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Compactness in the space of p-continuous vector-valued functions

by

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Abstract. Let X be a Banach space and let Ω be a compact Hausdorff space. We prove a characterization of compact sets in the spaces $\mathcal{C}_p(\Omega, X)$ of p-continuous functions on Ω, $1 \le p \le \infty$, with $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$. We also establish a necessary condition and a sufficient condition for a set in $\mathcal{C}_p(\Omega, X)$ to be p-compact.

1. Introduction. Let X be a Banach space. Let $1 \le p \le \infty$. In [35], the notion of p-compact set is introduced. A set $K \subset X$ is *relatively p-compact* if there exists $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ if $p = \infty$) such that $K \subset p\text{-co}(x_n) := \{\sum_n a_n x_n : \|(a_n)\|_{p'} \le 1\}$ (here p' denotes the conjugate index of p). Notice that, according to the Grothendieck compactness principle, relatively ∞-compact sets are precisely relatively compact sets. Let Y be a Banach space. A bounded linear operator $T \in \mathcal{L}(Y, X)$ is called *p-compact* if T maps bounded sets to relatively p-compact sets. The set of such operators is denoted by $\mathcal{K}_p(Y, X)$. It is known [35, Theorem 4.2] that $\mathcal{K}_p$, endowed with an appropriate norm, is a Banach operator ideal. A suitable formula for the Banach operator ideal norm in $\mathcal{K}_p(Y, X)$ was given by Delgado, Serrano, and the second author [12] (see [1, Theorem 3.4 and Remark 3.7]) as follows: for every $T \in \mathcal{K}_p(Y, X)$,

$$\|T\|_{\mathcal{K}_p} = \inf \|(x_n)\|_p,$$

where the infimum is taken over all sequences $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$) such that $T(B_Y) \subset p\text{-co}(x_n)$. In the last decade, p-compact sets and p-compact operators have been studied quite intensively (see, e.g., [1, 2, 4–6, 9, 11–13, 18, 22–28, 31, 32, 36]).

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In recent years, several authors studied different types of functions (such as linear operators, polynomials, holomorphic and continuous functions) between Banach spaces in relation with the class of p-compact sets. For instance, in 2011, Aron and Rueda [6] showed that continuous homogeneous polynomials map p-compact sets to p-compact sets. The same result for holomorphic functions was proved by Aron, Çalişkan, García, and Maestre [4] and Lassalle and Turco [23].

Denote by Ω a compact Hausdorff space. The space of continuous functions from Ω into X is denoted by $\mathcal{C}(\Omega, X)$, and by $\mathcal{C}(\Omega)$ when $X = \mathbb{K}$. In [24], Oja and the present authors considered the Banach space of *p-continuous* X-valued functions $\mathcal{C}_p(\Omega, X)$ formed by all $f \in \mathcal{C}(\Omega, X)$ such that $f(\Omega)$ is p-compact. It follows from properties of p-compactness that $\mathcal{C}_p(\Omega, X) \subset \mathcal{C}_q(\Omega, X)$ if $p \leq q$, and $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$. The space $\mathcal{C}_p(\Omega, X)$ becomes a Banach space when endowed with the norm $\|f\|_{\mathcal{C}_p(\Omega,X)} = \inf \|(x_n)\|_p$, where the infimum is taken over all sequences $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$) such that $f(\Omega) \subset \{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_{p'}}\}$, and $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$ as Banach spaces (see [24, Proposition 3.6]). One of the main results of [24] is that $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ as Banach spaces, where d_p denotes the right Chevet–Saphar tensor norm (see [34] or, e.g., [33]). More precisely, every $f \in \mathcal{C}_p(\Omega, X)$ can be represented as

$$f(\omega) = \sum_{n=1}^{\infty} \varphi_n(\omega) x_n, \qquad \omega \in \Omega,$$

where $(\varphi_n) \in \ell_{p'}^w(\mathcal{C}(\Omega))$ and $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$), where the series converges in $\mathcal{C}_p(\Omega, X)$ (see [24, Theorem 3.8]).

This paper aims to study compactness and p-compactness in the Banach space $\mathcal{C}_p(\Omega, X)$. In Section 2, we prove a characterization of compactness in $\mathcal{C}_p(\Omega, X)$ (Theorem 2.2).

Sections 3 and 4 are devoted to the study of p-compactness in $\mathcal{C}_p(\Omega, X)$. In Section 3, we prove a sufficient condition (Proposition 3.2) and a necessary condition (Proposition 3.3) for a subset of $\mathcal{C}_p(\Omega, X)$ to be p-compact. In Section 4, the notion of *collectively p-compact* subset of $\mathcal{C}_p(\Omega, X)$ is introduced (see Remark 4.2 for background). Such sets appear in the study of p-compact sets in $\mathcal{C}_p(\Omega, X)$. We prove a characterization of collectively p-compact subsets of $\mathcal{C}_p(\Omega, X)$ (Proposition 4.3).

Our notation is standard. Let $1 \leq p \leq \infty$, and denote by p' the conjugate index of p (i.e., $1/p + 1/p' = 1$ with the convention $1/\infty = 0$). We consider Banach spaces over the same, either real or complex, field $\mathbb{K}$. The closed unit ball of X is denoted by B_X. The Banach space of all *absolutely p-summable sequences* in X is denoted by $\ell_p(X)$ and its norm by $\|\cdot\|_p$. By $\ell_p^w(X)$ we mean the Banach space of *weakly p-summable sequences* in X

with the norm $\|\cdot\|_p^w$ (see, e.g., [15, pp. 32–33]). Denote by $\ell_p^u(X)$ the Banach space of all *unconditionally p-summable sequences* in X, which is the closed subspace of $\ell_p^w(X)$ formed by the sequences $(x_n) \in \ell_p^w(X)$ satisfying $(x_n) = \lim_{N\to\infty}(x_1, \dots, x_N, 0, 0, \dots)$ in $\ell_p^w(X)$ (see [17] or, e.g., [10, 8.2, 8.3]). We denote by $\mathcal{L}$ the operator ideal of bounded linear operators.

2. Compact sets in $\mathcal{C}_p(\Omega, X)$. In this section, we prove a characterization of compactness in $\mathcal{C}_p(\Omega, X)$ (Theorem 2.2). Firstly, we shall present a lemma that will be used in the proof.

