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# $L_p$ regular sparse hypergraphs: box norms

by

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**Abstract.** We consider some variants of the Gowers box norms, introduced by Hatami, and show their relevance in the context of sparse hypergraphs. Our main results are the following. Firstly, we prove a generalized von Neumann theorem for $L_p$ graphons. Secondly, we give natural examples of pseudorandom families, that is, sparse weighted uniform hypergraphs which satisfy relative versions of the counting and removal lemmas.

## 1. Introduction

**1.1. Overview.** Let $\langle (X_i, \Sigma_i, \mu_i) : i \in e\rangle$ be a nonempty finite family of probability spaces and let $(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ denote their product. Recall that the *box norm* of a random variable $f\colon \boldsymbol{X}_e \to \mathbb{R}$ is the quantity

$$\|f\|_{\square(\boldsymbol{X}_e)} := \mathbb{E}\Big[\prod_{\omega\in\{0,1\}^e} f(\mathbf{x}_e^{(\omega)}) \,\Big|\, \mathbf{x}_e^{(0)}, \mathbf{x}_e^{(1)} \in \boldsymbol{X}_e\Big]^{1/2^{|e|}}. \tag{1.1}$$

(For unexplained notation see Subsection 1.2 below.) These norms were introduced by Gowers [9, 11] and are important tools in arithmetic and extremal combinatorics. There are some slight variants of the box norms which first appeared [1] in [14, 15]: for every even integer $\ell \geq 2$ we define the *$\ell$-box norm* of $f\colon \boldsymbol{X}_e \to \mathbb{R}$ by

$$\|f\|_{\square_\ell(\boldsymbol{X}_e)} := \mathbb{E}\Big[\prod_{\omega\in\{0,\dots,\ell-1\}^e} f(\mathbf{x}_e^{(\omega)}) \,\Big|\, \mathbf{x}_e^{(0)}, \dots, \mathbf{x}_e^{(\ell-1)} \in \boldsymbol{X}_e\Big]^{1/\ell^{|e|}}. \tag{1.2}$$

Clearly, the $\square_2(\boldsymbol{X}_e)$-norm coincides with the $\square(\boldsymbol{X}_e)$-norm. As the parameter $\ell$ increases, the quantity $\|f\|_{\square_\ell(\boldsymbol{X}_e)}$ also increases and depends on the inte-

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2010 *Mathematics Subject Classification*: Primary 05C35, 05C65; Secondary 46B20, 46B25.
*Key words and phrases*: sparse hypergraphs, Gowers box norms.
Received 2 July 2018.
Published online 17 June 2019.

[1] Actually, the framework in [14, 15] is more general and includes several other variants of (1.1).

 

grability properties of $f$. In particular, for bounded functions all these norms are essentially equivalent (see [6, Proposition A.1]), but for unbounded functions they behave quite differently.

The starting point of this paper is the observation that the $\ell$-box norms can serve as the proper higher-complexity ([2]) analogues of the box norms in the context of sparse hypergraphs and related structures. A strong indication which supports this point of view is that the Gowers–Cauchy–Schwarz inequality also holds for the $\square_\ell(\boldsymbol{X}_e)$-norms. This fact together with several elementary properties are discussed in Section 2.

The rest of this paper is devoted to the proof of our main results which use the $\ell$-box norms in an essentially way (further examples showing the relevance of these norms are given in [6]). In Section 3 we present a version of the *generalized von Neumann theorem* for $L_p$ graphons (Theorem 3.1 in the main text); as we shall discuss in more detail in Section 3, the main point of this result is that it can be applied to $L_p$ graphons for any $p > 1$. The second part of this paper deals with *pseudorandom families* [7], a class of sparse weighted uniform hypergraphs whose most important feature is that they satisfy relative versions of the counting and removal lemmas. Their definition is recalled in Section 4, but for a more complete discussion of their properties we refer the reader to [7]. We present two different types of examples of pseudorandom families (see Theorems 4.2 and 4.3 in the main text). They can both be seen as deviations (in an $L_p$-sense) of hypergraphs which satisfy the linear forms condition, a well-known pseudorandomness condition originating from [12].

**1.2. Background material.** Our general notation and terminology is standard. By $\mathbb{N} = \{0, 1, \dots\}$ we denote the set of all natural numbers. As usual, for every positive integer $n$ we set $[n] := \{1, \dots, n\}$. If $f$ is an integrable real-valued random variable defined on a probability space $(X, \Sigma, \mu)$, then by $\mathbb{E}[f(x) \mid x \in X]$ we shall denote the expected value of $f$; if the sample space $X$ is understood from the context, then we write simply $\mathbb{E}[f]$. All the necessary background from probability theory can be found, e.g., in [1].

As already mentioned, the box norms and their variants are associated with finite products of probability spaces. It is more convenient, however, to work with the following more general structures.

DEFINITION 1.1 ([18]). A *hypergraph system* is a triple

$$\mathscr{H} = (n, \langle (X_i, \Sigma_i, \mu_i) : i \in [n] \rangle, \mathcal{H}) \tag{1.3}$$

where $n$ is a positive integer, $\langle (X_i, \Sigma_i, \mu_i) : i \in [n] \rangle$ is a finite sequence of probability spaces and $\mathcal{H}$ is a hypergraph on $[n]$. If $\mathcal{H}$ is $r$-uniform, then $\mathscr{H}$ will be called an *$r$-uniform hypergraph system*.

([2]) Note here that if $s, r$ are positive integers with $s > r$, then there is no analogue of the Gowers $U^s$-norm for $r$-uniform hypergraphs.

For every hypergraph system $\mathscr{H} = (n, \langle (X_i, \Sigma_i, \mu_i) : i \in [n]\rangle, \mathcal{H})$, we shall denote by $(\boldsymbol{X}, \boldsymbol{\Sigma}, \boldsymbol{\mu})$ the product of the spaces $\langle (X_i, \Sigma_i, \mu_i) : i \in [n]\rangle$. More generally, let $e \subseteq [n]$ be nonempty and let $(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ denote the product of the spaces $\langle (X_i, \Sigma_i, \mu_i) : i \in e\rangle$. (By convention, we set $\boldsymbol{X}_\emptyset$ to be the empty set.) The $\sigma$-algebra $\boldsymbol{\Sigma}_e$ is not comparable with $\boldsymbol{\Sigma}$, but it can be "lifted" to $\boldsymbol{X}$ by setting

$$\mathcal{B}_e = \{\pi_e^{-1}(\mathbf{A}) : \mathbf{A} \in \boldsymbol{\Sigma}_e\} \tag{1.4}$$

where $\pi_e \colon \boldsymbol{X} \to \boldsymbol{X}_e$ is the natural projection. Note that if $f \in L_1(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$, then there exists a unique random variable $\mathbf{f} \in L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ such that

$$f = \mathbf{f} \circ \pi_e \tag{1.5}$$

and note that the map $L_1(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu}) \ni f \mapsto \mathbf{f} \in L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ is a linear isometry. We will also deal with products of the space $(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$. Specifically, let $\ell \in \mathbb{N}$ with $\ell \geq 2$. For every $\mathbf{x}_e^{(0)} = (x_i^{(0)})_{i\in e}, \dots, \mathbf{x}_e^{(\ell-1)} = (x_i^{(\ell-1)})_{i\in e}$ in $\boldsymbol{X}_e$ and every $\omega = (\omega_i)_{i\in e} \in \{0, \dots, \ell-1\}^e$ we set

$$\mathbf{x}_e^{(\omega)} = (x_i^{(\omega_i)})_{i\in e} \in \boldsymbol{X}_e. \tag{1.6}$$

Notice that if $\omega = m^e$ for some $m \in \{0, \dots, \ell-1\}$ (that is, $\omega = (\omega_i)_{i\in e}$ with $\omega_i = m$ for every $i \in e$), then $\mathbf{x}_e^{(\omega)} = \mathbf{x}_e^{(m)}$.

## 2. $\ell$-box norms.

In this section we present several elementary properties of the $\ell$-box norms. We follow the exposition in [13, Appendix B] quite closely. In what follows, let $\mathscr{H} = (n, \langle (X_i, \Sigma_i, \mu_i) : i \in [n]\rangle, \mathcal{H})$ denote a hypergraph system.

**2.1. Basic properties.** Let $e \subseteq [n]$ be nonempty and let $\ell \geq 2$ be an even integer. Also let $f \in L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$. We first observe that the $\ell$-box norm of $f$ can be recursively defined as follows. If $e = \{j\}$ is a singleton, then by (1.2) we have

$$\begin{aligned} \|f\|_{\square_\ell(\boldsymbol{X}_e)} &= \mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} f(x_j^{(\omega)}) \;\Big|\; x_j^{(0)}, \dots, x_j^{(\ell-1)} \in X_j\Big]^{1/\ell^{|e|}} \\ &= (\mathbb{E}[f]^\ell)^{1/\ell} = |\mathbb{E}[f]|. \end{aligned} \tag{2.1}$$

On the other hand, if $|e| \geq 2$, then for every $j \in e$ we have

$$\|f\|_{\square_\ell(\boldsymbol{X}_e)} = \mathbb{E}\Big[\Big\|\prod_{\omega=0}^{\ell-1} f(\,\cdot\,, x_j^{(\omega)})\Big\|_{\square_\ell(\boldsymbol{X}_{e\setminus\{j\}})}^{\ell^{|e|-1}} \;\Big|\; x_j^{(0)}, \dots, x_j^{(\ell-1)} \in X_j\Big]^{1/\ell^{|e|}}. \tag{2.2}$$

We have the following proposition.

Proposition 2.1. *Let $e \subseteq [n]$ be nonempty and let $\ell \geq 2$ be an even integer.*

(a) (Gowers–Cauchy–Schwarz inequality) *For every* $\omega \in \{0,\dots,\ell-1\}^e$ *let* $f_\omega \in L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$. *Then*

$$\Big|\mathbb{E}\Big[\prod_{\omega\in\{0,\dots,\ell-1\}^e} f_\omega(\mathbf{x}_e^{(\omega)}) \ \Big|\ \mathbf{x}_e^{(0)},\dots,\mathbf{x}_e^{(\ell-1)} \in \boldsymbol{X}_e\Big]\Big| \le \prod_{\omega\in\{0,\dots,\ell-1\}^e} \|f_\omega\|_{\square_\ell(\boldsymbol{X}_e)}. \tag{2.3}$$

(b) *Let* $f \in L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$. *Then* $|\mathbb{E}[f]| \le \|f\|_{\square_\ell(\boldsymbol{X}_e)}$. *Moreover, if* $\ell_1 \le \ell_2$ *are even positive integers, then* $\|f\|_{\square_{\ell_1}(\boldsymbol{X}_e)} \le \|f\|_{\square_{\ell_2}(\boldsymbol{X}_e)}$.

(c) *If* $|e| \ge 2$, *then* $\|\cdot\|_{\square_\ell(\boldsymbol{X}_e)}$ *is a norm on the vector subspace of* $L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ *consisting of all* $f \in L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ *with* $\|f\|_{\square_\ell(\boldsymbol{X}_e)} < \infty$.

(d) *Let* $1 < p \le \infty$ *and let* $q$ *denote the conjugate exponent of* $p$. *Assume that* $\ell \ge q$ *and* $e = \{i,j\}$ *is a doubleton. Then for every* $f \in L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$, *every* $u \in L_p(X_i, \Sigma_i, \mu_i)$ *and every* $v \in L_p(X_j, \Sigma_j, \mu_j)$,

$$|\mathbb{E}[f(x_i,x_j)u(x_i)v(x_j) \mid x_i \in X_i,\ x_j \in X_j]| \le \|f\|_{\square_\ell(\boldsymbol{X}_e)} \|u\|_{L_p} \|v\|_{L_p}. \tag{2.4}$$

*Proof.* (a) We follow the proof of [13, Lemma B.2] which proceeds by induction on the cardinality of $e$. The case "$|e| = 1$" is straightforward, and so let $r \ge 2$ and assume that the result has been proved for every $e' \subseteq [n]$ with $1 \le |e'| \le r-1$. Let $e \subseteq [n]$ with $|e| = r$ be arbitrary. Fix $j \in e$, set $e' = e \setminus \{j\}$ and for every $\omega \in \{0,\dots,\ell-1\}^e$ let $f_\omega \in L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$. Moreover, for every $\omega_j \in \{0,\dots,\ell-1\}$ define $G_{\omega_j} \colon \boldsymbol{X}_{e'}^\ell \to \mathbb{R}$ by

$$G_{\omega_j}(\mathbf{x}_{e'}^{(0)},\dots,\mathbf{x}_{e'}^{(\ell-1)}) = \mathbb{E}\Big[\prod_{\omega_{e'}\in\{0,\dots,\ell-1\}^{e'}} f_{(\omega_{e'},\omega_j)}(\mathbf{x}_{e'}^{(\omega_{e'})}, x_j) \ \Big|\ x_j \in X_j\Big] \tag{2.5}$$

where $(\omega_{e'},\omega_j)$ is the unique element $\omega$ of $\{0,\dots,\ell-1\}^e$ such that $\omega(j) = \omega_j$ and $\omega(i) = \omega_{e'}(i)$ for every $i \in e'$. Observe that

$$\Big|\mathbb{E}\Big[\prod_{\omega\in\{0,\dots,\ell-1\}^e} f_\omega(\mathbf{x}_e^{(\omega)}) \ \Big|\ \mathbf{x}_e^{(0)},\dots,\mathbf{x}_e^{(\ell-1)} \in \boldsymbol{X}_e\Big]\Big| = \Big|\mathbb{E}\Big[\prod_{\omega_j=0}^{\ell-1} G_{\omega_j}\Big]\Big|$$

and, by Hölder's inequality ([3]), $|\mathbb{E}[\prod_{\omega_j=0}^{\ell-1} G_{\omega_j}]| \le \prod_{\omega_j=0}^{\ell-1} \mathbb{E}[G_{\omega_j}^\ell]^{1/\ell}$. Therefore, it is enough to show that for every $\omega_j \in \{0,\dots,\ell-1\}$ we have

$$\mathbb{E}[G_{\omega_j}^\ell] \le \prod_{\omega_{e'}\in\{0,\dots,\ell-1\}^{e'}} \|f_{(\omega_{e'},\omega_j)}\|_{\square_\ell(\boldsymbol{X}_e)}^\ell. \tag{2.6}$$

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([3]) Here, and in the rest of the proof, we use the following form of Hölder's inequality: if $(X,\Sigma,\mu)$ is a probability space, then for every integer $k \ge 2$, every $p_1,\dots,p_k > 1$ with $\sum_{i=1}^k 1/p_i = 1$, and every $f_1,\dots,f_k \colon X \to \mathbb{R}$ with $f_i \in L_{p_i}(X,\Sigma,\mu)$ for all $i \in [k]$, we have

$$\Big|\mathbb{E}\Big[\prod_{i=1}^k f_i\Big]\Big| \le \prod_{i=1}^k \|f_i\|_{L_{p_i}}.$$

Indeed, fix $\omega_j \in \{0,\dots,\ell-1\}$ and notice that, by (2.5),

$$G^{\ell}_{\omega_j}(\mathbf{x}^{(0)}_{e'},\dots,\mathbf{x}^{(\ell-1)}_{e'}) = \mathbb{E}\Big[\prod_{\omega_{e'}\in\{0,\dots,\ell-1\}^{e'}}\prod_{\omega=0}^{\ell-1} f_{(\omega_{e'},\omega_j)}(\mathbf{x}^{(\omega_{e'})}_{e'}, x^{(\omega)}_j)\Big] \tag{2.7}$$

where the expectation is over all $x^{(0)}_j,\dots,x^{(\ell-1)}_j \in X_j$. By (2.7) and Fubini's theorem, we see

$$\mathbb{E}[G^{\ell}_{\omega_j}] = \mathbb{E}\Big[\mathbb{E}\Big[\prod_{\omega_{e'}\in\{0,\dots,\ell-1\}^{e'}}\prod_{\omega=0}^{\ell-1} f_{(\omega_{e'},\omega_j)}(\mathbf{x}^{(\omega_{e'})}_{e'}, x^{(\omega)}_j)\,\Big|\,\mathbf{x}^{(0)}_{e'},\dots,\mathbf{x}^{(\ell-1)}_{e'}\in \boldsymbol{X}_{e'}\Big]\Big]$$

where the outer expectation is over all $x^{(0)}_j,\dots,x^{(\ell-1)}_j \in X_j$. Thus, applying the induction hypothesis and Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}[G^{\ell}_{\omega_j}] &\le \mathbb{E}\Big[\prod_{\omega_{e'}\in\{0,\dots,\ell-1\}^{e'}}\Big\|\prod_{\omega=0}^{\ell-1} f_{(\omega_{e'},\omega_j)}(\,\cdot\,, x^{(\omega)}_j)\Big\|_{\square_\ell(\boldsymbol{X}_{e'})}\Big] \\ &\le \prod_{\omega_{e'}\in\{0,\dots,\ell-1\}^{e'}} \mathbb{E}\Big[\Big\|\prod_{\omega=0}^{\ell-1} f_{(\omega_{e'},\omega_j)}(\,\cdot\,, x^{(\omega)}_j)\Big\|^{\ell^{|e'|}}_{\square_\ell(\boldsymbol{X}_{e'})}\Big]^{1/\ell^{|e'|}}.\end{aligned} \tag{2.8}$$

By (2.2) and (2.8), we conclude that (2.6) is satisfied.

