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Knot-theoretic ternary groups

by

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Abstract. We describe various properties and give several characterizations of ternary groups satisfying two axioms derived from the third Reidemeister move in knot theory. Using special attributes of such ternary groups, such as semi-commutativity, we construct a ternary invariant of curves immersed in compact surfaces, considered up to flat Reidemeister moves.

1. Introduction and motivation. The subject of our paper, knot-theoretic ternary groups, is at the intersection of two classical areas: ternary group theory and knot theory. Ternary groups have a long history. As far back as 1904, E. Kasner considered generalizing the properties of binary groups [15]. The complete formulation of the concept of n-ary groups, generalizing binary groups, appeared in Dörnte's paper [3]. Independently, in 1932, Lehmer [11] considered a structure he termed *triplex*, which in Dörnte's terminology is an abelian ternary group. For a thorough introduction to the subject of n-ary groups, see [15].

Knot theory has its roots in the study of embeddings of simple closed curves in a three-dimensional space $\mathbb{R}^3$ (or $\mathbb{S}^3$). A single such curve is called a *knot*, and a collection of knots is a *link*. Knots and links are analyzed using *diagrams*, that is, projections on the plane that involve double points. Two link diagrams represent the same link if and only if one can be obtained from the other by a finite sequence of *Reidemeister moves* of type I, II and III, and planar isotopy. A *link invariant* is a function defined on diagrams that is not changed by Reidemeister moves and planar isotopy. For

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more details about these notions, and their various generalizations, see for example [9].

One of the main goals of knot theory is to find strong and at the same time computable link invariants. A possible approach to this problem is to use assignments (called *colorings*) of elements of some binary groupoid $(X, *)$ to the arcs of a given link diagram. If the number of colorings is to be unchanged by Reidemeister moves, the binary operation $*$ has to satisfy certain conditions. In this way one obtains the axioms for racks and quandles, with self-distributivity corresponding to the Reidemeister move of type III (see e.g. [8]).

Instead of coloring arcs of a link diagram using a binary operation $*$, one can color the regions of the complement of the diagram using a ternary algebra $(A, [\,])$. Again, if the number of ternary colorings is to be an invariant, the operation $[\,]$ has to satisfy some axioms. The first and second Reidemeister move require $(A, [\,])$ to be a ternary quasigroup. The third Reidemeister move yields two conditions:

$$[[abc]cd] = [[ab[bcd]][bcd]d], \tag{1.1}$$

$$[ab[bcd]] = [a[abc][[abc]cd]], \tag{1.2}$$

for all $a, b, c, d \in A$. They are illustrated in Fig. 1, where the colors of the regions in the bottom left part of the figure have to be equal to the colors of the corresponding regions on the right, and the flat version of the third Reidemeister move is shown. These axioms appeared in [13], and in [14] a homology theory was developed for the structures in which they hold. Knot-theoretic ternary groups are ternary groups satisfying (1.1) and (1.2).

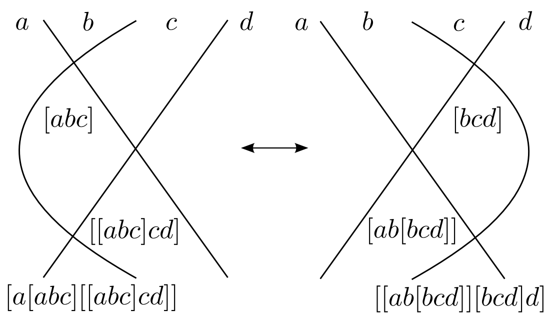

Fig. 1. Ternary colorings and the third Reidemeister move (flat version)

The paper is organized as follows. In Section 2 we collect basic results on ternary groups and characterize knot-theoretic ternary groups. In Section 3 we focus on idempotent ternary groups (all of them are knot-theoretic ones). Section 4 is devoted to representation and isomorphism problems for knot-theoretic ternary groups. In particular, we enumerate such ternary groups with a small number of elements. Section 5 provides a further characterization of knot-theoretic ternary groups. Finally, in Section 6 we construct an invariant of curves immersed in compact surfaces, considered up to flat Reidemeister moves.

2. Basic properties of knot-theoretic ternary groups. We begin with some preliminary definitions.

A *ternary groupoid* is a non-empty set A equipped with a ternary operation $[\,]\colon A^3 \to A$. It is denoted by $(A, [\,])$.

A ternary groupoid $(A, [\,])$ is called a *ternary quasigroup* if for any $a, b, c \in A$ each of the following equations is uniquely solvable for $z \in A$:

$$[zab] = c, \tag{2.1}$$

$$[azb] = c, \tag{2.2}$$

$$[abz] = c. \tag{2.3}$$

Remark 2.1. Each ternary quasigroup $(A, [\,])$ has the left, middle and right cancellation property. For example, by the unique solvability of (2.1) one finds that for all $x, y, a, b \in A$, $[xab] = [yab]$ always implies $x = y$ (*left cancellativity*). *Middle cancellativity* follows from (2.2), and *right cancellativity* from (2.3).

In a ternary quasigroup $(A, [\,])$, for every $a \in A$, the unique solution of the equation $[aza] = a$ is called *a skewed* and denoted by $\bar{a}$.

The property of associativity of binary operations can be generalized to the ternary case. We say that an operation $[\,]\colon A^3 \to A$ is *associative* if for all $a, b, c, d, e \in A$,

$$[[abc]de] = [a[bcd]e] = [ab[cde]].$$

From this condition it follows that in expressions (of length $2k-1$) obtained by applying the operation $[\,]$ k times, different placements of parentheses yield the same final result. An algebra $(A, [\,])$ with one ternary associative operation is called a *ternary semigroup*. An associative ternary quasigroup is called a *ternary group*.

If $(A, [\,])$ is a ternary group, then by [1, Theorem 10] we have, for all $a, b, c \in A$,

$$[\bar{a}aa] = [a\bar{a}a] = [aa\bar{a}] = a, \tag{2.4}$$

$$[ba\bar{a}] = [b\bar{a}a] = [a\bar{a}b] = [\bar{a}ab] = b, \tag{2.5}$$

$$\overline{[abc]} = [\bar{c}\bar{b}\bar{a}], \tag{2.6}$$

$$\bar{\bar{a}} = a. \tag{2.7}$$

The pair $(a, \bar{a})$ (or $(\bar{a}, a)$) plays the role of a *binary identity* [17].

A ternary operation $P\colon A^3 \to A$ is called a *Mal'tsev operation* if $P(a,b,b) = P(b,b,a) = a$ for all $a,b \in A$. Then the ternary algebra (A,P) is called a *Mal'tsev algebra*. In this way, a ternary group $(A,[\,])$ is Mal'tsev if for all $a,b \in A$,

$$[abb] = [bba] = a.$$

In general, each ternary group $(A,[\,])$ has at least one Mal'tsev *derived* operation $P(x,y,z) = [x\bar{y}z]$.