LEMMA 2.1. *Let X be a Banach space. Let $p \geq 1$. Assume that $K \subset \ell_p(X)$ is a compact set. Then there exists $\eta = (\eta_n) \in B_{c_0}$, $\eta \neq 0$, such that $(x_n/\eta_n) \in \ell_p(X)$ for all $(x_n) \in K$.*

Proof. By the characterization of compact sets in $\ell_p(X)$ (see, e.g., [16] for $p = 2$ or [20, Theorem 4] and [14, p. 6, Exercise 6] for $p \geq 1$), we can choose $n_1 \in \mathbb{N}$ such that $\sum_{m=n_1}^{\infty} \|x_m\|^p < 1/1^{3/p}$ for all $(x_m) \in K$. Now, choose $n_2 \in \mathbb{N}$, $n_2 > n_1$, such that $\sum_{m=n_2}^{\infty} \|x_m\|^p < 1/2^{3/p}$ for all $(x_m) \in K$. Inductively, we can choose an increasing sequence (n_k) in $\mathbb{N}$ such that $\sum_{m=n_k}^{\infty} \|x_m\|^p < 1/k^{3/p}$ for all $(x_m) \in K$. We choose $\eta_m = 1$ if $1 \leq m < n_1$, and $\eta_m = (1/k)^{1/p}$ if $n_{k-1} \leq m < n_k$. It is clear that $(\eta_m) \in B_{c_0}$. Then, for all $(x_m) \in K$, we have

$$\sum_{m=1}^{\infty} \left\| \frac{x_m}{\eta_m} \right\|^p = \sum_{m=1}^{n_1-1} \left\| \frac{x_m}{\eta_m} \right\|^p + \sum_{k=1}^{\infty} \sum_{m=n_k}^{n_{k+1}-1} \left\| \frac{x_m}{\eta_m} \right\|^p$$

$$= \sum_{m=1}^{n_1-1} \left\| \frac{x_m}{\eta_m} \right\|^p + \sum_{k=1}^{\infty} k \sum_{m=n_k}^{n_{k+1}-1} \|x_m\|^p \leq \sum_{m=1}^{n_1-1} \left\| \frac{x_m}{\eta_m} \right\|^p + \sum_{k=1}^{\infty} \frac{k}{k^3} < \infty. \blacksquare$$

Let us now present a characterization of compactness in $\mathcal{C}_p(\Omega, X)$. In the proof we shall use Proposition 3.2, which will be proved in the next section.

THEOREM 2.2. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $p \geq 1$. Then $H \subset \mathcal{C}_p(\Omega, X)$ is relatively compact if and only if there exist compact sets $K_1 \subset \ell_{p'}^u(\mathcal{C}(\Omega))$ and $K_2 \subset \ell_p(X)$ such that*

$$H \subset \left\{ \sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in K_1,\, (x_n) \in K_2 \right\}.$$

Proof. For the "if" part, note that by Proposition 3.2 below, the set $\{\sum_n \varphi_n x_n : (\varphi_n) \in K_1,\, (x_n) \in K_2\}$ is compact.

For the "only if" part, let (f_n) be a null sequence in $\mathcal{C}_p(\Omega, X)$ such that $H \subset \overline{\mathrm{co}}(f_n)$ (we denote by $\overline{\mathrm{co}}(A)$ the closed convex hull of A). In [24, Theorem 3.8], it is proved that $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$. By [33, Proposition 6.10], if $f \in \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ and $\varepsilon > 0$, then there exist sequences $(\varphi_m) \in \ell_{p'}^w(\mathcal{C}(\Omega))$ and $(x_m) \in \ell_p(X)$ such that the series $\sum_{m=1}^{\infty} \varphi_m \otimes x_m$ converges to f and $d_p(f) \le \|(\varphi_m)\|_{p'}^w \|(x_m)\|_p \le d_p(f) + \varepsilon$. Therefore, for each $n \in \mathbb{N}$, there exists a representation $f_n = \sum_{m=1}^{\infty} \varphi_m^n x_m^n$ satisfying

$$\|(\varphi_m^n)_m\|_{p'}^w \le 1 \quad \text{and} \quad \|(x_m^n)_m\|_p < d_p(f_n) + 1/n.$$

Then $\delta_n := \|(x_m^n)_m\|_p \to 0$ as $n \to \infty$; we may assume that $\delta_n \neq 0$ for all n. Consider $\phi_m^n := \delta_n^{1/2} \varphi_m^n$ and $z_m^n := \delta_n^{-1/2} x_m^n$. Then

$$\lim_{n\to\infty} \|(\phi_m^n)_m\|_{p'}^w = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(z_m^n)_m\|_p = 0.$$

Take $K_1 := \overline{\text{aco}}(\hat{\phi}_n)$ and $K_2 := \overline{\text{aco}}(\hat{z}_n)$, where $\hat{\phi}_n := (\phi_m^n)_m$ and $\hat{z}_n := (z_m^n)_m$ (we denote by $\overline{\text{aco}}(A)$ the closed absolutely convex hull of A). Obviously,

$$f_n \in \Big\{ \sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in K_1,\ (x_n) \in K_2 \Big\}$$

for all $n \in \mathbb{N}$. Hence

$$\text{co}(f_n) \subset \text{co}\Big\{ \sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in K_1,\ (x_n) \in K_2 \Big\}.$$

The set $\{\sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in K_1,\ (x_n) \in K_2\}$ is balanced, hence so is $\text{co}\{\sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in K_1,\ (x_n) \in K_2\}$. Since every balanced set A satisfies $\lambda A + (1-\lambda)A \subset A + A$ for all $\lambda \in [0,1]$, we obtain

$$\begin{aligned} &\text{co}\Big\{ \sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in K_1,\ (x_n) \in K_2 \Big\} \\ &\quad \subset \Big\{ \sum_{n=1}^{\infty} \psi_n y_n : (\psi_n) \in K_1,\ (y_n) \in K_2 \Big\} + \Big\{ \sum_{n=1}^{\infty} \phi_n z_n : (\phi_n) \in K_1,\ (z_n) \in K_2 \Big\}. \end{aligned}$$

Finally, define $\hat{K}_1 := \{(\psi_1, \phi_1, \psi_2, \phi_2, \ldots) : (\psi_n), (\phi_n) \in K_1\}$ and $\hat{K}_2 := \{(y_1, z_1, y_2, z_2, \ldots) : (y_n), (z_n) \in K_2\}$. As $\hat{K}_1$ and $\hat{K}_2$ are the images of continuous functions on compact sets, they are also compact. Then it is straightforward to check that

$$\begin{aligned} &\Big\{ \sum_{n=1}^{\infty} \psi_n y_n : (\psi_n) \in K_1,\ (y_n) \in K_2 \Big\} + \Big\{ \sum_{n=1}^{\infty} \phi_n z_n : (\phi_n) \in K_1,\ (z_n) \in K_2 \Big\} \\ &\qquad \subset \Big\{ \sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in \hat{K}_1,\ (x_n) \in \hat{K}_2 \Big\}. \end{aligned}$$

Thus, we have proved that

$$H \subset \Big\{ \sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in \hat{K}_1,\ (x_n) \in \hat{K}_2 \Big\},$$

where $\hat{K}_1$ is a compact set in $\ell_p^w(\mathcal{C}(\Omega))$ and $\hat{K}_2$ is a compact set in $\ell_p(X)$.