(b) This is a consequence of the Gowers–Cauchy–Schwarz inequality. Specifically, for every $\omega \in \{0,\dots,\ell-1\}^e$ let $f_\omega = f$ if $\omega = 0^e$ and $f_\omega = 1$ otherwise. By (2.3), we see that $|\mathbb{E}[f]| \le \|f\|_{\square_\ell(\boldsymbol{X}_e)}$. Next, let $\ell_1 \le \ell_2$ be even positive integers. As before, for every $\omega \in \{0,\dots,\ell_2-1\}^e$ let $f_\omega = f$ if $\omega \in \{0,\dots,\ell_1-1\}^e$; otherwise, let $f_\omega = 1$. Then

$$\begin{aligned}\|f\|^{\ell_1^{|e|}}_{\square_{\ell_1}(\boldsymbol{X}_e)} &= \mathbb{E}\Big[\prod_{\omega\in\{0,\dots,\ell_1-1\}^e} f(\mathbf{x}^{(\omega)}_e)\,\Big|\,\mathbf{x}^{(0)}_e,\dots,\mathbf{x}^{(\ell_1-1)}_e \in \boldsymbol{X}_e\Big] \\ &= \mathbb{E}\Big[\prod_{\omega\in\{0,\dots,\ell_2-1\}^e} f_\omega(\mathbf{x}^{(\omega)}_e)\,\Big|\,\mathbf{x}^{(0)}_e,\dots,\mathbf{x}^{(\ell_2-1)}_e \in \boldsymbol{X}_e\Big] \overset{(2.3)}{\le} \|f\|^{\ell_1^{|e|}}_{\square_{\ell_2}(\boldsymbol{X}_e)},\end{aligned}$$

which implies that $\|f\|_{\square_{\ell_1}(\boldsymbol{X}_e)} \le \|f\|_{\square_{\ell_2}(\boldsymbol{X}_e)}$.

(c) Absolute homogeneity is straightforward. The triangle inequality

$$\|f+g\|_{\square_\ell(\boldsymbol{X}_e)} \le \|f\|_{\square_\ell(\boldsymbol{X}_e)} + \|g\|_{\square_\ell(\boldsymbol{X}_e)}$$

follows by raising both sides to the power $\ell^{|e|}$ and then applying (2.3). Finally, let $f \in L_1(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ with $\|f\|_{\square_\ell(\boldsymbol{X}_e)} = 0$ and observe that it suffices to show that $f = 0$ $\boldsymbol{\mu}_e$-almost everywhere. First we note that using (2.3) and arguing precisely as in [13, Corollary B.3] we infer that $\mathbb{E}[f\cdot \mathbf{1}_R] = 0$ for every measurable rectangle $R$ of $\boldsymbol{X}_e$ (that is, every set $R$ of the form

$\prod_{i\in e} A_i$ where $A_i \in \Sigma_i$ for every $i \in e$). We claim that this implies that $\mathbb{E}[f \cdot \mathbf{1}_A] = 0$ for every $A \in \boldsymbol{\Sigma}_e$; this is enough to complete the proof. Indeed, fix $A \in \boldsymbol{\Sigma}_e$ and let $\varepsilon > 0$ be arbitrary. Since $f$ is integrable, there exists $\delta > 0$ such that $\mathbb{E}[\,|f| \cdot \mathbf{1}_C] < \varepsilon$ for every $C \in \boldsymbol{\Sigma}_e$ with $\boldsymbol{\mu}_e(C) < \delta$. Moreover, by Carathéodory's extension theorem, there exists a finite family $R_1, \dots, R_m$ of pairwise disjoint measurable rectangles of $\boldsymbol{X}_e$ such that, setting $B = \bigcup_{k=1}^m R_k$, we have $\boldsymbol{\mu}_e(A \triangle B) < \delta$ (see, e.g., [1, Theorem 11.4]). Hence, $\mathbb{E}[f \cdot \mathbf{1}_B] = 0$ and so

$$|\mathbb{E}[f \cdot \mathbf{1}_A]| = |\mathbb{E}[f \cdot \mathbf{1}_A] - \mathbb{E}[f \cdot \mathbf{1}_B]| \le \mathbb{E}[\,|f| \cdot \mathbf{1}_{A\triangle B}] < \varepsilon.$$

Since $\varepsilon$ was arbitrary, we conclude that $\mathbb{E}[f \cdot \mathbf{1}_A] = 0$.

(d) Set $I = \mathbb{E}[f(x_i, x_j)u(x_i)v(x_j) \mid x_i \in X_i,\, x_j \in X_j]$ and let $\ell'$ denote the conjugate exponent of $\ell$. Notice that $1 < \ell' \le p$. By Hölder's inequality,

$$\begin{aligned} (2.9)\qquad |I| &= \big|\mathbb{E}\big[\mathbb{E}[f(x_i, x_j)v(x_j) \mid x_j \in X_j]u(x_i) \bigm| x_i \in X_i\big]\big| \\ &\le \mathbb{E}\big[\mathbb{E}[f(x_i, x_j)v(x_j) \mid x_j \in X_j]^{\ell} \bigm| x_i \in X_i\big]^{1/\ell} \cdot \|u\|_{L_{\ell'}} \\ &\le I_1^{1/\ell} \cdot \|u\|_{L_p} \end{aligned}$$

where $I_1 = \mathbb{E}\big[\prod_{\omega=0}^{\ell-1} f(x_i, x_j^{(\omega)})v(x_j^{(\omega)}) \mid x_i \in X_i,\, x_j^{(0)}, \dots, x_j^{(\ell-1)} \in X_j\big]$. Moreover,

$$\begin{aligned} (2.10)\quad I_1 &= \mathbb{E}\Bigg[\mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} f(x_i, x_j^{(\omega)}) \Bigm| x_i \in X_i\Big] \\ &\qquad\qquad \times \prod_{\omega=0}^{\ell-1} v(x_j^{(\omega)}) \Bigm| x_j^{(0)}, \dots, x_j^{(\ell-1)} \in X_j\Bigg] \\ &\le \mathbb{E}\Bigg[\mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} f(x_i, x_j^{(\omega)}) \Bigm| x_i \in X_i\Big]^{\ell} \Bigm| x_j^{(0)}, \dots, x_j^{(\ell-1)} \in X_j\Bigg]^{1/\ell} \cdot \|v\|_{L_{\ell'}}^{\ell} \\ &\overset{(2.2)}{=} \|f\|_{\square_\ell(\boldsymbol{X}_e)}^{\ell} \cdot \|v\|_{L_{\ell'}}^{\ell} \le \|f\|_{\square_\ell(\boldsymbol{X}_e)}^{\ell} \cdot \|v\|_{L_p}^{\ell}. \end{aligned}$$

By (2.9) and (2.10), the result follows. ∎

**2.2. The $(\ell, p)$-box norms.** We will need the following $L_p$ versions of the $\ell$-box norms. We remark that closely related norms appear ([4]) in [3]. Recall that by $\mathscr{H} = (n, \langle (X_i, \Sigma_i, \mu_i) : i \in [n]\rangle, \mathcal{H})$ we denote a hypergraph system.

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([4]) Precisely, in [3], for every finite abelian group $Z$, every integer $s \geqslant 2$ and every $f\colon Z \to \mathbb{R}$ the quantity $\|\,|f|^2\|_{U^s(Z)}^{1/2}$ was considered. (Here, $\|\cdot\|_{U^s(Z)}$ stands for the $s$th Gowers uniformity norm for the group $Z$.) It is noted in [3] that this quantity is indeed a norm. The $(\ell, p)$-box norms defined above are the analogues of these norms. in the hypergraph setting.

Definition 2.2. Let $e \subseteq [n]$ be nonempty and let $\ell \geq 2$ be an even integer. Also let $1 \leq p < \infty$ and $f \in L_p(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$. The $(\ell, p)$-*box norm* of $f$ is defined by

$$\|f\|_{\square_{\ell,p}(\boldsymbol{X}_e)} := \big\||f|^p\big\|^{1/p}_{\square_\ell(\boldsymbol{X}_e)}. \tag{2.11}$$

Moreover, for every $f \in L_\infty(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ we set

$$\|f\|_{\square_{\ell,\infty}(\boldsymbol{X}_e)} := \|f\|_{L_\infty}. \tag{2.12}$$

We have the following analogue of Proposition 2.1.

Proposition 2.3. *Let $e \subseteq [n]$ be nonempty and let $\ell \geq 2$ be an even integer.*

(a) *Let $1 \leq p < \infty$. If $f_\omega \in L_p(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ for every $\omega \in \{0, \dots, \ell-1\}^e$, then*

$$\mathbb{E}\Big[\prod_{\omega\in\{0,\dots,\ell-1\}^e} |f_\omega|^p(\mathbf{x}_e^{(\omega)}) \,\Big|\, \mathbf{x}_e^{(0)}, \dots, \mathbf{x}_e^{(\ell-1)} \in \boldsymbol{X}_e\Big] \leq \prod_{\omega\in\{0,\dots,\ell-1\}^e} \|f_\omega\|^p_{\square_{\ell,p}(\boldsymbol{X}_e)}. \tag{2.13}$$

(b) *Let $1 < p, q < \infty$ be conjugate exponents, that is, $1/p + 1/q = 1$. Then for every $f \in L_p(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ and every $g \in L_q(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ we have*

$$\|fg\|_{\square_\ell(\boldsymbol{X}_e)} \leq \|f\|_{\square_{\ell,p}(\boldsymbol{X}_e)} \cdot \|g\|_{\square_{\ell,q}(\boldsymbol{X}_e)}. \tag{2.14}$$

(c) *Assume that $|e| \geq 2$ and let $1 \leq p < \infty$. Then $\|\cdot\|_{\square_{\ell,p}(\boldsymbol{X}_e)}$ is a norm on the vector subspace of $L_p(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ consisting of all $f \in L_p(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ with $\|f\|_{\square_{\ell,p}(\boldsymbol{X}_e)} < \infty$. Moreover, the following hold.*
   (i) *For every $f \in L_p(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ we have $\|f\|_{L_p} \leq \|f\|_{\square_{\ell,p}(\boldsymbol{X}_e)}$.*
   (ii) *For every $1 \leq p_1 \leq p_2 < \infty$ and every $f \in L_{p_2}(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ we have $\|f\|_{\square_{\ell,p_1}(\boldsymbol{X}_e)} \leq \|f\|_{\square_{\ell,p_2}(\boldsymbol{X}_e)}$.*
   (iii) *For every $f \in L_\infty(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ we have $\lim_{p\to\infty} \|f\|_{\square_{\ell,p}(\boldsymbol{X}_e)} = \|f\|_{L_\infty}$.*

*Proof.* Part (a) follows immediately by (2.3). For (b) fix a pair $1 < p, q < \infty$ of conjugate exponents, and let $f \in L_p(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ and $g \in L_q(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$ be arbitrary. We define $F, G\colon \boldsymbol{X}_e^\ell \to \mathbb{R}$ by setting

$$F(\mathbf{x}_e^{(0)}, \dots, \mathbf{x}_e^{(\ell-1)}) = \prod_{\omega\in\{0,\dots,\ell-1\}^e} f(\mathbf{x}_e^{(\omega)}),$$
$$G(\mathbf{x}_e^{(0)}, \dots, \mathbf{x}_e^{(\ell-1)}) = \prod_{\omega\in\{0,\dots,\ell-1\}^e} g(\mathbf{x}_e^{(\omega)}).$$

By Hölder's inequality, we have

$$\|fg\|^{\ell^{|e|}}_{\square_\ell(\boldsymbol{X}_e)} \leq \mathbb{E}[|F \cdot G|] \leq \mathbb{E}[|F|^p]^{1/p} \cdot \mathbb{E}[|G|^q]^{1/q}.$$

Noticing that $\mathbb{E}[|F|^p]^{1/p} = \|f\|^{\ell^{|e|}}_{\square_{\ell,p}(\boldsymbol{X}_e)}$ and $\mathbb{E}[|G|^q]^{1/q} = \|g\|^{\ell^{|e|}}_{\square_{\ell,q}(\boldsymbol{X}_e)}$ we conclude that (2.14) is satisfied.

We proceed to show (c). Arguing as in the proof of the classical Minkowski inequality we see that the $(\ell,p)$-box norm satisfies the triangle inequality. Absolute homogeneity is clear and so, by Proposition 2.1, $\|\cdot\|_{\square_{\ell,p}(\boldsymbol{X}_e)}$ is indeed a norm. Next, (c.i) follows from (2.13) applied for $f_\omega = f$ if $\omega = \{0\}^e$ and $f_\omega = 1$ otherwise. For (c.ii) set $p = p_2/p_1$ and notice that

$$\|f\|^{p_1}_{\square_{\ell,p_1}(\boldsymbol{X}_e)} = \big\||f|^{p_1}\big\|_{\square_{\ell}(\boldsymbol{X}_e)} \overset{(2.14)}{\le} \big\||f|^{p_1}\big\|_{\square_{\ell,p}(\boldsymbol{X}_e)} = \|f\|^{p_1}_{\square_{\ell,p_2}(\boldsymbol{X}_e)}.$$

Finally, let $f \in L_\infty(\boldsymbol{X}_e, \boldsymbol{\Sigma}_e, \boldsymbol{\mu}_e)$. By (c.i), we have

$$\|f\|_{L_p} \le \|f\|_{\square_{\ell,p}(\boldsymbol{X}_e)} \le \|f\|_{L_\infty}.$$

Since $\lim_{p\to\infty}\|f\|_{L_p} = \|f\|_{L_\infty}$, we obtain $\lim_{p\to\infty}\|f\|_{\square_{\ell,p}(\boldsymbol{X}_e)} = \|f\|_{L_\infty}$ and the proof is complete. ∎

**3. A generalized von Neumann theorem for $L_p$ graphons.** Let $(X,\Sigma,\mu)$ be a probability space and recall that a *graphon* ([5]) is an integrable random variable $W\colon X\times X\to\mathbb{R}$ which is symmetric, that is, $W(x,y) = W(y,x)$ for every $x,y\in X$. If, in addition, $W$ belongs to $L_p$ for some $p>1$, then $W$ is said to be an *$L_p$ graphon* (see [2]).