We say that a ternary algebra $(A,[\,])$ is

- *idempotent* if for every $a \in A$,
$$[aaa] = a;$$
- $(1,2)$-*associative* if for all $a,b,c,d,e \in A$,
$$[[abc]de] = [a[bcd]e];$$
- $(2,3)$-*associative* if for all $a,b,c,d,e \in A$,
$$[a[bcd]e] = [ab[cde]];$$
- *commutative* if for all $a,b,c \in A$,
$$[abc] = [bac] = [acb] = [cba];$$
- *semi-commutative* (or *semi-abelian*) if for all $a,b,c \in A$,
$$[abc] = [cba];$$
- *entropic* (or *medial*) if for all $a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3 \in A$,
$$[[a_1a_2a_3][b_1b_2b_3][c_1c_2c_3]] = [[a_1b_1c_1][a_2b_2c_2][a_3b_3c_3]].$$

An element $e \in A$ is said to be *left* (resp. *middle*, *right*) *neutral* if for every $a \in A$,

$$[eea] = a \quad (\text{resp. } [eae] = a,\ [aee] = a). \tag{2.8}$$

An element $e \in A$ is called *neutral* if it is left, middle and right neutral.

Remark 2.2. In a semi-commutative ternary groupoid the conditions (1.1) and (1.2) are equivalent. Indeed, by (1.1) and semi-commutativity we have

$$[ab[bcd]] = [[bcd]ba] = [[dcb]ba] = [[dc[cba]][cba]a] = [a[abc][[abc]cd]].$$

On the other hand, by (1.2) and semi-commutativity,

$$[[abc]cd] = [dc[cba]] = [d[dcb][[dcb]ba]] = [[ab[bcd]][bcd]d].$$

Remark 2.3. A semi-commutative $(1,2)$-associative ternary groupoid is a ternary semigroup.

REMARK 2.4. Each Mal'tsev algebra is idempotent. Moreover, a ternary group $(A,[\,])$ is idempotent if and only if $a=\bar{a}$ for each $a\in A$. In particular, each idempotent ternary group is semi-commutative:

$$[abc]=\overline{[abc]}=[\bar{c}\bar{b}\bar{a}]=[cba].$$

THEOREM 2.5 ([4, Corollary 9]). *A ternary group $(A,[\,])$ is commutative if and only if for all $a,b\in A$,*

$$[ab\bar{a}]=b \quad \text{or} \quad [\bar{a}ba]=b.$$

THEOREM 2.6 ([7, Theorem 3]). *A ternary group is semi-commutative if and only if it is entropic.*

LEMMA 2.7. *Let $(A,[\,])$ be a ternary group. Then the following conditions are equivalent for all $a,b,c\in A$:*

$$[abb]=\bar{a}=[bba], \tag{2.9}$$

$$[\bar{a}bc]=[a\bar{b}c]=[ab\bar{c}]. \tag{2.10}$$

Proof. By (2.9),

$$[\bar{a}cd]=[[abb]cd]=[a[bbc]d]=[a\bar{c}d]=[a[cdd]d]=[ac[ddd]]=[ac\bar{d}].$$

On the other hand, by (2.10),

$$[abb]=[\bar{\bar{a}}bb]=[\bar{a}\bar{b}b]=\bar{a}, \qquad [bba]=[bb\bar{\bar{a}}]=[b\bar{b}\bar{a}]=\bar{a}. \ \blacksquare$$

LEMMA 2.8. *Let $(A,[\,])$ be a semi-commutative ternary group which satisfies, for all $a,b\in A$,*

$$[abb]=\bar{a}. \tag{2.11}$$

Then $(A,[\,])$ satisfies the conditions (1.1) and (1.2).

Proof. By Remark 2.2, it suffices to check only one condition, e.g. (1.1). By (2.11), Lemma 2.7 and semi-commutativity we have

$$\begin{aligned}[][[ab[bcd]][bcd]d]&=[[abb][cdb][cdd]]=[\bar{a}[cdb]\bar{c}]\\&=[\bar{c}[cdb]\bar{a}]=[db\bar{a}]=[ab\bar{d}]=[ab[ccd]]=[[abc]cd].\ \blacksquare\end{aligned}$$

DEFINITION 2.9. A ternary group $(A,[\,])$ which satisfies (1.1) and (1.2) is called a *knot-theoretic ternary group*.

LEMMA 2.10. *Let $(A,[\,])$ be a knot-theoretic ternary group. Then $(A,[\,])$ is semi-commutative and satisfies (2.11).*

Proof. By associativity, (1.1), (2.4), and the middle and right cancellation properties we obtain

$$[ab[ccc]]=[[abc]cc]=[[ab[bcc]][bcc]c]=[a[b[bcc]b][ccc]], \quad \text{hence}$$
$$b=[b[bcc]b]=[bb[ccb]], \quad \text{hence} \quad [ccb]=\bar{b}.$$

Similarly, by associativity, (1.2), (2.4), middle and left cancellation properties we obtain

$$[[bbb]cd] = [bb[bcd]] = [b[bbc][[bbc]cd]] = [[bbb][c[bbc]c]d], \quad \text{hence}$$
$$c = [c[bbc]c] = [[cbb]cc], \quad \text{hence} \quad [cbb] = \bar{c}.$$

Finally, by Lemma 2.7,

$$[abc] = [\overline{[abc]}aa] = [[\bar{c}\bar{b}\bar{a}]aa] = [\bar{c}\bar{b}[\bar{a}aa]] = [\bar{c}\bar{b}a] = [cba],$$

which means that $(A,[\,])$ is semi-commutative. ∎

Corollary 2.11. *Let $(A,[\,])$ be a ternary group. $(A,[\,])$ is a knot-theoretic ternary group if and only if $(A,[\,])$ is semi-commutative and satisfies* (2.11). *In particular, in each knot-theoretic group,*

$$[ccc] = \bar{c} \quad \textit{for every } c \in A.$$

Example 2.12. Let $k > 1$ be a natural number. Consider a cyclic group $(\mathbb{Z}_k, +)$ and let $a \in \mathbb{Z}_k$. Define on $\mathbb{Z}_k$ the ternary operation $[xyz] = x - y + z + a \pmod{k}$. Then $(\mathbb{Z}_k,[\,])$ is a knot-theoretic ternary group if and only if $2a = 0 \pmod{k}$ in $\mathbb{Z}_k$. For each even k, there are exactly two knot-theoretic groups constructed in this way:

(1) idempotent for $a = 0$,
(2) non-idempotent for a being the element of order 2 in $\mathbb{Z}_k$.

3. Idempotent ternary groups. By results of Section 2 one obtains the relation between idempotent (knot-theoretic) ternary groups and Mal'tsev ones.

Corollary 3.1. *Let $(A,[\,])$ be a ternary group. The following conditions are equivalent:*

(1) *$(A,[\,])$ is idempotent,*
(2) *$(A,[\,])$ is idempotent knot-theoretic,*
(3) *$(A,[\,])$ is Mal'tsev.*

But it is also possible to obtain an idempotent (knot-theoretic) ternary group from any semi-commutative ternary group. The aim of this section is to provide details of that construction.

Lemma 3.2. *Let $(A,[\,])$ be a ternary group, and define a new (derived) ternary operation by*

$$P(a,b,c) = [a\bar{b}c]$$

for $a,b,c \in A$. Then (A,P) is a Mal'tsev algebra which satisfies the conditions (1.1) *and* (1.2) *for all $a,b,c,d \in A$.*

Proof. It is easy to show that in the algebra (A, P) the following identities hold:

$$P(P(x,y,z),z,t) = P(x,y,t), \tag{3.1}$$

$$P(x,y,P(y,z,t)) = P(x,z,t), \tag{3.2}$$

for all $x, y, z, t \in A$. ∎

In fact, the identities (3.1), (3.2) are satisfied in the algebra (A, P) independently of the order of variables in the definition of P, e.g. one can also take $P(a,b,c) = [b\bar{a}c]$, etc.