By Lemma 2.1, let $\eta = (\eta_n) \in B_{c_0}$, $\eta \neq 0$, be such that $(z_n) := (x_n/\eta_n) \in \ell_p(X)$ for all $(x_n) \in \hat{K}_2$. Then

$$\begin{aligned} H &\subset \Big\{ \sum_{n=1}^{\infty} \eta_n \varphi_n z_n : (\varphi_n) \in \hat{K}_1,\ (x_n) \in \hat{K}_2 \Big\} \\ &= \Big\{ \sum_{n=1}^{\infty} \tau_n z_n : (\tau_n) \in \hat{K}_0, (x_n) \in \hat{K}_2 \Big\}, \end{aligned}$$

where $\hat{K}_0 := \{(\eta_n \varphi_n)_n : (\varphi_n)_n \in \hat{K}_1\}$. Using the continuous linear map $V : \ell_{p'}^w(\mathcal{C}(\Omega)) \to \ell_{p'}^u(\mathcal{C}(\Omega))$ defined by $V((\rho_n)_n) = (\eta_n \rho_n)_n$ for all $(\rho_n)_n \in \ell_{p'}(\mathcal{C}(\Omega))$, we obtain $V(\hat{K}_1) = \hat{K}_0$, and thus $\hat{K}_0$ is compact in $\ell_{p'}^u(\mathcal{C}(\Omega))$. ∎

3. p-Compactness in $\mathcal{C}_p(\Omega, X)$. In this section, we study p-compactness in $\mathcal{C}_p(\Omega, X)$, providing a sufficient condition (Proposition 3.2) and a necessary condition (Proposition 3.3) for a subset of $\mathcal{C}_p(\Omega, X)$ to be p-compact.

We shall start with a lemma which summarizes some useful results. The proof is very easy and it is omitted. Let $A \subset \mathcal{C}(\Omega)$ and $B \subset X$. We denote $A \otimes B := \{\varphi x : \varphi \in A,\ x \in B\} \subset \mathcal{C}_p(\Omega, X)$.

LEMMA 3.1. *Let X, Y, and Z be Banach spaces. Let $1 \leq p \leq \infty$.*

(a) *For every $T \in \mathcal{L}(X, Y)$, if $K \subset X$ is a p-compact set, then $T(K)$ is p-compact in Y.*
(b) *Let $A \subset \mathcal{C}(\Omega)$ and $B \subset X$. Then $A \otimes B$ is p-compact in $\mathcal{C}_p(\Omega, X)$ if and only if A and B are p-compact.*
(c) *For every bounded bilinear map $S : X \times Y \to Z$, if $K_1 \subset X$ and $K_2 \subset Y$ are p-compact, then $S(K_1 \times K_2)$ is p-compact in Z.*

The next proposition gives a sufficient condition for a subset of $\mathcal{C}_p(\Omega, X)$ to be p-compact. In other words, it provides a way to build p-compact sets in $\mathcal{C}_p(\Omega, X)$.

PROPOSITION 3.2. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $p \geq 1$. Assume that $H \subset \{\sum_n \varphi_n x_n : (\varphi_n) \in K_1, (x_n) \in K_2\} \subset \mathcal{C}_p(\Omega, X)$ with $K_1 \subset \ell_{p'}^w(\mathcal{C}(\Omega))$ and $K_2 \subset \ell_p(X)$. If K_1 and K_2 are p-compact (compact, respectively), then H is p-compact (compact, respectively) in $\mathcal{C}_p(\Omega, X)$.*

Proof. The bilinear map

$$S : \ell_{p'}^{w}(\mathcal{C}(\Omega)) \times \ell_p(X) \to \mathcal{C}_p(\Omega, X), \qquad ((\varphi_n), (x_n)) \mapsto f = \sum_n \varphi_n x_n,$$

is continuous, since

$$d_p(S((\varphi_n), (x_n))) = d_p(f) \le \|(\varphi_n)\|_{p'}^{w} \|(x_n)\|_p$$

for all $(\varphi_n) \in \ell_{p'}^{w}(\mathcal{C}(\Omega))$ and $(x_n) \in \ell_p(X)$. So, by Lemma 3.1(c), $S(K_1 \times K_2)$ is p-compact whenever K_1 and K_2 are p-compact. Since $H \subset S(K_1 \times K_2)$, H is p-compact.

It is well known that if K_1 and K_2 are compact, then $S(K_1 \times K_2)$ is also compact, therefore $H \subset S(K_1 \times K_2)$ is compact. ∎

The next theorem gives a necessary condition for a subset of $\mathcal{C}_p(\Omega, X)$ to be p-compact. As Proposition 3.8 will show, this condition is not sufficient in general.

Proposition 3.3. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $p \ge 1$. If $H \subset \mathcal{C}_p(\Omega, X)$ is p-compact, then there exist $(x_n) \in \ell_p(X)$ and a compact set $K \subset \ell_{p'}^{u}(\mathcal{C}(\Omega))$ such that*

$$H \subset \Big\{ \sum_{n=1}^{\infty} \psi_n x_n : (\psi_n) \in K \Big\}.$$

Proof. First, there exists $(f_n) \in \ell_p(\mathcal{C}_p(\Omega, X))$ such that $H \subset \{\sum_{n=1}^{\infty} \alpha_n f_n : (\alpha_n) \in M\}$ with some $M \subset B_{\ell_{p'}}$ compact. Indeed, $H \subset p\text{-co}(f_n) = \{\sum_{n=1}^{\infty} \hat\alpha_n f_n : (\hat\alpha_n) \in B_{\ell_{p'}}\}$. By Lemma 2.1, there exists $\delta = (\delta_n) \in B_{c_0}$, $\delta \ne 0$, such that $(g_n) := (f_n/\delta_n) \in \ell_p(\mathcal{C}_p(\Omega, X))$. Then

$$H \subset \Big\{ \sum_{n=1}^{\infty} \hat\alpha_n \delta_n g_n : (\hat\alpha_n) \in B_{\ell_{p'}} \Big\} = \Big\{ \sum_{n=1}^{\infty} \beta_n g_n : (\beta_n) \in M \Big\},$$

where $M := \{(\hat\alpha_n \delta_n)_n : (\hat\alpha_n)_n \in B_{\ell_{p'}}\}$. Obviously, $M \subset B_{\ell_{p'}}$ is compact since

$$\sum_{n \ge n_0} |\beta_n|^{p'} = \sum_{n \ge n_0} |\hat\alpha_n \delta_n|^{p'} \le \Big(\sup_{n \ge n_0} |\delta_n|^{p'} \Big) \sum_{n \ge n_0} |\hat\alpha_n|^{p'} \le \sup_{n \ge n_0} |\delta_n|^{p'} \to 0$$

as $n_0 \to \infty$.