Now let $n$ be a positive integer and let $\mathcal{G}$ be a nonempty graph on $[n]$. Recall that the *maximum degree* of $\mathcal{G}$ is the number $\Delta(\mathcal{G}) := \max\{|\{e\in\mathcal{G} : i\in e\}| : i\in[n]\}$. Given two $L_p$ graphons $W$ and $U$, a natural problem (which is of importance in the context of graph limits—see, e.g., [17]) is to estimate the quantity

$$\Big|\mathbb{E}\Big[\prod_{\{i,j\}\in\mathcal{G}} W(x_i,x_j)\ \Big|\ x_1,\dots,x_n\in X\Big] - \mathbb{E}\Big[\prod_{\{i,j\}\in\mathcal{G}} U(x_i,x_j)\ \Big|\ x_1,\dots,x_n\in X\Big]\Big|.$$

Note that this problem essentially boils down to analysing the boundedness of the multilinear operator

$$\Lambda_{\mathcal{G}}((f_e)_{e\in\mathcal{G}}) := \mathbb{E}\Big[\prod_{e=\{i,j\}\in\mathcal{G}} f_e(x_i,x_j)\ \Big|\ x_1,\dots,x_n\in X\Big]$$

where the functions $(f_e)_{e\in\mathcal{G}}$ belong to $L_p$. Not surprisingly, the behavior of this operator depends heavily on the range of $p$ one is working with. Undoubtedly, the simplest case is $p=\infty$; indeed, using Fubini's theorem, it is not hard to see that for bounded functions the operator $\Lambda_{\mathcal{G}}$ is controlled

([5]) We remark that in several places in the literature, graphons are required to be $[0,1]$-valued, and the term *kernel* is used for (not necessarily bounded) integrable, symmetric random variables.

by the cut norm (⁶). The next critical range for the behavior of $\Lambda_{\mathcal{G}}$ is $\Delta(\mathcal{G}) \le p < \infty$. In this case, Hölder's inequality implies that $\Lambda_{\mathcal{G}}$ is bounded in $L_p$. This was used in [2, Theorem 2.20] to show that $\Lambda_{\mathcal{G}}$ is also controlled by the cut norm when $p > \Delta(\mathcal{G})$.

Unfortunately, in the regime $1 < p < \Delta(\mathcal{G})$ the operator $\Lambda_{\mathcal{G}}$ is not bounded but merely densely defined in $L_p$. Nevertheless, experience from arithmetic combinatorics (see, e.g., [9, 20]) and harmonic analysis (see, e.g., [16]) indicates that one can still obtain nontrivial information provided that one replaces the $L_p$-norm with a suitable box norm. It turns out that this intuition is correct as is shown in the following theorem.

Theorem 3.1 (Generalized von Neumann theorem for $L_p$ graphons). *Let $\Delta$ be a positive integer, $C \ge 1$ and $1 < p \le \infty$. If $p = \infty$ or $\Delta = 1$, then we set $\ell = 2$; otherwise, let*

$$\ell = \min\{2n : n \in \mathbb{N} \text{ and } 2n \ge p^{(\Delta-1)^{-1}}(p^{(\Delta-1)^{-1}} - 1)^{-1}\}. \tag{3.1}$$

*Also let $\mathscr{G} = (n, \langle (X_i, \Sigma_i, \mu_i) : i \in [n]\rangle, \mathcal{G})$ be a 2-uniform hypergraph system with $\Delta(\mathcal{G}) = \Delta$. For every $e \in \mathcal{G}$ let $f_e \in L_p(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$ be such that*

$$\|\mathbf{f}_e\|_{\square_{\ell,p}(\boldsymbol{X}_e)} \le 1 \tag{3.2}$$

*where $\mathbf{f}_e$ is as in* (1.5). *Assume that for every* (*possibly empty*) $\mathcal{G}' \subseteq \mathcal{G}$ *we have*

$$\Big\| \prod_{e\in\mathcal{G}'} f_e \Big\|_{L_p} \le C. \tag{3.3}$$

(*Here, we follow the convention that the product of an empty family of functions is equal to the constant function* 1.) *Then*

$$\Big| \mathbb{E}\Big[\prod_{e\in\mathcal{G}} f_e\Big]\Big| \le C \cdot \min_{e\in\mathcal{G}} \|\mathbf{f}_e\|_{\square_\ell(\boldsymbol{X}_e)}. \tag{3.4}$$

Observe that (3.2) is an integrability condition; as already noted, this condition is necessary if $p < \Delta$. On the other hand, condition (3.3) is the analogue of the "linear forms condition" appearing in several versions of the generalized von Neumann theorem (see, e.g., [12, Proposition 5.3] and [19, Theorem 3.8]).

*Proof of Theorem 3.1.* Let $e \in \mathcal{G}$ be arbitrary, and set

$$I := \mathbb{E}\Big[f_e \prod_{e'\in\mathcal{G}\setminus\{e\}} f_{e'}\Big]$$

Clearly, it suffices to show that $|I| \le C \cdot \|\mathbf{f}_e\|_{\square_\ell(\boldsymbol{X}_e)}$.

---

(⁶) We recall the definition of the cut norm in (4.2). We note, however, that we will not use the cut norm in this section.

To this end, we first observe that if $\varDelta = 1$, then the result is straightforward. Indeed, in this case we have $\ell = 2$ and the edges of $\mathcal{G}$ are pairwise disjoint. Hence, by Propositions 2.1(b) and 2.3(c.ii), we see that

$$|I| = |\mathbb{E}[\mathbf{f}_e]| \cdot \prod_{e' \in \mathcal{G}\setminus\{e\}} |\mathbb{E}[\mathbf{f}_{e'}]| \leq \|\mathbf{f}_e\|_{\square_2(\boldsymbol{X}_e)} \cdot \prod_{e' \in \mathcal{G}\setminus\{e\}} \|\mathbf{f}_{e'}\|_{\square_{2,p}(\boldsymbol{X}_{e'})} \overset{(3.2)}{\leq} C \cdot \|\mathbf{f}_e\|_{\square_2(\boldsymbol{X}_e)}.$$

Therefore, in what follows we will assume that $\varDelta \geq 2$. To simplify the exposition we will also assume that $p \neq \infty$. (The proof for $p = \infty$ is similar.) Write $e = \{i, j\}$, and set $\mathcal{G}(i) = \{e' \in \mathcal{G} \setminus \{e\} : i \in e'\}$ and $\mathcal{G}^*(i) = \{e' \in \mathcal{G} \setminus \{e\} : i \notin e'\}$; notice that $\mathcal{G} \setminus \{e\} = \mathcal{G}(i) \cup \mathcal{G}^*(i)$. Let $\ell'$ be the conjugate exponent of $\ell$ and observe that, by (3.1), we have $\ell \geq q'$ where $q'$ is the conjugate exponent of $p^{(\varDelta-1)^{-1}}$. Hence,

$$1 < \ell' \leq p^{(\varDelta-1)^{-1}} \leq p. \tag{3.5}$$

We set

$$I_{e,\mathcal{G}(i)} = \mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} \mathbf{f}_e(x_i^{(\omega)}, x_j) \prod_{e' \in \mathcal{G}(i)} \mathbf{f}_{e'}(x_i^{(\omega)}, x_{e'\setminus\{i\}})\Big], \tag{3.6}$$

$$I_{\mathcal{G}(i)} = \mathbb{E}\Big[\prod_{e' \in \mathcal{G}(i)} \prod_{\omega=0}^{\ell-1} |\mathbf{f}_{e'}|^{\ell'}(x_i^{(\omega)}, x_{e'\setminus\{i\}})\Big], \tag{3.7}$$

where both expectations are over $x_i^{(0)}, \ldots, x_i^{(\ell-1)} \in X_i$ and $\mathbf{x}_{[n]\setminus\{i\}} \in \boldsymbol{X}_{[n]\setminus\{i\}}$.

Claim 3.2. *We have* $|I| \leq C \cdot I_{e,\mathcal{G}(i)}^{1/\ell}$.

*Proof of Claim 3.2.* Since $i \notin e'$ for every $e' \in \mathcal{G}^*(i)$, we have

$$I = \mathbb{E}\bigg[\mathbb{E}\Big[\mathbf{f}_e(x_i, x_j) \prod_{e' \in \mathcal{G}(i)} \mathbf{f}_{e'}(x_i, x_{e'\setminus\{i\}}) \,\Big|\, x_i \in X_i\Big] \cdot \prod_{e' \in \mathcal{G}^*(i)} \mathbf{f}_{e'}(\mathbf{x}_{e'})\bigg].$$

By Hölder's inequality, (3.3), (3.5) and (3.6), we obtain

$$\begin{aligned} |I| &\leq \mathbb{E}\bigg[\mathbb{E}\Big[\mathbf{f}_e(x_i, x_j) \prod_{e' \in \mathcal{G}(i)} \mathbf{f}_{e'}(x_i, x_{e'\setminus\{i\}}) \,\Big|\, x_i \in X_i\Big]^{\ell}\bigg]^{1/\ell} \cdot \Big\| \prod_{e' \in \mathcal{G}^*(i)} f_{e'} \Big\|_{L_{\ell'}} \\ &\leq I_{e,\mathcal{G}(i)}^{1/\ell} \cdot \Big\| \prod_{e' \in \mathcal{G}^*(i)} f_{e'} \Big\|_{L_p} \leq C \cdot I_{e,\mathcal{G}(i)}^{1/\ell} \end{aligned}$$

as desired. ∎

Claim 3.3. *We have* $I_{e,\mathcal{G}(i)} \leq \|\mathbf{f}_e\|_{\square_\ell(\boldsymbol{X}_e)}^{\ell} \cdot I_{\mathcal{G}(i)}^{1/\ell'}$.

*Proof of Claim 3.3.* Note that $j \notin e'$ for every $e' \in \mathcal{G}(i)$, and so

$$I_{e,\mathcal{G}(i)} = \mathbb{E}\Big[\mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} \mathbf{f}_e(x_i^{(\omega)}, x_j) \;\Big|\; x_j \in X_j\Big] \cdot \prod_{e' \in \mathcal{G}(i)} \prod_{\omega=0}^{\ell-1} \mathbf{f}_{e'}(x_i^{(\omega)}, x_{e' \setminus \{i\}})\Big].$$

Thanks to this observation the claim follows by Hölder's inequality and arguing precisely as in the proof of Claim 3.2. ∎

Claim 3.4. *We have $I_{\mathcal{G}(i)} \le 1$.*

*Proof of Claim 3.4.* We may assume, of course, that $\mathcal{G}(i)$ is nonempty. We set $m = |\mathcal{G}(i)|$ and observe that $1 \le m \le \Delta - 1$. Therefore, by (3.5),

$$1 < (\ell')^r \le (\ell')^{\Delta-1} \le p \tag{3.8}$$

for every $r \in [m]$. Write $\mathcal{G}(i) = \{e'_1, \dots, e'_m\}$ and for every $r \in [m]$ let $j_r \in [n]$ be such that $e'_r = \{i, j_r\}$. For every $d \in [m]$ set

$$Q_d = \mathbb{E}\Big[\prod_{r=d}^{m} \prod_{\omega=0}^{\ell-1} |\mathbf{f}_{e'_r}|^{(\ell')^d}(x_i^{(\omega)}, x_{j_r})\Big] \tag{3.9}$$

and note that

$$Q_1 = I_{\mathcal{G}(i)} \quad \text{and} \quad Q_m = \mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} |\mathbf{f}_{e'_m}|^{(\ell')^m}(x_i^{(\omega)}, x_{j_m})\Big]. \tag{3.10}$$

(Here, the expectation is over all $x_i^{(0)}, \dots, x_i^{(\ell-1)} \in X_i$ and $\mathbf{x}_{[n]\setminus\{i\}} \in \boldsymbol{X}_{[n]\setminus\{i\}}$.) Now observe that it is enough to show that for every $d \in [m-1]$ we have

$$Q_d \le Q_{d+1}^{1/\ell'}. \tag{3.11}$$

Indeed, by (3.11), we see that $Q_1 \le Q_m^{1/(\ell')^{m-1}}$. Hence, by (3.10), the monotonicity of the $L_p$ norms and Proposition 2.3(a), we obtain

$$\begin{aligned} I_{\mathcal{G}(i)} \;&\le\; \mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} |\mathbf{f}_{e'_m}|^{(\ell')^m}(x_i^{(\omega)}, x_{j_m})\Big]^{\ell'/(\ell')^m} \\ &\overset{(3.8)}{\le} \mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} |\mathbf{f}_{e'_m}|^{p}(x_i^{(\omega)}, x_{j_m})\Big]^{\ell'/p} \le \|\mathbf{f}_{e'_m}\|_{\square_{\ell,p}(\boldsymbol{X}_{e'_m})}^{\ell\ell'} \overset{(3.2)}{\le} 1. \end{aligned} \tag{3.12}$$

It remains to show (3.11). Fix $d \in [m-1]$ and notice that $j_d \notin e'_r$ for every $r \in \{d+1, \dots, m\}$. Thus,

$$Q_d = \mathbb{E}\Big[\mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} |\mathbf{f}_{e'_d}|^{(\ell')^d}(x_i^{(\omega)}, x_{j_d}) \;\Big|\; x_{j_d} \in X_{j_d}\Big] \cdot \prod_{r=d+1}^{m} \prod_{\omega=0}^{\ell-1} |\mathbf{f}_{e'_r}|^{(\ell')^d}(x_i^{(\omega)}, x_{j_r})\Big].$$

By Hölder's inequality and arguing as in the proof of (3.12), we see that

$$Q_d \leq \mathbb{E}\Big[ \prod_{\omega \in \{0,\dots,\ell-1\}^{e'_d}} |\mathbf{f}_{e'_d}|^{(\ell')^d}(\mathbf{x}^{(\omega)}_{e'_d})\Big]^{1/\ell} \cdot Q_{d+1}^{1/\ell'} \leq \|\mathbf{f}_{e'_d}\|^{\ell(\ell')^d}_{\square_{\ell,p}(\boldsymbol{X}_{e'_d})} \cdot Q_{d+1}^{1/\ell'}$$

as desired. ∎

By Claims 3.2–3.4, we conclude that $|I| \leq C \cdot \|\mathbf{f}_e\|_{\square_\ell(\boldsymbol{X}_e)}$, and so the entire proof of Theorem 3.1 is complete. ∎

We close this section with the following counting lemma for $L_p$ graphons. It follows readily by Theorem 3.1 and a telescopic argument.

Corollary 3.5. *Let $\Delta$, $C$, $p$, $\ell$ and $\mathscr{G}$ be as in Theorem 3.1. For every $e \in \mathcal{G}$ let $f_e, g_e \in L_p(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$ be such that $\|\mathbf{f}_e\|_{\square_{\ell,p}(\boldsymbol{X}_e)}, \|\mathbf{g}_e\|_{\square_{\ell,p}(\boldsymbol{X}_e)} \leq 1$ where $\mathbf{f}_e$ and $\mathbf{g}_e$ are as in (1.5) for $f_e$ and $g_e$ respectively. Assume that for every $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{G}$ with $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$ we have $\| \prod_{e\in\mathcal{G}_1} f_e \prod_{e\in\mathcal{G}_2} g_e\|_{L_p} \leq C$. Then*

$$\Big|\mathbb{E}\Big[\prod_{e\in\mathcal{G}} f_e\Big] - \mathbb{E}\Big[\prod_{e\in\mathcal{G}} g_e\Big]\Big| \leq C \cdot \sum_{e\in\mathcal{G}} \|\mathbf{f}_e - \mathbf{g}_e\|_{\square_\ell(\boldsymbol{X}_e)}. \tag{3.13}$$

## 4. Pseudorandom families

**4.1.** We begin by introducing some notation. Let $n, r$ be integers with $n \geq r \geq 2$ and let $\mathscr{H} = (n, \langle (X_i, \Sigma_i, \mu_i) : i \in [n]\rangle, \mathcal{H})$ be an $r$-uniform hypergraph system. Given $e \in \mathcal{H}$ let $\partial e = \{e' \subseteq e : |e'| = |e| - 1\}$ and set

$$\mathcal{S}_{\partial e} := \Big\{ \bigcap_{e'\in\partial e} A_{e'} : A_{e'} \in \mathcal{B}_{e'} \text{ for every } e' \in \partial e\Big\} \subseteq \mathcal{B}_e. \tag{4.1}$$

Also recall that for every $f \in L_1(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$ the *cut norm* of $f$ is defined by

$$\|f\|_{\mathcal{S}_{\partial e}} = \sup\Big\{\Big|\int_A f\, d\boldsymbol{\mu}\Big| : A \in \mathcal{S}_{\partial e}\Big\}. \tag{4.2}$$

The cut norm is a standard tool in extremal combinatorics (see [8, 17, 18]). It is weaker than the box norm, but for bounded functions these two norms are essentially equivalent (see [10, Theorem 4.1]).