LEMMA 3.3. *Let $(A, [\,])$ be a ternary group, and let*

$$P(a,b,c) = [a\bar{b}c].$$

Then $(A, [\,])$ is semi-commutative if and only if (A, P) is a ternary semi-group.

Proof. For all $a, b, c, d, e \in A$ we have

$$P(P(a,b,c),d,e) = [[a\bar{b}c]\bar{d}e] = [a\bar{b}[c\bar{d}e]] = P(a,b,P(c,d,e)).$$

Moreover,

$$P(a,P(b,c,d),e) = P(P(a,b,c),d,e) \iff [a\overline{[b\bar{c}d]}e] = [[a\bar{b}c]\bar{d}e] \iff$$
$$[a[\bar{d}c\bar{b}]e] = [a[\bar{b}c\bar{d}]e] \iff [\bar{d}c\bar{b}] = [\bar{b}c\bar{d}]. \;∎$$

COROLLARY 3.4. *Let $(A, [\,])$ be a semi-commutative ternary group, and let*

$$P(a,b,c) = [a\bar{b}c].$$

Then (A, P) is a Mal'tsev ternary group.

Proof. Since $\bar{\bar{x}} = x$, the map $x \mapsto \bar{x}$ is 1-1, and the quasigroup property of (A, P) follows from the fact that $(A, [\,])$ is a ternary quasigroup. By Lemma 3.3, the ternary quasigroup (A, P) is a ternary group. ∎

EXAMPLE 3.5. Consider the two types of knot-theoretic ternary groups introduced in Example 2.12. For even k, we take the cyclic group $(\mathbb{Z}_k, +)$ with the ternary operation

$$[xyz] = x - y + z + k/2 \pmod{k},$$

and the unary operation

$$\bar{y} = y + k/2 \pmod{k}$$

for any $x, y, z \in \mathbb{Z}_k$. Now defining

$$P(x,y,z) = x - \bar{y} + z + k/2 = x - y + z \pmod{k},$$

we obtain an idempotent ternary group $(\mathbb{Z}_k, P)$.

4. Isomorphisms of knot-theoretic ternary groups. The aim of this section is to recognize when two knot-theoretic ternary groups are isomorphic. We start by formulating representation theorems for knot-theoretic ternary groups.

DEFINITION 4.1. Let $(A,[\,])$ be a ternary algebra. The binary groupoid $(A,*)$, where $x*y=[xay]$ for some fixed $a\in A$, is called a *retract* of $(A,[\,])$, and is denoted by $\mathrm{ret}_a(A,[\,])$.

REMARK 4.2. It is well known (see e.g. [5]) that a retract $\mathrm{ret}_a(A,[\,])$ of a semi-commutative ternary group is an abelian group with neutral element $\bar{a}$ and the inverse of $x\in A$ given by $\overline{[axa]}$.

LEMMA 4.3 ([1, Corollary 15]). *Let $(A,[\,])$ be a semi-commutative ternary group and let $a\in A$. Then for the abelian group $(A,+)=\mathrm{ret}_a(A,[\,])$ and its involutive automorphism $\phi(x)=[\bar{a}xa]$, one has*

$$[xyz]=x+\phi(y)+z+b,\quad \textit{where}\quad b=[\bar{a}\bar{a}\bar{a}].$$

THEOREM 4.4. *Let $(A,[\,])$ be a ternary group. The following conditions are equivalent:*

(1) $(A,[\,])$ *is a knot-theoretic ternary group;*
(2) *there is an abelian group $(A,+)$ with neutral element $e\in A$ such that*

$$[xyz]=x-y+z+\bar{e},$$

and the mapping $\bar{\ }: A\to A$ is an involutive bijection such that $\bar{x}=x+\bar{e}$ for every $x\in A$.

Proof. Let $(A,[\,])$ be a knot-theoretic ternary group. Then it is enough to take $(A,+)=\mathrm{ret}_{\bar{e}}(A,[\,])$, which, by Remark 4.2, is an abelian group with neutral element $\bar{\bar{e}}=e$ and $-x=[e\bar{x}e]$. Clearly, $x+\bar{e}=[x\bar{e}\bar{e}]=\bar{x}$. The statement follows because $\phi(y)=[\bar{\bar{e}}y\bar{e}]=[e\bar{y}e]=-y$ and $b=[eee]=\bar{e}$.

Conversely, let $(A,+)$ be as in (2). It is evident that $(A,[\,])$ with $[xyz]=x-y+z+\bar{e}$ is a knot-theoretic ternary group. ∎

COROLLARY 4.5. *Each knot-theoretic ternary group $(A,[\,])$ is determined by an abelian group $(A,+)$ and an element $a\in A$ of order one or two in $(A,+)$. Then for all $x,y,z\in A$,*

$$[xyz]=x-y+z+a\quad \textit{and}\quad \bar{x}=x+a.$$

In particular, for all $x,y\in A$,

$$[xy\bar{x}]=x-y+\bar{x}+a=x-y+x+a+a=2x-y. \tag{4.1}$$

Deviatov [2] constructed combinatorial invariants of classical knots based on colorings of regions of a knot diagram by elements of some finite ring R. His coloring requirements involve the equation $pa+b-c-pd=0$ for a, b, c, $d\in R$ and an invertible element $p\in R$. If we extract the ternary operation

$f(a,b,c) = p^{-1}(b-c)+a$ from this equation, then for $p=-1$ one obtains our idempotent case $[abc] = a-b+c$ for $R = \mathbb{Z}_n$.

We denote by $\mathcal{T}((A,+),a)$ the knot-theoretic ternary group $(A,[\,])$ described in Corollary 4.5, where we always assume that $(A,+)$ is an abelian group and a is its fixed element of order one or two. Then $\mathrm{ret}_a(\mathcal{T}((A,+),a)) = (A,+)$. The group $(A,+)$ is called the *associated group* of $\mathcal{T}((A,+),a)$.

LEMMA 4.6. *Let $(A,[\,]) = \mathcal{T}((A,+),a)$ be a knot-theoretic ternary group. Then for each $b \in A$, the groups $ret_b(A,[\,])$ and $(A,+)$ are isomorphic.*

Proof. The bijection $h\colon A \to A$, $x \mapsto x-b+a$, is an isomorphism of the groups $(A,*) = \mathrm{ret}_b(A,[\,])$ and $(A,+)$. Indeed, for $x,y \in A$,

$$\begin{aligned} h(x*y) = h([xby]) &= h(x-b+y+a) = x-b+y+a-b+a \\ &= (x-b+a)+(y-b+a) = h(x)+h(y). \ \blacksquare \end{aligned}$$

DEFINITION 4.7. A ternary group $(A,[\,])$ is *g-derived from* a binary group $(A,*)$ if for all $x,y,z \in A$, $[xyz] = [xyz]_g := x*y*z*g$ for some $g \in A$. If g is a neutral element in $(A,*)$, then we simply say that $(A,[\,])$ is *derived from* (or *reducible to*) $(A,*)$.