Given $\varepsilon > 0$, for each $n \in \mathbb{N}$ we can choose a representation of f_n of the form $f_n = \sum_{m=1}^{\infty} \psi_m^n x_m^n$ satisfying

$$\|(\psi_m^n)_m\|_{p'}^{w} = 1 \quad \text{and} \quad \|(x_m^n)_m\|_p < d_p(f_n) + \varepsilon/2^n$$

(see the proof of Theorem 2.2). Then, consider the sequence $(x_m^n)_{m,n}$ where the indices (m, n) are considered, for instance, with the square ordering. We have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\|x_m^n\|^p < \sum_{n=1}^{\infty}\Big(d_p(f_n)+\frac{\varepsilon}{2^n}\Big)^p \le \Big(\sum_{n=1}^{\infty} d_p(f_n)^p\Big)^{1/p} + \Big(\sum_{n=1}^{\infty}\frac{\varepsilon^p}{2^{pn}}\Big)^{1/p} < \infty.$$

Therefore, $(x_m^n)_{m,n} \in \ell_p(X)$. On the other hand, for every $f \in H$, there exists $(\alpha_n^f) \in M$ such that

$$f = \sum_{n=1}^{\infty}\alpha_n^f f_n = \sum_{n=1}^{\infty}\alpha_n^f\Big(\sum_{m=1}^{\infty}\psi_m^n x_m^n\Big) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\phi_m^n x_m^n,$$

where $\phi_m^n := \alpha_n^f\psi_m^n$ for all $m, n \in \mathbb{N}$. Thus,

$$H \subset \Big\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\phi_m^n x_m^n : (\phi_m^n) \in K\Big\},$$

where $K := \{(\alpha_n^f\psi_m^n) : f \in H\}$. The set K is inside the image of a continuous function on the compact set M, so K is also compact.

So far we have proved that if $H \subset \mathcal{C}_p(\Omega, X)$ is p-compact, then there exist $(x_n) \in \ell_p(X)$ and a compact set $K \subset \ell_{p'}^w(\mathcal{C}(\Omega))$ such that

$$H \subset \Big\{\sum_{n=1}^{\infty}\psi_n x_n : (\psi_n) \in K\Big\}.$$

By Lemma 2.1, let $\eta = (\eta_n) \in B_{c_0}$, $\eta \neq 0$, be such that $(z_n) := (x_n/\eta_n) \in \ell_p(X)$. Then

$$H \subset \Big\{\sum_{n=1}^{\infty}\eta_n\psi_n z_n : (\psi_n) \in K\Big\} = \Big\{\sum_{n=1}^{\infty}\tau_n z_n : (\tau_n) \in K_0\Big\},$$

where $K_0 := \{(\eta_n\psi_n)_n : (\psi_n)_n \in K\}$. Using the continuous linear map $V : \ell_{p'}^w(\mathcal{C}(\Omega)) \to \ell_{p'}^u(\mathcal{C}(\Omega))$ defined by $V((\rho_n)) = (\eta_n\rho_n)$ for all $(\rho_n)_n \in \ell_{p'}(\mathcal{C}(\Omega))$, we obtain $V(K) = K_0$, and thus K_0 is compact in $\ell_{p'}^u(\mathcal{C}(\Omega))$. ■

Let $(x_n) \in \ell_p(X)$ and consider the linear map $U : \ell_{p'}^u(\mathcal{C}(\Omega)) \to \mathcal{C}_p(\Omega, X)$ defined by

$$U((\psi_n)) = \sum_{n=1}^{\infty}\psi_n x_n \quad \text{for all } (\psi_n) \in \ell_{p'}^u(\mathcal{C}(\Omega)).$$

The map U is continuous. Indeed, for all $(\psi_n) \in \ell_{p'}^u(\mathcal{C}(\Omega))$, we have

$$d_p\Big(\sum_{n=1}^{\infty}\psi_n x_n\Big) \le \|(\psi_n)\|_{p'}^w\,\|(x_n)\|_p,$$

and thus $\|U\| \le \|(x_n)\|_p$. Therefore, for a p-compact subset $K \subset \ell^u_{p'}(\mathcal{C}(\Omega))$, the set

$$U(K) = \Big\{ \sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in K \Big\}$$

is p-compact in $\mathcal{C}_p(\Omega, X)$ (by Lemma 3.1(a)). That can be expressed in the following result.

Proposition 3.4. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. If $(x_n) \in \ell_p(X)$ and $K \subset \ell^u_{p'}(\mathcal{C}(\Omega))$ is a p-compact set, then the set $\{\sum_{n=1}^{\infty} \psi_n x_n : (\psi_n) \in K\}$ is p-compact in $\mathcal{C}_p(\Omega, X)$.*

Since $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$, Propositions 3.3 and 3.4 yield the following characterization of compactness in $\mathcal{C}(\Omega, X)$.

