam of A_1 and A_2 over B. Then $\lambda(D/A_1) = \lambda(A_2/B)$.

For $A, B \in \mathcal{C}_0^{\mathrm{clq}}$, say that an embedding $f : A \to B$ of $\mathcal{L}$-structures is *strong* if $f[A] \leqslant B$. The class $\mathcal{C}^{\mathrm{clq}}$ is closed under taking substructures and has the JEP. By the above observation, $\mathcal{C}^{\mathrm{clq}}$ also has AP with respect to strong embeddings. Since $\mathcal{C}_0^{\mathrm{clq}}$ has countably many isomorphism types, it follows from Fraïssé's Theorem that it has a unique countable generic structure $\mathbb{M}^{\mathrm{clq}}$ defined by the following property:

(∗) Whenever $A \leqslant B \in \mathcal{C}^{\mathrm{clq}}$ and $A \leqslant \mathbb{M}^{\mathrm{clq}}$, there exists a strong embedding $f : B \to \mathbb{M}^{\mathrm{clq}}$ fixing A pointwise.

As pointed out by the referee, if $r = 1$ the above construction can be easily viewed as a standard Hrushovski construction of a bi-partite graph. Namely, work in a two-sorted language with sorts P for points and C for cliques and a unique relation, $R \subseteq P \times C$. If cliques are given weight $s-1$ then the standard predimension function associated with such graphs is $\lambda(P, C) = |P| + (s-1)|C| - |R(P, C)|$—precisely the predimension function associated with $\mathbb{M}^{\mathrm{clq}}$ in case $r = 1$ and provided each clique is related to at least s points.

In case $r > 1$ some adaptations are needed for a similar construction to work. E.g., add a binary relation I on P (to be interpreted as "the domains of $\bar{x}$ and $\bar{y}$ intersect"). In that case points are precisely maximal I-cliques. If we require that elements in P are sets of size r, rather than r-tuples ([2]), then the restriction on the structure should be that every $p \in P$ belongs to exactly r maximal I-cliques.

We hope that a deeper study of this construction and its many possible variants may shed new light on the inner structure of Hrushovski's constructions and their reducts.

3. Relation to Hrushovski's ab initio construction. As above, we fix some natural $n \geq 2$ and $0 < r < n$, and denote $s = n-r+1$. Let $\mathcal{L}_n = \{R\}$ be the language of a single n-ary relation. For a finite $\mathcal{L}_n$-structure A, define $\delta(A) = |A| - |R^A|$. Define $\leqslant$ for $\mathcal{L}_n$-structures as defined above with respect to λ. Let $\mathcal{C}_n$ be the class of finite $\mathcal{L}_n$-structures A with $\emptyset \leqslant A$. It is closed under substructures and free amalgamation, and thus is an amalgamation class. Let $\mathbb{M}_n$ be the generic structure for the class $\mathcal{C}_n$.

3.1. $\mathbb{M}^{\mathrm{clq}}$ is a proper reduct of $\mathbb{M}_n$. For every natural $k \geq s$ we let $\varphi_{T_k}(\bar{x}_1, \ldots, \bar{x}_k)$, where $|\bar{x}_i| = r$, be the $\mathcal{L}_n$-formula specifying that there exists an $(s-1)$-tuple $\bar{y}$ whose elements are distinct from the elements of $\bar{x}_1, \ldots, \bar{x}_k$ such that, denoting by X the set of all elements appearing in $\bar{x}_1, \ldots, \bar{x}_k, \bar{y}$,

- $\bar{x}_i \neq \bar{x}_j$ for all $1 \leq i < j \leq k$,
- $R^X = \{(\bar{y}, \bar{x}_i) : 1 \leq i \leq k\}$,

([2]) If we allow several relations in the construction, this can be achieved by decomposing the relation R.

- $X \leqslant B$ in any superstructure B (in the ambient structure) with $|B| \leq |X| + s$.

Note that the last item above guarantees that $\bar{y}$ is unique. For every $\mathcal{L}_n$-structure A, denote by A^T the reduct $\langle A, \varphi_{T_s}(A), \varphi_{T_{s+1}}(A), \dots \rangle$. In the present subsection we prove:

Proposition 3.1. *The structure $\mathbb{M}^{\mathrm{clq}}$ is isomorphic to $\mathbb{M}_n^T$. Moreover, M_n^T is a proper reduct of M_n.*

In order to show that $\mathbb{M}_n^T \cong \mathbb{M}^{\mathrm{clq}}$ and $R^{\mathbb{M}_n}$ is not definable in $\mathbb{M}_n^T$, we need the following lemma:

Lemma 3.2.