The following class of sparse weighted uniform hypergraphs was introduced in [7, Definition 6.1].

Definition 4.1. Let $n, r \in \mathbb{N}$ with $n \geq r \geq 2$, and let $C \geq 1$ and $0 < \eta < 1$. Also let $1 < p \leq \infty$ and let $q$ denote the conjugate exponent of $p$. Finally, let $\mathscr{H} = (n, \langle (X_i, \Sigma_i, \mu_i) : i \in [n]\rangle, \mathcal{H})$ be an $r$-uniform hypergraph system. For every $e \in \mathcal{H}$ let $\nu_e \in L_1(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$ be a nonnegative random variable. We say that the family $\langle \nu_e : e \in \mathcal{H}\rangle$ is $(C, \eta, p)$-*pseudorandom* if the following hold:

(C1) For every nonempty $\mathcal{G} \subseteq \mathcal{H}$ we have $\mathbb{E}[\prod_{e\in\mathcal{G}} \nu_e] \geq 1-\eta$.

(C2) For every $e \in \mathcal{H}$ there exists $\psi_e \in L_p(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$ with $\|\psi_e\|_{L_p} \leq C$ and with the following properties:

(a) $\|\nu_e - \psi_e\|_{\mathcal{S}_{\partial e}} \leq \eta$.

(b) For every $e' \in \mathcal{H} \setminus \{e\}$ and $\omega \in \{0,1\}$ let $g_{e'}^{(\omega)} \in L_1(\boldsymbol{X}, \mathcal{B}_{e'}, \boldsymbol{\mu})$ be such that either $0 \leq g_{e'}^{(\omega)} \leq \nu_{e'}$ or $0 \leq g_{e'}^{(\omega)} \leq 1$. Let $\boldsymbol{\nu}_e$ and $\boldsymbol{\psi}_e$ be as in (1.5) for $\nu_e$ and $\psi_e$ respectively. Then

$$\left|\mathbb{E}\Big[(\boldsymbol{\nu}_e - \boldsymbol{\psi}_e)(\mathbf{x}_e) \prod_{\omega\in\{0,1\}} \mathbb{E}\Big[\prod_{e'\in\mathcal{H}\setminus\{e\}} g_{e'}^{(\omega)}(\mathbf{x}_e, \mathbf{x}_{[n]\setminus e}) \;\Big|\; \mathbf{x}_{[n]\setminus e} \in \boldsymbol{X}_{[n]\setminus e}\Big] \;\Big|\; \mathbf{x}_e \in \boldsymbol{X}_e\Big]\right| \leq \eta.$$

(C3) Let $e \in \mathcal{H}$, let $\mathcal{G} \subseteq \mathcal{H} \setminus \{e\}$ be nonempty, and define $\boldsymbol{\nu}_{e,\mathcal{G}} \colon \boldsymbol{X}_e \to \mathbb{R}$ by $\boldsymbol{\nu}_{e,\mathcal{G}}(\mathbf{x}_e) = \mathbb{E}[\prod_{e'\in\mathcal{G}} \nu_{e'}(\mathbf{x}_e, \mathbf{x}_{[n]\setminus e}) \mid \mathbf{x}_{[n]\setminus e} \in \boldsymbol{X}_{[n]\setminus e}]$. Then, setting

$$\ell := \min\{2n : n \in \mathbb{N} \text{ and } 2n \geq 2q + (1-1/C) + 1/p\} \tag{4.3}$$

(where $1/p = 0$ if $p = \infty$), we have $\mathbb{E}[\boldsymbol{\nu}_{e,\mathcal{G}}^{\ell}] \leq C + \eta$.

We note that closely related definitions were introduced in [4, 19] and we refer the reader to [7, Section 6] for a detailed discussion of conditions (C1)–(C3) and their relation to the notions of pseudorandomness appearing in [4, 19]. As mentioned in the introduction, the most important property of pseudorandom families is that they satisfy relative versions of the counting and removal lemmas (see, in particular, [7, Theorems 2.2 and 7.1]).

**4.2. Motivation.** In the second part of this paper our goal is to give examples of pseudorandom families. We have already pointed out that these examples can be seen as deviations (in an $L_p$-sense) of weighted hypergraphs which satisfy the linear forms condition. This fact is not accidental. Indeed, by [7, Theorem 2.1], under quite general hypotheses one can decompose a nonnegative random variable $\nu_e$ as $s_e + u_e$ where $s_e$ belongs to $L_p$ and $u_e$ has negligible cut norm. Unfortunately, this information is not strong enough to guarantee that the weighted hypergraph $\langle \nu_e : e \in \mathcal{H}\rangle$ satisfies relative versions of the counting and removal lemmas. However, as we shall see, this problem can be bypassed by imposing slightly stronger integrability conditions on each $s_e$ and assuming that the random variables $\langle u_e : e \in \mathcal{H}\rangle$ are very mildly correlated.

**4.3. The first main result.** The following theorem is our first result in this section. Its proof is given in Section 5.

Theorem 4.2. *Let $n \in \mathbb{N}$ with $n \geq 3$, $C \geq 1$ and $1 < p \leq \infty$, and let $\ell$ be as in (4.3). Also let $0 < \eta \leq (4C)^{-n\ell^n}$ and let $\mathscr{H} = (n, \langle (X_i, \Sigma_i, \mu_i) :$*

*$i \in [n]\rangle, \mathcal{H})$ be a hypergraph system with $\mathcal{H} = K_n^{(n-1)} = \binom{[n]}{n-1}$. (In particular, $\mathscr{H}$ is $(n-1)$-uniform.) For every $e \in \mathcal{H}$ let $\lambda_e \in L_1(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$ and $\varphi_e \in L_p(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$ be nonnegative random variables, and let $\boldsymbol{\lambda}_e$ and $\boldsymbol{\varphi}_e$ be as in (1.5) for $\lambda_e$ and $\varphi_e$ respectively. Assume that:*

(I) *We have*

$$1-\eta \le \mathbb{E}\Big[\prod_{e\in\mathcal{H}} \prod_{\omega\in\{0,\dots,\ell-1\}^e} \boldsymbol{\lambda}_e^{n_{e,\omega}}(\mathbf{x}_e^{(\omega)}) \;\Big|\; \mathbf{x}^{(0)}, \dots, \mathbf{x}^{(\ell-1)} \in \boldsymbol{X}\Big] \le 1+\eta \tag{4.4}$$

*for any choice of $n_{e,\omega} \in \{0,1\}$.*

(II) *For every $e \in \mathcal{H}$ we have $\|\boldsymbol{\varphi}_e\|_{\square_{\ell,p}(\boldsymbol{X}_e)} \le C$.*

*Then the family $\langle \lambda_e + \varphi_e : e \in \mathcal{H}\rangle$ is $(C', \eta', p)$-pseudorandom where $C' = (4C)^{n\ell}$ and $\eta' = (4C)^{n\ell}\eta^{1/\ell^{n-1}}$.*

We remark that condition (I) in Theorem 4.2 is a modification of the *linear forms condition* introduced in [19, Definition 2.8]; it expresses the fact that the weighted hypergraph $\langle \lambda_e : e \in \mathcal{H}\rangle$ contains roughly the expected number of copies of the $\ell$-blow-up of $\mathcal{H}$ and its subhypergraphs. On the other hand, (II) is an integrability condition; in particular, using Hölder's inequality, it is easy to see that $\|\boldsymbol{\varphi}_e\|_{\square_{\ell,p}(\boldsymbol{X}_e)} \le C$ provided that $\|\boldsymbol{\varphi}_e\|_{L_q} \le C$ for some $q$ sufficiently large. Thus we see that the family $\langle \nu_e + \varphi_e : e \in \mathcal{H}\rangle$ is a perturbation of a system of measures which appears in [19], the main point being that only integrability conditions are imposed on each "noise" $\varphi_e$.

We proceed to give some concrete examples of weighted graphs and hypergraphs which can be obtained by using Theorem 4.2. They are the simplest type of examples for which the results obtained in [7] can be applied, yet they are out of the scope of the counting lemmas developed by Tao [19] and by Conlon Fox and Zhao [4].

**4.3.1.** *Example: weighted graphs.* Let $V_1, V_2, V_3$ be three pairwise disjoint nonempty sets; we view $V_1, V_2$ and $V_3$ as discrete probability spaces equipped with their uniform probability measures. Also let $\mathcal{H}$ denote the graph $K_3 = \{\{1,2\},\{2,3\},\{1,3\}\}$ (that is, $\mathcal{H}$ is the complete graph on three vertices). For every $e = \{i < j\} \in \mathcal{H}$ let $\boldsymbol{\varphi}_e \colon V_i \times V_j \to \mathbb{R}^+$ be *any* function satisfying ([7])

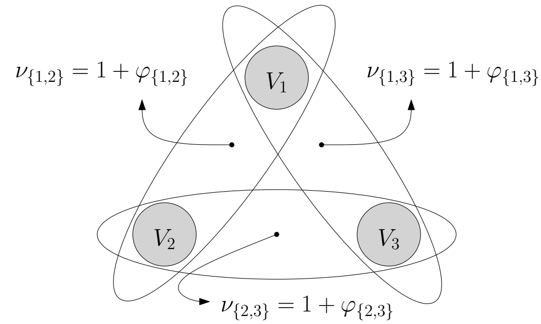
$$\|\boldsymbol{\varphi}_e\|_{L_{64}} = \mathbb{E}[\boldsymbol{\varphi}_e(x,y)^{64} \mid (x,y) \in V_i \times V_j]^{1/64} \le 1 \tag{4.5}$$

and define

$$\boldsymbol{\lambda}_e = 1 \quad \text{and} \quad \boldsymbol{\nu}_e = 1 + \boldsymbol{\varphi}_e. \tag{4.6}$$

---

([7]) We do not know whether the estimate (4.5) is optimal; in fact, it is likely that the exponent 64 can be improved. We point out, however, that an integrability condition like (4.5) is necessary in order to have a sparse version of the counting lemma.




We will apply Theorem 4.2 for $n = 3$, $C = 1$ and $p = 4$ in order to show that the weighted graph $\langle \nu_e : e \in \mathcal{H} \rangle$ is a $(4^{12}, \eta, 4)$-pseudorandom family for every $\eta > 0$. (Here, $\nu_e$ is as in (1.5) for $\boldsymbol{\nu}_e$.) To this end notice first that, by (4.3), we have $\ell = 4$. On the other hand, it is clear that (4.4)—that is, condition (I) in Theorem 4.2—is satisfied for every $\eta > 0$. Finally, for (II) observe that, by Hölder's inequality ([8]), for every $e = \{i < j\} \in \mathcal{H}$ we have

$$\|\boldsymbol{\varphi}_e\|_{\square_{4,4}(V_i \times V_j)} \leq \|\boldsymbol{\varphi}_e^4\|_{L_{16}} = \|\boldsymbol{\varphi}_e\|_{L_{64}}^4 \overset{(4.5)}{\leq} 1.$$

Thus, condition (II) is also satisfied, which implies that $\langle \nu_e : e \in \mathcal{H} \rangle$ is indeed a $(4^{12}, \eta, 4)$-pseudorandom family for every $\eta > 0$.

Our next goal is to show that the weighted graphs defined above cannot be realized as dense subgraphs of weighted graphs which satisfy the aforementioned "linear forms condition", a pseudorandomness condition which forms the basis of the sparse counting lemmas developed in [4, 19]. The framework in [4, 19] is asymptotic; consequently, for every integer $N \geq 1$ we select, recursively, a positive integer $m_N$, three pairwise disjoint nonempty sets $V_1^N, V_2^N, V_3^N$ and families $\langle A_{1,l}^N : l \in [N] \rangle$, $\langle A_{2,l}^N : l \in [N] \rangle$, $\langle A_{3,l}^N : l \in [N] \rangle$ of nonempty sets such that:

($\mathcal{P}1$) $m_{N+1}^{64} \geq 2^{N+2} m_N$; moreover, $m_1 = 2$.
($\mathcal{P}2$) For every $i \in \{1,2,3\}$ and every $l \in [N]$ we have $A_{i,l}^N \subseteq V_i^N$.
($\mathcal{P}3$) For every $e = \{i < j\} \in \mathcal{H}$ and every $l \in [N]$ we have

$$|A_{i,l}^N \times A_{j,l}^N| = \frac{1}{m_l^{64^2}} |V_i^N \times V_j^N|.$$

In particular, $\|\mathbf{1}_{A_{i,l}^N \times A_{j,l}^N}\|_{L_{64}} = 1/m_l^{64}$.

Given the above data, for every integer $N \geq 1$ and every $e = \{i < j\} \in \mathcal{H}$

([8]) Note that here, as in the proof of Proposition 2.1(a), we use the generalized form of Hölder's inequality.

we define $\boldsymbol{\varphi}_e^N, \boldsymbol{\lambda}_e^N, \boldsymbol{\nu}_e^N \colon V_i^N \times V_j^N \to \mathbb{R}^+$ by setting

$$\boldsymbol{\varphi}_e^N = \sum_{l=1}^N \frac{1}{2^l} \frac{\mathbf{1}_{A_{i,l}^N \times A_{j,l}^N}}{\|\mathbf{1}_{A_{i,l}^N \times A_{j,l}^N}\|_{L_{64}}} = \sum_{l=1}^N \frac{m_l^{64}}{2^l} \mathbf{1}_{A_{i,l}^N \times A_{j,l}^N}, \tag{4.7}$$
$$\boldsymbol{\lambda}_e^N = 1 \quad \text{and} \quad \boldsymbol{\nu}_e^N = 1 + \boldsymbol{\varphi}_e^N.$$

Since $\|\boldsymbol{\varphi}_e^N\|_{L_{64}} \le 1$, by the previous discussion we see that the weighted graph $\langle \nu_e^N : e \in \mathcal{H}\rangle$ is a $(4^{12}, \eta, 4)$-pseudorandom family for every $\eta > 0$ and every $N \ge 1$.