Let $(A,[\,]) = \mathcal{T}((A,+),a)$ be a commutative knot-theoretic ternary group. Then in the associated abelian group $(A,+)$,

$$-x = [e\bar{x}e] = [ex\bar{e}] = [xe\bar{e}] = x$$

for every $x \in A$. This immediately implies

COROLLARY 4.8. *Let $(A,[\,]) = \mathcal{T}((A,+),a)$ be a commutative knot-theoretic ternary group. The associated group $(A,+)$ is an elementary 2-group (Boolean group). Moreover, $(A,[\,])$ is a-derived from $(A,+)$.*

Recall that every elementary 2-group is abelian and is a direct sum of cyclic groups of order 2. Hence, in the finite case, every such group is isomorphic to $\mathbb{Z}_2^k$ for some natural number k.

COROLLARY 4.9. *Each finite commutative knot-theoretic ternary group is a-derived from $\mathbb{Z}_2^k$ for some natural k and some $a \in \mathbb{Z}_2^k$. In particular, each finite commutative knot-theoretic ternary group has 2^k elements for some natural k.*

LEMMA 4.10. *Let $(A,[\,])$ be a ternary group derived from a group $(A,*)$. Then $(A,[\,])$ satisfies (2.11) if and only if $(A,*)$ is an elementary 2-group.*

Proof. By [5] in a ternary group derived from the binary group $(A,*)$, the skewed element $\bar{a}$ coincides with the inverse of a in $(A,*)$. Let $e \in A$ be the neutral element in $(A,*)$. Let $(A,[\,])$ satisfy (2.11). Then for every $x \in A$,

$$x^{-1} = \bar{x} = [xee] = x*e*e = x, \quad \text{hence} \quad x^2 = e.$$

If $(A, *)$ is an elementary 2-group, then

$$[abb] = a * b * b = a * e = a = a^{-1} = \bar{a}. \blacksquare$$

COROLLARY 4.11. *The following conditions are equivalent:*

(1) $(A, [\,])$ *is a commutative ternary Mal'tsev group;*
(2) $(A, [\,])$ *is a knot-theoretic ternary group derived from a binary group* $(A, *)$*;*
(3) $(A, [\,])$ *is a knot-theoretic ternary group derived from an elementary* 2*-group.*

EXAMPLE 4.12. The ternary group $(\mathbb{Z}_2, [\,])$ with $[xyz] = x+y+z \pmod 2$ is the only two-element knot-theoretic ternary group derived from a binary group. The knot-theoretic ternary groups from Example 2.12 for $k > 2$ are not derived from any binary group.

Having the representation theorems for knot-theoretic ternary groups, we can move to isomorphism problems. We start by recalling a general definition of a homomorphism of ternary groups.

DEFINITION 4.13. Let $(A_1, [\,]_1)$ and $(A_2, [\,]_2)$ be ternary groups with unary operations denoted by $^{-1}$ and $^{-2}$, respectively. A mapping $h\colon A_1 \to A_2$ is a *homomorphism* of ternary groups if h preserves both fundamental operations, i.e.

$$h([abc]_1) = [h(a)h(b)h(c)]_2, \tag{4.2}$$

$$h(\bar{a}^1) = \overline{h(a)}^2 \tag{4.3}$$

for all $a, b, c \in A_1$. A bijective homomorphism is an *isomorphism* and an isomorphism from $(A_1, [\,]_1)$ to itself is an *automorphism*.

Note that the property (4.3) follows from (4.2) and the definition of a skewed element. More precisely,

$$h(a) = h([a\bar{a}^1 a]_1) = [h(a)h(\bar{a}^1)h(a)]_2,$$

so $h(\bar{a}^1)$ must be $h(a)$ skewed. This means that (4.2) suffices to define a homomorphism of ternary groups.

The following result concerning isomorphisms between ternary groups was proved in [6].

THEOREM 4.14 ([6, Corollary 7]). *The following statements are equivalent for ternary groups* $(A_1, [\,]_1)$ *and* $(A_2, [\,]_2)$*:*

(1) $(A_1, [\,]_1)$ *and* $(A_2, [\,]_2)$ *are isomorphic,*
(2) *for every* $c \in A_1$ *there exists a group isomorphism*

$$h\colon \operatorname{ret}_c(A_1, [\,]_1) \to \operatorname{ret}_d(A_2, [\,]_2),$$

such that $d = h(c)$, $h([\bar{c}\bar{c}\bar{c}]_1) = [\bar{d}\bar{d}\bar{d}]_2$ *and* $h([cx\bar{c}^1]_1) = [dh(x)\overline{d}^2]_2$ *for each* $x \in A_1$,

(3) *for some* $c \in A_1$ *there exists a group isomorphism*

$$h\colon \operatorname{ret}_c(A_1, [\,]_1) \to \operatorname{ret}_d(A_2, [\,]_2),$$

such that $d = h(c)$, $h([\bar{c}\bar{c}\bar{c}]_1) = [\bar{d}\bar{d}\bar{d}]_2$ *and* $h([cx\bar{c}^1]_1) = [dh(x)\overline{d}^2]_2$ *for each* $x \in A_1$.

Lemma 4.6 and the conditions (2.11) and (4.1) satisfied in each knot-theoretic ternary group simplify Theorem 4.14. Now it becomes:

THEOREM 4.15. *Suppose that* $(A_1, [\,]_1) = \mathcal{T}((A_1, +_1), a)$ *and* $(A_2, [\,]_2) = \mathcal{T}((A_2, +_2), b)$ *are two knot-theoretic ternary groups. Then the following statements are equivalent:*

(1) $(A_1, [\,]_1)$ *and* $(A_2, [\,]_2)$ *are isomorphic;*

(2) *there exists a group isomorphism* $h\colon (A_1, +_1) \to (A_2, +_2)$ *such that* $b = h(a)$.

In particular, using Corollary 4.9, we can characterize all non-isomorphic finite commutative knot-theoretic ternary groups.

COROLLARY 4.16. *Let* k *be a natural number and* $a, b \in \mathbb{Z}_2^k$. *Two commutative knot-theoretic ternary groups* $(\mathbb{Z}_2^k, [\,]_a)$ *and* $(\mathbb{Z}_2^k, [\,]_b)$ *are isomorphic if and only if there is a group automorphism* $h\colon \mathbb{Z}_2^k \to \mathbb{Z}_2^k$ *such that* $b = h(a)$.

Enumerating knot-theoretic ternary groups. We can apply the above results to enumerate isomorphism classes of knot-theoretic ternary groups. By Theorem 4.15, knot-theoretic ternary groups associated with non-isomorphic abelian groups are non-isomorphic. Moreover, if there is a group automorphism $h\colon (A, +) \to (A, +)$, then two knot-theoretic ternary groups $(A, [\,]) = \mathcal{T}((A, +), a)$ and $(A, \langle\,\rangle) = \mathcal{T}((A, +), h(a))$ are isomorphic.

EXAMPLE 4.17. For $(A, +) = (\mathbb{Z}_2 \times \mathbb{Z}_4, +)$, there are three non-isomorphic knot-theoretic ternary groups: $\mathcal{T}((A, +), (0, 2))$, $\mathcal{T}((A, +), (1, 0))$ (isomorphic to $\mathcal{T}((A, +), (1, 2))$) and idempotent $\mathcal{T}((A, +), (0, 0))$.

For each abelian group there is exactly one idempotent knot-theoretic ternary group. By Corollary 4.5, these are the only knot-theoretic ternary groups with an odd number of elements.