(1) *If $A \in \mathcal{C}_n$ then $A^T \in \mathcal{C}^{\mathrm{clq}}$.*
(2) *Whenever M is an $\mathcal{L}_n$-structure, and $A \in \mathcal{C}_n$ and $A \leqslant M$, the substructure induced on the set A by M^T is exactly A^T.*
(3) *For every $A \in \mathcal{C}_n$ and $B_c \in \mathcal{C}^{\mathrm{clq}}$ such that $A^T \leqslant B_c$, there exists some $C \in \mathcal{C}_n$ such that $A \leqslant C$ and $B_c \leqslant C^T$.*
(4) *For any $F \in \mathcal{C}_n$, there exist $A, B \in \mathcal{C}_n$ with $F \leqslant A, B$ such that A, B are not isomorphic over F, but A^T, B^T are isomorphic over F.*

The proof of Proposition 3.1 follows the procedure of [Mer16, 3.6.7, 3.6.9], and is repeated below for completeness. We will first prove the proposition assuming Lemma 3.2 and then prove the lemma.

Proof of Proposition 3.1.

Claim 1. *If $A \leqslant B \in \mathcal{C}_n$, then $A^T \leqslant B^T$. Consequently, if $A \leqslant \mathcal{M}$ where $\mathcal{M}$ is an $\mathcal{L}_n$-structure each of whose finite substructures is an element of $\mathcal{C}_n$, then $A^T \leqslant \mathcal{M}^T$.*

Proof. By (1) we have $A^T, B^T \in \mathcal{C}^{\mathrm{clq}}$, and by Lemma 3.2(2) we have $A^T \subseteq B^T$. Since B is arbitrary, it will suffice to show that $\lambda(B^T/A^T) \geq 0$.

Let $\mathbb{B}$ be the free join of r copies of B over A, for some natural r. Then $A \leqslant \mathbb{B}$ so $A^T \subseteq \mathbb{B}^T \in \mathcal{C}^{\mathrm{clq}}$. By submodularity,

$$\lambda(\mathbb{B}^T/A^T) \leq \sum_{i=1}^{r} \lambda(B^T/A^T) = r \cdot \lambda(B^T/A^T).$$

Since $\mathbb{B}^T \in \mathcal{C}^{\mathrm{clq}}$, it must be that $\lambda(\mathbb{B}^T/A^T) \geq -\lambda(A^T)$. As r is arbitrary, this means $\lambda(B^T/A^T)$ is non-negative. ∎

We can now show that $\mathbb{M}^{\mathrm{clq}}$ is isomorphic to $\mathbb{M}_n^T$. To do that it will suffice to show:

Claim 2. *The reduct $\mathbb{M}_n^T$ is a generic structure for $\mathcal{C}^{\mathrm{clq}}$.*

Proof. We must show that every finite substructure of $\mathbb{M}_n^T$ is an element of $\mathcal{C}^{\mathrm{clq}}$. Let X be some finite subset of the universe of $\mathbb{M}_n$. Choose some finite $A \leqslant \mathbb{M}_n$ containing X. Then by Claim 1, $A^T \leqslant \mathbb{M}_n^T$. By Lemma 3.2(1) we know $A^T \in \mathcal{C}^{\mathrm{clq}}$, so the substructure that A^T induces on X is also in $\mathcal{C}^{\mathrm{clq}}$.