Corollary 3.5. *Let X be a Banach space and let Ω be a compact Hausdorff space. A set $H \subset \mathcal{C}(\Omega, X)$ is relatively compact if and only if there exist $(x_n) \in c_0(X)$ and a compact set $K \subset \ell^u_1(\mathcal{C}(\Omega))$ such that*

$$H \subset \Big\{ \sum_{n=1}^{\infty} \varphi_n x_n : (\varphi_n) \in K \Big\}.$$

We shall show that in general the converse of Proposition 3.3 does not hold. First of all, we recall that, for $1 < p \le \infty$, the Banach spaces $\ell^u_{p'}(\mathcal{C}(\Omega))$ and $\mathcal{C}(\Omega, \ell_{p'})$ are isometric by the isometry $J : \mathcal{C}(\Omega, \ell_{p'}) \to \ell^u_{p'}(\mathcal{C}(\Omega))$ defined by $Jf = (\langle f(\omega), e^*_n\rangle)_n$, $\omega \in \Omega$, (e^*_n) being the vector unit basis of ℓ_p (see, e.g., [10, p. 92, Proposition 8.2]). Indeed, for all $f \in \mathcal{C}(\Omega, \ell_{p'})$, we have

$$\|Jf\|^w_{p'} = \sup_{\omega\in\Omega} \Big(\sum_{n=1}^{\infty} |\langle f(\omega), e^*_n\rangle|^{p'} \Big)^{1/p'} = \sup_{\omega\in\Omega} \|f(\omega)\|_{p'} = \|f\|_\infty.$$

In the proof, we also use the well-known fact (see, e.g., [33, p. 142]) that the dual space operator ideal (we follow the terminology of [29]) of the Chevet–Saphar tensor norm d_p coincides with $\mathcal{P}_{p'}$, i.e., $(Z \hat{\otimes}_{d_p} X)^* = \mathcal{P}_{p'}(Z, X^*)$ as Banach spaces (here Z is an arbitrary Banach space). (Recall that $\mathcal{P}_q = (\mathcal{P}_q, \|\cdot\|_{\mathcal{P}_q})$, $1 \le q \le \infty$, denotes the Banach operator ideal of absolutely q-summing operators.) Since $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ as Banach spaces, we have

$$\mathcal{C}_p(\Omega, X)^* = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$$

as Banach spaces, under the duality

$$\langle \varphi x, T\rangle = \langle x, T\varphi\rangle, \qquad \varphi \in \mathcal{C}(\Omega),\ x \in X,\ T \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*).$$

Proposition 3.6. *Let Ω be an infinite compact Hausdorff space. Let $1 < p \le \infty$. Let $\alpha = (\alpha_n) \in \ell_p$, $\alpha \ne 0$, and define $U \in \mathcal{L}(\ell^u_{p'}(\mathcal{C}(\Omega)), \mathcal{C}(\Omega))$ by*

$U((\psi_n)) = \sum_{n=1}^{\infty} \alpha_n \psi_n$ for $(\psi_n) \in \ell_{p'}^u(\mathcal{C}(\Omega))$. Then the adjoint operator U^* is not absolutely p-summing.

Proof. For contradiction, suppose that $U^* \in \mathcal{P}_p(\mathcal{C}(\Omega)^*, \ell_{p'}^u(\mathcal{C}(\Omega))^*)$. Since $\ell_{p'}^u(\mathcal{C}(\Omega))$ and $\mathcal{C}(\Omega, \ell_{p'})$ are isometric (see above), and

$$\mathcal{C}(\Omega, \ell_{p'})^* = \mathcal{C}_\infty(\Omega, \ell_{p'})^* = \mathcal{P}_1(\mathcal{C}(\Omega), \ell_p)$$

as Banach spaces (the first identity holds because $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$, see the Introduction; for the second, see above), we see that U^* is in $\mathcal{P}_p(\mathcal{C}(\Omega)^*, \mathcal{P}_1(\mathcal{C}(\Omega), \ell_p))$. We claim that $U^*\mu = \mu \otimes \alpha$ for all $\mu \in \mathcal{C}(\Omega)^*$. In fact, for all $\beta = (\beta_n) \in \ell_{p'}$ and $\psi \in \mathcal{C}(\Omega)$, we have

$$\langle U((\beta_n \psi)), \mu\rangle = \langle (\beta_n \psi), U^*\mu\rangle.$$

On the one hand,

$$\Big\langle U((\beta_n\psi)), \mu \Big\rangle = \Big\langle \sum_{n=1}^{\infty} \alpha_n \beta_n \psi, \mu \Big\rangle = \Big(\sum_{n=1}^{\infty} \alpha_n \beta_n \Big) \int_{\Omega} \psi\, d\mu = \langle \alpha, \beta\rangle \int_{\Omega} \psi\, d\mu.$$

On the other hand,

$$\langle (\beta_n \psi), U^*\mu\rangle = \langle (U^*\mu)\psi, (\beta_n)\rangle.$$

This yields $(U^*\mu)\psi = (\int_\Omega \psi\, d\mu)\alpha$ for all $\psi \in \mathcal{C}(\Omega)$. Finally, as $\|U^*\mu\| = \|\mu\|\,\|\alpha\|_p$, the identity map in $\mathcal{C}(\Omega)^*$ is absolutely p-summing: Indeed, since $U^* \in \mathcal{P}_p(\mathcal{C}(\Omega)^*, \mathcal{P}_1(\mathcal{C}(\Omega), \ell_p))$, we have

$$\sum_{n=1}^{\infty} \|(U^*\mu_n)\|_{\mathcal{P}_1}^p = \sum_{n=1}^{\infty} \|\mu_n \otimes \alpha\|_{\mathcal{P}_1}^p = \sum_{n=1}^{\infty} \|\mu_n\|^p \|\alpha\|^p = \|\alpha\|^p \sum_{n=1}^{\infty} \|\mu_n\|^p < \infty$$

for all $(\mu_n) \in \ell_p^w(\mathcal{C}(\Omega)^*)$, where the last series converges because U^* is absolutely p-summing. Thus, every sequence $(\mu_n) \in \ell_p^w(\mathcal{C}(\Omega)^*)$ is in $\ell_p(\mathcal{C}(\Omega)^*)$, and this contradicts the Dvoretzky–Rogers theorem (see, e.g., [10, p. 143, Ex. 11.9]) because Ω is infinite. ■

REMARK 3.7. The chain of identities

$$\mathcal{C}(\Omega, \ell_{p'})^* = \mathcal{C}_\infty(\Omega, \ell_{p'})^* = \mathcal{P}_1(\mathcal{C}(\Omega), \ell_p)$$

can also be justified using the following argument:

$$\mathcal{C}(\Omega, \ell_{p'})^* = (\mathcal{C}(\Omega) \mathbin{\hat{\otimes}_\varepsilon} \ell_{p'})^* = \mathcal{I}(\mathcal{C}(\Omega), \ell_p) = \mathcal{P}_1(\mathcal{C}(\Omega), \ell_p),$$

where the first identity holds by Grothendieck's classical result [19] (see, e.g., [33, pp. 49–50] or [10, p. 48, 4.4(2)]), the second can be found, for instance, in [10, p. 119, Proposition 10.1] (here $\mathcal{I}$ denotes the operator ideal of integral operators), and the last identity can be found, for instance, in [10, p. 132, Corollary 11.3.3(2)].