Further, it is known that for each $k \in \mathbb{N}$ and two non-neutral elements $a \neq b \in \mathbb{Z}_2^k$ there is a group automorphism $h\colon \mathbb{Z}_2^k \to \mathbb{Z}_2^k$ such that $b = h(a)$. Then by Corollaries 4.9 and 4.16, for each $k \in \mathbb{N}$ there are exactly two non-isomorphic commutative knot-theoretic ternary groups: idempotent $(\mathbb{Z}_2^k, [\,]_e)$ and non-idempotent $(\mathbb{Z}_2^k, [\,]_a)$, where e is the neutral element in the group $(\mathbb{Z}_2^k, +)$ and a is an arbitrary element in $\mathbb{Z}_2^k - \{e\}$. In Table 1 we list the numbers of knot-theoretic ternary groups of size ≤ 64 up to isomorphism.

Table 1. The number of knot-theoretic ternary groups of size n, up to isomorphism

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
all	1	2	1	4	1	2	1	7	2	2	1	4	1	2	1	12	1
idempotent	1	1	1	2	1	1	1	3	2	1	1	2	1	1	1	5	1
commutative	1	2	0	2	0	0	0	2	0	0	0	0	0	0	0	2	0

n	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
all	4	1	4	1	2	1	7	2	2	3	4	1	2	1	19
idempotent	2	1	2	1	1	1	3	2	1	3	2	1	1	1	7
commutative	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2

n	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
all	1	2	1	10	1	2	1	7	1	2	1	4	2	2	1	10
idempotent	1	1	1	5	1	1	1	3	1	1	1	2	2	1	1	4
commutative	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

n	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
all	2	4	1	4	1	6	1	7	1	2	1	4	1	2	2	30
idempotent	2	2	1	2	1	3	1	3	1	1	1	2	1	1	2	11
commutative	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2

5. Further characterization of knot-theoretic ternary groups. Now we present some characterizations of knot-theoretic ternary groups using sets of properties that do not include associativity and unique solvability.

LEMMA 5.1 ([5, Corollary 4, for $n = 3$]). *A ternary semigroup $(A, [\,])$ is a ternary group if and only if for all $a, b \in A$ there are $x, y \in A$ such that*

$$[xaa] = [aay] = b. \tag{5.1}$$

LEMMA 5.2. *Let $(A, [\,])$ be a semi-commutative, $(1, 2)$-associative ternary groupoid. Assume that there exists a unary operation $^{-}\colon x \mapsto \bar{x}$ on A such that for all $a, b \in A$ the conditions (2.11) and*

$$[\bar{a}ab] = b \tag{5.2}$$

are satisfied. Then $(A, [\,])$ is a knot-theoretic ternary group.

Proof. By Remark 2.3, $(A, [\,])$ is a ternary semigroup. By (2.11) and (5.2),

$$\bar{\bar{a}} = [\bar{a}aa] = a.$$

Thus, by semi-commutativity and Lemma 5.1 (substituting $x = y = \bar{b}$ in (5.1)), $(A, [\,])$ is a ternary semigroup. By Corollary 2.11 it is knot-theoretic. ■

THEOREM 5.3. *A ternary groupoid $(A, [\,])$ is a knot-theoretic ternary group if and only if it satisfies, for all $x, a, b, c, d, e \in A$, the following conditions:*

$$[[abc]de] = [[adc]be], \tag{5.3}$$

$$[a[bcd]e] = [a[bed]c], \tag{5.4}$$

and there exists on A *a unary operation* $^{-}\colon x \mapsto \bar{x}$ *such that* (2.11) *holds and*

$$[\bar{a}ax] = [xa\bar{a}] = x. \tag{5.5}$$

Proof. Suppose that $(A, [\,])$ is a knot-theoretic ternary group. Then by Corollary 2.11 it is semi-commutative and satisfies the conditions (5.5) and (2.11). Associativity and semi-commutativity imply the exchange conditions (5.3) and (5.4).

Now let $(A, [\,])$ satisfy (5.3)–(5.5) and (2.11). By (2.11) and (5.5), as in the proof of Lemma 5.2, $\bar{\bar{a}} = a$. Thus, from (5.5),

$$[a\bar{a}x] = [\bar{\bar{a}}\bar{a}x] = x \quad \text{and} \quad [x\bar{a}a] = [x\bar{a}\bar{\bar{a}}] = x.$$

From (5.3) we get semi-commutativity:

$$[abc] = [[c\bar{c}a]bc] = [[cba]\bar{c}c] = [cba].$$

Now, from (5.3) and (5.4), we obtain $(1,2)$-associativity:

$$[a[bcd]e] = [[ab\bar{b}][bcd]e] = [[a[bcd]\bar{b}]be] = [[a[b\bar{b}d]c]be] = [[adc]be] = [[abc]de].$$

By Lemma 5.2, $(A, [\,])$ is a knot-theoretic ternary group. ∎

Theorem 5.4. *A ternary groupoid* $(A, [\,])$ *is a knot-theoretic ternary group if and only if it satisfies the conditions* (5.5) *and* (2.11) *and is entropic.*

Proof. If $(A, [\,])$ is a knot-theoretic ternary group then, by Corollary 2.11, it is a semi-commutative ternary group, and therefore it is entropic. Also, from the same corollary, it satisfies (2.11), and each ternary group satisfies (5.5).

Conversely, suppose that an entropic ternary groupoid $(A, [\,])$ satisfies (5.5) and (2.11). Then from (2.11) and (5.5), as in the proof of Theorem 5.3, we obtain $[a\bar{a}x] = [x\bar{a}a] = x$ for all $a, x \in A$. From this and from the entropic property we get $(1,2)$-associativity and semi-commutativity:

$$[[abc]de] = [[abc][dc\bar{c}][\bar{d}de]] = [[ad\bar{d}][bcd][c\bar{c}e]] = [a[bcd]e],$$

$$[abc] = [[ab\bar{b}][\bar{a}ab][c\bar{a}a]] = [[a\bar{a}c][ba\bar{a}][\bar{b}ba]] = [cba].$$

By Lemma 5.2, $(A, [\,])$ is a knot-theoretic ternary group. ∎

Theorem 5.5. *A ternary groupoid* $(A, [\,])$ *is a knot-theoretic ternary group derived from a binary group if and only if it satisfies the condition* (5.3) *and each element in* A *is neutral.*

Proof. If $(A, [\,])$ is a knot-theoretic ternary group derived from a binary group, then it is a commutative and idempotent ternary group and by Corollary 2.11 it satisfies (2.11). This implies that (5.3) follows, and for all $a, b \in A$ we have $a = \bar{a}$ and

$$[abb] = [bab] = [bba] = a. \tag{5.6}$$

Now, let $(A, [\,])$ be a ternary groupoid satisfying (5.3) and (5.6). Then $(A, [\,])$ is idempotent, and (5.3) implies semi-commutativity:

$$[abc] = [[cca]bc] = [[cba]cc] = [cba].$$

Further, from semi-commutativity we get

$$[abc] = [[bab]bc] = [[bbb]ac] = [bac] = [cab] = [[aca]ab] = [acb],$$

and so $(A, [\,])$ is commutative. Now we show $(1, 2)$-associativity using commutativity and (5.3):

$$[[abc]de] = [[bac]de] = [[bdc]ae] = [a[bdc]e] = [a[bcd]e].$$

As before, by Lemma 5.2, $(A, [\,])$ is a knot-theoretic ternary group. From Corollary 4.11 it follows that $(A, [\,])$ is derived from a binary group. ∎

Theorem 5.6. *A ternary groupoid* $(A, [\,])$ *is a knot-theoretic ternary group if and only if it is semi-commutative, satisfies the conditions* (1.1) *and* (5.4)*, and there exists on* A *an involutive unary operation* $\bar{\ }: x \mapsto \bar{x}$ *such that* $[a\bar{a}x] = x$ *for all* $a, x \in A$.