Now we show that $\mathbb{M}_n^T$ has the extension property. Let $A_c \leqslant \mathbb{M}_n^T$ and $A_c \leqslant B_c \in \mathcal{C}^{\mathrm{clq}}$. We need to find a strong embedding of B_c into $\mathbb{M}_n^T$ over A_c. Denote by A the universe of A_c and choose some $A \subseteq \bar{A} \leqslant \mathbb{M}_n$. By Claim 1 we have $\bar{A}^T \leqslant \mathbb{M}_n^T$, so $A_c \leqslant \bar{A}^T$. Let D_c be a standard amalgam of $\bar{A}^T$ and B_c over A_c. Note that $\bar{A}^T \leqslant D_c$, because $A_c \leqslant B_c$. Let $C \in \mathcal{C}_n$ be as guaranteed by Lemma 3.2(3), i.e. $\bar{A} \leqslant C$ and $D \leqslant C^T$. Strongly embed C into $\mathbb{M}_n$ over $\bar{A}$. Without loss of generality assume $C \leqslant \mathbb{M}_n$. Then by Claim 1 we have $C^T \leqslant \mathbb{M}_n^T$. Since $B_c \leqslant C^T$, we have found a strong embedding of B_c into $\mathbb{M}_n^T$ over A_c. ∎

To complete the proof of the proposition it will suffice to show:

Claim 3. *The relation R of $\mathbb{M}_n$ is not definable in $\mathbb{M}_n^T$.*

Proof. Let $F \subseteq \mathbb{M}_n$. By increasing it, we may assume $F \leqslant \mathbb{M}_n$. Choose some A, B as guaranteed by Lemma 3.2(4). Since $\mathbb{M}_n$ is generic, we may assume that $F \leqslant A, B \leqslant \mathbb{M}_n$. In a generic, every finite partial isomorphism between strong substructures extends to a full automorphism of the structure. Claim 1 asserts that $A^T, B^T \leqslant \mathbb{M}_n^T$, so there is some automorphism α of $\mathbb{M}_n^T$ taking A to B and fixing F pointwise. We know that A, B are not isomorphic over F, so α does not preserve the relation R. Hence, R is not definable from the parameter set F in $\mathbb{M}_n^T$. ∎

This concludes the proof of Proposition 3.1. ∎

We now turn to the proof of Lemma 3.2.

Proof of (1). Let $A \in \mathcal{C}_n$. Denote $A_c = A^T$. Clearly, $A_c \in \mathcal{C}_0^{\mathrm{clq}}$. Let $B \subseteq A$ be an arbitrary non-empty substructure of A. Denote by B_c the substructure induced on B by A_c. Consider

$$\bar{B} = B \cup \big\{\bar{y} \in A : \{\bar{x} : B \models R(\bar{y}, \bar{x})\} \in \mathrm{M}(B_c)\big\}$$

as a substructure of A. Then

$$\begin{aligned} 0 \leq \delta(\bar{B}) &= (|B| + |\bar{B} \setminus B|) - |R^{\bar{B}}| \\ &\leq |B| + (s-1) \cdot |\,\mathrm{M}(B_c)| - \sum_{K \in \mathrm{M}(B_c)} |K| \\ &= |B| - \sum_{K \in \mathrm{M}(B_c)} |K|_* = \lambda(B_c). \end{aligned}$$

Thus, $\lambda(B_c) \geq 0$ and $A_c \in \mathcal{C}^{\mathrm{clq}}$. ∎

Proof of (2). Let M and A be $\mathcal{L}_n$-structures such that $A \in \mathcal{C}_n$ and $A \leqslant M$. Let $(\bar{a}_1, \ldots, \bar{a}_k) \in \varphi_{T_k}(A^r)$ and assume $\bar{y} \in M$ is such that $M \models$

$\bigwedge_{1\le i\le k} R(\bar y,\bar a_i)$. It must be that $\bar y \in A^{s-1}$, for otherwise by definition $k \geq s$ and $\delta(\bar y/A) < 0$, in contradiction to $A \leqslant M$.

Also by $A \leqslant M$, if $\{\bar a_1,\dots,\bar a_k,\bar y\} \nleqslant B$ where $B \subseteq M$, then already within A we have $\bar a_1 \dots \bar a_k \bar y \nleqslant B \cap A$ with $|B\cap A| \le |B|$.

Thus, $M \models \varphi_{T_k}(\bar a_1,\dots,\bar a_k)$ if and only if $A \models \varphi_{T_k}(\bar a_1,\dots,\bar a_k)$. ■

Proof of (3). Let $A \in \mathcal{C}_n$ and $B_c \in \mathcal{C}^{\mathrm{clq}}$ be such that $A^T \leqslant B_c$. Denote $A_c = A^T$; by (1) we know $A_c \in \mathcal{C}^{\mathrm{clq}}$. Denote by B the underlying set of B_c.