PROPOSITION 3.8. *Let Ω be an infinite compact Hausdorff space. Let $1 < p \le \infty$. For every Banach space X and every $x = (x_n) \in \ell_p(X)$, $x \neq 0$,* *there exists a compact set $K \subset \ell^u_{p'}(\mathcal{C}(\Omega))$ such that the set $\{\sum_{n=1}^\infty \psi_n x_n : (\psi_n) \in K\}$ is not p-compact in $\mathcal{C}_p(\Omega, X)$.*

Proof. For contradiction, assume that there exist a Banach space X and a sequence $x = (x_n) \in \ell_p(X)$, $x \neq 0$, such that $\{\sum_{n=1}^\infty \psi_n x_n : (\psi_n) \in K\}$ is p-compact in $\mathcal{C}_p(\Omega, X)$ whenever $K \subset \ell^u_{p'}(\mathcal{C}(\Omega))$ is compact. Choose $x^* \in X^*$ such that $\alpha_n := \langle x_n, x^*\rangle \neq 0$ for some $n \in \mathbb{N}$. We claim that the operator $U \in \mathcal{L}(\ell^u_{p'}(\mathcal{C}(\Omega)), \mathcal{C}(\Omega))$ defined by $U((\psi_n)) = \sum_{n=1}^\infty \alpha_n \psi_n$ for $(\psi_n) \in \ell^u_{p'}(\mathcal{C}(\Omega))$ maps compact sets into p-compact sets. To prove it, take a compact set $K \subset \ell^u_{p'}(\mathcal{C}(\Omega))$ and notice that

$$\begin{aligned} U(K) &= \Big\{\sum_{n=1}^\infty \alpha_n \psi_n : (\psi_n) \in K\Big\} = \Big\{\sum_{n=1}^\infty \langle x_n, x^*\rangle \psi_n : (\psi_n) \in K\Big\} \\ &= \{x^* f : f \in H\}, \end{aligned}$$

where $H := \{\sum_{n=1}^\infty \psi_n x_n : (\psi_n) \in K\}$. By hypothesis, H is p-compact, so $\{x^* f : f \in H\} = U(K)$ is also p-compact.

Therefore, U maps compact sets into p-compact sets, and this implies that $U \in \mathcal{P}^{\mathrm{dual}}_p(\ell^u_{p'}(\mathcal{C}(\Omega)), \mathcal{C}(\Omega))$ (see [12, Theorem 3.14] or [31, Theorems 12 and 13]). Thus U^* is absolutely p-summing, contradicting Proposition 3.6. ∎

4. Collectively p-continuous sets of functions. Given $(x_n) \in \ell_p(X)$, we shall study the following subsets of $\mathcal{C}_p(\Omega, X)$:

$$\Big\{\sum_{n=1}^\infty \psi_n x_n : (\psi_n) \in \ell^w_{p'}(\mathcal{C}(\Omega)),\ \|(\psi_n)\|^w_{p'} \le 1\Big\}.$$

These sets have the property that $f(\Omega) \subset p\text{-co}(x_n)$ for all f in such a set. To be more precise consider the following:

Definition 4.1. Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. A subset $H \subset \mathcal{C}_p(\Omega, X)$ is *collectively p-compact* if $\bigcup_{f\in H} f(\Omega)$ is p-compact in X. That is, $H \subset \mathcal{C}_p(\Omega, X)$ is collectively p-compact if and only if there exists a sequence $(x_n) \in \ell_p(X)$ such that $\bigcup_{f\in H} f(\Omega) \subset p\text{-co}(x_n)$.

Remark 4.2. In [3], Anselone and Palmer study the notion of collectively compact subset of $\mathcal{L}(X, Y)$. A subset $H \subset \mathcal{L}(X, Y)$ is *collectively compact* if $\bigcup_{T\in H} T(B_X)$ is relatively compact in Y. Very recently, Çalişkan and Keten [7] extended this notion. Let $p \ge 1$. A subset $H \subset \mathcal{L}(X, Y)$ is *collectively p-compact* if $\bigcup_{T\in H} T(B_X)$ is relatively p-compact in Y (see [7, Definition 3.10]). In [24, Section 3], Oja and the present authors proved that every function $f \in \mathcal{C}_p(\Omega, X)$ can be identified with an operator $U_f \in \mathcal{K}_p(\ell_1(\Omega), X)$ (see [24, Proposition 3.6]). Thus, Definition 4.1 can be seen as a reformulation of [7, Definition 3.10]. In [7, Proposition 3.11], it is proved that every relatively p-compact subset $H \subset \mathcal{K}_p(X,Y)$ is collectively p-compact. We can reformulate this result: every relatively p-compact subset $H \subset \mathcal{C}_p(\Omega, X)$ is collectively p-compact.

The next proposition gives a characterization of collectively p-compact sets of $\mathcal{C}_p(\Omega, X)$.

PROPOSITION 4.3. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $p \geq 1$. Assume that $H \subset \mathcal{C}_p(\Omega, X)$ and $(x_n) \in \ell_p(X)$. The following statements are equivalent:*

(i) $\bigcup_{f\in H} f(\Omega) \subset p\text{-co}(x_n)$.

(ii) $H \subset \overline{\{\sum_n \psi_n x_n : (\psi_n) \in \ell_{p'}^w(\mathcal{C}(\Omega)),\ \|(\psi_n)\|_{p'}^w \leq 1\}}^{\|\cdot\|_\infty}$.

Proof. (i)$\Rightarrow$(ii). Set $A := \{\sum_{n=1}^\infty \psi_n x_n : (\psi_n) \in \ell_{p'}^w(\mathcal{C}(\Omega)),\ \|(\psi_n)\|_{p'}^w \leq 1\}$. For contradiction, suppose that there exists $f \in H$ such that $f \notin \overline{A}^{\|\cdot\|_\infty}$. Since $\overline{A}^{\|\cdot\|_\infty}$ is absolutely convex and closed in $\mathcal{C}(\Omega, X)$, we can strictly separate f and $\overline{A}^{\|\cdot\|_\infty}$. That is, there exist a constant $\alpha > 0$ and $\Phi \in \mathcal{C}(\Omega, X)^*$ such that

$$|\langle f, \Phi\rangle| > \alpha \quad \text{and} \quad |\langle g, \Phi\rangle| < \alpha \ \text{ for all } g \in \overline{A}^{\|\cdot\|_\infty} \tag{4.1}$$