Now assume, towards a contradiction, that there exist a constant $M > 0$ and a sequence $\langle \kappa_e^N : e \in \mathcal{H}\rangle$ of weighted graphs ([9]) which satisfy the "linear forms condition" (see [4, Definition 2.8] or [19, Definition 2.8]) such that $\nu_{e_0}^N \le M \cdot \kappa_{e_0}^N$ for some $e_0 = \{i < j\} \in \mathcal{H}$ and infinitely many $N$. Setting $l_0 := \min\{l \ge 1 : M \le m_l\}$, we thus have ([10])

$$0 \le \boldsymbol{\varphi}_{e_0}^N \le m_{l_0} \cdot \boldsymbol{\kappa}_{e_0}^N \tag{4.8}$$

for infinitely many $N$. Next, set

$$\sigma = \frac{1}{2} \frac{m_{l_0+1}^{64}/2^{l_0+1} - m_{l_0}}{m_{l_0+1}^{64^2}}$$

and notice that, by $(\mathcal{P}1)$ above, we have $\sigma > 0$. By (4.8) and [4, Lemma 2.15 and Theorem 2.16], for every integer $N \ge 1$ there exists $\boldsymbol{h}_{e_0}^N \colon V_i^N \times V_j^N \to \mathbb{R}$ with $0 \le \boldsymbol{h}_{e_0}^N \le m_{l_0}$ and such that

$$\sup\{|\mathbb{E}[(\boldsymbol{\varphi}_{e_0}^N - \boldsymbol{h}_{e_0}^N)\, \mathbf{1}_{A\times B}]| : A \subseteq V_i^N,\ B \subseteq V_j^N\} \le \sigma \tag{4.9}$$

for infinitely many $N$. On the other hand, by $(\mathcal{P}2)$, $(\mathcal{P}3)$, (4.7) and the fact that $0 \le \boldsymbol{h}_{e_0}^N \le m_{l_0}$, for every $N \ge l_0 + 1$ we have

$$\frac{m_{l_0+1}^{64}/2^{l_0+1} - m_{l_0}}{m_{l_0+1}^{64^2}} \le |\mathbb{E}[(\boldsymbol{\varphi}_{e_0}^N - \boldsymbol{h}_{e_0}^N)\mathbf{1}_{A_{i,l_0+1}^N \times A_{j,l_0+1}^N}]|,$$

which clearly leads to a contradiction by (4.9) and the choice of $\sigma$.

**4.3.2.** *Example: weighted hypergraphs.* It is the hypergraph analogue of the previous example. Specifically, let $n \ge 3$ be an integer, and let $V_1, \dots, V_n$ be pairwise disjoint nonempty sets which we view as discrete probability spaces equipped with their uniform probability measures. Also let $\mathcal{H}$ denote the graph $K_n^{(n-1)}$ (thus, $\mathcal{H}$ is the complete $(n-1)$-uniform hypergraph on $n$

([9]) Specifically, for every $e = \{i < j\} \in \mathcal{H}$ and every $N \ge 1$ we have $\boldsymbol{\kappa}_e^N : V_i^N \times V_j^N \to \mathbb{R}^+$, and $\kappa_e^N$ is as in (1.5) for $\boldsymbol{\kappa}_e^N$.

([10]) Note that here we do not use the fact that $\boldsymbol{\lambda}_e^N = 1$ for every $e \in \mathcal{H}$ and every integer $N \ge 1$. Actually, the same argument can be applied if $\langle \boldsymbol{\lambda}_e^N : e \in \mathcal{H}\rangle$ is *any* sequence of weighted graphs which satisfy condition (I) in Theorem 4.2 for $\ell = 4$ and $\eta > 0$ sufficiently small.

vertices). For every $e \in \mathcal{H}$ set $\mathbf{V}_e = \prod_{i\in e} V_i$, let $\boldsymbol{\varphi}_e \colon \mathbf{V}_e \to \mathbb{R}^+$ be a function satisfying

$$\|\boldsymbol{\varphi}_e\|_{L_{4^n}} = \mathbb{E}\big[\boldsymbol{\varphi}_e(\mathbf{x}_e)^{4^n} \;\big|\; \mathbf{x}_e \in \mathbf{V}_e\big]^{1/4^n} \le 1 \tag{4.10}$$

and define

$$\boldsymbol{\lambda}_e = 1 \quad \text{and} \quad \boldsymbol{\nu}_e = 1 + \boldsymbol{\varphi}_e. \tag{4.11}$$

By Theorem 4.2 and Hölder's inequality, arguing precisely as in the previous example we see that the weighted hypergraph $\langle \nu_e : e \in \mathcal{H}\rangle$ is a $(4^{4n}, \eta, 4)$-pseudorandom family for every $\eta > 0$. Moreover, a straightforward modification of the argument in the previous example shows that there exist weighted hypergraphs of this form which cannot be realized as dense subhypergraphs of weighted hypergraphs satisfying the "linear forms condition".

**4.4. The second main result.** Our second main result provides a somewhat different type of examples of pseudorandom families.

Theorem 4.3. *Let $n \in \mathbb{N}$ with $n \ge 3$, $C \ge 1$ and $1 < p \le \infty$, and let $\ell$ be as in* (4.3). *Also let $0 < \eta \le 1/(n\ell)$ and let $\mathscr{H} = (n, \langle (X_i, \Sigma_i, \mu_i) : i \in [n]\rangle, \mathcal{H})$ be a hypergraph system with $\mathcal{H} = K_n^{(n-1)} = \binom{[n]}{n-1}$. (Again observe that $\mathscr{H}$ is $(n-1)$-uniform.) For every $e \in \mathcal{H}$ let $\nu_e \in L_1(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$ and $\psi_e \in L_p(\boldsymbol{X}, \mathcal{B}_e, \boldsymbol{\mu})$ be nonnegative random variables, and let $\boldsymbol{\nu}_e$ and $\boldsymbol{\psi}_e$ be as in* (1.5) *for $\nu_e$ and $\psi_e$ respectively. Assume that:*

(I) *We have*

$$1-\eta \le \mathbb{E}\Big[\prod_{e\in\mathcal{H}} \prod_{\omega\in\{0,\dots,\ell-1\}^e} \boldsymbol{\psi}_e^{n_{e,\omega}}(\mathbf{x}_e^{(\omega)}) \;\Big|\; \mathbf{x}^{(0)}, \dots, \mathbf{x}^{(\ell-1)} \in \boldsymbol{X}\Big] \le C+\eta \tag{4.12}$$

*for any choice of $n_{e,\omega} \in \{0,1\}$.*

(II) *For every $e \in \mathcal{H}$ we have $1 \le \|\boldsymbol{\nu}_e\|_{\square_{\ell,p}(\boldsymbol{X}_e)} < \infty$, $\|\boldsymbol{\psi}_e\|_{\square_{\ell,p}(\boldsymbol{X}_e)} \le C$ and*

$$\|\boldsymbol{\nu}_e - \boldsymbol{\psi}_e\|_{\square_\ell(\boldsymbol{X}_e)} \le \eta (C\cdot M)^{-(n-1)\ell} \tag{4.13}$$

*where $M = \max\{\|\boldsymbol{\nu}_e\|_{\square_{\ell,p}(\boldsymbol{X}_e)} : e \in \mathcal{H}\}$.*

*Then the family $\langle \nu_e \colon e \in \mathcal{H}\rangle$ is $(C, \eta', p)$-pseudorandom where $\eta' = n\ell\eta$.*

Notice that in Theorem 4.3 each $\nu_e$ is decomposed as $\psi_e + (\nu_e - \psi_e)$. Here, the condition on the first components—that is, condition (4.12)—is weaker than that in Theorem 4.2, but this is offset by the stronger condition on the pseudorandom components. We also remark that Theorem 4.3 was motivated by [5, Lemmas 5 and 6] which dealt ([11]) with the case $C = 1$, $p = \infty$ and $\psi_e = 1$ for every $e \in \mathcal{H}$. Its proof is given in Section 6.

([11]) We notice that if $\psi_e = 1$ for every $e \in \mathcal{H}$, then a slight weakening of (4.13) is only needed. Specifically, one can assume that $\|\boldsymbol{\nu}_e - 1\|_{\square_2(\boldsymbol{X}_e)} \le \eta M^{-(n-1)}$ where $M = \max\{\|\boldsymbol{\nu}_e\|_{L_\infty} : e \in \mathcal{H}\}$; see [5] for details.

**5. Proof of Theorem 4.2.** Let $n$, $C$, $p$, $\ell$ and $\eta$ be as in the statement of the theorem and set

$$\bar{C}=(2C)^{n\ell},\ \ \bar{\eta}=(2C)^{n\ell}\eta^{1/\ell^{n-1}},\ \ C'=(4C)^{n\ell},\ \ \eta'=(4C)^{n\ell}\eta^{1/\ell^{n-1}}. \tag{5.1}$$

Also let $\mathscr{H}=(n,\langle(X_i,\Sigma_i,\mu_i):i\in[n]\rangle,\mathcal{H})$ be a hypergraph system with $\mathcal{H}=\binom{n}{n-1}$ and for every $e\in\mathcal{H}$ let $\lambda_e\in L_1(\boldsymbol{X},\mathcal{B}_e,\boldsymbol{\mu})$ and $\varphi_e\in L_p(\boldsymbol{X},\mathcal{B}_e,\boldsymbol{\mu})$ be nonnegative random variables satisfying (I) and (II).

We will need the following lemma. Its proof is given in Subsection 5.1.

Lemma 5.1. *Let $e\in\mathcal{H}$ and let $i\in[n]$ be unique such that $e=[n]\setminus\{i\}$. For every $e'\in\mathcal{H}\setminus\{e\}$ and every $\omega\in\{0,\dots,\ell-1\}$ let $g_{e'}^{(\omega)}\in L_1(\boldsymbol{X},\mathcal{B}_{e'},\boldsymbol{\mu})$ be such that either* (i) $0\le g_{e'}^{(\omega)}\le\lambda_{e'}$, *or* (ii) $0\le g_{e'}^{(\omega)}\le\varphi_{e'}$, *or* (iii) $0\le g_{e'}^{(\omega)}\le 1$. *Then*

$$\Big|\mathbb{E}\Big[(\boldsymbol{\lambda}_e-1)(\mathbf{x}_e)\prod_{\omega=0}^{\ell-1}\prod_{e'\in\mathcal{H}\setminus\{e\}}g_{e'}^{(\omega)}(\mathbf{x}_e,x_i^{(\omega)})\ \Big|\ \mathbf{x}_e\in\boldsymbol{X}_e,\,x_i^{(0)},x_i^{(1)}\in X_i\Big]\Big|\le\bar{\eta} \tag{5.2}$$

*and*

$$\mathbb{E}\Big[\prod_{\omega=0}^{\ell-1}\prod_{e'\in\mathcal{H}\setminus\{e\}}g_{e'}^{(\omega)}(\mathbf{x}_e,x_i^{(\omega)})\ \Big|\ \mathbf{x}_e\in\boldsymbol{X}_e,\,x_i^{(0)},\dots,x_i^{(\ell-1)}\in X_i\Big]\le\bar{C} \tag{5.3}$$

*where $\bar{\eta}$ and $\bar{C}$ are as in* (5.1).

After this preliminary discussion we turn to the main part of the proof. For every $e\in\mathcal{H}$ set $\nu_e=\lambda_e+\varphi_e$ and let $\boldsymbol{\nu}_e$ be as in (1.5) for $\nu_e$. Recall that we need to verify conditions (C1)–(C3) in Definition 4.1 for the family $\langle\nu_e:e\in\mathcal{H}\rangle$.

First, let $\mathcal{G}\subseteq\mathcal{H}$ be nonempty. Since $0\le\lambda_e\le\nu_e$, by (4.4) we have

$$1-\eta\le\mathbb{E}\Big[\prod_{e\in\mathcal{G}}\lambda_e\Big]\le\mathbb{E}\Big[\prod_{e\in\mathcal{G}}\nu_e\Big],$$

and so condition (C1) is satisfied.

Next, for every $e\in\mathcal{H}$ let $\psi_e=\varphi_e+1$. By Proposition 2.3(c.i),

$$\|\psi_e\|_{L_p}\le\|\varphi_e\|_{L_p}+1\le\|\boldsymbol{\varphi}_e\|_{\square_{\ell,p}(\boldsymbol{X}_e)}+1\le C+1\overset{(5.1)}{\le}C'.$$

Fix $e\in\mathcal{H}$ and for every $f\in\partial e$ let $A_f\in\mathcal{B}_f$. For every $e'\in\mathcal{H}\setminus\{e\}$ and every $\omega\in\{0,\dots,\ell-1\}$ we define $g_{e'}^{(\omega)}\in L_1(\boldsymbol{X},\mathcal{B}_{e'},\boldsymbol{\mu})$ by setting $g_{e'}^{(\omega)}=1$ if $\omega\in\{1,\dots,\ell-1\}$ and $g_{e'}^{(0)}=\mathbf{1}_{A_f}$ where $f=e'\cap e$. By (5.2), we have

$$\Big|\mathbb{E}\Big[(\boldsymbol{\lambda}_e-1)(\mathbf{x}_e)\prod_{f\in\partial e}\mathbf{1}_{A_f}(\mathbf{x}_e,x_i)\ \Big|\ \mathbf{x}_e\in\boldsymbol{X}_e,\,x_i^{(0)},x_i^{(1)}\in X_i\Big]\Big|\le\bar{\eta}. \tag{5.4}$$

Hence, by (5.4), the definition of the cut norm and the fact that $\nu_e - \psi_e = \lambda_e - 1$, we conclude that $\|\nu_e - \psi_e\|_{\mathcal{S}_{\partial e}} \le \eta'$. That is, condition (C2.a) is satisfied.

We proceed to verify (C2.b). Let $e \in \mathcal{H}$ be arbitrary and let $i \in [n]$ be unique such that $e = [n] \setminus \{i\}$. Also let $\omega \in \{0,1\}$. For every $e' \in \mathcal{H} \setminus \{e\}$ let $g_{e'}^{(\omega)} \in L_1(\boldsymbol{X}, \mathcal{B}_{e'}, \boldsymbol{\mu})$ be such that either $0 \le g_{e'}^{(\omega)} \le \nu_{e'}$ or $0 \le g_{e'}^{(\omega)} \le 1$. We set

$$\begin{aligned}\mathcal{G}_{e,\nu}^{(\omega)} &= \{e' \in \mathcal{H} \setminus \{e\} : 0 \le g_{e'}^{(\omega)} \le \nu_{e'}\},\\ \mathcal{G}_{e,1}^{(\omega)} &= \{e' \in \mathcal{H} \setminus \{e\} : 0 \le g_{e'}^{(\omega)} \le 1\},\end{aligned}$$

and for every $e' \in \mathcal{G}_{e,\nu}^{(\omega)}$ let

$$g_{e',\lambda}^{(\omega)} = g_{e'}^{(\omega)}\, \mathbf{1}_{[g_{e'}^{(\omega)} \le \lambda_{e'}]} \quad \text{and} \quad g_{e',\varphi}^{(\omega)} = (g_{e'}^{(\omega)} - \lambda_{e'})\, \mathbf{1}_{[g_{e'}^{(\omega)} > \lambda_{e'}]}. \tag{5.5}$$

Finally, for every $\mathcal{G} \subseteq \mathcal{G}_{e,\nu}^{(\omega)}$ set

$$A_{\mathcal{G}}^{(\omega)} = \prod_{e' \in \mathcal{G}} g_{e',\lambda}^{(\omega)} \prod_{e' \in \mathcal{G}_{e,\nu}^{(\omega)} \setminus \mathcal{G}} g_{e',\varphi}^{(\omega)} \prod_{e' \in \mathcal{G}_{e,1}^{(\omega)}} g_{e'}^{(\omega)}. \tag{5.6}$$

(Recall that, by convention, the product of an empty family of functions is the constant function 1.) The following properties are straightforward consequences of the relevant definitions:

(a) For every $\omega \in \{0,1\}$ and every $e' \in \mathcal{G}_{e,\nu}^{(\omega)}$ we have $g_{e',\lambda}^{(\omega)}, g_{e',\varphi}^{(\omega)} \in L_1(\boldsymbol{X}, \mathcal{B}_{e'}, \boldsymbol{\mu})$, $0 \le g_{e',\lambda}^{(\omega)} \le \lambda_{e'}$, $0 \le g_{e',\varphi}^{(\omega)} \le \varphi_{e'}$ and $g_{e'}^{(\omega)} = g_{e',\lambda}^{(\omega)} + g_{e',\varphi}^{(\omega)}$.