For each $K \in \mathrm{M}(A_c)$ let $\bar y_K \in A^{s-1}$ be the unique tuple such that $\{\bar a \in A^r : (\bar y_K, \bar a) \in R^A\} = K$, and let $\widehat{K} \in \mathrm{M}(B_c)$ be the unique maximal clique in B_c extending K. Let

$$R_0 = \bigcup_{K\in\mathrm{M}(A_c)} \{(\bar y_K, \bar b) : \bar b \in \widehat{K} \setminus K\}.$$

For each $L \in \mathrm{M}(B_c)$ such that $L \cap A^r \notin \mathrm{M}(A_c)$, let $\bar z_L$ be an $(s-1)$-tuple of new elements. Let

$$\begin{aligned} R_1 &= \{(\bar z_L, \bar b) : \bar b \in L,\ L \in \mathrm{M}(B_c),\ L\cap A^r \notin \mathrm{M}(A_c)\},\\ Z &= \{\bar z_L : L \in \mathrm{M}(B_c),\ L\cap A^r \notin \mathrm{M}(A_c)\}. \end{aligned}$$

Define C to be the $\mathcal{L}_n$-structure with underlying set $B \cup Z$ and

$$R^C = R^A \cup R_0 \cup R_1$$

and denote $C_c = C^T$. Clearly B_c is a substructure of C_c. Moreover, $B_c \leqslant C_c$. Indeed, as $\mathrm{M}(C_c) = \mathrm{M}(B_c)$, by construction we get $\lambda(B \cup Z_0/B) = |Z_0|$ for any $Z_0 \subseteq Z$.

It remains to show that $A \leqslant C$, i.e., $\delta(X/A) \geq 0$ for any intermediate $A \subseteq X \subseteq C$. Note that if $L \in \mathrm{M}(B_c)$ with $L\cap A^r \notin \mathrm{M}(A_c)$, then $\delta(X\bar z_L/A) < \delta((X\setminus \bar z_L)/A)$ if and only if $|L \cap X^r| \geq s$. Thus, it will suffice to prove the inequality under the assumption that for all such L, $\bar z_L \in X^{(s-1)}$ if and only if $|L\cap X^r| \geq s$. Indeed, in that case

$$\begin{aligned} \delta(X/A) &= \big(|X\setminus (X\cap B)| + |(X\cap B)\setminus A|\big) - \big(|R_0 \cap X^n| + |R_1\cap X^n|\big)\\ &\geq |(X\cap B)\setminus A| - \sum_{K\in\mathrm{M}(A_c)} |(\widehat{K}\cap X^r)\setminus K| - \sum_{\substack{L\in\mathrm{M}(B_c)\\ |L\cap X^r|\geq s\\ L\cap A^r\notin \mathrm{M}(A_c)}} |L\cap X^r|_*\\ &= \lambda(X\cap B_c/A_c) \geq 0. \ ■ \end{aligned}$$

Proof of (4). Let $F \in \mathcal{C}_n$. Define A and B to be the $\mathcal{L}_n$-structures with underlying set $F\bar a$, where $\bar a$ is an n-tuple of new elements, and

$$R^A = R^F, \qquad R^B = R^F \cup \{\bar a\}. \ ■$$

3.2. The pregeometry of $\mathbb{M}^{\mathrm{clq}}$. Recall that for an $\mathcal{L}$-structure A and some substructure $B \subseteq A$,

$$\mathrm{cl}^A(B) = \bigcup\{X \subseteq A : \lambda(X/X \cap B) \leq 0\}$$

is a closure operator giving rise to a pregeometry on the underlying set of A. The dimension function associated to this closure operator is

$$\Lambda^A(B) = \min\{|X| : X \subseteq B,\ \mathrm{cl}^A(X) = \mathrm{cl}^A(B)\},$$

and we say that a finite B is *independent* in A if $\Lambda^A(B) = |B|$. A pregeometry is defined in a similar way on $\mathcal{L}_k$-structures, for any natural k. For an $\mathcal{L}$-structure or an $\mathcal{L}_k$-structure A, we denote its associated pregeometry by $\mathrm{PG}(A)$.

A pregeometry is uniquely determined by any one of the following: its closure operator on finite sets, its dimension function, or its collection of finite independent subsets. We say that two pregeometries are isomorphic if there exists a bijection between the two, preserving any one of these in both directions.