Proof. If $(A, [\,])$ is a knot-theoretic ternary group, then by definition it satisfies (1.1). The skewing operation is involutive and satisfies $[a\bar{a}x] = x$ for all $a, x \in A$. Property (5.4) follows from associativity and semi-commutativity.

Conversely, suppose that a ternary groupoid $(A, [\,])$ has the properties from the second part of the theorem. Then semi-commutativity gives

$$[xa\bar{a}] = [\bar{a}ax] = [\bar{a}\bar{\bar{a}}x] = x \quad \text{and} \quad [x\bar{a}a] = [a\bar{a}x] = x.$$

By Remark 2.2 the second knot-theoretic property (1.2) follows from (1.1). Now as consequence of (1.2) for $c = \bar{b}$ and $d = b$ we obtain, for any $a, b \in A$,

$$[abb] = [ab[b\bar{b}b]] = [a[ab\bar{b}][[ab\bar{b}]\bar{b}b]] = [aaa], \quad \text{so} \quad [bba] = [abb] = [a\bar{a}\bar{a}] = \bar{a}.$$

From (1.1) for $c = b$ it follows that for $a, b, d \in A$,

$$[\bar{a}bd] = [[abb]bd] = [[ab[bbd]][bbd]d] = [[ab\bar{d}]\bar{d}d] = [ab\bar{d}].$$

From (5.4) we have

$$[\bar{a}bc] = [ab\bar{c}] = [a[\bar{b}cc]\bar{c}] = [a[\bar{b}\bar{c}c]c] = [a\bar{b}c].$$

Thus,

$$[\bar{a}\bar{b}c] = [a\bar{\bar{b}}c] = [abc].$$

Finally, we obtain $(2, 3)$-associativity:

$$\begin{aligned} [ab[cde]] &= [a[\bar{b}cc][cde]] = [a[\bar{b}[cde]c]c] = [a[\bar{b}[cce]d]c] \\ &= [a[\bar{b}\bar{e}d]c] = [a[bed]c] = [a[bcd]e]. \end{aligned}$$

From $(2, 3)$-associativity and property (5.4), we get $(1, 2)$-associativity:

$$[[abc]de] = [ed[abc]] = [e[dab]c] = [e[bad]c] = [e[bcd]a] = [a[bcd]e].$$

By Lemma 5.2, $(A, [\,])$ is a knot-theoretic ternary group. ∎

6. Geometric application of knot-theoretic ternary groups. We now use knot-theoretic ternary groups to study curves immersed in compact surfaces. We do it in two ways: first abstractly, via unoriented flat virtual links, and then concretely, by considering curves immersed in compact surfaces that need not be orientable, and can have boundary components.

6.1. Knot-theoretic ternary groups and flat virtual links. Here we recall basic definitions concerning flat virtual links; for a deeper introduction to virtual knot theory, see [10]. We only mention that in virtual link diagrams there are both classical and virtual crossings (marked by little circles). Virtual crossings appear when a diagram placed on a surface (of some genus g) is projected on the plane. In flat virtual knot diagrams classical crossings are replaced with their flat version, in which there is no indication which string of the crossing was higher before the projection; such flat crossings appear in the figures in this paper.

We now give a formal definition of flat virtual links.

Definition 6.1. A *flat virtual link diagram* is a 4-regular plane graph with two types of vertices (also called crossings): flat and virtual.

A *flat virtual link* is an equivlence class of flat virtual link diagrams generated by planar isotopy and the set of moves illustrated in Fig. 2. The left of the figure represents the *flat Reidemeister moves*, and the right the

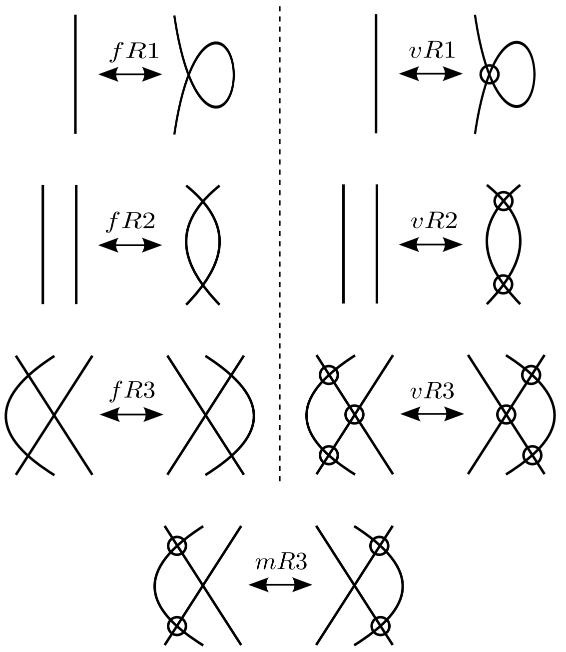

Fig. 2. Flat virtual moves

virtual Reidemeister moves; the last move is the *mixed Reidemeister type three move.*

In [12] two ternary operations were utilized, satisfying mixed axioms, for oriented virtual links. One operation was for the classical crossings, and the other for the virtual ones. Orientation is helpful in distinguishing the four regions around a crossing, but in our construction we will not assume it. Because of this lack of orientation, we need a simple way of collecting inputs for ternary operations, and this is achieved by using the structure of knot-theoretic ternary groups.

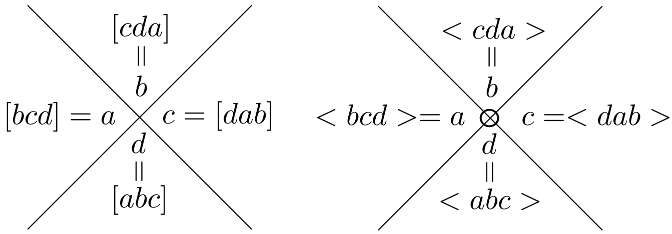

Fig. 3. Coloring of the regions around flat and virtual crossings

In our invariant, we use two knot-theoretic ternary group structures $(A,[\,])$ and $(A,\langle\,\rangle)$ on the same set A, the first for the flat crossings and the second for the virtual crossings. Let D be a diagram of a flat virtual link. Coloring conventions for flat and virtual crossings are shown in Fig. 3. More specifically, a color is a ternary product of the other three colors near the crossing taken in a cyclic order; it does not matter whether this order is clockwise or counter-clockwise, since any knot-theoretic ternary group is semi-commutative. If there is a consistent way of assigning group elements to the regions in the entire diagram D, then we call it a *knot-theoretic ternary group coloring* of D. Note that the properties of knot-theoretic ternary groups ensure that the four relations around the crossings in Fig. 3 are equivalent, for example:

$$a=[bcd] \iff b=[a\bar{d}\bar{c}]=[a\bar{\bar{d}}c]=[adc]=[cda],$$
$$a=[bcd] \iff c=[\bar{b}a\bar{d}]=[\bar{\bar{b}}ad]=[bad]=[dab],$$
$$a=[bcd] \iff d=[\bar{c}\bar{b}a]=[\bar{\bar{c}}ba]=[cba]=[abc].$$

Definition 6.2. We say that two ternary groups $(A,[\,])$ and $(A,\langle\,\rangle)$ are *compatible* if

$$[ab\langle bcd\rangle]=\langle a\langle abc\rangle[\langle abc\rangle cd]\rangle \tag{6.1}$$

for any $a,b,c,d\in A$.