(b) For every $\mathbf{x}_e \in \boldsymbol{X}_e$ and every $x_i^{(0)}, x_i^{(1)} \in X_i$ we have

$$\prod_{\omega \in \{0,1\}} \prod_{e' \in \mathcal{H} \setminus \{e\}} g_{e'}^{(\omega)}(\mathbf{x}_e, x_i^{(\omega)}) = \sum_{\mathcal{G}_0 \subseteq \mathcal{G}_{e,\nu}^{(0)}} \sum_{\mathcal{G}_1 \subseteq \mathcal{G}_{e,\nu}^{(1)}} \prod_{\omega \in \{0,1\}} A_{\mathcal{G}_\omega}^{(\omega)}(\mathbf{x}_e, x_i^{(\omega)}). \tag{5.7}$$

By (a), every factor of $A_{\mathcal{G}}^{(\omega)}$ satisfies the assumptions of Lemma 5.1. Therefore,

$$\begin{aligned}&\mathbb{E}\Big[(\boldsymbol{\nu}_e - \boldsymbol{\psi}_e)(\mathbf{x}_e) \prod_{\omega \in \{0,1\}} \mathbb{E}\Big[\prod_{e' \in \mathcal{H} \setminus \{e\}} g_{e'}^{(\omega)}(\mathbf{x}_e, x_i) \,\Big|\, x_i \in X_i\Big] \,\Big|\, \mathbf{x}_e \in \boldsymbol{X}_e\Big]\\ &= \mathbb{E}\Big[(\boldsymbol{\lambda}_e - 1)(\mathbf{x}_e) \prod_{\omega \in \{0,1\}} \prod_{e' \in \mathcal{H} \setminus \{e\}} g_{e'}^{(\omega)}(\mathbf{x}_e, x_i^{(\omega)}) \,\Big|\, \mathbf{x}_e \in \boldsymbol{X}_e,\, x_i^{(0)}, x_i^{(1)} \in X_i\Big]\\ &\overset{(5.7)}{=} \sum_{\mathcal{G}_0 \subseteq \mathcal{G}_{e,\nu}^{(0)}} \sum_{\mathcal{G}_1 \subseteq \mathcal{G}_{e,\nu}^{(1)}} \mathbb{E}\Big[(\boldsymbol{\lambda}_e - 1)(\mathbf{x}_e) \prod_{\omega \in \{0,1\}} A_{\mathcal{G}_\omega}^{(\omega)}(\mathbf{x}_e, x_i^{(\omega)}) \,\Big|\, \mathbf{x}_e \in \boldsymbol{X}_e,\, x_i^{(0)}, x_i^{(1)} \in X_i\Big]\\ &\overset{(5.2)}{\le} 2^{|\mathcal{G}_{e,\nu}^{(0)}|}\, 2^{|\mathcal{G}_{e,\nu}^{(1)}|}\, \bar{\eta} \le 4^{n-1} \bar{\eta} \overset{(5.1)}{\le} \eta',\end{aligned}$$

which implies, of course, that condition (C2.b) is satisfied.

It remains to verify (C3). Fix $e \in \mathcal{H}$ and, as above, let $i \in [n]$ be such that $e = [n] \setminus \{i\}$. Also let $\mathcal{G} \subseteq \mathcal{H} \setminus \{e\}$ be nonempty and let $\boldsymbol{\nu}_{e,\mathcal{G}} \colon \boldsymbol{X}_e \to \mathbb{R}$ be as in Definition 4.1. Then notice that

$$\begin{aligned}\mathbb{E}[\boldsymbol{\nu}_{e,\mathcal{G}}^{\ell}] &= \mathbb{E}\Big[\mathbb{E}\Big[\prod_{e'\in\mathcal{G}} \nu_{e'}(\mathbf{x}_e, x_i) \,\Big|\, x_i \in X_i\Big]^{\ell} \,\Big|\, \mathbf{x}_e \in \boldsymbol{X}_e\Big] \\ &= \mathbb{E}\Big[\prod_{\omega=0}^{\ell-1}\prod_{e'\in\mathcal{G}} \nu_{e'}(\mathbf{x}_e, x_i^{(\omega)}) \,\Big|\, \mathbf{x}_e \in \boldsymbol{X}_e,\, x_i^{(0)}, \dots, x_i^{(\ell-1)} \in X_i\Big].\end{aligned}$$

Next, observe that for every $\mathbf{x}_e \in \boldsymbol{X}_e$ and every $x_i^{(0)}, \dots, x_i^{(\ell-1)} \in X_i$,

$$\begin{aligned}\prod_{\omega=0}^{\ell-1}\prod_{e'\in\mathcal{G}} \nu_{e'}(\mathbf{x}_e, x_i^{(\omega)}) &= \prod_{\omega=0}^{\ell-1}\prod_{e'\in\mathcal{G}} (\lambda_{e'} + \varphi_{e'})(\mathbf{x}_e, x_i^{(\omega)}) \\ &= \sum_{\mathcal{G}_0\subseteq\mathcal{G}} \cdots \sum_{\mathcal{G}_{\ell-1}\subseteq\mathcal{G}} \prod_{\omega=0}^{\ell-1}\prod_{e'\in\mathcal{G}_\omega} \lambda_{e'}(\mathbf{x}_e, x_i^{(\omega)}) \prod_{e'\in\mathcal{G}\setminus\mathcal{G}_\omega} \varphi_{e'}(\mathbf{x}_e, x_i^{(\omega)}).\end{aligned}$$

Therefore, setting

$$B_{\mathcal{G}'} = \prod_{e'\in\mathcal{G}'} \lambda_{e'} \prod_{e'\in\mathcal{G}\setminus\mathcal{G}'} \varphi_{e'}$$

for every $\mathcal{G}' \subseteq \mathcal{G}$, we obtain

$$\begin{aligned}\mathbb{E}[\boldsymbol{\nu}_{e,\mathcal{G}}^{\ell}] &= \sum_{\mathcal{G}_0\subseteq\mathcal{G}} \cdots \sum_{\mathcal{G}_{\ell-1}\subseteq\mathcal{G}} \mathbb{E}\Big[\prod_{\omega=0}^{\ell-1} B_{\mathcal{G}_\omega}(\mathbf{x}_e, x_i^{(\omega)}) \,\Big|\, \mathbf{x}_e \in \boldsymbol{X}_e,\, x_i^{(0)}, \dots, x_i^{(\ell-1)} \in X_i\Big] \\ &\overset{(5.3)}{\leq} 2^{|\mathcal{G}|\ell}\bar{C} \leq 2^{(n-1)\ell}\bar{C} \overset{(5.1)}{\leq} C'.\end{aligned}$$

This shows that condition (C3) is satisfied, and so the entire proof of Theorem 4.2 is complete.

**5.1. Proof of Lemma 5.1.** The argument is similar to that in the proofs of [4, Lemma 6.3] and [19, Proposition 5.1]. Proofs of this sort originate from the work of Green and Tao [12, 13].

We proceed to the details. First we need to introduce some notation. Let $d$ be a (possibly empty) subset of $e$ and write $\boldsymbol{X} = \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}_{d\cup\{i\}}$. (Recall that $i \in [n]$ is such that $e = [n] \setminus \{i\}$.) Notice that every element of $\boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}_{d\cup\{i\}}^{\ell}$ is written as $(\mathbf{x}_{e\setminus d}, \mathbf{x}_{d\cup\{i\}}^{(0)}, \dots, \mathbf{x}_{d\cup\{i\}}^{(\ell-1)})$ where $\mathbf{x}_{e\setminus d} \in \boldsymbol{X}_{e\setminus d}$ and $\mathbf{x}_{d\cup\{i\}}^{(0)}, \dots, \mathbf{x}_{d\cup\{i\}}^{(\ell-1)} \in \boldsymbol{X}_{d\cup\{i\}}$. On the other hand, for every $\mathbf{x}_{d\cup\{i\}}^{(0)} = (x_j^{(0)})_{j\in d\cup\{i\}}, \dots, \mathbf{x}_{d\cup\{i\}}^{(\ell-1)} = (x_j^{(\ell-1)})_{j\in d\cup\{i\}} \in \boldsymbol{X}_{d\cup\{i\}}$, every $d' \subseteq d \cup \{i\}$ and every $\omega = (\omega_j)_{j\in d'} \in \{0, \dots, \ell-1\}^{d'}$ we shall denote by $\mathbf{x}_{d'}^{(\omega)}$ the unique element of $\boldsymbol{X}_{d'}$ defined by

$$\mathbf{x}_{d'}^{(\omega)} = (x_j^{(\omega_j)})_{j\in d'}. \tag{5.8}$$

Next, for every $d \subseteq e$ we define $F_d, G_d \colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ as follows. First, set

$$F_d(\mathbf{x}_{e\setminus d}, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \prod_{\omega\in\{0,\dots,\ell-1\}^d} (\boldsymbol{\lambda}_e(\mathbf{x}_{e\setminus d}, \mathbf{x}^{(\omega)}_d) - 1) \tag{5.9}$$

and notice that $F_d$ does not depend on $x_i^{(0)}, \dots, x_i^{(\ell-1)} \in X_i$. The definition of the function $G_d$ is somewhat more involved. For every $e' \in \mathcal{H}\setminus\{e\}$ and every $\omega \in \{0, \dots, \ell-1\}$ let $g^{(\omega)}_{e'}$ be as in the statement of the lemma and let $\mathbf{g}^{(\omega)}_{e'}$ be as in (5.1) for $g^{(\omega)}_{e'}$. Given $e' \in \mathcal{H} \setminus \{e\}$ and $\omega \in \{0, \dots, \ell-1\}^{e'\cap(d\cup\{i\})}$ it is convenient to introduce an auxiliary function $g_{e',d,\omega} \colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ by defining $g_{e',d,\omega}(\mathbf{x}_{e\setminus d}, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}})$ to be

(5.10)

$$\begin{cases} \mathbf{g}^{(\omega_i)}_{e'}(\mathbf{x}_{e'\setminus(d\cup\{i\})}, \mathbf{x}^{(\omega)}_{d\cup\{i\}}) & \text{if } d \subseteq e', \\ \boldsymbol{\lambda}_{e'}(\mathbf{x}_{e'\setminus(d\cup\{i\})}, \mathbf{x}^{(\omega)}_{e'\cap(d\cup\{i\})}) & \text{if } d \nsubseteq e' \text{ and } g^{(\omega_i)}_{e'} \le \lambda_{e'}, \\ 1 & \text{if } d \nsubseteq e' \text{ and either } g^{(\omega_i)}_{e'} \le \varphi_{e'} \text{ or } g^{(\omega_i)}_{e'} \le 1. \end{cases}$$

(Here, $\omega_i$ is the $i$th coordinate of $\omega$. Moreover, $\mathbf{x}_{e'\setminus(d\cup\{i\})}$ stands for the natural projection of $\mathbf{x}_{e\setminus d}$ into $\boldsymbol{X}_{e'\setminus(d\cup\{i\})}$; note that this projection is well-defined since $e' \setminus (d \cup \{i\}) \subseteq e \setminus d$.) We now define

$$G_d = \prod_{e'\in\mathcal{H}\setminus\{e\}} \prod_{\omega\in\{0,\dots,\ell-1\}^{e'\cap(d\cup\{i\})}} g_{e',d,\omega}. \tag{5.11}$$

Finally, we set

$$Q_d = \mathbb{E}[F_d G_d] \quad \text{and} \quad R_d = \mathbb{E}[G_d]. \tag{5.12}$$

Claim 5.2.

(a) *$Q_\emptyset$ and $R_\emptyset$ coincide with the quantities on the left-hand side of* (5.2) *and* (5.3) *respectively.*
(b) $|Q_e| \le 2\eta$ *and* $0 \le R_e \le 1 + \eta$.

*Proof of Claim 5.2.* For part (a) it is enough to observe that

$$\begin{aligned} F_\emptyset(\mathbf{x}_e, x_i^{(0)}, \dots, x_i^{(\ell-1)}) &= \boldsymbol{\lambda}_e(\mathbf{x}_e) - 1, \\ G_\emptyset(\mathbf{x}_e, x_i^{(0)}, \dots, x_i^{(\ell-1)}) &= \prod_{e'\in\mathcal{H}\setminus\{e\}} \prod_{\omega\in\{0,\dots,\ell-1\}^{\{i\}}} \mathbf{g}^{(\omega_i)}_{e'}(\mathbf{x}_{e'\setminus\{i\}}, x_i^{(\omega)}) \\ &= \prod_{\omega=0}^{\ell-1} \prod_{e'\in\mathcal{H}\setminus\{e\}} g^{(\omega)}_{e'}(\mathbf{x}_e, x_i^{(\omega)}). \end{aligned}$$

For (b) notice that

$$Q_e = \mathbb{E}\Big[\prod_{\omega\in\{0,\dots,\ell-1\}^e} (\lambda_e(\mathbf{x}_e^{(\omega)}) - 1) \prod_{e'\in\mathcal{H}\setminus\{e\}} \prod_{\omega\in\{0,\dots,\ell-1\}^{e'}} \boldsymbol{\lambda}_{e'}^{n_{e',\omega}}(\mathbf{x}_e^{(\omega)}) \ \Big| \ \mathbf{x}^{(0)},\dots,\mathbf{x}^{(\ell-1)} \in \boldsymbol{X}\Big],$$

$$R_e = \mathbb{E}\Big[\prod_{e'\in\mathcal{H}\setminus\{e\}} \prod_{\omega\in\{0,\dots,\ell-1\}^{e'}} \boldsymbol{\lambda}_{e'}^{n_{e',\omega}}(\mathbf{x}_e^{(\omega)}) \ \Big| \ \mathbf{x}^{(0)},\dots,\mathbf{x}^{(\ell-1)} \in \boldsymbol{X}\Big],$$

where for every $e' \in \mathcal{H} \setminus \{e\}$ and every $\omega \in \{0,\dots,\ell-1\}^{e'}$ we have $n_{e',\omega} \in \{0,1\}$, and $n_{e',\omega} = 1$ if and only if $g_{e'}^{(\omega_i)} \le \lambda_{e'}$. Therefore, by (4.4), we conclude that $|Q_e| \le 2\eta$ and $0 \le R_e \le 1+\eta$. ∎

Claim 5.3. *For every $d \subsetneq e$ and every $j \in e\setminus d$ we have $Q_{d\cup\{j\}} \ge 0$, and*

$$|Q_d|^{1/\ell^{|d|}} \le (2C)^\ell\, Q_{d\cup\{j\}}^{1/\ell^{|d|+1}} \quad \textit{and} \quad R_d^{1/\ell^{|d|}} \le (2C)^\ell\, R_{d\cup\{j\}}^{1/\ell^{|d|+1}}. \tag{5.13}$$

*Proof of Claim 5.3.* We will only show the first inequality; the proof of the other one is identical. (In particular, it follows by setting $F_d = 1$ below.)

Fix $d \subsetneq e$ and $j \in e\setminus d$, and set $e_j = [n]\setminus\{j\}$ and $f = [n]\setminus(d\cup\{i,j\})$. Also write $\boldsymbol{X}_{e_j} = \boldsymbol{X}_f \times \boldsymbol{X}_{d\cup\{i\}}$ and $\boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^\ell_{d\cup\{i\}} = X_j \times \boldsymbol{X}_f \times \boldsymbol{X}^\ell_{d\cup\{i\}}$, and let $\pi\colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^\ell_{d\cup\{i\}} \to \boldsymbol{X}_f \times \boldsymbol{X}^\ell_{d\cup\{i\}}$ denote the natural projection.