The following facts are standard and easy:

- If $\mathrm{cl}^A(X) = X$, then $X \leqslant A$.
- If $\mathrm{cl}^A(X) = X$ and $B \subseteq A$, then $\mathrm{cl}^B(X \cap B) = X \cap B$.
- If $X \leqslant A$ is finite, then $\lambda(X) = \Lambda^A(X)$.

For amalgamation classes $(\mathcal{D}_1, \leqslant)$ and $(\mathcal{D}_2, \leqslant)$ of either $\mathcal{L}$-structures or $\mathcal{L}_k$-structures, write $\mathcal{D}_1 \overset{*}{\rightsquigarrow} \mathcal{D}_2$ if

($*$) whenever $A_1 \in \mathcal{D}_1$ and $A_2 \in \mathcal{D}_2$, if $f : \mathrm{PG}(A_1) \to \mathrm{PG}(A_2)$ is an isomorphism of pregeometries, and $A_1 \leqslant B_1 \in \mathcal{D}_1$, then there exist some $B_1 \leqslant C_1 \in \mathcal{D}_1$ and $C_2 \in \mathcal{D}_2$ with $A_2 \leqslant C_2$ and an isomorphism $\widehat{f} : \mathrm{PG}(C_1) \to \mathrm{PG}(C_2)$ extending f.

By a standard back-and-forth argument [EF12, Lemma 2.3], assuming that $\emptyset \in \mathcal{D}_1, \mathcal{D}_2$, if $\mathcal{D}_1 \overset{*}{\rightsquigarrow} \mathcal{D}_2$ and $\mathcal{D}_2 \overset{*}{\rightsquigarrow} \mathcal{D}_1$, then $\mathrm{PG}(\mathbb{D}_1) \cong \mathrm{PG}(\mathbb{D}_2)$, where $\mathbb{D}_i$ is the countable generic structure of $\mathcal{D}_i$.

The following proposition will conclude the proof of our main theorem:

Proposition 3.3. $\mathrm{PG}(\mathbb{M}^{\mathrm{clq}}) \cong \mathrm{PG}(\mathbb{M}_{rs})$.

The proof of the proposition is split into the next four lemmas. Lemma 3.4 is taken from [Mer16], and we repeat the proof for completeness. Throughout, we think of a relation in $\mathcal{L}_{rs}$-structures as an rs-tuple as well as an s-tuple of r-tuples.

Lemma 3.4. *Let $A_1 \leqslant B_1$ and $A_2 \leqslant B_2$ be such that B_1 and B_2 have the same underlying set B and $\mathrm{PG}(A_1) = \mathrm{PG}(A_2)$ with underlying set A. For $X \subseteq B$, denote by X_i the substructure induced on X by B_i. Assume* *that for every* $X \subseteq B$*, there is some extension* $\widehat{X}$ *such that* $\lambda(\widehat{X}_1/X_1) \leq 0$, $\lambda(\widehat{X}_2/X_2) \leq 0$ *and* $\lambda(\widehat{X}_1/\widehat{X}_1 \cap A) = \lambda(\widehat{X}_2/\widehat{X}_2 \cap A)$*. Then* $\mathrm{PG}(B_1) = \mathrm{PG}(B_2)$*.*

Proof. Take some X closed in B_1. Then $\widehat{X} = X$, because $\lambda(\widehat{X}_i/X_i) \leq 0$. We know $X \cap A$ is closed in A_1, and as $\mathrm{PG}(A_1) = \mathrm{PG}(A_2)$ it is also closed in A_2. Hence, $X_i \cap A \leqslant A_i$ for $i = 1, 2$. Since $\Lambda^{A_1}(X \cap A) = \Lambda^{A_2}(X \cap A)$, we have $\lambda(X_1 \cap A) = \lambda(X_2 \cap A)$.

By assumption $\lambda(X_1/X_1 \cap A) = \lambda(X_2/X_2 \cap A)$, so $\Lambda^{A_1}(X) = \lambda(X_1) = \lambda(X_2) \geq \Lambda^{A_2}(X)$. Taking $\bar{X} = \mathrm{cl}^{B_2}(X)$ and applying the argument symmetrically, we find that $\Lambda^{A_1}(X) \leq \lambda(\bar{X}_1) = \lambda(\bar{X}_2) = \Lambda^{A_2}(X)$. So $\Lambda^{A_1}(X) = \Lambda^{A_2}(X)$. Hence, $\bar{X}$ cannot properly extend X because $\lambda(\bar{X}_1) = \Lambda^{A_1}(X)$ and X is closed in B_1.