Example 6.3. Any knot-theoretic ternary group is compatible with itself.

EXAMPLE 6.4. Let $(A,+)$ be a finite abelian binary group (of an even rank) with neutral element 0 and let x be an element of order 2. Consider two (non-isomorphic) knot-theoretic ternary groups: $(A,[\,]) = \mathcal{T}((A,+),0)$ and $(A,\langle\,\rangle) = \mathcal{T}((A,+),x)$. These groups are compatible. Specifically, for any $a,b,c,d \in A$, the condition (6.1) holds:

$$[ab\langle bcd\rangle] = a - b + (b - c + d + x) = a - c + d + x = \langle acd\rangle,$$
$$\langle a\langle abc\rangle[\langle abc\rangle cd]\rangle = a - \langle abc\rangle + \langle abc\rangle - c + d + x$$
$$= a - c + d + x = \langle acd\rangle.$$

Now we will show that there may be non-compatible knot-theoretic ternary groups defined on the same set.

EXAMPLE 6.5. Let $k > 1$ be a natural number. Let $(A,\cdot) \simeq (\mathbb{Z}_{2^k}, +_{2^k})$ and $(A,+) \simeq (\mathbb{Z}_2^k, +)$ be two non-isomorphic abelian groups of rank 2^k, and let 0 be the neutral element in $(A,+)$, b an element of order 2 in $(A,\cdot)$, and c an element of order 2 in $(A,+)$ such that $c^2 \neq 0^2$. Take two knot-theoretic ternary groups $(A,[\,]) = \mathcal{T}((A,\cdot),b)$ and $(A,\langle\,\rangle) = \mathcal{T}((A,+),c)$.

By (2.9) and Corollary 4.5 we have

$$[0c\langle ccc\rangle] = [0c(c+c)] = [0c0] = 0\cdot c^{-1}\cdot 0\cdot b = 0^2\cdot c^{-1}\cdot b$$

and

$$\langle 0\langle 0cc\rangle[\langle 0cc\rangle cc]\rangle = \langle 0(0+c)[(0+c)cc]\rangle = \langle 0(0+c)((0+c)\cdot b)\rangle$$
$$= 0 + (0+c) + (0+c)\cdot b + c = c\cdot b.$$

Since by assumption $c^2 \neq 0^2$, the condition (6.1) does not hold.

THEOREM 6.6. *For a pair of compatible knot-theoretic ternary groups* $(A,[\,])$ *and* $(A,\langle\,\rangle)$, *and a diagram* D *of a flat virtual link, the number of knot-theoretic ternary group colorings of* D *is not changed by Reidemeister moves used for flat virtual links.*

Proof. The argument is analogous for strictly flat moves and strictly virtual moves (Fig. 2). The first flat/virtual move does not change the number of colorings, because the color of the corner in the kink is uniquely determined by the colors of the other three corners of the crossing. The invariance under the second flat/virtual move is a consequence of the involutions $[a[abc]c] = b$ and $\langle a\langle abc\rangle c\rangle = b$ that hold in knot-theoretic ternary groups, see Fig. 4. The axioms (1.1) and (1.2) (for both $[\,]$ and $\langle\,\rangle$) make the number of colorings invariant under the third flat/virtual Reidemeister move; see Fig. 1 for an illustration for the operation $[\,]$. Finally, the invariance under the mixed move is shown in Fig. 5. It requires the condition (6.1) and another condition that follows from it by semi-commutativity:

$$[\langle abc\rangle cd] = \langle [ab\langle bcd\rangle]\langle bcd\rangle d\rangle. \blacksquare$$

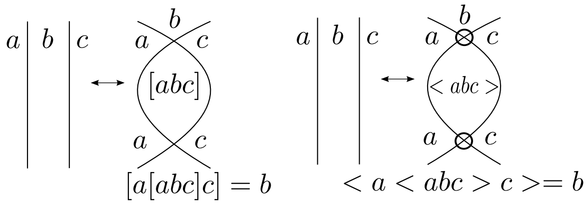

Fig. 4. Involutions from the second flat and virtual Reidemeister moves

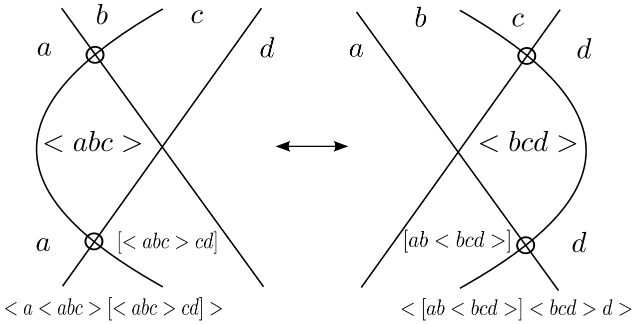

Fig. 5. A flat mixed move and knot-theoretic ternary group coloring

Example 6.7. Fig. 6 illustrates how knot-theoretic ternary group colorings can be used to establish that a given diagram represents a non-trivial flat virtual link. The knot-theoretic ternary groups $(A, [\,])$ and $(A, \langle\,\rangle)$ used here are the ones from Example 6.4, more precisely, we take the cyclic group $(\mathbb{Z}_2, +)$ and use the operations $[abc] = a - b + c \pmod 2$ and $\langle abc \rangle = a - b + c + 1 \pmod 2$. As is clear from Fig. 6, there are no colorings using this pair of compatible knot-theoretic ternary groups. The number of colorings for the unlink (two disjoint unknotted loops) is $2^3 = 8$. Thus, the diagram represents a non-trivial flat virtual link.

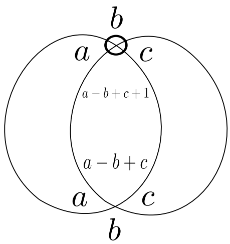

Fig. 6. The flat-virtual Hopf link is distinguished from the unlink by two-element ternary groups

6.2. Knot-theoretic ternary groups and curves immersed in compact surfaces. Let F be a compact surface, possibly with boundary. Let $f\colon \bigsqcup_i S^1_i \sqcup \bigsqcup_k I_k \to F$ be a local embedding of n disjoint circles and

m disjoint arcs into F such that precisely the endpoints of the arcs are sent to the boundary components of F, if there are any (if not, we only take immersions of circles). Assume that $|f^{-1}f(x)| < 3$ for all $x \in \bigsqcup_i S_i^1 \sqcup \bigsqcup_k I_k$, that is, we allow at most double points. For simplicity, we let $|f^{-1}f(x)| = 1$ if x is an arc endpoint. We consider images of such maps (diagrams on F) up to flat Reidemeister moves and isotopy on F. We let the images of the arc endpoints slide on the boundary of F and pass through each other. We call such an equivalence class a *relative flat link* on F if it involves any arcs, or simply a *flat link* on F if there are no arcs.

To distinguish (relative) flat link diagrams on a given surface F, we use the same coloring convention as in the previous subsection (Fig. 3). Now, however, only one knot-theoretic ternary group $(A, [\,])$ is needed, as there are no virtual crossings. We assign elements of $(A, [\,])$ to the regions of the complement of the (relative) flat link diagram in F. Then we have a theorem analogous to Theorem 6.6.