For every $\omega \in \{0,\dots,\ell-1\}^{d\cup\{i\}}$ we define $\mathbf{g}_{e_j,d,\omega}\colon \boldsymbol{X}_f \times \boldsymbol{X}^\ell_{d\cup\{i\}} \to \mathbb{R}$ by

$$\mathbf{g}_{e_j,d,\omega}(\mathbf{x}_f, \mathbf{x}^{(0)}_{d\cup\{i\}},\dots,\mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \mathbf{g}_{e_j}^{(\omega_i)}(\mathbf{x}_f, \mathbf{x}^{(\omega)}_{d\cup\{i\}}). \tag{5.14}$$

(Recall that $\omega_i$ is the $i$th coordinate of $\omega$ and $\mathbf{g}_{e_j}^{(\omega_i)}$ is as in (5.1) for $g_{e_j}^{(\omega_i)}$.) Observe that, by (5.10), for every $\omega \in \{0,\dots,\ell-1\}^{d\cup\{i\}}$ we have

$$g_{e_j,d,\omega} = \mathbf{g}_{e_j,d,\omega} \circ \pi. \tag{5.15}$$

Next, let $\Omega_\lambda$, $\Omega_\varphi$ and $\Omega_1$ denote the subsets of $\{0,\dots,\ell-1\}^{d\cup\{i\}}$ consisting of all $\omega$ such that $g_{e_j}^{(\omega_i)} \le \lambda_{e_j}$, $g_{e_j}^{(\omega_i)} \le \varphi_{e_j}$ and $g_{e_j}^{(\omega_i)} \le 1$ respectively. Set

$$\mathbf{G}_{j,d,\lambda} = \prod_{\omega\in\Omega_\lambda} \mathbf{g}_{e_j,d,\omega}, \quad \mathbf{G}_{j,d,\varphi} = \prod_{\omega\in\Omega_\varphi} \mathbf{g}_{e_j,d,\omega}, \quad \mathbf{G}_{j,d,1} = \prod_{\omega\in\Omega_1} \mathbf{g}_{e_j,d,\omega}, \tag{5.16}$$

and notice that these functions are defined on $\boldsymbol{X}_f \times \boldsymbol{X}^\ell_{d\cup\{i\}}$. We also define $G'_{j,d}\colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^\ell_{d\cup\{i\}} \to \mathbb{R}$ by

$$G'_{j,d} = \prod_{e'\in\mathcal{H}\setminus\{e,e_j\}} \prod_{\omega\in\{0,\dots,\ell-1\}^{e'\cap(d\cup\{i\})}} g_{e',d,\omega}. \tag{5.17}$$

By (5.11) and (5.15)–(5.17), we have $G_d = G'_{j,d} \cdot [(\mathbf{G}_{j,d,\lambda}\mathbf{G}_{j,d,\varphi}\mathbf{G}_{j,d,1}) \circ \pi]$ and so

$$Q_d = \mathbb{E}\big[\mathbb{E}[F_d\, G'_{j,d} \mid x_j \in X_j] \cdot (\mathbf{G}_{j,d,\lambda}\mathbf{G}_{j,d,\varphi}\mathbf{G}_{j,d,1})\big] \tag{5.18}$$

where the outer expectation is over $\boldsymbol{X}_f \times \boldsymbol{X}^{\ell}_{d\cup\{i\}}$. Denote by $\ell'$ the conjugate exponent of $\ell$ and recall that $\ell$ is an even positive integer. By (5.18), Hölder's inequality and the fact that $0 \le \mathbf{G}_{j,d,1} \le 1$, we obtain

$$(5.19)\qquad \begin{aligned} |Q_d| &= \big|\mathbb{E}\big[\mathbb{E}[F_d G'_{j,d} \mid x_j \in X_j]\cdot((\mathbf{G}^{1/\ell}_{j,d,\lambda}\mathbf{G}^{1/\ell'}_{j,d,\lambda})\mathbf{G}_{j,d,\varphi}\mathbf{G}_{j,d,1})\big]\big| \\ &\le \mathbb{E}\big[\mathbb{E}[F_d G'_{j,d} \mid x_j \in X_j]^{\ell}\cdot \mathbf{G}_{j,d,\lambda}\big]^{1/\ell}\cdot \mathbb{E}[\mathbf{G}_{j,d,\lambda}\mathbf{G}^{\ell'}_{j,d,\varphi}]^{1/\ell'}. \end{aligned}$$

Now define $\overline{\mathbf{G}}_{j,d,\lambda}, \overline{\mathbf{G}}_{j,d,\varphi}\colon \boldsymbol{X}_f \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ by

$$(5.20)\qquad \overline{\mathbf{G}}_{j,d,\lambda}(\mathbf{x}_f, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \prod_{\omega\in\Omega_\lambda} \boldsymbol{\lambda}_{e_j}(\mathbf{x}_f, \mathbf{x}^{(\omega)}_{d\cup\{i\}}),$$

$$(5.21)\qquad \overline{\mathbf{G}}_{j,d,\varphi}(\mathbf{x}_f, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \prod_{\omega\in\Omega_\varphi} \boldsymbol{\varphi}_{e_j}(\mathbf{x}_f, \mathbf{x}^{(\omega)}_{d\cup\{i\}}).$$

By (5.14) and (5.16), we see that $0 \le \mathbf{G}_{j,d,\lambda} \le \overline{\mathbf{G}}_{j,d,\lambda}$ and $0 \le \mathbf{G}_{j,d,\varphi} \le \overline{\mathbf{G}}_{j,d,\varphi}$. Therefore, by (5.19),

$$(5.22)\qquad |Q_d| \le \mathbb{E}\big[\mathbb{E}[F_d G'_{j,d} \mid x_j \in X_j]^{\ell}\cdot \overline{\mathbf{G}}_{j,d,\lambda}\big]^{1/\ell}\cdot \mathbb{E}[\overline{\mathbf{G}}_{j,d,\lambda}\overline{\mathbf{G}}^{\ell'}_{j,d,\varphi}]^{1/\ell'}.$$

It is easy to see that

$$(5.23)\qquad \mathbb{E}\big[\mathbb{E}[F_d G'_{j,d} \mid x_j \in X_j]^{\ell}\cdot \overline{\mathbf{G}}_{j,d,\lambda}\big] = Q_{d\cup\{j\}},$$

which implies in particular that $Q_{d\cup\{j\}} \ge 0$. On the other hand, by (5.20), (5.21) and Proposition 2.1(a),

$$(5.24)\qquad \mathbb{E}[\overline{\mathbf{G}}_{j,d,\lambda}\overline{\mathbf{G}}^{\ell'}_{j,d,\varphi}] \le \|\boldsymbol{\lambda}_{e_j}\|^{|\Omega_\lambda|}_{\square_\ell(\boldsymbol{X}_{e_j})}\cdot \|\boldsymbol{\varphi}^{\ell'}_{e_j}\|^{|\Omega_\varphi|}_{\square_\ell(\boldsymbol{X}_{e_j})}.$$

By (4.4), it is clear that $\|\boldsymbol{\lambda}_{e_j}\|_{\square_\ell(\boldsymbol{X}_{e_j})} \le 1+\eta \le 2$. Moreover, by (4.3), we see that $1 < \ell' < p$. Hence, by Proposition 2.3(c.ii) and condition (II) in Theorem 4.2, we have $\|\boldsymbol{\varphi}^{\ell'}_{e_j}\|_{\square_\ell(\boldsymbol{X}_{e_j})} \le \|\boldsymbol{\varphi}_{e_j}\|^{\ell'}_{\square_{\ell,p}(\boldsymbol{X}_{e_j})} \le C^{\ell'}$. Thus, by (5.24),

$$(5.25)\qquad \mathbb{E}[\overline{\mathbf{G}}_{j,d,\lambda}\overline{\mathbf{G}}^{\ell'}_{j,d,\varphi}]^{1/\ell'} \le 2^{|\Omega_\lambda|/\ell'}\cdot C^{|\Omega_\varphi|} \le (2C)^{|\Omega_\lambda|+|\Omega_\varphi|} \le (2C)^{\ell^{|d|+1}}.$$

Combining (5.22), (5.23) and (5.25), we get $|Q_d| \le Q^{1/\ell}_{d\cup\{j\}}(2C)^{\ell^{|d|+1}}$, which is equivalent to $|Q_d|^{1/\ell^{|d|}} \le (2C)^{\ell} Q^{1/\ell^{|d|+1}}_{d\cup\{j\}}$. ■

By induction and using Claim 5.3, we see that

$$(5.26)\qquad |Q_\emptyset| \le (2C)^{(n-1)\ell} Q_e^{1/\ell^{n-1}} \quad\text{and}\quad R_\emptyset \le (2C)^{(n-1)\ell} R_e^{1/\ell^{n-1}}.$$

Invoking (5.26) and Claim 5.2, we conclude that (5.2) and (5.3) are satisfied, and so the proof of Lemma 5.1 is complete.

**6. Proof of Theorem 4.3.** Let $\mathscr{H}$ and $\langle \nu_e, \psi_e : e \in \mathcal{H}\rangle$ be as in the statement of the theorem, and for every $e \in \mathcal{H}$ let $\boldsymbol{\nu}_e$ and $\boldsymbol{\psi}_e$ be as in (1.5) for $\nu_e$ and $\psi_e$ respectively.

The following lemma is the first main step of the proof.

LEMMA 6.1. *Let $e \in \mathcal{H}$ and let $i \in [n]$ be such that $e = [n]\setminus\{i\}$. For every $e' \in \mathcal{H} \setminus \{e\}$ and every $\omega \in \{0,\dots,\ell-1\}$ let $g_{e'}^{(\omega)} \in L_1(\boldsymbol{X}, \mathcal{B}_{e'}, \boldsymbol{\mu})$ be such that either* (i) $0 \le g_{e'}^{(\omega)} \le \nu_{e'}$, *or* (ii) $0 \le g_{e'}^{(\omega)} \le \psi_{e'}$, *or* (iii) $0 \le g_{e'}^{(\omega)} \le 1$. *Then*

(6.1)
$$\Big|\mathbb{E}\Big[(\boldsymbol{\nu}_e - \boldsymbol{\psi}_e)(\mathbf{x}_e)\prod_{\omega=0}^{\ell-1}\prod_{e'\in\mathcal{H}\setminus\{e\}} g_{e'}^{(\omega)}(\mathbf{x}_e, x_i^{(\omega)}) \,\Big|\, \mathbf{x}_e \in \boldsymbol{X}_e,\, x_i^{(0)},\dots,x_i^{(\ell-1)} \in X_i\Big]\Big| \le \eta.$$

*Proof.* The argument is similar to the proof of Lemma 5.1. Specifically, let $d \subseteq e$ be arbitrary. For every $e' \in \mathcal{H} \setminus \{e\}$ with $d \subseteq e'$ and every $\omega \in \{0,\dots,\ell-1\}^{d\cup\{i\}}$ we define $g_{e',d,\omega}\colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ by setting

$$g_{e',d,\omega}(\mathbf{x}_{e\setminus d}, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \mathbf{g}_{e'}^{(\omega_i)}(\mathbf{x}_{e'\setminus(d\cup\{i\})}, \mathbf{x}^{(\omega)}_{d\cup\{i\}}) \tag{6.2}$$

where $\omega_i$ is the $i$th coordinate of $\omega$ and $\mathbf{g}_{e'}^{(\omega_i)}$ is as in (1.5) for $g_{e'}^{(\omega_i)}$. Also define $F_d, G_d\colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ by

$$F_d(\mathbf{x}_{e\setminus d}, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \prod_{\omega\in\{0,\dots,\ell-1\}^d} (\boldsymbol{\nu}_e - \boldsymbol{\psi}_e)(\mathbf{x}_{e\setminus d}, \mathbf{x}^{(\omega)}_d), \tag{6.3}$$

$$G_d = \prod_{\substack{e'\in\mathcal{H}\setminus\{e\}\\ d\subseteq e'}} \prod_{\omega\in\{0,\dots,\ell-1\}^{d\cup\{i\}}} g_{e',d,\omega}. \tag{6.4}$$

(Here, as in Section 3, we follow the convention that the product of an empty family of functions is the constant function 1.) Finally, let

$$Q_d = \mathbb{E}[F_d G_d]. \tag{6.5}$$

Note that $Q_\emptyset$ coincides with the quantity on the left-hand side of (6.1). Moreover,

$$Q_e = \|\boldsymbol{\nu}_e - \boldsymbol{\psi}_e\|^{\ell^{n-1}}_{\square_\ell(\boldsymbol{X}_e)}. \tag{6.6}$$

CLAIM 6.2. *For every $d \subsetneq e$ and every $j \in e \setminus d$ we have $Q_{d\cup\{j\}} \ge 0$ and*

$$|Q_d|^{1/\ell^{|d|}} \le (C\cdot M)^{\ell} \cdot Q_{d\cup\{j\}}^{1/\ell^{|d|+1}}. \tag{6.7}$$

*Proof of Claim 6.2.* As in the proof of Claim 5.3, fix $d \subsetneq e$ and $j \in e\setminus d$, and set $e_j = [n]\setminus\{j\}$ and $f = [n]\setminus(d\cup\{i,j\})$. Write $\boldsymbol{X}_{e_j} = \boldsymbol{X}_f \times \boldsymbol{X}_{d\cup\{i\}}$ and $\boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} = X_j \times (\boldsymbol{X}_f \times \boldsymbol{X}^{\ell}_{d\cup\{i\}})$, and for every $\omega \in \{0,\dots,\ell-1\}^{d\cup\{i\}}$ define $\mathbf{g}_{e_j,d,\omega}\colon \boldsymbol{X}_f \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ by

$$\mathbf{g}_{e_j,d,\omega}(\mathbf{x}_f, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \mathbf{g}_{e_j}^{(\omega_i)}(\mathbf{x}_f, \mathbf{x}^{(\omega)}_{d\cup\{i\}}). \tag{6.8}$$

If $\pi\colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \boldsymbol{X}_f \times \boldsymbol{X}^{\ell}_{d\cup\{i\}}$ is the natural projection map, then by (6.2) we see that $g_{e_j,d,\omega} = \mathbf{g}_{e_j,d,\omega} \circ \pi$ for every $\omega \in \{0,\dots,\ell-1\}^{d\cup\{i\}}$.