The pregeometries $\mathrm{PG}(B_1)$ and $\mathrm{PG}(B_2)$ have the same lattice of closed sets, so they are equal. ■

Lemma 3.5. *For every* $A \leqslant B \in \mathcal{C}_{rs}$ *there exist some* $B \leqslant C \in \mathcal{C}_{rs}$ *and* $A \leqslant D \in \mathcal{C}_{rs}$ *such that* $\mathrm{PG}(D) = \mathrm{PG}(C)$ *and* $(\bar{b}_{\sigma(1)}, \ldots, \bar{b}_{\sigma(s)}) \notin R^D$ *for all* $(\bar{b}_1, \ldots, \bar{b}_s) \in R^D \setminus R^A$ *and* $\sigma \in S_s \setminus \{\mathrm{id}\}$*.*

Proof. For each $\bar{a} = (a_1, \ldots, a_{rs}) \in R^B \setminus R^A$ let $\{x_{\bar{a}}, y_{\bar{a}}\}$ be two new elements and let

$$R^C_{\bar{a}} = \{(a_1, \ldots, a_{rs-2}, x_{\bar{a}}, y_{\bar{a}}), (a_3, \ldots, a_{rs}, y_{\bar{a}}, x_{\bar{a}})\},$$
$$R^D_{\bar{a}} = \{(a_1, \ldots, a_{rs-1}, x_{\bar{a}}), (a_2, \ldots, a_{rs}, y_{\bar{a}}), (a_1, a_3, \ldots, a_{rs-2}, a_{rs}, x_{\bar{a}}, y_{\bar{a}})\}.$$

Define C, D to be the structures with universe

$$V := B \cup \bigcup_{\bar{a} \in R^B \setminus R^A} \{x_{\bar{a}}, y_{\bar{a}}\}$$

and

$$R^C = R^B \cup \bigcup_{\bar{a} \in R^B \setminus R^A} R^C_{\bar{a}}, \qquad R^D = R^A \cup \bigcup_{\bar{a} \in R^B \setminus R^A} R^D_{\bar{a}}.$$

We show that $\mathrm{PG}(C) = \mathrm{PG}(D)$. For $\bar{a} = (a_1, \ldots, a_{rs}) \in R^B \setminus R^A$ write $S_{\bar{a}} = \{a_1, \ldots, a_{rs}, x_{\bar{a}}, y_{\bar{a}}\}$. Let $X \subseteq V$: it will suffice to show that X is closed in C if and only if it is closed in D. If X is closed in C and $\bar{a} \in R^B \setminus R^A$, then $S_{\bar{a}} \subseteq X$ if and only if $|S_{\bar{a}} \cap X| \geq rs - 1$. The same holds if we assume X is closed in D. So if X is closed in C we immediately see that $\delta_C(X) = |X \setminus A| - 3 \cdot |\{\bar{a} : S_{\bar{a}} \subseteq X\}| + \delta_A(X \cap A) = \delta_D(X)$, implying that X is closed in D, since the exact same calculation holds in the opposite direction as well. Indeed, if $Y = \mathrm{cl}^D(X)$ then our argument shows that $\delta_D(Y) = \delta_C(Y) \geq \delta_C(X) = \delta_D(X)$, so $X = Y$ is already closed in D. A similar argument shows that if X is closed in D it is also closed in C. ■

Lemma 3.6. $(\mathcal{C}_{rs}, \leqslant) \overset{*}{\rightsquigarrow} (\mathcal{C}^{\mathrm{clq}}, \leqslant)$.

Proof. Let $A \in \mathcal{C}_{rs}$ and $A_c \in \mathcal{C}^{\mathrm{clq}}$ have isomorphic pregeometries. Without loss of generality, assume that A and A_c have the same underlying set and $\mathrm{PG}(A) = \mathrm{PG}(A_c)$. Let $B \in \mathcal{C}_{rs}$ be such that $A \leqslant B$. We have to show that there exist $B \leqslant D \in \mathcal{C}_{rs}$ and $A \leqslant C_c \in \mathcal{C}^{\mathrm{clq}}$ such that $\mathrm{PG}(D) \cong_A \mathrm{PG}(C_c)$. Let $B \leqslant D$ be as provided by Lemma 3.5. We will now construct $C_c \in \mathcal{C}^{\mathrm{clq}}$ with the same universe as D whose clique structure captures exactly the R-relations not already in A.