THEOREM 6.8. *Given a compact surface F, a knot-theoretic ternary group $(A, [\,])$ and a (relative) flat link diagram D on F, the number of knot-theoretic ternary group colorings of D is a (relative) flat link invariant.*

Due to the semi-commutative nature of knot-theoretic ternary groups, we do not require that the surface F be orientable. On a non-orientable surface, we cannot distinguish between "clockwise" and "counter-clockwise", but that corresponds exactly to the property $[abc] = [cba]$.

EXAMPLE 6.9. Fig. 7 shows a realization of the flat Kishino knot on a connected sum of a torus and a Klein bottle. There are two connected surface regions in the complement of this curve. Therefore, given a knot-theoretic ternary group $(A, [\,]) = \mathcal{T}((A, +), \bar{e})$, any coloring of this diagram can use at most two colors: a and b. There are four flat crossings, but the relations assigned to them are all the same: $a = b - b + b + \bar{e}$. Thus, the color b

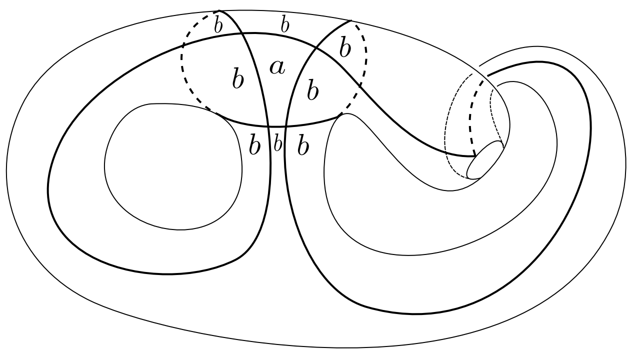

Fig. 7. The Kishino curve on a non-orientable surface

determines the color a, and the number of colorings of the diagram is equal to the order of $(A, [\,])$. This distinguishes the Kishino curve from a trivial (separating) loop on this surface, for which the number of colorings would be $|A|^2$.

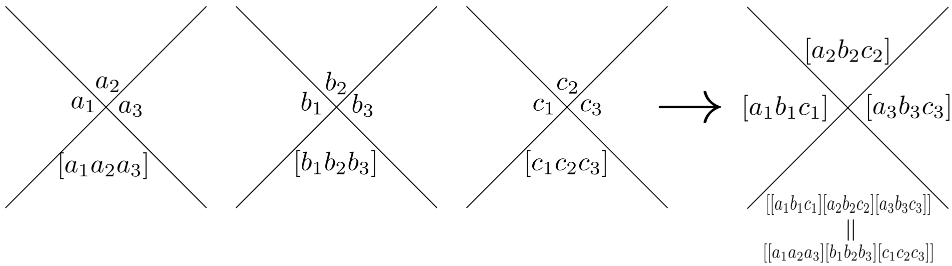

Fig. 8. Entropic property yields a ternary product on the set of colorings

Remark 6.10. The fact that knot-theoretic ternary groups are entropic allows one to introduce a knot-theoretic ternary group structure on the set of colorings of a given diagram D. This is a consequence of the general theory of entropic algebras presented in [16]. The ternary operation on colorings is performed region-wise, as in Fig. 8. The right side of the figure shows the consistency between the ternary product on colorings and the relation assigned to a crossing, thanks to the entropic condition. The isomorphism class of such a knot-theoretic ternary group of colorings is a (relative) flat link invariant.

Example 6.11. In Fig. 9 there are four diagrams of relative flat links, each consisting of an arc and an immersed circle, placed on a Möbius band with a handle (the sides of the figures are identified according to the arrows). Up to isomorphism, there are four knot-theoretic ternary groups of order four, and we will check how efficient they are in distinguishing these links. We will use $\mathcal{T}((\mathbb{Z}_4,+),0)$, $\mathcal{T}((\mathbb{Z}_4,+),2)$, $\mathcal{T}((\mathbb{Z}_2\times\mathbb{Z}_2,+),(0,0))$ and $\mathcal{T}((\mathbb{Z}_2\times\mathbb{Z}_2,+),(1,1))$. In this order of groups, the numbers of colorings of the diagrams are as follows:

$$(A) : [8, 0, 16, 0],$$
$$(B) : [8, 8, 16, 0],$$
$$(C) : [8, 8, 16, 16],$$
$$(D) : [16, 16, 16, 16].$$

Thus, we see that the non-idempotent ternary groups $\mathcal{T}((\mathbb{Z}_4,+),2)$ and $\mathcal{T}((\mathbb{Z}_2\times\mathbb{Z}_2,+),(1,1))$ are sufficient to prove that the four diagrams represent different relative flat links. The figures also show examples of colorings. Fig. 9(A) indicates a coloring with $\mathcal{T}((\mathbb{Z}_2\times\mathbb{Z}_2,+),(0,0))$, and Fig. 9(B) a

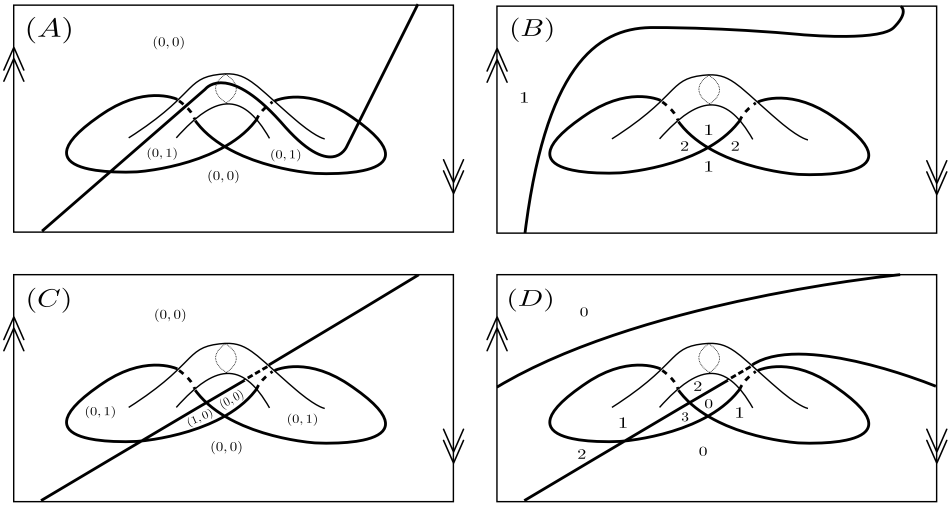

Fig. 9. Four different relative flat links on a Möbius band with a handle

coloring with $\mathcal{T}((\mathbb{Z}_4, +), 2)$. In Fig. 9(C) $\mathcal{T}((\mathbb{Z}_2 \times \mathbb{Z}_2, +), (1, 1))$ is used, and as an example we list the equations involved (in the order: left, middle, and right crossing):

$$(0, 0) = (0, 1) - (1, 0) + (0, 0) + (1, 1),$$
$$(1, 0) = (0, 1) - (0, 0) + (0, 0) + (1, 1),$$
$$(1, 0) = (0, 0) - (0, 1) + (0, 0) + (1, 1).$$

Finally, Fig. 9(D) shows a coloring with $\mathcal{T}((\mathbb{Z}_4, +), 0)$.

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