Let $\Omega_\nu$, $\Omega_\psi$ and $\Omega_1$ denote the subsets of $\{0,\dots,\ell-1\}^{d\cup\{i\}}$ consisting of all $\omega$ such that $g^{(\omega_i)}_{e_j} \le \nu_{e_j}$, $g^{(\omega_i)}_{e_j} \le \psi_{e_j}$ and $g^{(\omega_i)}_{e_j} \le 1$ respectively. We set

$$\mathbf{G}_{j,d,\nu} = \prod_{\omega\in\Omega_\nu} \mathbf{g}_{e_j,d,\omega}, \quad \mathbf{G}_{j,d,\psi} = \prod_{\omega\in\Omega_\psi} \mathbf{g}_{e_j,d,\omega}, \quad \mathbf{G}_{j,d,1} = \prod_{\omega\in\Omega_1} \mathbf{g}_{e_j,d,\omega}. \tag{6.9}$$

Moreover, let $G'_{j,d}\colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ be defined by

$$G'_{j,d} = \prod_{\substack{e'\in\mathcal{H}\setminus\{e,e_j\}\\ d\subseteq e'}} \ \prod_{\omega\in\{0,\dots,\ell-1\}^{d\cup\{i\}}} g_{e',d,\omega}. \tag{6.10}$$

Observe that $G_d = G'_{j,d} \cdot [(\mathbf{G}_{j,d,\nu}\mathbf{G}_{j,d,\varphi}\mathbf{G}_{j,d,1}) \circ \pi]$. Hence, if $\ell'$ denotes the conjugate exponent of $\ell$, then, by Hölder's inequality,

$$\begin{aligned} |Q_d| &= \big|\mathbb{E}\big[\mathbb{E}[F_d G'_{j,d} \mid x_j \in X_j] \cdot (\mathbf{G}_{j,d,\nu}\mathbf{G}_{j,d,\psi}\mathbf{G}_{j,d,1})\big]\big| \\ &\le \mathbb{E}\big[\mathbb{E}[F_d G'_{j,d} \mid x_j \in X_j]^{\ell}\big]^{1/\ell} \cdot \mathbb{E}[(\mathbf{G}_{j,d,\nu}\mathbf{G}_{j,d,\psi})^{\ell'}]^{1/\ell'} \end{aligned} \tag{6.11}$$

where the outer expectation is over $\boldsymbol{X}_f \times \boldsymbol{X}^{\ell}_{d\cup\{i\}}$. Note that

$$\mathbb{E}\big[\mathbb{E}[F_d G'_{j,d} \mid x_j \in X_j]^{\ell}\big] = Q_{d\cup\{j\}}, \tag{6.12}$$

and so $Q_{d\cup\{j\}} \ge 0$. Next, define $\overline{\mathbf{G}}_{j,d,\nu}, \overline{\mathbf{G}}_{j,d,\psi}\colon \boldsymbol{X}_f \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ by

$$\overline{\mathbf{G}}_{j,d,\nu}(\mathbf{x}_f, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \prod_{\omega\in\Omega_\nu} \boldsymbol{\nu}_{e_j}(\mathbf{x}_f, \mathbf{x}^{(\omega)}_{d\cup\{i\}}), \tag{6.13}$$

$$\overline{\mathbf{G}}_{j,d,\psi}(\mathbf{x}_f, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \prod_{\omega\in\Omega_\psi} \boldsymbol{\psi}_{e_j}(\mathbf{x}_f, \mathbf{x}^{(\omega)}_{d\cup\{i\}}). \tag{6.14}$$

By (6.8) and (6.9), we see that $0 \le \mathbf{G}_{j,d,\nu} \le \overline{\mathbf{G}}_{j,d,\nu}$ and $0 \le \mathbf{G}_{j,d,\varphi} \le \overline{\mathbf{G}}_{j,d,\varphi}$. On the other hand, by (4.3), we have $1 < \ell' < p$. Therefore, by (6.13), (6.14) and Proposition 2.3(a, c.ii), we obtain

$$\begin{aligned} \mathbb{E}[(\mathbf{G}_{j,d,\nu}\mathbf{G}_{j,d,\psi})^{\ell'}]^{1/\ell'} &\le \mathbb{E}[(\overline{\mathbf{G}}_{j,d,\nu}\overline{\mathbf{G}}_{j,d,\psi})^{\ell'}]^{1/\ell'} \\ &\le \|\boldsymbol{\nu}_{e_j}\|^{|\Omega_\nu|}_{\square_{\ell,p}(\boldsymbol{X}_{e_j})} \cdot \|\boldsymbol{\psi}_{e_j}\|^{|\Omega_\psi|}_{\square_{\ell,p}(\boldsymbol{X}_{e_j})} \\ &\le (C\cdot M)^{\ell^{|d|+1}}. \end{aligned} \tag{6.15}$$

By (6.11), (6.12) and (6.15), we see that $|Q_d| \le Q^{1/\ell}_{d\cup\{j\}} (C\cdot M)^{\ell^{|d|+1}}$. ■

By the above claim, we have

$$|Q_\emptyset| \le (C\cdot M)^{(n-1)\ell} Q_e^{1/\ell^{n-1}}.$$

As already mentioned, $Q_\emptyset$ coincides with the quantity on the left-hand side of (6.1). Thus, combining the previous estimate with (4.13) and (6.6), we conclude that (6.1) is satisfied, and so the proof of Lemma 5.1 is complete. ■

The following lemma is the second main step of the proof.

LEMMA 6.3. *Let $e \in \mathcal{H}$ and let $i \in [n]$ be such that $e = [n] \setminus \{i\}$. Also let $e' \in \mathcal{H} \setminus \{e\}$ and $\omega' \in \{0, \dots, \ell-1\}$. For every $e'' \in \mathcal{H} \setminus \{e\}$ and every $\omega \in \{0, \dots, \ell-1\}$ with $(e'', \omega) \neq (e', \omega')$ let $g^{(\omega)}_{e''} \in L_1(\boldsymbol{X}, \mathcal{B}_{e''}, \boldsymbol{\mu})$ be such that either* (i) $0 \le g^{(\omega)}_{e''} \le \nu_{e''}$, *or* (ii) $0 \le g^{(\omega)}_{e''} \le \psi_{e''}$, *or* (iii) $0 \le g^{(\omega)}_{e''} \le 1$. *Then*

$$\Big|\mathbb{E}\Big[(\boldsymbol{\nu}_{e'} - \boldsymbol{\psi}_{e'})(\mathbf{x}_{e'\setminus\{i\}}, x_i^{(\omega')}) \prod_{\substack{e''\in\mathcal{H}\setminus\{e\} \\ \omega\in\{0,\dots,\ell-1\} \\ (e'',\omega)\neq(e',\omega')}} \mathbf{g}^{(\omega)}_{e''}(\mathbf{x}_{e''\setminus\{i\}}, x_i^{(\omega)})\Big]\Big| \le \eta \tag{6.16}$$

*where the expectation is over all $\mathbf{x}_e \in \boldsymbol{X}_e$ and $x_i^{(0)}, \dots, x_i^{(\ell-1)} \in X_i$. (Here, $\mathbf{x}_{e'\setminus\{i\}}$ and $\mathbf{x}_{e''\setminus\{i\}}$ denote the projections of $\mathbf{x}_e$ into $\boldsymbol{X}_{e'\setminus\{i\}}$ and $\boldsymbol{X}_{e''\setminus\{i\}}$ respectively.)*

*Proof.* Without loss of generality, and to simplify the exposition, we will assume that $\omega' = 0$. For every $d \subseteq e' \setminus \{i\}$ let $F_d, G_d \colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ be defined by

$$F_d(\mathbf{x}_{e\setminus d}, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \prod_{\omega\in\{0,\dots,\ell-1\}^d} (\boldsymbol{\nu}_{e'} - \boldsymbol{\psi}_{e'})(\mathbf{x}_{e'\setminus(d\cup\{i\})}, \mathbf{x}^{(\omega)}_d, x_i^{(0)}),$$

$$G_d = \prod_{(e'',\omega)\in\Gamma_d} g_{e'',d,\omega},$$

where (a) the set $\Gamma_d$ consists of all pairs $(e'', \omega) \in \mathcal{H}\setminus\{e\} \times \{0, \dots, \ell-1\}^{d\cup\{i\}}$ such that $d \subseteq e'' \setminus \{i\}$ and $(e'', \omega_i) \neq (e', 0)$, and (b) for every $(e'', \omega) \in \Gamma_d$ the function $g_{e'',d,\omega} \colon \boldsymbol{X}_{e\setminus d} \times \boldsymbol{X}^{\ell}_{d\cup\{i\}} \to \mathbb{R}$ is defined by

$$g_{e'',d,\omega}(\mathbf{x}_{e\setminus d}, \mathbf{x}^{(0)}_{d\cup\{i\}}, \dots, \mathbf{x}^{(\ell-1)}_{d\cup\{i\}}) = \mathbf{g}^{(\omega_i)}_{e''}(\mathbf{x}_{e''\setminus(d\cup\{i\})}, \mathbf{x}^{(\omega_d)}_d, x_i^{(\omega_i)}).$$

(As before, $\omega_i$ is the $i$th coordinate of $\omega$ and $\mathbf{g}^{(\omega_i)}_{e''}$ is as in (1.5) for $g^{(\omega_i)}_{e''}$. Moreover, $\mathbf{x}_{e'\setminus(d\cup\{i\})}$ and $\mathbf{x}_{e''\setminus(d\cup\{i\})}$ are the projections of $\mathbf{x}_{e\setminus d}$ into $\boldsymbol{X}_{e'\setminus(d\cup\{i\})}$ and $\boldsymbol{X}_{e''\setminus(d\cup\{i\})}$ respectively.)

Next, setting $Q_d = \mathbb{E}[F_d G_d]$ and arguing precisely as in the proof of Lemma 6.1, we obtain $Q_{d\cup\{j\}} \ge 0$ and $|Q_d|^{1/\ell^{|d|}} \le (C \cdot M)^{\ell} Q^{1/\ell^{|d|+1}}_{d\cup\{j\}}$ for every $d \subsetneq e' \setminus \{i\}$ and every $j \in e' \setminus (d \cup \{i\})$. Therefore,

$$|Q_\emptyset| \le (C \cdot M)^{(n-2)\ell} Q^{1/\ell^{n-2}}_{e'\setminus\{i\}}. \tag{6.17}$$

Now observe that $|Q_\emptyset|$ coincides with the quantity on the left-hand side of (6.16). On the other hand,

$$Q_{e'\setminus\{i\}} = \mathbb{E}[\mathbf{F}_{e'\setminus\{i\}}\mathbf{G}_{e'\setminus\{i\}} \mid \mathbf{x}_{e'}^{(0)},\dots,\mathbf{x}_{e'}^{(\ell-1)} \in \boldsymbol{X}_{e'}]$$

where

$$\mathbf{F}_{e'\setminus\{i\}}(\mathbf{x}_{e'}^{(0)},\dots,\mathbf{x}_{e'}^{(\ell-1)}) = \prod_{\omega\in\{0,\dots,\ell-1\}^{e'\setminus\{i\}}} (\boldsymbol{\nu}_{e'} - \boldsymbol{\psi}_{e'})(\mathbf{x}_{e'\setminus\{i\}}^{(\omega)}, x_i^{(0)}),$$

$$\mathbf{G}_{e'\setminus\{i\}}(\mathbf{x}_{e'}^{(0)},\dots,\mathbf{x}_{e'}^{(\ell-1)}) = \prod_{\omega\in\{0,\dots,\ell-1\}^{e'\setminus\{i\}}\times[\ell-1]^{\{i\}}} \mathbf{g}_{e'}^{(\omega_i)}(\mathbf{x}_{e'\setminus\{i\}}^{(\omega_{e'\setminus\{i\}})}, x_i^{(\omega_i)}).$$

(The arguments of the functions in the definitions of $\mathbf{F}_{e'\setminus\{i\}}$ and $\mathbf{G}_{e'\setminus\{i\}}$ follow from our previous conventions, mutatis mutandis.) Thus, by Proposition 2.1(a), we obtain

$$Q_{e'\setminus\{i\}} \le \|\boldsymbol{\nu}_{e'} - \boldsymbol{\psi}_{e'}\|_{\square_\ell(\boldsymbol{X}_{e'})}^{\ell^{n-2}} \cdot \prod_{\omega=1}^{\ell-1} \|\mathbf{g}_{e'}^{(\omega)}\|_{\square_\ell(\boldsymbol{X}_{e'})}^{\ell^{n-2}},$$

and consequently, by (6.17),

$$|Q_\emptyset| \le (C\cdot M)^{(n-2)\ell} \cdot \|\boldsymbol{\nu}_{e'} - \boldsymbol{\psi}_{e'}\|_{\square_\ell(\boldsymbol{X}_{e'})} \cdot \prod_{\omega=1}^{\ell-1} \|\mathbf{g}_{e'}^{(\omega)}\|_{\square_\ell(\boldsymbol{X}_{e'})}. \tag{6.18}$$

By Proposition 2.3(c.ii), for every $\omega\in[\ell-1]$ we have

$$\|\mathbf{g}_{e'}^{(\omega)}\|_{\square_\ell(\boldsymbol{X}_{e'})} \le \|\mathbf{g}_{e'}^{(\omega)}\|_{\square_{\ell,1}(\boldsymbol{X}_{e'})} \le \|\mathbf{g}_{e'}^{(\omega)}\|_{\square_{\ell,p}(\boldsymbol{X}_{e'})}.$$

Hence, by (6.18) and condition (II),

$$\begin{aligned}|Q_\emptyset| &\le (C\cdot M)^{(n-2)\ell} \cdot \|\boldsymbol{\nu}_{e'} - \boldsymbol{\psi}_{e'}\|_{\square_\ell(\boldsymbol{X}_{e'})} \cdot (C\cdot M)^{\ell-1} \\ &\le (C\cdot M)^{(n-1)\ell} \cdot \|\boldsymbol{\nu}_{e'} - \boldsymbol{\psi}_{e'}\|_{\square_\ell(\boldsymbol{X}_{e'})} \le \eta,\end{aligned}$$

and the proof of Lemma 6.3 is complete. ■

We are now in a position to complete the proof of Theorem 4.3. Recall that we need to show that the family $\langle \nu_e : e\in\mathcal{H}\rangle$ satisfies conditions (C1)–(C3) in Definition 4.1 for the constants $C$ and $\eta' = n\ell\eta$. For (C1) let $\mathcal{G}\subseteq\mathcal{H}\setminus\{e\}$ be nonempty. Set $m=|\mathcal{G}|$ and let $e'_1,\dots,e'_m$ be an enumeration of $\mathcal{G}$. Notice that

$$\begin{aligned}\Big|\mathbb{E}\Big[\prod_{e'\in\mathcal{G}}\nu_{e'}\Big] - \mathbb{E}\Big[\prod_{e'\in\mathcal{G}}\psi_{e'}\Big]\Big| &\le \sum_{j=1}^{m}\Big|\mathbb{E}\Big[\prod_{k<j}\psi_{e'_k}(\nu_{e'_j} - \psi_{e'_j})\prod_{k>j}\nu_{e'_k}\Big]\Big| \\ &\overset{(6.1)}{\le} (n-1)\eta\end{aligned} \tag{6.19}$$

and so, by condition (I), we obtain

$$\mathbb{E}\Big[\prod_{e'\in\mathcal{G}}\nu_{e'}\Big] \ge \mathbb{E}\Big[\prod_{e'\in\mathcal{G}}\psi_{e'}\Big] - (n-1)\cdot\eta \ge 1-\eta'.$$

That is, (C1) is satisfied. Condition (C2.a) follows by arguing precisely as in the proof of Theorem 4.2, while (C2.b) is an immediate consequence of (6.1). Finally, for (C3) fix $e \in \mathcal{H}$ and let $i \in [n]$ be such that $e = [n] \setminus \{i\} \in \mathcal{H}$. Also let $\mathcal{G} \subseteq \mathcal{H} \setminus \{e\}$ be nonempty. By the choice of $\eta'$, it is enough to show that

$$\mathbb{E}[\boldsymbol{\nu}^{\ell}_{e,\mathcal{G}}] \leq C + (|\mathcal{G}| \cdot \ell + 1)\eta. \tag{6.20}$$

To this end, set

$$\varDelta := \Big| \mathbb{E}\Big[ \prod_{\omega=0}^{\ell-1} \prod_{e' \in \mathcal{G}} \boldsymbol{\nu}_{e'}(\mathbf{x}_{e' \setminus \{i\}}, x_i^{(\omega)}) \Big] - \mathbb{E}\Big[ \prod_{\omega=0}^{\ell-1} \prod_{e' \in \mathcal{G}} \boldsymbol{\psi}_{e'}(\mathbf{x}_{e' \setminus \{i\}}, x_i^{(\omega)}) \Big] \Big|$$

(both expectations are over the space $\boldsymbol{X}_e \times X_i^{\ell}$) and note that, by (I),

$$\mathbb{E}[\boldsymbol{\nu}^{\ell}_{e,\mathcal{G}}] \leq \varDelta + C + \eta.$$

Next, by enumerating the set $\mathcal{G} \times \{0, \dots, \ell - 1\}$ and applying a telescoping argument as in (6.19), we see that $\varDelta$ is bounded by a sum of $|\mathcal{G}| \cdot \ell$ terms each of which has the form of the quantity on the left-hand side of (6.16). Therefore, by Lemma 6.3, (6.20) is satisfied, and so the entire proof of Theorem 4.3 is complete.

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