Given $\varepsilon > 0$, using a standard argument, we can choose a covering of Ω by open sets in Ω, $(V_i)_{i=1}^m$, such that $\|f(\omega') - f(\omega'')\| < \varepsilon$ for all $\omega', \omega'' \in V_i$ with $1 \leq i \leq m$. Now, we choose a partition of unity $(\psi_i)_{i=1}^m$ subordinate to the covering $(V_i)_{i=1}^m$. For each $1 \leq i \leq m$, take $\omega_i \in V_i$ and put $g := \sum_{i=1}^m \psi_i f(\omega_i)$. It is clear that $\|f - g\|_\infty < \varepsilon$. We are going to show that $g \in A$. As $f(\Omega) \subset p\text{-co}(x_n)$, for every $1 \leq i \leq m$ there exists $(\alpha_n^i) \in B_{\ell_{p'}}$ such that $f(\omega_i) = \sum_n \alpha_n^i x_n$. Then

$$g(\omega) = \sum_{i=1}^m \psi_i(\omega) f(\omega_i) = \sum_{i=1}^m \psi_i(\omega)\Big(\sum_{n=1}^\infty \alpha_n^i x_n\Big) = \sum_{n=1}^\infty \Big(\sum_{i=1}^m \psi_i(\omega)\alpha_n^i\Big) x_n.$$

Let $\phi_n := \sum_{i=1}^m \alpha_n^i \psi_i$. It suffices to prove $\|(\phi_n)\|_{p'}^w \leq 1$. For every $\omega \in \Omega$,

$$\begin{aligned}\Big(\sum_{n=1}^\infty |\phi_n(\omega)|^{p'}\Big)^{1/p'} &= \Big(\sum_{n=1}^\infty \Big|\sum_{i=1}^m \alpha_n^i \psi_i(\omega)\Big|^{p'}\Big)^{1/p'} \leq \sum_{i=1}^m \Big(\sum_{n=1}^\infty |\alpha_n^i \psi_i(\omega)|^{p'}\Big)^{1/p'} \\ &= \sum_{i=1}^m \psi_i(\omega)\Big(\sum_{n=1}^\infty |\alpha_n^i|^{p'}\Big)^{1/p'} = \sum_{i=1}^m \psi_i(\omega) = 1,\end{aligned}$$

and this yields $g \in A$. Finally, using (4.1), we have

$$|\langle f, \Phi\rangle| \leq |\langle f - g, \Phi\rangle| + |\langle g, \Phi\rangle| \leq \|f - g\|_\infty \|\Phi\| + \alpha \leq \varepsilon\|\Phi\| + \alpha.$$

As $|\langle f, \Phi\rangle| \leq \varepsilon\|\Phi\| + \alpha$ for every $\varepsilon > 0$, we see that $|\langle f, \Phi\rangle| \leq \alpha$, and this contradicts (4.1).

(ii)⇒(i). Let $f \in H$. Then $f \in \overline{A}^{\|\cdot\|_\infty}$, so there exists $(f_n) \in A$ such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. For all $n \in \mathbb{N}$, $f_n(\omega) = \sum_{k=1}^{\infty} \varphi_k^n x_k$ with $\|(\varphi_k^n)_k\|_{p'}^w \le 1$. Therefore, $f_n(\Omega) \subset p\text{-co}(x_n)$ and, for all $\omega \in \Omega$, $f(\omega) = \lim_{n\to\infty} f_n(\omega) \in p\text{-co}(x_n)$, because $p\text{-co}(x_n)$ is closed. Thus $f(\Omega) \subset p\text{-co}(x_n)$. This yields $\bigcup_{f\in H} f(\Omega) \subset p\text{-co}(x_n)$. ∎

The next proposition shows how one can build collectively p-compact sets in $\mathcal{C}_p(\Omega, X)$.

Proposition 4.4. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $p \ge 1$. Assume that $A \subset \mathcal{C}(\Omega)$ and $B \subset X$. The following statements are equivalent:*

(i) *$A \otimes B$ is collectively p-compact in $\mathcal{C}_p(\Omega, X)$.*
(ii) *A is bounded in $\mathcal{C}(\Omega)$ and B is p-compact in X.*

Proof. (i)⇒(ii). If $A \otimes B$ is collectively p-compact, then there exists $(x_n) \in \ell_p(X)$ such that $\bigcup_{\varphi\in A,\, x\in B}\{\varphi(\omega)x : \omega \in \Omega\} \subset p\text{-co}(x_n)$.

On the one hand, for fixed $x_0 \in X$, we have

$$\bigcup_{\varphi\in A}\{\varphi(\omega)x_0 : \omega \in \Omega\} \subset \bigcup_{\varphi\in A,\, x\in B}\{\varphi(\omega)x : \omega \in \Omega\} \subset p\text{-co}(x_n),$$

and since $p\text{-co}(x_n)$ is compact (in particular), this implies that $\bigcup_{\varphi\in A}\{\varphi(\omega)x_0 : \omega \in \Omega\}$ is bounded, or equivalently $\{\varphi(\omega) : \omega \in \Omega, \varphi \in A\}$ is bounded.

On the other hand, for fixed $\varphi_0 \in A$ such that there exists $\omega_0 \in \Omega$ with $\varphi_0(\omega_0) \neq 0$, we have

$$\bigcup_{x\in B}\{\varphi_0(\omega_0)x\} \subset \bigcup_{\varphi\in A,\, x\in B}\{\varphi(\omega)x : \omega \in \Omega\} \subset p\text{-co}(x_n),$$

and since $p\text{-co}(x_n)$ is p-compact, so is $\bigcup_{x\in B}\{\varphi_0(\omega_0)x\}$, or equivalently B is p-compact.

(ii)⇒(i). Since B is p-compact in X, there exists $(x_n) \in \ell_p(X)$ such that $B \subset p\text{-co}(x_n)$. Then there exists $(\alpha_n) \in B_{\ell_{p'}}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$ for all $x \in B$. Since $A \subset \mathcal{C}(\Omega)$ is bounded, there exists $\lambda > 0$ such that $A \subset \lambda B_{\mathcal{C}(\Omega)}$. Therefore, for all $\varphi \in A$ and all $\omega \in \Omega$,

$$\varphi(\omega)x = \sum_{n=1}^{\infty} \alpha_n \varphi(\omega) x_n,$$

where $(\alpha_n\varphi(\omega))_n \in \lambda B_{\ell_{p'}}$. Thus, for all $\varphi \in A$ and all $x \in B$, $\varphi(\omega)x \subset p\text{-co}(\lambda x_n)$, and thus $A \otimes B$ is collectively p-compact. ∎

The next proposition characterizes when some subsets of $\mathcal{C}_p(\Omega, X)$ are conditionally weakly compact. Recall that a subset A of a Banach space X is *conditionally weakly compact* whenever each sequence in A contains a weak-Cauchy subsequence.