For each $\bar{a} = (\bar{a}_1, \dots, \bar{a}_s) \in R^D \setminus R^A$ let $K_{\bar{a}} = \{\bar{a}_1, \dots, \bar{a}_s\}$. By Remark 2.2 an $\mathcal{L}$-structure with a given universe can be defined by specifying its maximal cliques. So we define the $\mathcal{L}$-structure C_c with universe D and

$$\mathrm{M}(C_c) = \mathrm{M}(A_c) \cup \{K_{\bar{a}} : \bar{a} \in R^D \setminus R^A\}.$$

Observe that for any $L_1, L_2 \in \mathrm{M}(C_c)$ distinct, $|L_1 \cap L_2| < s$. Indeed, whenever $L_1, L_2 \in \mathrm{M}(A_c)$, this follows from $A_c \in \mathcal{C}^{\mathrm{clq}}$. Otherwise, say if $L_1 \notin \mathrm{M}(C_c)$, by $L_1 \nsubseteq L_2$ we have $|L_1 \cap L_2| < |L_1| = s$.

For a substructure $X \subseteq D$, denote by X_c the substructure induced by C_c on the universe of X. Then for $X \subseteq D$ we have $\delta(X/X \cap A) = \lambda(X_c/X_c \cap A_c)$. Thus, $A_c \leqslant C_c$, and $\mathrm{PG}(D) = \mathrm{PG}(C_c)$ by Lemma 3.4. ∎

Lemma 3.7. $(\mathcal{C}^{\mathrm{clq}}, \leqslant) \overset{*}{\rightsquigarrow} (\mathcal{C}_{rs}, \leqslant)$.

Proof. Let $A_c \in \mathcal{C}^{\mathrm{clq}}$ and $A_{rs} \in \mathcal{C}_{rs}$ with $\mathrm{PG}(A_c) = \mathrm{PG}(A_{rs})$ and underlying set A. Let $B_c \in \mathcal{C}^{\mathrm{clq}}$ with $A_c \leqslant B_c$ and underlying set B. Fix a function $f : \mathrm{M}(B_c) \to (B^r)^{s-1}$ such that, with $f(K) = (\bar{a}_1^K, \dots, \bar{a}_{s-1}^K)$ and $E_K = \{\bar{a}_i^K : 1 \le i \le s-1\}$, we have:

(1) $E_K \in [K^r]^{s-1}$.
(2) If $K \cap A^r \in \mathrm{M}(A_c)$, then $E_K \subseteq (A^r)$.

We can require f to be injective, but this is not necessary. Denote by $f(K)\bar{b}$ the concatenation of the tuple $f(K)$ with the tuple $\bar{b}$. Define

$$R_0 = \{f(K)\bar{b} : K \in \mathrm{M}(B_c),\ K \cap A^r \in \mathrm{M}(A_c),\ \bar{b} \in K \setminus A^r\},$$
$$R_1 = \{f(K)\bar{b} : K \in \mathrm{M}(B_c),\ K \cap A^r \notin \mathrm{M}(A_c),\ \bar{b} \in K \setminus E_K\},$$

and note that each $\bar{a}\bar{b} \in R_0 \cup R_1$ has a unique $K \in \mathrm{M}(B_c)$ such that $\bar{a} = f(K)$ and $\bar{b} \in K$. Let B_{rs} be the structure with underlying set B and

$$R^{B_{rs}} = R^{A_{rs}} \cup R_0 \cup R_1.$$

Say that a set X is *good* if for every $K \in \mathrm{M}(B_c)$, we have $E_K \subseteq X^r$ if and only if $|K \cap X^r| \ge s$. Then for a good X, the equality $\lambda(X_c/X_c \cap A_c) = \delta(X_{rs}/X_{rs} \cap A)$ holds. Every non-good set X has a good extension $\bar{X}$ with $\delta(\bar{X}/X) < 0$, so $A_{rs} \leqslant B_{rs}$. This finishes the proof, as in the previous lemma. ∎

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