Proposition 4.5. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $p \geq 1$. The following statements are equivalent:*

(i) *The set $H = \{\sum_{n=1}^{\infty} \psi_n x_n : (\psi_n) \in \ell_{p'}^w(\mathcal{C}(\Omega)), \|(\psi_n)\|_{p'}^w \leq 1\}$ is conditionally weakly compact in $\mathcal{C}_p(\Omega, X)$ for all $(x_n) \in \ell_p(X)$.*

(ii) *$B_{\mathcal{C}(\Omega)}$ is conditionally weakly compact.*

Proof. (i)⇒(ii). Let $(x_n) \in \ell_p(X)$ be a non-null sequence and suppose that H as in (i) is conditionally weakly compact. Let $n_0 \in \mathbb{N}$ be such that $x_{n_0} \neq 0$ and consider the set $B_{\mathcal{C}(\Omega)} x_{n_0} = \{\psi x_{n_0} : \psi \in B_{\mathcal{C}(\Omega)}\} \subset H$. As H is conditionally weakly compact, so is $B_{\mathcal{C}(\Omega)} x_{n_0}$. Since $B_{\mathcal{C}(\Omega)}$ is isometrically isomorphic to $B_{\mathcal{C}(\Omega)} x_{n_0}$, it follows that $B_{\mathcal{C}(\Omega)}$ is conditionally weakly compact.

(ii)⇒(i). We have to prove that every sequence in H has a weak-Cauchy subsequence. So, take a sequence (f_n) in H. For each $n \in \mathbb{N}$, we have $f_n = \sum_{m=1}^{\infty} \psi_m^n x_m$ with $\|(\psi_m^n)_m\|_{p'}^w \leq 1$. The sequence $(\psi_1^n)_n$ is bounded in $\mathcal{C}(\Omega)$, and therefore there exists a weak-Cauchy subsequence $(\psi_1^{k_1(n)})_n$. Now, we consider the bounded sequence $(\psi_2^{k_1(n)})_n$. Again it has a weak-Cauchy subsequence $(\psi_2^{k_2(n)})_n$. As $(\psi_1^{k_2(n)})_n$ is a subsequence of $(\psi_1^{k_1(n)})_n$, the two sequences $(\psi_1^{k_2(n)})_n$ and $(\psi_2^{k_2(n)})_n$ are weak-Cauchy. Inductively, we can produce a subsequence $(\psi_{h+1}^{k_{h+1}(n)})_n$ of $(\psi_{h+1}^{k_h(n)})_n$ such that the sequences $(\psi_r^{k_{h+1}(n)})_n$ are weak-Cauchy for all $r \leq h+1$. Consider the subsequence $(f_{k_n(n)})_n$ of $(f_n)_n$. For each $n \in \mathbb{N}$, we have $f_{k_n(n)} = \sum_m \psi_m^{k_n(n)} x_m$ and the sequences $(\psi_m^{k_n(n)})_n$ are weak-Cauchy for all $m \in \mathbb{N}$. For simplicity, we denote $f_{k_n(n)}$ by g_n and assume that $g_n = \sum_m \psi_m^n x_m$, where $(\psi_m^n)_n$ is weak-Cauchy for all $m \in \mathbb{N}$. To prove that (g_n) is weak-Cauchy, take $S \in \mathcal{C}_p(\Omega, X)^* = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$. Given $\varepsilon > 0$, we choose $m_0 \in \mathbb{N}$ so that

$$\Big(\sum_{m>m_0} \|x_m\|^p\Big)^{1/p} < \frac{\varepsilon}{4\|S\|_{\mathcal{P}_{p'}}}.$$

Then, for all $h, k \in \mathbb{N}$, we have

$$\begin{aligned}
&|\langle g_h - g_k, S\rangle| \leq \sum_{m=1}^{m_0} |\langle S(\psi_m^h - \psi_m^k), x_m\rangle| + \sum_{m>m_0} |\langle S(\psi_m^h - \psi_m^k), x_m\rangle| \\
&\leq \sum_{m=1}^{m_0} |\langle \psi_m^h - \psi_m^k, S^*(x_m)\rangle| + \Big(\sum_{m>m_0} \|S(\psi_m^h - \psi_m^k)\|^{p'}\Big)^{1/p'} \Big(\sum_{m>m_0} \|x_m\|^p\Big)^{1/p} \\
&\leq \sum_{m=1}^{m_0} |\langle \psi_m^h - \psi_m^k, S^*(x_m)\rangle| + \|S\|_{\mathcal{P}_{p'}} \|(\psi_m^h - \psi_m^k)_m\|_{p'}^w \Big(\sum_{m>m_0} \|x_m\|^p\Big)^{1/p} \\
&\leq \sum_{m=1}^{m_0} |\langle \psi_m^h - \psi_m^k, S^*(x_m)\rangle| + \varepsilon/2,
\end{aligned}$$

since $\|(\psi_m^h - \psi_m^k)_m\|_{p'}^w \leq 2$ (here, we have considered X embedded in X^{**}). Finally, there exists $k_0 \in \mathbb{N}$ such that

$$|\langle \psi_m^h - \psi_m^k, S^*(x_m)\rangle| < \frac{\varepsilon}{2m_0}$$

for all $1 \leq m \leq m_0$ and all $h, k > k_0$. ∎

Remark 4.6. Recall that if $\mathcal{C}(\Omega)$ does not contain any subspace isomorphic to ℓ_1 (which is equivalent to Ω being dispersed (see [30, Section 2, Main Theorem], or, e.g., [8, Theorem 3.1.1] or [21, p. 114])), then $B_{\mathcal{C}(\Omega)}$ is conditionally weakly compact (by the Rosenthal ℓ_1 Theorem).

The next proposition connects the compactness in $\mathcal{C}_p(\Omega, X)$ with the notion of collective p-compactness.

Proposition 4.7. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $p \geq 1$. There exist compact sets in $\mathcal{C}_p(\Omega, X)$ that are not collectively p-compact.*

Proof. Let $A \subset \mathcal{C}(\Omega)$ be relatively compact and let $B \subset X$ be relatively compact but not p-compact. Then, by Lemma 3.1(b), $A \otimes B$ is relatively compact. However, by Proposition 4.4, $A \otimes B$ is not collectively p-compact.

Let $A \otimes B$ (see the beginning of Section 3), where $A \subset \mathcal{C}(\Omega)$ and $B \subset X$ are relatively compact but B is not p-compact. By Lemma 3.1(b), H is relatively compact. However, by Proposition 4.4, $A \otimes B$ is not collectively p-compact. ∎

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